Classification of closure operators for categories of topological spaces*

Dikran Dikranjan,
Dipartimento di Matematica e Informatica,
Università di Udine, Via delle Scienze 206,
33100 Udine, Italy
dikranjan@dimi.uniud.it

Walter Tholen
Department of Mathematics and Statistics,
York University, 4700 Keele Street
Toronto ON, Canada M3J 1P3
tholen@pascal.math.yorku.ca

Stephen Watson
Department of Mathematics and Statistics,
York University, 4700 Keele Street
Toronto ON, Canada M3J 1P3
stephen.watson@mathstat.yorku.ca

To Horst Herrlich, a great friend and mathematician

Abstract

We describe some features of the (very) large lattice of all closure operators of the category **Top** of topological spaces and exhibit the central role of the Kuratowski closure, as the only proper hereditary closure operator which behaves well on products. Additivity alone characterizes the Kuratowski closure in some subcategories of **Top**, such as all connected and locally connected metrizable spaces or all zero-dimensional metrizable spaces.

1 Introduction

Ever since the appearance of Salbany's paper [13] closure operators have become a standard tool in categorical topology, predominantly in the study of epimorphisms of full subcategories of the category **Top** of topological spaces, via their so called regular closures (namely, the closure operators introduced by Salbany). There is therefore a profusion of closure operators on **Top** (at least as many as there are subclasses of a proper class), especially since general Dikranjan-Giuli closure operators are not even required to be idempotent (all regular operators are idempotent).

By definition, a categorical closure operator c is given by a coherent family of closure operations c_X, one for every space X. The main purpose of this paper is to show that, in the presence of good properties of c_X, such as additivity ($c_X(M \cup N) = c_X(M) \cup c_X(N)$ for all $M, N \subseteq X$) or hereditariness ($c_Y(M) = c_X(M) \cap Y$ for $M \subseteq Y \subseteq X$), the values of c_X for very special spaces X determine the values of c_X for all other spaces X. Of particular importance are the cases when X is the Sierpiński dyad or the real line. Among other things these considerations lead us to a characterization of the

*This work has been supported by the Natural Sciences and Engineering Research Council of Canada and the Ministry of Research and University of Italy

AMS classification numbers: Primary 18A20, 18B30, 54B30, 54D25;
Secondary 54A10, 54B17, 54D10, 54D15, 54D25, 54D30.
Key words and phrases: *closure operator, θ-closure, sequential closure, metrizable space, Alexandroff-Tucker space*

usual Kuratowski closure operator of **Top**, as the only proper additive and hereditary closure operator with the so called Tychonoff property. In certain subcategories of **Top**, additivity alone suffices to characterize the Kuratowski closure. In none of these characterizations idempotency plays an explicit role; rather, it comes "for free".

Along the way we exhibit many known and new closure operators of **Top** which shed further light on its "lattice" of closure operators.

Here is a brief outline of the contents of the paper.

In Section 2 we give the definition of closure operator and several examples. In Section 3 we study three properties of closure operators and the interrelations between them: idempotency, productivity and hereditariness. Section 4 is devoted to various degrees of additivity, the lowest one being the usual Čech additivity property.

Then we describe closure operators solely depending on its effect on "simple" spaces such as Sierpiński's dyad or the real line $\mathbb{R}$. In Section 5 we "test" Sierpiński's dyad, which helps us to describe all fully additive hereditary closure operators of **Top** and, in particular, all Čech hereditary closure operators of the category **FinTop** of finite topological spaces. Let us note that full additivity together with hereditariness presents a rather strong restriction on the closure operator. There are many closure operators on finite spaces; for example there are four Čech clops which are distinct on the Sierpiński dyad while only three Čech closure operators are distinct on the real line. In Section 5.2 we classify closure operators on finite spaces. More on closure operators on finite spaces can be found in the forthcoming paper [9]. In Section 5.2 we continue the study of hereditary closure operators. Here we discuss the Tychonoff property (formerly called *the finite structure property for products*, or briefly f.s.p.p. [9]), naturally related to products, and we show that the Kuratowski closure is the only hereditary Čech closure operator with the Tychonoff property on **Top**.

In Section 6 we trade Sierpiński's dyad for a richer space, namely the real line $\mathbb{R}$. This permits us to obtain a classification of all Čech closure operators into three classes: $\mathbb{R}$-fine, $\mathbb{R}$-coarse, and $\mathbb{R}$-tame closure operators. Finally, in Section 7, we use this classification to characterize the Kuratowski closure among Čech closure operators on metrizable-like spaces. We show here that locally "good" spaces admit few closure operators, in fact only k in most cases. It should be stressed that now the good properties of the spaces under consideration permit us to relax the strong restraints on the closure operators used earlier. Actually, most often Čech additivity is sufficient.

The following full subcategories of **Top** will be discussed in this note: $\mathbf{Top_0}$ (T_0-spaces), $\mathbf{Top_1}$ (T_1-spaces), **Tych** (Tychonoff spaces), **FU** (Fréchet-Urysohn spaces), **Metr** (metrizable spaces).

2 The large lattice of closure operators of Top and its subcategories

Let $\mathcal{A}$ be a full subcategory of **Top**, the category of topological spaces and continuous maps, which contains a singleton space 1 and which is closed under subspaces (so that $Y \subseteq X \in \mathcal{A}$ implies $Y \in \mathcal{A}$).

Definition 2.1 (Dikranjan and Giuli [4]) A *closure operator* c of $\mathcal{A}$ is given by a family of maps $c_X : 2^X \longrightarrow 2^X$, $X \in \mathcal{A}$, such that for every X in $\mathcal{A}$

- c_X is *extensive* : $M \subseteq c_X(M)$ for all $M \in 2^X$,
- c_X is *monotone* : $c_X(M) \subseteq c_X(N)$ for all $M \subseteq N \subseteq X$,

and for every continuous map $f : X \to Y$ in $\mathcal{A}$

- f is *c-continuous* : $f(c_X(M)) \subseteq c_Y(f(M))$ for all $M \subseteq X$.

The conglomerate $CL(\mathcal{A})$ of all closure operators in **Top** is partially ordered by

$$c \leq d \Leftrightarrow c_X(M) \subseteq d_X(M) \text{ for all } M \subseteq X \in \mathcal{A};$$

one says that c *is finer than* d, or that d *is coarser than* c in this case. Arbitrary meets and joins of families of closure operators exist in $CL(\mathcal{A})$, and they are given by set-theoretic intersections and unions, respectively; we use the lattice-theoretic notations $\wedge$, $\bigwedge$, $\vee$, $\bigvee$.

Not surprisingly, the most important closure operator on **Top** is given by the usual Kuratowksi closure $k_X(M) = \overline{M}$ of a subset M in X. The *discrete* operator j with $j_X(M) = M$ for all $M \subseteq X \in \mathcal{A}$ is the finest closure operator, and the *trivial* operator t with $t_X(M) = X$ for all $M \subseteq X \in \mathcal{A}$ is the coarsest closure operator. We note that $t_X(\emptyset) \neq \emptyset$ unless $X = \emptyset$; however, for every other closure operator c of $\mathcal{A}$ one has $c_X(\emptyset) = \emptyset$ for all $X \in \mathcal{A}$. (Exploit the continuity condition for maps $X \to 1$ and $1 \to X$ to see that $c_X(\emptyset) = \emptyset$ or $c_X(\emptyset) = X$ for all $X \in \mathcal{A}$, depending on whether $c_1(\emptyset) = \emptyset$ or $c_1(\emptyset) = 1$ for a singleton space 1.) The trivial closure operator t has an immediate predecessor, the *indiscrete* closure operator g, given by $g_X(\emptyset) = \emptyset$ and $g_X(M) = X$ for each $X \in$ **Top** and $\emptyset \neq M \subseteq X \in \mathcal{A}$. These names are motivated by the fact that a space X is discrete (indiscrete) precisely when $k_X = j_X$ ($k_X = g_X$, respectively). Any closure operator different from j, g, t is called *proper* on $\mathcal{A}$. We say that c is *proper on a space* $X \in \mathcal{A}$ if $c_X \neq t_X, g_X, j_X$. Next we shall show that there is a finest and a coarsest proper closure operator of **Top**.

