CHARACTERIZATION OF TORSION THEORIES IN GENERAL CATEGORIES

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Dedicated to Ross Street on the occasion of his sixtieth birthday

Abstract. In a pointed category with kernels and cokernels, we characterize torsion–free classes in terms of their closure under extensions. They are also described as indexed reflections. We obtain a corresponding characterization of torsion classes by formal categorical dualization.

1. Introduction

Torsion theories and equivalent concepts serve as a standard tool in module theory and in abelian category theory. They already appear, under varying terminology, explicitly or implicitly, in [B], [G], [D], and in many other papers of the 1960s and 1970s. The Lecture Notes [L] by Lambek give a comprehensive account of these developments and present an embedding theorem for abelian categories as their main application.

Recently Bourn and Gran [BG] developed a basic theory of torsion theories in so-called semi–abelian categories [JMT], which in part works even in so-called homological categories [BB]. These include categories like groups, (non–unital) rings, (various types of) algebras, and even topological algebras. In the more general context of so–called (E, M)–normal categories, which includes categories quite far removed from the realm of algebra, such as the category of pointed topological spaces, the paper [CDT] characterizes torsion and torsion–free classes in terms of their closure under extensions. Unfortunately, when the conditions on the ambient category (like being semi–abelian, homological, or (E, M)–normal) are not self–dual, the self– duality of these characterizations becomes unduly hidden. Hence, in this note, under minimal and self–dual hypotheses on the ambient category, we give necessary and sufficient conditions for a full subcategory to appear as the torsion–free class of a torsion theory. Formal categorical dualization gives the corresponding characterization for torsion classes. We provide two examples showing that, even in the realm of semi–abelian or of additive homological categories, closure under extensions is not sufficient for a normal–epireflective subcategory to be torsion–free.

2. Definitions and Theorems

Let $\mathcal{C}$ be a category with kernels and cokernels. A morphism in $\mathcal{C}$ is a *normal monomorphism* (*normal epimorphism*) if it is the kernel (cokernel, respectively) of

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some morphism. A *short exact sequence* in $\mathcal{C}$ is given by a pair (k, q) of composable morphisms

$$0 \longrightarrow B \overset{k}{\rightarrowtail} A \overset{q}{\twoheadrightarrow} C \longrightarrow 0$$

such that k is the kernel of q, and q is the cokernel of k. A *torsion theory* of $\mathcal{C}$ is given by a pair $(\mathcal{T}, \mathcal{F})$ of full, replete (= isomorphism–closed) subcategories of $\mathcal{C}$ such that

(1) every morphism $B \to C$ with $B \in \mathcal{T}$, $C \in \mathcal{F}$ is zero;
(2) for every object A in $\mathcal{C}$ there exists a short exact sequence

$$0 \longrightarrow B \rightarrowtail A \twoheadrightarrow C \longrightarrow 0$$

with $B \in \mathcal{T}$, $C \in \mathcal{F}$.

$\mathcal{T}$ is the torsion part of the theory, $\mathcal{F}$ is the torsion–free part. Any full, replete subcategory is *torsion* (*torsion–free*) if it is the torsion (torsion-free) part of a torsion theory.

In order to characterize them, we need some more terminology. A full replete subcategory $\mathcal{B}$ of $\mathcal{C}$ is *normal–epireflective*, if the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{C}$ has a left adjoint, such that all reflections (= adjunction units) $\rho_A : A \to RA$ ($A \in \operatorname{ob}\mathcal{C}$) are normal epimorphisms. With the notation $\kappa_A = \ker \rho_A : rA \to A$, one says that *the induced radical of $\mathcal{B}$ is idempotent* if $\kappa_{rA} : rrA \to rA$ is an isomorphism, for every object A of $\mathcal{C}$. The *$\mathcal{B}$–reflections are pullback stable* if, in any pullback diagram

