The minimalist's approach (Hébert 2007)

(*) Have: $\{\eta_X : X \to TX\}_X$ with $\{\eta_X\}_X \triangleright \{TX\}_X$

Put $\mathcal{H} := \{h : A \to B \mid \exists g : g \cdot h = \eta_A\}$,

$\mathcal{I} := \{ I \mid \exists r : r \cdot \eta_I = 1 \}$

Then:

$$\mathcal{H} = {}^{\triangleright}\mathcal{I} \text{ and } \mathcal{I} = \mathcal{H}^{\triangleright}$$

How to get (*) ?

The ideal case : $\exists$ injective cogenerator

$\mathcal{G} \subseteq \mathrm{ob}\,\mathcal{C}$ set with:

- $\mathcal{G}$ is coseparating

$$\forall \underset{f \neq g}{f, g : X \to Y} \; \exists y : Y \to C \in \mathcal{G} : yf \neq yg$$

Better: $\begin{aligned} y : \mathcal{C}^{op} &\to \underline{Set}^{\mathcal{G}} \\ A &\mapsto \mathcal{C}(A,-) \end{aligned}$ faithful

- $\forall C \in \mathcal{G} : C$ is injective ($\mathcal{H} = \mathrm{Mono}$)

Then:

$$A \xrightarrow{\eta_A} \prod_{C \in \mathcal{G}} C^{\mathcal{C}(A,C)} =: TA$$

satisfies (*) (and is functorial/natural).

Relativized versions: $\mathcal{H}$-cogenerator

$\forall A: \mathcal{C}(A, \mathcal{G})$ jointly in $\mathcal{H}$

- $\mathcal{H} = \text{Mono}$: $y: \mathcal{C}^{op} \to \underline{\mathrm{Set}}^{\mathcal{G}}$ faithful : cogenerator
- $\mathcal{H} = \text{ExtMono}$: y ff & refl. isos: strong cogenerator
- $\mathcal{H} = \text{StrictMono}$: y fully faithful : codense cogenerator

Examples:

$\mathbb{Q}/\mathbb{Z}$ in $\underline{\mathrm{AbGrp}}$, $\hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ in R-$\underline{\mathrm{Mod}}$

$\mathbb{R}, \mathbb{C}$ in $\underline{\mathrm{Ban}}_1$ (Hahn-Banach)

$[0,1]$ in $\underline{\mathrm{Tych}}$, $\underline{\mathrm{CompHaus}}$ (Tietze-Urysohn)

$[0,1]^2$ in $\underline{\mathrm{CompHaus}}$

Nullstellensatz

NSS 1 (Hilbert 1893) $f_i \in k[\overbrace{x_1, \dots, x_n}^{x}]$ $(i = 1, \dots, r)$

$k \leq F$ algebraically closed. Then:

$\exists a \in F^n \ \forall i : f_i(a) = 0 \iff \nexists g_i \in k[x] : \sum_{i=1}^{r} g_i f_i = 1$

$\Uparrow$ Take $P = (f_1, \dots, f_r)$

NSS 2 $P \trianglelefteq k[x]$, $P \neq k[x]$, $k \leq F$ alg. cl. $\Rightarrow \sqrt{P} = \mathcal{J}(S(P))$

$\sqrt{P} := \{ f \mid \exists m \geq 1 : f^m \in P \}$; $\quad P \longmapsto S(P) := \{ a \in F^n \mid \forall f \in P : f(a) = 0 \}$

$\mathcal{J}(X) = \{ f \in k[x] \mid \forall a \in X : f(a) = 0 \} \longleftarrow X \subseteq F^n$

<u>Note</u>: Hilbert's Basis Theorem: $k[x]$ is Noetherian

$\Uparrow$ Take $A = k[x]/\sqrt{P}$

NSS 3 A finitely generated k-algebra with 1, no nilpotent elts, $0 \neq u \in A$, $k \leq F$ alg. cl.
Then: $\exists\ \varphi : A \to F$ k-homom., $\varphi(u) \neq 0$