Recall that the *quasi-component* $q_X(M)$ of $M \subseteq X$ in $X \in$ **Top** is the intersection of all clopen (=closed and open) sets of X containing M. "Dually" define the *minimal proper closure operator* $\mu_X(M)$ to be the union of all indiscrete sets of X which meet M. It is easy to check that $q = \{q_X\}_{X \in \mathbf{Top}}$ and $\mu = \{\mu_X\}_{X \in \mathbf{Top}}$ are proper closure operators of **Top** (see also 2.5 below). Moreover:

Proposition 2.2 *For every proper closure operator c of* **Top** *one has*

$$j < \mu \leq c \leq q < g < t.$$

Moreover, for every topological space X one has $\mu_X = j_X$ iff X is T_0 and $q_X = g_X$ iff X is connected, while $q_X = k_X$ means that X is zero-dimensional.

Proof. In order to show $\mu \leq c$, provide $2 = \{0, 1\}$ with the indiscrete topology. If one of the two points, 1 say, were c-closed (so that $c_2(\{1\}) = \{1\}$), then c would be the discrete operator (apply the continuity condition to the characteristic function $\chi_M : X \to 2$ with $M = \chi_M^{-1}(\{1\})$). Hence we must have $c_2 = g_2$. Now consider $y \in \mu_X(M)$, for any space X and $M \subseteq X$. Then $y \in \mu_X(\{x\})$ for some $x \in M$. The subspace $Y = \{x, y\}$ of X is indiscrete. Hence $y \in c_Y(\{x\}) \subseteq c_X(\{x\}) \subseteq \mu_X(M)$. This proves $\mu_X(M) \subseteq c_X(M)$.

For the proof of $c \leq q$ one provides 2 with the discrete topology and proceeds "dually". QED

When dealing with closure operators of a subcategory $\mathcal{A}$ it is often convenient to know that they may be extended to **Top**. This is always possible when $\mathcal{A}$ is *epireflective* in **Top**, i. e., when $\mathcal{A}$ is not only closed under subspaces, but also under direct products. In fact, given $c \in CL(\mathcal{A})$, for any space X, let $r_X : X \to RX$ be the reflection into $\mathcal{A}$. Denoting by $\hat{M}$ the range of the map $R(M \hookrightarrow X) : RM \to RX$, we may define

$$c_X(M) = r_X^{-1}(c_{RX}(\hat{M})). \tag{1}$$

One can then easily prove (cf. Dikranjan-Tholen [6]):

Proposition 2.3 *If $\mathcal{A}$ is an epireflective subcategory of* **Top**, *formula* (1) *gives for every closure operator c of $\mathcal{A}$ an extension to* **Top**. *This way, $CL(\mathcal{A})$ gets reflectively embedded into $CL($***Top***$)$, with the reflector given by restriction. QED*

In case $\mathcal{A} = \mathbf{Top_0}$ is the category of T_0-spaces, one has a kind of converse assertion (see Dikranjan, Giuli and Tholen [5]):

Proposition 2.4 *Every non-discrete closure operator c of* **Top** *is an extension of a closure operator of* $\mathbf{Top_0}$*, i. e. satisfies formula* (1) *with $r_X : X \to RX$ the T_0-reflection of a space X.* *QED*

3 Idempotency, hereditariness and productivity

Definition 3.1 *Let c be a closure operator of a full and (for simplicity) epireflective subcategory $\mathcal{A}$ of* **Top***, and let κ be a cardinal number. One calls c*

- *idempotent at $X \in \mathcal{A}$ if $c_X(c_X(M)) = c_X(M)$ for all $M \subseteq X$,*
- *hereditary at $X \in \mathcal{A}$ if $c_Y(M) = c_X(M) \cap Y$ for all $M \subseteq Y \subseteq X$.*

Furthermore, c is idempotent (hereditary) if it is idempotent (hereditary, resp.) at every $X \in \mathcal{A}$, and c is called

- *κ-productive if $c_X(\prod_{i\in I} M_i) = \prod_{i\in I} c_{X_i}(M_i)$ holds for all $M_i \subseteq X_i \in \mathcal{A}$, $i \in I$, $\mathrm{card}\, I < \kappa$, with $X = \prod_{i\in I} X_i$.*

Finally, c is

- *finitely productive if it is ω-productive, and*
- *productive if it is κ-productive for every κ.*

Each of the properties of idempotency, hereditariness and κ-productivity is stable under meet in $CL(\mathcal{A})$; hereditariness is also stable under join in $CL(\mathcal{A})$. Consequently, for each closure operator c one can find a finest idempotent (hereditary) closure operator in $CL(\mathcal{A})$ coarser than c, called the *idempotent* (*hereditary*) *hull* of c and denoted by c^∞ (c^{he}, resp.). Similarly, there is a coarsest hereditary closure operator in $CL(\mathcal{A})$ finer than c, called the *hereditary core* of c and denoted by c_{he}. Of these hulls and cores, in general only c^∞ seems to be computationally accessible. One way is to consider the ordinal powers c^α (recursively defined by $c^0 = j$, $c^{\alpha+1} = cc^\alpha$ and $c^\alpha = \bigvee\{c^\beta : \beta < \alpha\}$ for a limit α); then c^∞ is the join of all c^α. Another way is discussed in Remark 3.5 below.

For $c, d \in CL(\mathbf{Top})$ one defines the *composite* dc and the *cocomposite* $d * c$ as follows. For $X \in \mathbf{Top}$ and a subspace M of X one puts

$$(dc)_X(M) = d_X(c_X(M)) \text{ and } (d * c)_X(M) = d_{c_X(M)}(M).$$

The verification of the fact that dc and $d*c$ are closure operators is straightforward. The discrete closure operator j is neutral with respect to composition and acts like a zero with respect to cocomposition, while for the trivial operator t these roles get interchanged:

$$cj = jc = c,\ c * j = j * c = j,\ ct = tc = t,\ c * t = t * c = c$$

hold for every $c \in CL(\mathbf{Top})$.

The following theorem gives some surprising interrelationships between the properties introduced in 3.1. It has been proved in a more general categorical context in Dikranjan-Tholen [6]; here we give *ad-hoc* proofs for the reader's convenience.

Lemma 3.1 *For $c, d \in CL(\mathcal{A})$, if the composite cd is hereditary, then $cd = c \vee d$.*

Proof. One always has $c \vee d \leq cd$. Now suppose that for some $M \subseteq X \in \mathcal{A}$ there is a point $x \in c_X(d_X(M))$ which is not in $c_X(M) \cup d_X(M)$. For the subspace $Y = M \cup \{x\}$ of X one then has $x \notin c_Y(M) \cup d_Y(M) \subseteq (c_X(M) \cup d_X(M)) \cap Y$. Hence $c_Y(M) = M = d_Y(M)$ and then $M = c_Y(d_Y(M))$. On the other hand $Y \subseteq c_X(d_X((M))$, so that the composition cd must fail to be hereditary. QED

Corollary 3.2 *If $c, d \in CL(\mathcal{A})$ and $cd \neq c \vee d$, then there exist $M \subset Y \subseteq X \in \mathcal{A}$ such that M is cd-closed in Y, but the cd-closure of M in X contains Y; in particular, cd is not hereditary.*

A closure operator c of $\mathcal{A}$ has *order* α if α is the least ordinal with $c^{\alpha+1} = c^{\alpha}$; if no such α exists, the order is ∞. We denote the order of c by $o(c)$. Note that $c^{o(c)} = c^{\infty}$.