(1)
$$\begin{array}{ccc} P & \xrightarrow{g} & A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \rho_A} \\ B & \xrightarrow{f} & RA \end{array}$$

with $B \in \mathcal{B}$ and a normal epimorphism p, that morphism p actually serves as a $\mathcal{B}$–reflection for P. The following Proposition is essentially known (see, for example, [BG], [CDT]), except that here we have dropped all restrictive assumptions on the category $\mathcal{C}$:

Proposition. *For a full replete subcategory $\mathcal{F}$ of $\mathcal{C}$, the following conditions are equivalent:*

(i) *$\mathcal{F}$ is torsion–free;*
(ii) *$\mathcal{F}$ is normal–epireflective, and the induced radical of $\mathcal{F}$ is idempotent;*
(iii) *$\mathcal{F}$ is normal–epireflective, and the $\mathcal{F}$–reflections are pullback–stable.*

The equivalence (i) $\Longleftrightarrow$ (iii) of the Proposition shows that, *if normal epimorphisms are stable under pullback in $\mathcal{C}$, the reflector of a torsion–free subcategory is semi–left exact in the sense of* [CHK] (admissible in the sense of [J1], see also Prop. 5.5.2 of [BJ]), *and conversely: any semi–left exact normal–epireflective subcategory is torsion–free.* In the presence of pullbacks in $\mathcal{C}$, and if normal epimorphisms are stable under pullbacks, condition (iii) is equivalently described by the requirement that the counits of all induced adjunctions

$$\mathcal{B}/RA \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} \mathcal{C}/A \qquad (A \in \operatorname{ob}\, \mathcal{C})$$

of the normal–epireflective subcategory $\mathcal{B}$ are isomorphisms. Hence, *torsion–free classes are, in fact, characterized as indexed reflections by the Proposition.*

The main point of this note is to refine the idempotency condition (ii) of the Proposition. To this end, let us point out that, under our assumptions on $\mathcal{C}$, the pushout k_A of κ_A along the normal epimorphism ρ_{rA} exists in $\mathcal{C}$: just form $q_A = \mathrm{coker}(\kappa_A \kappa_{rA})$ and let k_A be the induced morphism to obtain the commutative diagram

$$\begin{array}{ccccc} rrA & \xrightarrow{\kappa_{rA}} & rA & \xrightarrow{\kappa_A} & A \\ \downarrow & \boxed{1} & \downarrow \rho_{rA} & \boxed{2} & \downarrow q_A \\ 0 & \longrightarrow & RrA & \xrightarrow[k_A]{} & QA \end{array} \qquad (2)$$

Since $\boxed{1}$ and $\boxed{1}\boxed{2}$ are pushout diagrams, $\boxed{2}$ is also one. We say that *the reflections of* the normal–epireflective subcategory $\mathcal{B}$ *have stable kernels* if each morphism k_A is a normal monomorphism. Dual notions: *normal–monocoreflective, coreflections of* $\mathcal{B}$ *have stable cokernels.*

A full replete subcategory $\mathcal{B}$ is *closed under extensions* if, for every short exact sequence

$$0 \longrightarrow B \rhd\!\!\longrightarrow A \longrightarrow\!\!\rhd C \longrightarrow 0$$

with B, C in $\mathcal{B}$, the object A is also in $\mathcal{B}$.

Theorem. *A full replete subcategory $\mathcal{F}$ of $\mathcal{C}$ is torsion–free if, and only if,*

(1) *$\mathcal{F}$ is normal–epireflective,*
(2) *the $\mathcal{F}$–reflections have stable kernels,*
(3) *$\mathcal{F}$ is closed under extensions.*

Condition (2) of the Theorem may be eased if normal monomorphisms in $\mathcal{C}$ satisfy certain stability conditions under pushouts. We say that *normal monomorphisms are stable under normal quotients in* $\mathcal{C}$ if, for every pushout diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{s} & \cdot \\ p \downarrow & & \downarrow q \\ \cdot & \xrightarrow[k]{} & \cdot \end{array}$$

with normal epimorphisms p, q and a normal monomorphism s, the morphism k is also a normal monomorphism; they are *weakly stable* if this holds in the special case where p and q have isomorphic kernels, i. e., where $\ker q$ factors through $s \ker p$.

