$\Uparrow$ 1. K fin gen $\Rightarrow$ K/k alg
2. algebraically closed = inj wrt. {alg. ext's}

NSS4 A finitely gen. k-alg. with 1, no nilp. elts
$0 \neq u \in A \Rightarrow \exists K \geq k$, K fin. gen. (as a k-alg.),
$\exists \chi : A \to K$, $\chi(u) \neq 0$

$\Uparrow$ routine algebra, maximal ideals

NSS5 A comm. ring, no nilp. elts
$0 \neq u \in A \Rightarrow \exists Q \trianglelefteq A$ prime, $u \notin Q$
(easy proof)

Cor. R comm. ring; $P \trianglelefteq I$ radical ideal iff
P is intersection of prime ideals.

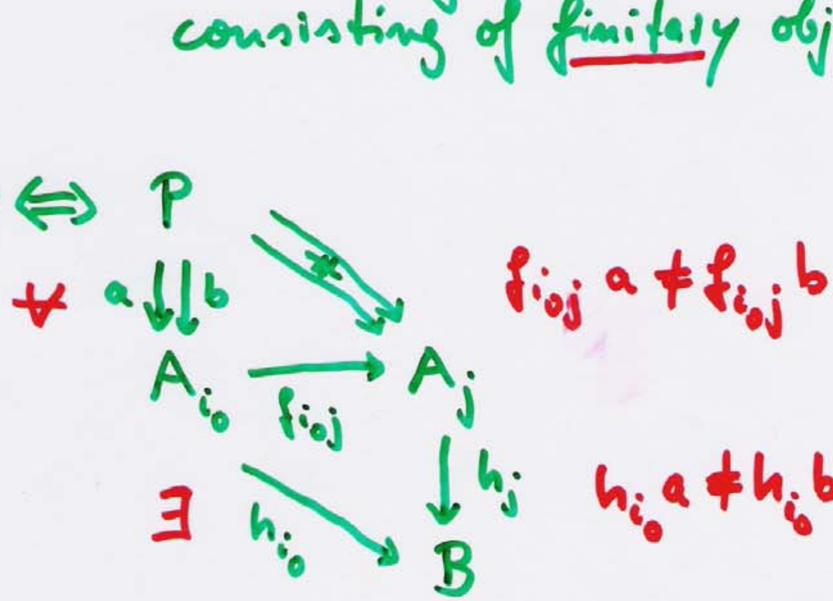
= Infinitary version of Noether Decomp. Theorem:
R comm, Noetherian, $P \trianglelefteq I \Rightarrow$
P is finite intersection of irreducible ideals.

1921

Hilbert-Noether-Birkhoff categories:

$\mathcal{C}$ HNB $\Leftrightarrow$ 1. $\mathcal{C}$ has (strong epi, mono)-facts
2. $\mathcal{C}$ is weakly cowellpowered
3. $\mathcal{C}$ has a generator consisting of finitary objects

P finitary $\Leftrightarrow$



Note: 1) $\mathcal{C}$ with dir. colims, P finitely presentable $\Rightarrow$ P finitary

2) $U : \mathcal{C} \to \underline{Set}^I$ faithful, right-adj., pres. $\varinjlim$ $\Rightarrow$ $\mathcal{C}$ satisfies 3.

3) $\mathcal{C}$ HNB, finite $\times$'s $\Rightarrow$ $\forall A$ $\mathcal{C}/A$ HNB

Birkhoff's Subdirect Representation Thm

A <u>sdi</u> $\Leftrightarrow \forall (f_i : A \to S_i)_i$ monic $\exists i_0 : f_{i_0}$ monic

$(f_i : A \to S_i)_i$ representation of A (by S_i's) $\Leftrightarrow$
$(f_i)_i$ monic, $\forall i\ f_i$ strong epi

$(f_i)_i$ rep. of A is <u>trivial</u> $\Leftrightarrow \exists i_0 : f_{i_0}$ iso

Birkhoff 1944, T1979/81 $\mathcal{C}$ HNB, $A \in \mathrm{ob}\,\mathcal{C}$
$\Rightarrow$ A has representation by sdi objects.

Cor. A sdi $\Leftrightarrow$ every rep. of A is trivial
$\Leftrightarrow \exists P \overset{a}{\underset{b}{\rightrightarrows}} A \quad \forall f : A \to B$
$(fa \neq fb \Rightarrow f \text{ monic})$

Residually small HNB categories

$\mathcal{C}$ residually small $\Leftrightarrow$ $\{S \mid S \text{ sdi}\}$ ess'ly small

Cor. 1) $\mathcal{C}$ res. small HNB $\Rightarrow$ $\mathcal{C}$ has cogen. consisting of sdi objects

2) $\mathcal{C}$ wellpwd HNB ($\Rightarrow$ $\mathcal{C}$ res. small with a cogenerator)

Cor. $\mathcal{C}$ residually small, HNB. Equivalent:

(i) $\mathcal{C}$ small-complete, large intersections of monos.

(ii) $\mathcal{C}$ total ($y : \mathcal{C} \to \underline{\mathrm{Set}}^{\mathcal{C}^{op}}$ has left adj.).

(iii) $\mathcal{C}$ small-cocomplete, large cointers. of epis.

(iv) $\mathcal{C}$ cototal.

Apply to: $\mathcal{C}^{op}$, $\mathcal{C}/A$; subd. rep. rank.

Questions: How to get sdi's to be injective?