Theorem 3.3 *Let $c \in CL(\mathbf{Top})$.*

(a) If c is not idempotent, then the powers c^{α} $(2 \leq \alpha \leq o(c))$ are not hereditary.

(b) If c has infinite order $o(c)$, then for every limit ordinal $\alpha \leq o(c)$ the power c^{α} is not α^{+}-productive; in particular, if $o(c)$ is a limit ordinal, then c^{∞} is not productive.

(c) c^{ω} is finitely productive.

(d) If c is idempotent (i. e., if $o(c) \leq 1$), then c is finitely productive.

(e) If c is finitely productive, then every power c^{α} $(0 \leq \alpha \leq \infty)$ is finitely productive.

Proof. (a) We may assume that $c \neq g, t$ and $\alpha \leq o(c)$. Fix any $1 \leq \beta < \alpha$ and apply Corollary 3.2 to the closure operators c and c^{β} with $cc^{\beta} = c^{\beta+1} > c^{\beta} = c \vee c^{\beta}$ to conclude that $c^{\beta+1}$ is not hereditary. This proves item (a) in case $\alpha = \beta + 1$ is a successor. In case α is a limit and $\beta < \alpha$, apply Corollary 3.2 to the closure operators c and c^{β} to get $M_{\beta} \subset Y_{\beta} \subseteq X_{\beta} \in \mathcal{A}$ such that M_{β} is cc^{β}-closed (and, hence c-closed) in Y_{β} and the cc^{β}-closure of M_{β} in X_{β} contains Y_{β}. Now let $X = \coprod_{\beta} X_{\beta}$ be the coproduct of the spaces X_{β}, $Y = \coprod_{\beta} Y_{\beta}$ and $M = \coprod_{\beta} M_{\beta}$. We shall show that M is c^{α}-closed in Y, and the c^{α}-closure of M in X contains Y, hence c^{α} fails to be hereditary also in this case. The proof if based on the following easy

Claim. *If $X = X_1 \coprod X_2$ and for $i = 1, 2$ M_i is a non-empty set of X_i, then $c_X(M_1 \cup M_2) = c_{X_1}(M_1) \cup c_{X_2}(M_2)$.*

To finish the proof of (a) note that by the Claim $c_X^{\gamma}(M) = \coprod_{\beta<\alpha} c_{X_{\beta}}^{\gamma}(M_{\beta})$ for every $\gamma < \alpha$. In particular, $c_X^{\alpha}(M) \supseteq Y$ and $c_Y^{\alpha}(M) = M$.

(b) By our assumption $\alpha \leq o(c)$, for every $\nu < \alpha$ there is a space $X_{\nu} \in \mathcal{A}$ with $c_{X_{\nu}}^{\nu+1} \neq c_{X_{\nu}}^{\nu}$. Choose a subset $M_{\nu} \subseteq X_{\nu}$ and a point $x_{\nu} \in c_{X_{\nu}}^{\nu+1}(M_{\nu}) \setminus c_{X_{\nu}}^{\nu}(M_{\nu})$. Consider the subspace $M = \prod_{\nu<\alpha} M_{\nu}$ of $X = \prod_{\nu<\alpha} X_{\nu}$ and let $\pi_{\nu} : X \to X_{\nu}$ be the projection. Since $x_{\nu} \in c_{X_{\nu}}^{\nu+1}(M_{\nu}) \subseteq c_{X_{\nu}}^{\alpha}(M_{\nu})$ for all $\nu < \alpha$, one has

$$x = (x_{\nu})_{\nu<\alpha} \in \prod_{\nu<\alpha} c_{X_{\nu}}^{\alpha}(M_{\nu}).$$

But if one had also $x \in c_X^{\alpha}(M) = \bigcup_{\nu<\alpha} c_X^{\nu}(M)$, hence $x \in c_X^{\nu}(M)$ for some $\nu < \alpha$, then c-continuity of π_{ν} would give

$$x_{\nu} = \pi_{\nu}(x) \in c_{X_{\nu}}^{\nu}(\pi_{\nu}(M)),$$

in contradiction to the choice of x_ν.

(c) For $M_1 \subseteq X_1$ and $M_2 \subseteq X_2$, first we show

$$c_{X_1}(M_1) \times M_2 \subseteq c_X(M_1 \times M_2), \tag{2}$$

where $X = X_1 \times X_2$. Indeed, for every $a_2 \in M_2$ we may apply the continuity condition to the map $X_1 \to X$, $x_1 \mapsto (x_1, a_2)$, to obtain $c_{X_1}(M_1) \times \{a_2\} \subseteq c_X(M_1 \times M_2)$ and therefore (2). Symmetrically to (2), one also has $M_1 \times c_{X_2}(M_2) \subseteq c_X(M_1 \times M_2)$. In this formula we can trade M_1 for $c_{X_1}(M_1)$ to obtain

$$c_{X_1}(M_1) \times c_{X_2}(M_2) \subseteq c_X(c_{X_1}(M_1) \times M_2) \subseteq c_X^2(M_1 \times M_2). \tag{3}$$

Since $c^\omega_{X_i}(M_i) = \bigcup_{n<\omega} c^n_{X_i}(M_i)$ for $i = 1, 2$, we get $c^\omega_{X_1}(M_1) \times c^\omega_{X_2}(M_2) = \bigcup_{n<\omega}(c^n)_{X_1}(M_1) \times (c^n)_{X_2}(M_2)$. By (3) we get

$$c^n_{X_1}(M_1) \times c^n_{X_2}(M_2) \subseteq c^{2n}_X(M_1 \times M_2) \subseteq c^\omega_X(M_1 \times M_2).$$

This proves that $c^\omega_{X_1}(M_1) \times c^\omega_{X_2}(M_2) \subseteq c^\omega_X(M_1 \times M_2)$, with the converse inclusion holding trivially.

(d) follows immediately from (c).

(e) is an easy induction proof. QED

Remark 3.4 (1) The assumption $o(c) < \infty$ in the concluding part of item (b) is essential. We shall show that c^∞ may be productive for a non-idempotent closure operator c with infinite $o(c)$ (necessarily, $o(c) = \infty$).

(2) In item (c) one can replace c^ω by limit ordinal powers c^α such that α is an indecomposable ordinal (i.e., $\beta + \beta < \alpha$ holds for every $\beta < \alpha$).

(3) Theorem 3.3 remains valid for $c \in CL(\mathcal{A})$ if the subcategory $\mathcal{A}$ is closed under the products and coproducts used in its proof.

For every space X and $M \subseteq X$, let $\sigma_X(M)$ be the set of points in X which are limits of sequences in M. This defines the *sequential closure* operator σ of **Top**. It is productive and hereditary, but not idempotent. In fact, $o(\sigma) = \omega_1$ (cf. Arhangel'skiĭ and Franklin [1]). The Theorem gives that for all α ($2 \le \alpha \le \omega_1$), σ^α is not hereditary, and that σ^ω and σ^{ω_1} are finitely productive, but not productive. In particular, $\sigma^{\omega_1} = \sigma^\infty$ gives an example of an idempotent operator which is not productive.