Corollary. *If normal monomorphisms are weakly stable under normal quotients in $\mathcal{C}$, then condition* (2) *of the Theorem may be replaced by:*

($2'$) *for all objects A, ($rrA \xrightarrow{\kappa_{rA}} rA \xrightarrow{\kappa_A} A$) is a normal monomorphism (with κ_A denoting the kernel of the $\mathcal{F}$–reflection of A).*

Condition (2) *may be omitted completely if normal monomorphisms are stable under normal quotients in $\mathcal{C}$.*

The Proposition, the Theorem and the Corollary are easily dualized. In the case of the Theorem one obtains:

Theorem*. *A full replete subcategory $\mathcal{T}$ of $\mathcal{C}$ is torsion if, and only if,*

(1) *$\mathcal{T}$ is normal–monocoreflective,*
(2) *the $\mathcal{T}$–coreflections have stable cokernels,*
(3) *$\mathcal{T}$ is closed under extensions.*

3. Proofs

Proof of the proposition. For (i) $\Longrightarrow$ (ii), let $\mathcal{F}$ be the torsion–free part of the torsion theory $(\mathcal{T}, \mathcal{F})$. Then, given A, in a short exact sequence

$$0 \longrightarrow B \rhd\!\!\longrightarrow A \longrightarrow\!\!\rhd C \longrightarrow 0$$

with $B \in \mathcal{T}$ and $C \in \mathcal{F}$, these objects depend functorially on A. In fact, for another such sequence

$$0 \longrightarrow B' \rhd\!\!\longrightarrow A' \longrightarrow\!\!\rhd C' \longrightarrow 0$$

and any morphism $f : A \to A'$, since the composite arrow

$$B \rhd\!\!\longrightarrow A \longrightarrow A' \longrightarrow\!\!\rhd C'$$

must be 0, the kernel and the cokernel properties give induced arrows $B \to B'$ and $C \to C'$, respectively. We put

$$0 \longrightarrow rA \overset{\kappa_A}{\rhd\!\!\longrightarrow} A \overset{\rho_A}{\longrightarrow\!\!\rhd} RA \longrightarrow 0$$

for the sequence associated with A (up to isomorphism). Obviously, ρ_A is the $\mathcal{F}$–reflection of A since, for any morphism $g : A \to F$ with $F \in \mathcal{F}$, the composite $g\kappa_A$ is zero, so that g factors uniquely through $\rho_A = \mathrm{coker}(\kappa_A)$. Consequently,

$$\mathcal{F} = \{C | \rho_C \,\mathrm{iso}\} = \{C | rC = 0\}.$$

Dually one obtains

$$\mathcal{T} = \{B | \kappa_B \,\mathrm{iso}\} = \{B | RB = 0\}.$$

Hence, $rA \in \mathcal{T}$ gives κ_{rA} iso, as desired.

(ii) $\Longrightarrow$ (iii): In the pullback diagram (1) with $B \in \mathcal{F}$ and a normal epimorphism p, by the reflection property p must actually factor through ρ_P. We need to show that ρ_P also factors through p; equivalently, that $\ker p$ factors through $\kappa_P = \ker \rho_P$. But as parallel arrows in the pullback diagram (1), p and ρ_A have isomorphic kernels. Hence, there is an arrow $u : rA \to P$ with $gu = \kappa_A$ that serves as the kernel of p. With the idempotency hypothesis one obtains now $rA = rrA \leq rP$ (as subobjects of P), as desired.

