

- \uparrow 1. K fin gen $\Rightarrow K/k$ alg
 \uparrow 2. algebraically closed $=$ inj wrt. $\{ \text{alg. ext's} \}$

NSS4 A finitely gen. k -alg. with 1, no nilp. elts
 $0 \neq u \in A \Rightarrow \exists K \supseteq k, K$ fin. gen. (as a k -alg.),
 $\exists \chi: A \rightarrow K, \chi(u) \neq 0$

\uparrow routine algebra, maximal ideals

NSS5 A comm. ring, no nilp. elts
 $0 \neq u \in A \Rightarrow \exists Q \subseteq A$ prime, $u \notin Q$
 (easy proof)

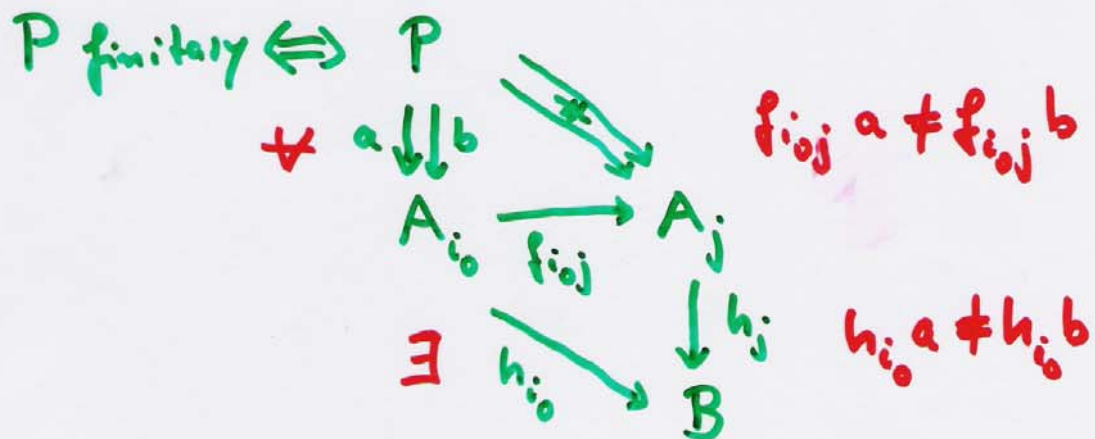
Cor. R comm. ring; $P \subseteq I$ radical ideal iff
 P is intersection of prime ideals.

= Infinitary version of Noether Decomp. Theorem:
 R comm, Noetherian, $P \subseteq I \Rightarrow$
 P is finite intersection of irreducible ideals.

1921

Hilbert-Noether-Birkhoff categories:

- \mathcal{C} HNB \Leftrightarrow
1. \mathcal{C} has (strong epi, mono)-facts
 2. \mathcal{C} is weakly cocomplete
 3. \mathcal{C} has a generator consisting of finitary objects



Note: 1) \mathcal{C} with dir. colims, P finitely presentable $\Rightarrow P$ finitary

2) $U: \mathcal{C} \rightarrow \underline{\text{Set}}^I$ faithful, right-adj., pres. \varinjlim $\Rightarrow \mathcal{C}$ satisfies 3.

3) \mathcal{C} HNB, $\left(\begin{array}{l} \Leftrightarrow \forall A \in \mathcal{C} \\ \text{finite } x\text{'s} \end{array} \right) \Rightarrow \mathcal{C}/A \text{ HNB}$

Birkhoff's Subdirect Representation Theorem

A sdi $\Leftrightarrow \forall (f_i: A \rightarrow S_i)_i$ monic $\exists i_0: f_{i_0}$ monic

$(f_i: A \rightarrow S_i)_i$ representation of A (by S_i 's) \Leftrightarrow
 $(f_i)_i$ monic, $\forall i$ f_i strong epi

$(f_i)_i$ rep. of A is trivial $\Leftrightarrow \exists i_0: f_{i_0}$ iso

Birkhoff 1944, T1979/81 \in HNB, Acob^o
 $\Rightarrow A$ has representation by sdi objects.

Cor. A sdi \Leftrightarrow every rep. of A is trivial
 $\Leftrightarrow \exists P \xrightarrow{a} A \quad \forall f: A \rightarrow B$
 $(fa \neq fb \Rightarrow f \text{ monic})$

Residually small HNB categories

\mathcal{C} residually small $\Leftrightarrow \{S \mid S \text{ sdi}\}$ essentially small

Cor. 1) \mathcal{C} res. small HNB $\Rightarrow \mathcal{C}$ has cogen. consisting of sdi objects

2) \mathcal{C} wellpvd HNB $\Rightarrow \mathcal{C}$ res. small with a cogenerator

Cor. \mathcal{C} residually small, HNB. Equivalent:

(i) \mathcal{C} small-complete, large intersections of monos.

(ii) \mathcal{C} total ($y: \mathcal{C} \rightarrow \underline{\text{Set}}^{\mathcal{C}^{\text{op}}}$ has left adj.).

(iii) \mathcal{C} small-cocomplete, large cointers. of epis.

(iv) \mathcal{C} cototal.

Apply to: \mathcal{C}^{op} , \mathcal{C}/A ; subd. rep. rank.

Questions: How to get sdi's to be injective?