Remark 3.5 For every $X \in \mathcal{A}$, let $\mathcal{K}_X$ be a set of subspaces of X such that $X \in \mathcal{K}_X$ and, for all $f : X \to Y$ in $\mathcal{A}$ and $N \in \mathcal{K}_Y$, $f^{-1}(N) \in \mathcal{K}_X$. Then

$$c^{\mathcal{K}}_X(M) = \bigcap\{N \in \mathcal{K}_X : M \subseteq N\}$$

defines an idempotent closure operator $c^{\mathcal{K}}$ of $\mathcal{A}$. This closure operator is hereditary if and only if for all $M \subseteq Y \subseteq X \in \mathcal{A}$ with $M \in \mathcal{K}_Y$ there exists $L \in \mathcal{K}_X$ with $M = L \cap Y$.

If $\mathcal{K}_X \subseteq \mathcal{L}_X$ holds for every $X \in \mathcal{A}$, then $c^{\mathcal{K}} \ge c^{\mathcal{L}}$.

For any $c \in CL(\mathcal{A})$ one may take $\mathcal{K}_X$ to be the set of c-*closed* subspaces of X, i. e., those $M \subseteq X$ with $c_X(M) = M$, and then obtain the corresponding idempotent closure operator $c^{\mathcal{K}}$. It is easy to see that $c^{\mathcal{K}}$ is the idempotent hull c^∞ of c in $CL(\mathcal{A})$.

Example 3.6 Each of the following properties for subspaces $M \subseteq X$ defines a suitable set $\mathcal{K}_X$ as required in Remark 3.5:

(a) M closed in X (in the usual topological sense), (b) M open in X, (c) M clopen in X, (d) M a zero-set in X (so that $M = f^{-1}(0)$ for some continuous function $f : X \to I$ into the unit interval),

(e) M an F_σ-set in X, (f) $M = f^{-1}(N)$ for some continuous function $f : X \to \mathbb{R}$ and some non-empty $N \subseteq \mathbb{R}$. The corresponding closure operators $c^{\mathcal{K}}$ are denoted by

(a) k (the *Kuratowski operator*), (b) k^* (the *inverse Kuratowski operator*), (c) q (see 1.2), (d) z (the *zero operator*), (e) p (the *p-closure operator*[1]) , and (f) ϱ (the *real closure*[2]) Since (b) $\Leftarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) $\Rightarrow$ (e), one has

$$k^* \leq q \geq z \geq k \geq p \text{ and } z \geq \varrho \geq p.$$

The inequality $p \leq \varrho$ follows from the fact that every subset M of $\mathbb{R}$ is both ϱ-closed and p-closed, thus every ϱ-closed set of X is p-closed as well. Each of the six operators is idempotent and productive in **Top**, but only k and k^* are hereditary.

Note that k^* is conveniently described also by

$$k^*_X(M) = \{x \in X : k_X(\{x\}) \cap M \neq \emptyset\}.$$

When replacing k by an arbitrary closure operator c we can define more generally *the inverse* c^* of c as

$$c^*_X(M) = \{x \in X : c_X(\{x\}) \cap M \neq \emptyset\}.$$

4 Additivity and symmetry

Definition 4.1 Let c be a closure operator of $\mathcal{A}$, with (for simplicity) $\mathcal{A}$ as in 3.1. For a cardinal number κ, c is called

- *κ-additive* at $X \in \mathcal{A}$ if $c_X(\bigcup_{i\in I} M_i) = \bigcup_{i\in I} c_X(M_i)$ for all $M_i \subseteq X$, $i \in I$ $card\, I < \kappa$.

c is

- *additive or a Čech closure operator* if c is ω-additive;
- *fully additive* if c is κ-additive for every κ.

Note that κ-additivity (with $\kappa \geq 1$) entails $c_X(\emptyset) = \emptyset$. Only the trivial closure operator of **Top** does not satisfy the last condition (see 2.1). A fully additive operator is completely determined by its point closures, i. e. $c_X(M) = \bigcup\{c_X(\{x\}) : x \in M\}$. It is easy to see that on a space X with at most two points every closure operator is fully additive. An example of a three-point space with a non-additive closure operator is given below (Example 4.1 (2)).

The property of κ-additivity is stable under join and composition in $CL(\mathcal{A})$. Consequently, every closure operator c has a *κ-additive core* and a *fully additive core*, denoted by $c^{+(\kappa)}$ and $c^{\oplus}$, respectively; we write $c^+ = c^{+(\omega)}$. These cores can be fairly easily computed as

$$c^{+(\kappa)}_X(M) = \bigcap\{\bigcup_{i<\kappa} c_X(M_i) : M \subseteq \bigcup_{i<\kappa} M_i \subseteq X\},$$

$$c^{\oplus}_X(M) = \bigcup\{c_X(\{x\}) : x \in M\}.$$

[1] Clearly $p_X = k_{PX}$, where the topological space PX is the P-modification of X. (Recall that a topological space X is called a P-space if every G_δ-set of X is open.)

[2] In detail $x \notin \varrho_X(M)$ for $X \in$ **Top**, $M \subseteq X$ and $x \in X$ iff there exists a continuous function $f \in C(X)$ such that $f(x) \notin f(M)$.

It is also easy to see that $c^{+(\kappa)}$ and $c^{\oplus}$ inherit the properties of idempotency and hereditariness from c; on the other hand, κ-additivity is preserved when passing from c to the idempotent hull c^{∞} (cf. Dikranjan-Tholen [6]).

The inverse closure operator c^* is always fully additive for every closure operator $c \in CL(\mathbf{Top})$. Using this fact one can easily show that $c^*c^* = c^{\oplus}$ and $(cd)^* = d^*c^*$ for any pair $c, d \in CL(\mathbf{Top})$, and trivially, $j^* = j$.

Recall that a topological space X is an *Alexandroff-Tucker space* if k_X is fully additive (then every point $x \in X$ has a smallest open neighbourhood, namely $k^*(\{x\})$). The category of T_0 Alexandroff-Tucker spaces is equivalent to the category of partially ordered sets with order preserving maps. The respective correspondences are defined by the Alexandroff-Tucker topology of a poset (having as non-empty open sets all downward closed sets), and by the specialization order of a topological space (let $x \leq y$ if $y \in k_X(\{x\})$, and note that this is equivalent to $x \in k^*(\{y\})$). In the sequel we describe Alexandroff-Tucker spaces by their partial order when this is more convenient.

Example 4.1 (1) The operators μ, k^* (see 2.2, 3.5) are fully additive and k, z, q are Čech, but not ω_1-additive in **Top**. p is the ω_1-additive core of k: $p = k^{+(\omega_1)}$.

(2) $k \wedge k^*$ is an idempotent, hereditary and productive operator in **Top** which fails to be Čech. Hence the meet of Čech operators need not necessarily be Čech. To see this, consider the Alexandroff-Tucker topology of the set $X = \{0, 1, 2\}$ with the usual order. Then the singleton $\{0\}$ is k-dense and k^*-closed, while the singleton $\{1\}$ is k^*-dense and k-closed. Therefore both singletons are $k \wedge k^*$-closed, while the set $M = \{0, 2\}$ is both k-dense and k^*-dense. Consequently, $1 \in k_X(M) \cap k^*_X(M) = (k \wedge k^*)_X(M)$, but $1 \notin (k \wedge k^*)_X(\{0\}) \cup (k \wedge k^*)_X(\{1\})$.