(iii) $\Longrightarrow$ (ii): An application of the hypothesis to the pullback diagram

$$\begin{array}{ccc} rA & \xrightarrow{\kappa_A} & A \\ \downarrow & & \downarrow{\scriptstyle \rho_A} \\ 0 & \longrightarrow & RA \end{array}$$

gives immediately $RrA = 0$ or, equivalently, κ_{rA} iso, for all A.

(ii) $\Longrightarrow$ (i): By hypothesis, rA belongs to the class

$$\mathcal{T} := \{B \mid \kappa_B \,\mathrm{iso}\} = \{B \mid RB = 0\}$$

Any morphism $B \to C$ with $B \in \mathcal{T}$ and $C \in \mathcal{F}$ must be 0, as its κ-naturality diagram shows at once. □

Proof of the Theorem. The necessity of conditions (1) and (2) follows from the Proposition since, when $RrA = 0$, the pushout of κ_A along ρ_{rA} can be formed trivially, as

$$\begin{array}{ccc} rA & \xrightarrow{\kappa_A} & A \\ \downarrow & & \downarrow{\scriptstyle \rho_A} \\ 0 & \longrightarrow & RA \end{array}$$

Since $0 \to RA$ is a normal monomorphism, we have indeed shown that the reflections of $\mathcal{F}$ have stable kernels. Closure under extensions follows from $\mathcal{F} = \{A \mid rA = 0\}$ and $\mathcal{T} := \{B \mid \kappa_B \,\text{iso}\}$. Indeed, if in the short exact sequence

$$0 \longrightarrow B \overset{k}{\rhd\!\!\longrightarrow} A \overset{q}{\longrightarrow\!\!\rhd} C \longrightarrow 0$$

we have $rB = 0 = rC$, the κ–naturality diagram for q shows that κ_A factors through k, by a monomorphism $t : rA \to B$. But $t = 0$ since $rA \in \mathcal{T}$ and $B \in \mathcal{F}$, which is possible only if $rA = 0$.

Conversely, let us first consider any normal–epireflective subcategory $\mathcal{F}$ and an object A in $\mathcal{C}$. We form the commutative diagram

(3)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & rrA & \xrightarrow{1} & rrA & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa_{rA}} & & \downarrow{\scriptstyle \kappa_A\kappa_{rA}} & & \downarrow & & \\
0 & \longrightarrow & rA & \xrightarrow{\kappa_A} & A & \xrightarrow{\rho_A} & RA & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_{rA}} & & \downarrow{\scriptstyle q_A} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & RrA & \xrightarrow{k_A} & QA & \xrightarrow{p_A} & RA & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with $q_A = \operatorname{coker}(\kappa_A \kappa_{rA})$ and k_A, p_A the induced morphisms between cokernels. Since cokernels preserve cokernels, the bottom row presents p_A as a cokernel of k_A. Consequently: *the bottom row of* (3) *is short exact if, and only if,* k_A *is a normal monomorphism.* In this case we may conclude $QA \in \mathcal{F}$ when $\mathcal{F}$ is closed under extensions, which then gives κ_{rA} iso by the Lemma below and, with the Proposition, concludes the proof of the Theorem. □

Lemma. *For a normal–epireflective subcategory* $\mathcal{B}$ *of* $\mathcal{C}$*, in the notation of Section 2, the following conditions are equivalent for every object* A *in* $\mathcal{C}$*:*

(i) κ_{rA} *is an isomorphism,*
(ii) $\ker k_A = 0$ *and* $QA \in \mathcal{B}$*,*
(iii) k_A *is a normal monomorphism, and* $QA \in \mathcal{B}$*.*

Proof of the Lemma. (i) $\Longrightarrow$ (iii) κ_{rA} iso means $RrA = 0$, so that k_A is trivially a normal monomorphism and $p_A = \operatorname{coker} k_A$ an isomorphism. Hence $QA \cong RA \in \mathcal{B}$. (iii) $\Longrightarrow$ (ii) is trivial. (ii) $\Longrightarrow$ (i) $QA \in \mathcal{B}$ makes q_A factor through ρ_A, so that $k_A \rho_{rA} = q_A \kappa_A = 0$. Since $\ker k_A = 0$, $\rho_{rA} = 0$ follows, and this means κ_{rA} iso. □