(3) Since $k \wedge k^*$ fails to be additive, its additive core

$$b = (k \wedge k^*)^+$$

is properly finer than $k \wedge k^*$. This is the well-known b- or *front closure*, cf. Baron [2] (see also [12]); for $M \subseteq X$, $b_X(M)$ is the set of all points x with $U \cap M \cap \overline{\{x\}} \neq \emptyset$ for every neighbourhood U of x in X. The closure operator b is, like the Kuratowski operator k, idempotent, hereditary, productive and Čech, but fails to be fully additive in **Top**. Consequently, the fully additive core $b^{\oplus}$ is properly finer than b. One has

$$b^{\oplus} = k^{\oplus} \wedge k^* = \mu.$$

Lemma 4.2 *The real closure ϱ is ω_1-additive with $p \leq \varrho \leq z^{+\omega_1}$. In particular, ϱ coincides with p for Tychonoff spaces.*

Proof. Assume $X \in \mathbf{Tych}$ has countable pseudocharacter at some point $x \in X$, i. e. $\{x\}$ is a G_δ-set. Then one can find a continuous function $f : X \to \mathbb{R}$ such that $\{x\} = f^{-1}(\{0\})$. Then also $X \setminus \{x\} = f^{-1}(\mathbb{R} \setminus \{0\})$. We conclude that $X \setminus \{x\}$ is ϱ-closed. Therefore, if $\psi(X) = \omega$ then $\varrho_X = s_X$. In particular, $\varrho_X = s_X$ for a metrizable space X.

To show that ϱ is ω_1-additive consider a family $\{M_n : n \in \mathbb{N}\}$ of ϱ-closed sets. Then for $M = \bigcup_n M_n$ and $x \in X \setminus \bigcup_n \varrho(M)$ there exist a $\{f_n : n \in \mathbb{N}\} \subseteq C(X)$ such that $f_n(x) \notin f_n(M_n)$. Consider the continuous product-map $f : X \to \mathbb{R}^{\mathbb{N}}$. Since $Y = \mathbb{R}^{\mathbb{N}}$ is metrizable, ϱ_Y is discrete, thus $f(x) \notin f(M) = \bigcup f(M_n)$, since $f_n(x) \notin f_n(M_n)$ (take the n-th projection !). By $f(x) \notin f(M) = \varrho(f(M))$ we conclude that $x \notin \varrho(M)$.

To finish the proof note that $\varrho \leq z$ is obvious, so that $\varrho \leq z^{+\omega_1}$ follows from ω_1-additivity of ϱ. QED

Remark 4.3 There exists a regular space X such that every continuous function $f \in C(X)$ is constant ([11]). Clearly for such a space X one has $p_X \leq k_X < \varrho_X = g_X$. Therefore, in the above corollary one cannot avoid the condition the X is a Tychonoff space. Actually, $p_X < \varrho_X$ may hold even for a functionally Hausdorff space X. Indeed, the Tychonoff reflection rX of such a space X satisfies $\rho_X = \rho_{rX}$ and $z_X = z_{rX}$, while in general only $p_X \leq p_{rX}$ holds. Hence, if X is a space with $p_X < p_{rX}$ (e.g., Tychonoff's corkscrew), then also $p_X < p_{rX} = \rho_{rX} = \rho_X$ as rX is a Tychonoff space.

Example 4.4 For a point $x \in X$ denote by $a_X(\{x\})$ (resp. $e_X(\{x\})$) the arcwise connected (resp. connected) component of x in X. Now setting $a_X(M) = \bigcup_{x \in M} a_X(x)$ (resp. $e_X(M) = \bigcup_{x \in M} e_X(x)$) we get a closure operator $a = \{a_X\}_{X \in \mathbf{Top}}$ $(e = \{e_X\}_{X \in \mathbf{Top}})$. They both fail to be hereditary, but they are still weakly hereditary. (A closure operator is *weakly hereditary* if $c_{c_X(M)}(M) = c_X(M)$ for all $M \subseteq X \in \mathbf{Top}$.)

It is easy to see that the closure operator $a * k$ is not additive. This answers negatively a question from [6, p.357, Table 1]: *additivity is not preserved by cocomposition even when one of the closure operators is* fully *additive.*

A closure operator $c \in CL(\mathcal{A})$ is called *symmetric* if $x \in c_X(\{y\})$ implies $y \in c_X(\{x\})$ for all $x, y \in X \in \mathcal{A}$. Symmetry is stable under taking meets and joins of families of closure operators. In particular, every closure operator has a symmetric hull, which is given by

$$c^{sym} = c \vee c^*.$$

Examples of symmetric closure operators of **Top** are a, q, p, z, b, μ, while k, σ, k^* fail to be symmetric. For every fully additive closure operator c both compositions c^*c and cc^* are symmetric closure operators. Moreover, if c is also idempotent, then $(c^{sym})^2 = c^*c \vee cc^*$.

Recall that a topological space X is *symmetric* if k_X is symmetric. It is easy to see that the symmetric Alexandroff-Tucker spaces are exactly the spaces X for which $\mu_X = q_X$.

Now we consider the symmetric hull $w = k^{sym} = k \vee k^*$ of the Kuratowski closure k and its fully additive core $w^{\oplus} = k^{\oplus} \vee k^*$.

Lemma 4.5 *Both w and $w^{\oplus}$ are hereditary and (fully) additive, but fail to be finitely productive in* **Top** *(and are therefore not idempotent either, see Theorem 3.3 (2)). More precisely, $o(w^{\oplus}) = \omega$.*

Proof. Since k and k^* are hereditary, also $w = k \vee k^*$ is hereditary. Analogously, $w^{\oplus} = k^{\oplus} \vee k^*$ is hereditary. For $M \subseteq X$ and $x \in X$, one has $x \in w_X k \vee k^*(M)$ if and only if $x \in \overline{\{y\}}$ or $y \in \overline{\{x\}}$ for some $y \in M$. In order to see that w is not finitely productive , we consider two copies of the (3-open set) Sierpiński space $S = \{0, 1\}$, with $\{1\}$ open, say. Then $0 \in w_S(\{1\})$ and $1 \in w_S(\{0\})$, but $(0, 1) \notin w_{S \times S}(\{(1, 0)\})$. Since $w^{\oplus}$ coincides with $k \vee k^*$ on finite spaces, this example shows that also $w^{\oplus}$ fails to be finitely productive.

Now Theorem 3.3 implies that neither w nor $w^{\oplus}$ is idempotent. A direct proof of the latter property can be obtained by observing that for every power $X = S^{\alpha}$ one has $w_X w_X = w^{\oplus}_X w^{\oplus}_X = (kk^*)_X = t_X$, while $w_X \neq t_X$ when $\alpha > 1$. Indeed, consider the points $\mathbf{0}, \mathbf{1} \in X$, where $\mathbf{0}$ (resp., $\mathbf{1}$) has all its coordinates 0 (resp., 1). Clearly, $\mathbf{0}$ is k^*-closed and k-dense, while $\mathbf{1}$ is k-closed and k^*-dense. Now choose an arbitrary point $x \in X$ distinct from $\mathbf{0}$ and $\mathbf{1}$. For $\nu = 0, 1$ let $I_\nu = \{i \in I : x_i = \nu\}$. Then both sets are non-empty by the choice of x and $x = (x^{(0)}, x^{(1)}) \in S^{I_0} \times S^{I_1}$, where all coordinates of $x^{(0)} \in S^{I_0}$ are 0 and all coordinates of $x^{(1)} \in S^{I_1}$ are 1. Then $x^{(0)} \in S^{I_0}$ is k^*-closed and k-dense, while $x^{(1)} \in S^{I_1}$ is k-closed and k^*-dense. Consequently,

$$k^*_X(x) = \{x^{(0)}\} \times S^{I_1} \text{ and } k_X(x) = S^{I_0} \times \{x^{(1)}\}.$$

Consequently, $w_X^{\oplus}(x) = (\{x^{(0)}\} \times S^{I_1}) \cup (S^{I_0} \times \{x^{(1)}\})$, in particular $w_X^{\oplus}(x) \ni \mathbf{0}, \mathbf{1}$. This implies that $w_X^{\oplus}(x)$ is $w^{\oplus}$-dense in X. Since the point $x \in X$ was chosen arbitrarily, this proves that $(w_X^{\oplus})^2 = g_X$.

To prove $o(w^{\oplus}) = \omega$ let us start by observing that $w^{\oplus}w^{\oplus} = \xi \vee \zeta$, where $\xi = k^{\oplus}k^*$ and $\zeta = k^*k^{\oplus}$. An easy induction proof gives $(w^{\oplus})^{2n} = \xi^n \vee \zeta^n$. Let L_n denote the partially ordered set $L_n = \{x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n\}$ with

$$x_0 \leq y_0 \geq x_1 \leq \dots \leq y_{i-1} \geq x_i \leq y_i \geq x_{i+1} \leq \dots \leq y_{n-1} \geq x_n \leq y_n$$

as the only non-trivial relations. Then $x_0 \in \zeta_{L_n}(x_n) \subseteq (w^{\oplus})^{2n}_{L_n}(x_n)$ and $y_0 \in \xi_{L_n}(y_n) \subseteq (w^{\oplus})^{2n}_{L_n}(x_n)$, but $x_0 \notin (w^{\oplus})^{2n-2}_{L_n}(x_n)$ and $y_0 \notin (w^{\oplus})^{2n-2}_{L_n}(y_n)$. This proves that $o(w^{\oplus}) \geq \omega$. As for every fully additive closure operator, $o(w^{\oplus}) \leq \omega$. This proves the lemma. QED

Remark 4.6 (a) We shall see in Corollary 5.12 below that $o(w) = \infty$.

(b) It follows from Lemma 4.5 and item (c) of Theorem 3.3 that $w = c^{\omega}$ for no closure operator c.

(c) The apparently fine closure operator $(w_{SI}^{\oplus})^2$ turns out to be indiscrete on the powers of S. Since every T_1 space is a subspace of such a power and since $(w_X^{\oplus})^2$ is obviously discrete on every T_1-space X, this shows how strongly the square $(w_X^{\oplus})^2$ fails to be hereditary (while $w_X^{\oplus}$ is hereditary). A similar comparison can be made for w: while w_{SI}^2 is indiscrete, $w_X = k_X$ for every T_1-space X.

Now we find another property of w that provides a characterization of w:

Proposition 4.7 *w is the hereditary core of q in* **Top**. *Consequently, w is the largest proper hereditary closure operator of* **Top**, *and k is the largest proper hereditary closure operator of* $\mathbf{Top_1}$.

Proof. Since w is hereditary, we get $w \leq q_{he}$. Now let $c \in CL(\mathbf{Top})$ be proper and hereditary. Then $c \leq q$ (see 2.2). Consider $x \in c_X(M)$ for $M \subseteq X$, and assume $x \notin k(M) \cup k^*(M)$. Then M is a clopen set in the subspace $Y = M \cup \{x\}$ of X, hence $x \notin M = q_Y(M)$. Since $c \leq q$ is hereditary, this would imply $x \notin c_Y(M)$ - contradiction. Since $k \vee k^*|_{\mathbf{Top_1}} = k|_{\mathbf{Top_1}}$, with Proposition 1.3, this proves also the second statement. QED

5 Testing the Sierpiński space

Let c be a closure operator of $\mathcal{A}$, in the setting of 2.1, and let $X \in \mathcal{A}$. We call c *discrete* (*tame*, *wild*, *indiscrete*) *on* X if $c_X = j_X$ ($c_X = k_X$, $c_X = k_X^*$, $c_X = g_X$), i. e., if c is discrete (the usual Kuratowski closure, k^*, indiscrete, respectively) on X.

It turns out that the behavior of a closure operator on a particular space X may govern its behavior globally. The first space that we consider under this aspect is the Sierpiński space $S = \{0, 1\}$, with $\emptyset, \{1\}, S$ open.

Proposition 5.1 *Let $c \in CL(\mathbf{Top})$. Then :*

(1) $c \leq k$ if and only if $c_S \leq k_S$, and $c \leq k^*$ if and only if $c_S \leq k_S^*$;

(2) if c is Čech, then $c \leq b$ if and only if $c_S = j_S$ (i. e., c is discrete on S);

(3) $k \geq c \geq k^{\oplus}$ if and only if $c_S = k_S$, and $c = k^*$ if and only if $c_S = k_S^*$;

(4) $c \geq w\oplus$ if and only if $c_S = g_S$ (i. e., c is indiscrete on S).

Proof. (a) Assume that $c_S \leq k_S$, hence $\{0\}$ is c-closed in S. Then for every $X \in \mathbf{Top}$ and $M \subseteq X$ the set $F = k_X(M)$ is closed in X, so that the characteristic function $f : X \to S$ defined by $f(F) = 0, f(X \setminus F) = 1$ is continuous. Thus $F = f^{-1}(0)$ is c-closed in X, cf. 2.4. Hence $c_X(M) \subseteq k_X(M) = F$. The proof of the second statement is similar.

(2) Follows from (1) since $b = (k \vee k^*)^+$.

(3) Assume that $c_S = k_S$. According to (1) it suffices to show that $c \geq k^{\oplus}$. For $X \in \mathbf{Top}$ and $x \in X$ we have to prove that $k_X(x) \subseteq c_X(x)$. Let $y \in k_X(x)$ and $D = \{x, y\}$. If D is T_0, then D is homeomorphic to S and $y \in k_D(x)$. Then our assumption gives $y \in c_D(x) \subseteq c_X(x)$. If D is indiscrete, then $y \in c_D(x)$ as $c_S = k_S$ and there exists a continuous bijection $S \to D$. Now we conclude $y \in c_X(x)$ as before. The proof for the case $c_S = k^*_S$ is similar.

(4) is left to the reader. QED

Theorem 5.2 *Let $c \in CL(\mathbf{Top})$ be proper, hereditary and fully additive. Then :*

(a) $c = k^{\oplus}$ if and only if c is tame on S.

(b) $c = k^*$ if and only if c is wild on S.

(c) $c = \mu$ if and only if c is discrete on S.

(d) $c = k^{\oplus} \vee k^*$ if and only if c is indiscrete on S.