Proof of the Corollary. Stability of normal monomorphisms under normal quotients applied to the right-hand side of diagram (2) will immediately give that k_A must be a normal monomorphism. The same conclusion can be reached under the weak stability hypothesis when $\kappa_A\kappa_{rA}$ is a normal monomorphism, since then $\kappa_A\kappa_{rA} \cong \ker q_A$, i.e., $\ker q_A$ factors through $\kappa_A \ker \rho_{rA}$. $\square$

Remarks. (1) For the Proposition, we need only kernels in $\mathcal{C}$, not cokernels. For the Theorem and its Corollary the general existence assumption for kernels and cokernels may be refined to the hypothesis that $\mathcal{C}$ has kernels of normal epimorphisms and cokernels of composites of pairs of normal monomorphisms.

(2) With respect to the proof of the Corollary we observe that, for any normal–epireflective subcategory $\mathcal{B}$ of $\mathcal{C}$, the implication

$$\ker k_A = 0 \implies \kappa_A\kappa_{rA} = \ker q_A$$

holds. Indeed, ρ_A factors through $q_A = \operatorname{coker}(\kappa_A\kappa_{rA})$ by a morphism $p_A : QA \to RA$, and the kernel $l : L \to A$ of q_A must factor through κ_A, by a morphism $h : L \to rA$. From $k_A\rho_{rA}h = q_A l = 0$ one has $\rho_{rA}h = 0$, so that h must factor through κ_{rA}. Hence, $L \leq rrA$ as subobjects of A, and trivially $rrA \leq L$.

(3) The reverse implication of (2) holds true, too, provided that pullbacks of normal epimorphisms have zero cokernel, in particular: when they are epic. Indeed, with $m = \ker k_A$ and $n = \ker(k_A\rho_{rA})$, we obtain an induced arrow e rendering

$$\begin{array}{ccccc} N & \xrightarrow{e} & M & \longrightarrow & 0 \\ {\scriptstyle n}\downarrow & \boxed{1} & \downarrow{\scriptstyle m} & \boxed{2} & \downarrow \\ rA & \xrightarrow[\rho_{rA}]{} & RrA & \xrightarrow[k_A]{} & QA \end{array}$$

commutative. Since $\boxed{2}$ and $\boxed{1}\boxed{2}$ are pullbacks, $\boxed{1}$ is also one, so that $\operatorname{coker} e = 0$, by hypothesis. Also, since $q_A\kappa_A n = k_A m e = 0$, and since $\ker q_A = \kappa_A\kappa_{rA}$ by hypothesis, we obtain $t : N \to rrA$ with $\kappa_A\kappa_{rA}t = \kappa_A n$, hence $\kappa_{rA}t = n$. Consequently, $me = \rho_{rA}\kappa_{rA}t = 0$, and then $m = 0$, as claimed.

(4) Whereas the Theorem deals essentially with the question when the rows of (3) are short exact, the previous two remarks show that even more often we can expect (3) to have exact columns.

(5) The argumentation used in (ii) $\implies$ (iii) of the proof of the Proposition follows the lines of the proof of Theorem 3.1 in [J2].