Proof. To show $c_X(M) \subseteq c'_X(M)$ for fully additive closure operators c, c' it suffices to consider singleton sets M, and when c, c' are also hereditary it even suffices to consider 2-point spaces X (since for arbitrary X one has $x \in c_X(\{y\})$ iff $x \in c_Y(\{y\})$ for $Y = \{x, y\} \subseteq X$). Therefore, if we now wish to show that a given proper hereditary closure operator c coincides with one of w, k, k^* or μ, then it suffices to show coincidence on 2-point spaces X since also these four closure operators are hereditary and fully additive. If X is discrete, then the coarsest proper operator q is discrete on X, hence all operators in question are discrete on X. If X is indiscrete, then the finest proper operator μ is indiscrete on X, hence all operators in question are indiscrete on X. Hence only the case $X \cong S$ matters, and there are four possible cases : $c_S = j_S = \mu_S$, $c_S = k_S = k^{\oplus}_S$, $c_S = k^*_S$, and $c_S = g_S = w^{\oplus}_S$. QED

Corollary 5.3 *The only proper hereditary and fully additive closure operators of* **Top** *are $w^{\oplus}$, $k^{\oplus}$, k^* and μ. All but w are idempotent and productive.*

Corollary 5.4 *The category* $\mathbf{Top_0}$ *of T_0-spaces has exactly three proper hereditary and fully additive closure operators $k^{\oplus}$, k^* and w, while the category* $\mathbf{Top_1}$ *of T_1 spaces has none.*

Proof. The proof of the above theorem shows $\mu|_{\mathbf{Top_0}} = S|_{\mathbf{Top_0}}$ and $k^{\oplus}|_{\mathbf{Top_1}} = k^*|_{\mathbf{Top_1}} = w|_{\mathbf{Top_1}} = Sj|_{\mathbf{Top_1}}$, so that these operators are no longer proper. In consideration of Proposition 2.3, this completes the proof. QED

Our next objective is to obtain a further application of Theorem 5.2 in the case of closure operators on finite spaces. Note that there can be at most 2^{ω} since there are ω finite spaces and each one can carry only a finite number of closure operators. Of course, the continuity condition in the definition of closure operator restricts severely this number, so that it would not be surprising if there were less than 2^{ω} closure operators in **FinTop**. Nevertheless, the following is proved in [9]:

Theorem 5.5 [9] *There are 2^{ω} distinct closure operators in* **FinTop**.

In Corollary 5.6 we carry out the classification in the case of hereditary and fully additive closure operators.

By Proposition 2.4 the non-discrete closure operators in **FinTop** are determined uniquely by the closure operators in $\mathbf{FinTop_0} = \mathbf{Top_0} \cap \mathbf{FinTop}$ which is equivalent to the category of finite partially ordered sets with order preserving maps. These spaces are subspaces of S^n, $n \in \mathbb{N}$, where S is the Sierpiński space $\{0,1\}$ with open sets $\emptyset, \{1\}$ and S. According to 5.5 there are 2^ω closure operators in **FinTop**, hence the restriction of closure operators to $\mathbf{Top_1}$, contrary to the case $\mathbf{Top_0}$, is strongly non injective (in fact, 2^ω closure operators which are non-discrete in **FinTop** will give the same restriction on $\mathbf{Top_1}$).

Next we note that according to Corollary 5.3 the space S classifies also all hereditary Čech closure operators on **FinTop** since full additivity coincides with additivity on finite spaces.

Corollary 5.6 *The only hereditary proper Čech closure operators on* **FinTop** *(resp.* $\mathbf{FinTop}_0$*), are* $k^\oplus \vee k^*$, k, k^* *and* μ *(resp.* $k^\oplus \vee k^*$, k *and* k^**).*

Next we give a partial result which does not make any additional assumption on the closure operator besides the necessary one, namely $c_S = k_S$.

Lemma 5.7 *If* $c \in CL(\mathbf{Top})$ *satisfies* $c_S = k_S$, *then* $c_X = k_X$ *for every finite space* X *whose specialization partial order is total i. e., S-tame yields X-tame for such spaces X.*

Proof. By Proposition 5.1 $c_X \leq k_X$. Since we have a totally ordered set $(X, \leq)$, for each pair $x < y$ in X there is a monotone map $f : S \to X$ with $f(0) = x$ and $f(1) = y$. Since $0 \in c_S(\{1\})$, this gives $x \in c_X(\{y\})$. Therefore, $c_X \geq k_X$. QED

In order to characterize the Kuratowski closure operator in **Top** we need a property that describes a typical feature of direct products in **Top**. Let c be a closure operator of an epireflective subcategory $\mathcal{A}$ of **Top**.

We say that *c has the Tychonoff property* (formerly, *the finite structure property for products*, abbreviated as f.s.p.p. [6]) if for every family $\{X_i : i \in I\}$ in **Top** and for every $M \subseteq X = \prod_i X_i$, a point $x = (x_i) \in X$ is in $c_X(M)$ precisely when for every finite $F \subseteq I$ one has $\pi_F(x) \in c_{X_F}(\pi_F(M))$, with $\pi_F : X \to X_F = \prod_{i \in F} X_i$ the canonical projection.

The Tychonoff property is stable under meet of closure operators. Therefore every closure operator c has a Tychonoff-hull c^{Tych}, namely a finest closure operator with the Tychonoff property that is coarser than c. It is easy to see that every finitely productive closure operator with the Tychonoff property is also productive. We have examples of closure operators without the Tychonoff property that are not finitely productive, such as w and $w^\oplus$(see 4.5), but no answer to:

Open Problem 1. *Does there exist a closure operator of* **Top** *with the Tychonoff property which fails to be finitely productive?*

The use of the Tychonoff property becomes clear by the following theorem which has been proved in [6] in a more general context. Again for the reader's convenience we give a proof in the context of this paper.

Theorem 5.8 *Let* $d \in CL(\mathcal{A})$ *have the Tychonoff property Then every idempotent closure operator* $c \geq d$ *is productive.*

Proof. In order to check productivity of c fix a product $X = \prod_{i \in I} X_i$ and a subspace $M = \prod_{i \in I} M_i$, where $M_i \subseteq X_i$ for each $i \in I$. For $J \subseteq I$, let $X_J = \prod_{i \in J} X_i$ and $M_J = \prod_{i \in J} M_i$ and write $c_J(M_J)$ instead of $c_{X_J}(M_J)$ $(J \subseteq I)$, and in particular $c_i(M_i)$ instead of $c_{X_i}(M_i)$ $(i \in I)$.

Now $d \le c$ implies $c \le cd \le cc$, hence $c = dc$ since c is idempotent. Therefore, in order to show $x \in c_X(M)$ for $x \in \prod_{i \in I} c_i(M_i)$, it suffices to show $x \in d_X(c_X(M))$. But since d has the Tychonoff property , for that we need to show only $\pi_F(x) \in d_F(\pi_F(c_X(M)))$ for every finite $F \subseteq I$.

In fact, since $M = M_F \times M_{I \setminus F}$ and since c is finitely productive (see Theorem 3.3(c)), one has

$$c_X(M) = c_F(M_F) \times c_{I \setminus F}(M_{I \setminus F})$$

and therefore $\pi_F(c_X(m)) \cong c_F(m_F)$. Furthermore, applying finite productivity again we conclude that $c_F(M_F) = \prod_{i \in F} c_i(M_i)$. Therefore, $x \in \prod_{i \in I} c_i(M_i)$ implies

$$\pi_F(x) \in c_F(M_F) = \pi_F(c_X(M)) \subseteq d_F(\pi_F(c_X(M))),$$

as required. QED

Obviously k has the Tychonoff property while the discrete closure operator does not have it.

Corollary 5.9 *Every idempotent closure operator $c \ge k$ is productive.*

Idempotency is essential in this corollary (e.g., for the θ-closure, as defined after Theorem 5.15 below, θ^ω is not productive, see Theorem 3.3).

Corollary 5.10 *Let $c \in CL(\mathbf{Top})$ have the Tychonoff property. Then c^∞ is productive.*

The above corollary shows again that the sequential closure σ does not have the Tychonoff property

Corollary 5.11 *If $c \in CL(\mathbf{Top})$ is a (non-idempotent) closure operator with $\omega \le o(c) < \infty$, then there exists no closure operator $d \le c$ with the Tychonoff property. In particular, $k \not\le c$, and no power of c can have the Tychonoff property.*

Proof. Arguing for a contradiction assume that c has the Tychonoff property. Then c^∞ is productive by Corollary 5.10. On the other hand, Theorem 3.3 implies that c^∞ cannot be productive as $\omega \le o(c) < \infty$, a contradiction. QED

Corollary 5.12 *Let a closure operator $c \ge k$ of* **Top** *have $o(c) \ge \omega$. Then $o(c) = \infty$. In particular, $o(w) = o(\theta) = \infty$.*

Proof. Follows directly from Corollary 5.11. QED

This corollary applies also to a proper class of closure operators $s(\eta) \ge k$ of infinite order, depending on a total order η, as defined in [6, 8].