4. Examples

A pointed category with finite limits is *homological* [BB] if it is Barr–regular [Ba] and Bourn–protomodular [Bo]; it is *semi–abelian* [JMT] if, in addition, it has finite coproducts, and if equivalence relations are effective (Barr–exactness). Examples include all varieties of Ω–groups, i.e, varieties of universal algebras with underlying group structure, such that the trivial subgroup is a subalgebra. In a semi–abelian category (like the category Grp of groups) and, more generally, in a so–called (E, M)–normal category in the sense of [CDT] (like the category TopGrp of topological groups), normal monomorphisms are weakly stable under normal quotients (a fact that follows from the so–called 3×3–Lemma), but not necessarily stable. Hence, in such a category a normal–epireflective subcategory is torsion free if, and only if, it is closed under extensions and satisfies condition (2′) of the Corollary. In fact, in many categories of this type (such as Grp and TopGrp), condition (2′) can be omitted entirely (since rrA is mapped into itself by every endomorphism of A). We point out that in a semi–abelian category condition (2) of the theorem may be formally weakened to "kernels of $\mathcal{F}$–reflections are pushout–stable monomorphisms", simply because (monomorphic) images of normal subobjects are always normal in such categories. Note also that, in the duals of homological categories, normal monomorphisms are stable under pushout, so that (2) comes for free.

In [JM], which contains a predecessor of the Proposition, with reference to [Ga] further categories are identified, in which closure under extensions alone guarantees that a normal–epireflective subcategory is torsion–free: associative (but not necessarily unital) rings, lattice–ordered groups, alternative rings, and Jordan algebras over a field of characteristic $\neq 2$.

This prompts the question: *Is condition* (2) *of the Theorem redundant*? We give two counter-examples of independent merit.

(1) *An additive homological category with a normal–epireflective subcategory, closed under extensions, which is not torsion-free.* The category $\mathcal{C}$ of abelian groups that satisfy the implication ($4x = 0 \implies 2x = 0$) is clearly additive and homological (but not semi–abelian). Although it is not closed under cokernels in the category of all abelian groups, its short exact sequences are formed as usual. The full subcategory $\mathcal{F}$ of groups satisfying $2x = 0$ is (abelian and) normal–epireflective and, as one easily checks, closed under extensions in $\mathcal{C}$. But it is not torsion–free: since the only quotients in $\mathcal{F}$ of the group $\mathbb{Z}$ of integers, namely $\mathbb{Z}_2$ and 0, both have kernels isomorphic to $\mathbb{Z}$, this group would have to lie in the torsion class corresponding to $\mathcal{F}$, but this is impossible since there is a non–zero morphism $\mathbb{Z} \to \mathbb{Z}_2$.

(2) *A semi–abelian category with a normal–epireflective subcategory, closed under extensions, which is not torsion–free.* Let $\mathcal{C}$ be the category of commutative, but not necessarily associative or unital rings, and $\mathcal{F}$ the full subcategory of objects without nilpotent elements (or even all those objects with $x^2 = 0 \implies x = 0$). Let A be the free abelian group with free generators x, y, z, made into an object of $\mathcal{C}$ via the multiplication table

	x	y	z
x	0	z	0
y	z	y	0
z	0	0	z

We claim that the $\mathcal{F}$–reflection of A can be described as the map $\rho_A : A \to \mathbb{Z}$ which sends x, z to 0 and y to 1; in fact, since $x^2 = 0$ and $z = xy$, any morphism $A \to B$ with $B \in \mathcal{F}$ must send x, z to 0 and therefore factor through ρ_A. Its kernel rA is the subring of A generated by x and z, and just as in the case of A, the $\mathcal{F}$–reflection of rA is isomorphic to $\mathbb{Z}$, not to 0. Hence $\mathcal{F}$ cannot be a torsion–free class.

We finally point out that, of course, it is impossible to find a counter–example to the redundancy conjecture for condition (2) of the Theorem in an additive semi–abelian category. Indeed, by "Tierney's equation", a Barr–exact additive category is already abelian, and in an abelian and, more generally, in every coregular category, regular monomorphisms are stable under pushouts, so that the Corollary applies.

Last, but not least, we should mention the interesting problem of investigating torsion theories in "richer" categorical contexts that are prominently studied in Ross Street's joint work with André Joyal. For example, braided monoidal categories should be a source of new examples of torsion theories, especially torsion theories of commutative monoids in braided monoidal categories. We note that these monoids form a protomodular category if the monoidal category is protomodular.

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