Open Problem 2. *Does there exist a closure operator d of* **Top** *with the Tychonoff property and $d < k$?*

Corollary 5.10 shows that the Tychonoff property, along with idempotency, implies productivity. But the Tychonoff property is surely not a necessary condition for productivity: σ does not have it since otherwise its idempotent hull σ^{ω_1} would have to be productive, but it is not (Theorem 3.3). In what follows we give explicit examples which show failure of the Tychonoff property not only for σ, but also for some other closure operators, including the hereditary, idempotent, productive and symmetric Čech closure operator b.

Example 5.13 *None of the closure operators j, μ, σ, b, $k^* \wedge k$, $w^\oplus$, w, $k^\oplus$ or k^* has the Tychonoff property*

Proof. The following example shows that k^* does not have the Tychonoff property. For each $n \in \mathbb{N}$, let X_n be the space of natural numbers $\mathbb{N}$ endowed with the discrete topology. Let $X = \prod_n X_n$ with projections $\pi_n : X \to X_1 \times \ldots \times X_n$ $(n \in \mathbb{N})$ and consider $y_k \in X$ defined by $y_k(i) = i$ for $1 \le i \le k$ and $y_k(i) = k$ for $k \le i < \infty$. For $M = \{y_k : k \in \mathbb{N}\}$ and $x \in X$ defined by $x(n) = n$ for each $n \in \mathbb{N}$, one has $\pi_n(x) = \pi_n(y_n) \in \pi_n(M)$. Hence

$$\pi_n(x) \in \pi_n(M) = j(\pi_n(M)) \tag{4}$$

for each $n \in \mathbb{N}$. On the other hand, $x \notin M$ and X is Hausdorff. This proves that

$$x \notin c_X(M), \tag{5}$$

for every closure operator c such that $c_X = j_X$, e.g., j, μ, b, $k^* \wedge k$, $w^\oplus$, w, $k^\oplus$ or k^*. By (4) and (5) none of these closure operators has the Tychonoff property.

To prove that w does not have the Tychonoff property we need a different example. Let $T = \{0, 1, 2\}$ be equipped with the Alexandroff-Tucker topology of its usual order. For every $n \in \mathbb{N}$ let $X_n = T$, $T^\omega = \prod_n X_n$, $T^n = \prod_{k=1}^n X_k$ and $\pi_n : T^\omega \to T^n$ be the canonical projection. Define $x = (x_n) \in T^\omega$ by letting $x_n = 1$ for every $n \in \mathbb{N}$ and $y^{(k)} = (y_n^{(k)}) \in T^\omega$ by letting $y_n^{(k)} = 2$ if $n \le k$, otherwise $y_n^{(k)} = 0$. Finally, let $M = \{y^{(k)} : k \in \mathbb{N}\}$. Then

$$\pi_n(x) \in k^*_{T^n}(\pi_n(\{y^{(n)}\}) \subseteq k^*_{T^n}(\pi_n(M)) \subseteq w^\oplus_{T^n}(\pi_n(M)) \text{ for all } n \in \mathbb{N}.$$

On the other hand, $x \notin w_{T^\omega}(M)$. Indeed, since $W = \{0, 1\} \times \prod_{k=2}^\infty X_k$ is an open neighbourhood of x missing M, we have $x \notin k_{T^\omega}(M)$. On the other hand, for every $n \in \mathbb{N}$ the set $U = T_n \times \{0\} \times \prod_{k=n+2}^\infty X_k$ is an open neighbourhood of $y^{(n)}$ in T^ω missing the point x, thus $y^{(n)} \notin k_{T^\omega}(\{x\})$. This proves that $x \notin k^*_{T^\omega}(M)$, and consequently $x \notin w_{T^\omega}(M)$.

This example cannot be used for σ since $x \in \sigma_X(M)$. To show that σ does not have the Tychonoff property consider the space $\mathbb{R}/\mathbb{Z}$ with the usual (compact) quotient topology and let X be the power $(\mathbb{R}/\mathbb{Z})^{\mathfrak{c}}$ equipped with the product topology. We note that X is actually a compact topological group. Let $x = (x_\beta) \in X$ be a point such that the coordinates x_β are rationally independent (to this end it suffices to choose a Hamel basis B of $\mathbb{R}$ over $\mathbb{Q}$ containing $1 \in \mathbb{R}$ and let $x_\beta \in \mathbb{R}/\mathbb{Z}$, $\beta \in B \setminus \{1\}$, be the coset of β w. r. t. $\mathbb{Z}$). Then the cyclic subgroup M of X generated by x is k-dense in X. On the other hand, the obvious inequalities $|\sigma_X(M)| \le |M|^\omega = 2^\omega = \mathfrak{c} < 2^{\mathfrak{c}} = |X|$ show that M cannot be σ-dense in X. To finish the proof it suffices to note that for each finite set F the subgroup $\pi_F(M)$ of $(\mathbb{R}/\mathbb{Z})^F$ is σ-dense since the latter group is metrizable so that σ and k coincide here. This shows that σ does not have the Tychonoff property. Note that in this example one can replace $\mathbb{R}/\mathbb{Z}$ by any separable metrizable non-singleton space. More precisely, if I is a set with $\mathfrak{c} < |I| \le 2^{\mathfrak{c}}$, and if and for each $i \in I$ the space X_i is separable metrizable and non-singleton, then by the Hewitt-Marczewski-Pondiczery theorem the product $X = \prod_{i \in I} X_i$ is still separable. Then for every k-dense countable subset M of X the above argument remains valid. QED

Remark 5.14 (a) The first example of 5.13 does not work for p, since $x \in p_X(M)$. It follows from Theorem 5.15 below that p does not have the Tychonoff property since it is hereditary. But the reader should note that p has the *countable Tychonoff property* defined in analogy with the Tychonoff property, but w.r.t. countable subproducts.

(b) The second example of 5.13 shows that every closure operator $c \in CL(\mathbf{Top})$ with $k^* \le c \le w = k^* \vee k$ does not have the Tychonoff property. This applies to any c of the form $c = k^* \vee d$ where $d \le k$ (e.g., $d = \sigma, \alpha \ldots$). In particular, this shows that the answer to Problem 2 is negative in $\mathbf{Top_1}$.

Theorem 5.15 *The closure operator k is the only proper, hereditary, Čech closure operator with the Tychonoff property in* **Top**.

Proof. Certainly, k has all four properties.

Consider now any closure operator c with the properties in question. According to Proposition 2.4, c is determined by its restriction to T_0 spaces. Hence consider $X \in \mathbf{Top_0}$ and $M \subseteq X$. Since X is a subspace of some power S^γ of the Sierpiński space S and c is hereditary, it suffices to consider the case $X = S^\gamma$. Since c has the Tychonoff property, a point $x = (x_i) \in X$ is in $c_X(M)$ iff $\pi_F(x) \in c_{S^F}(\pi_F(M))$ for each finite $F \subseteq I$, where $\pi_F : X = S^\gamma \to S^F$ is the canonical projection. Since S^F is finite, c is fully additive on S^F. Hence, according to Theorem 5.2 we have several cases depending on whether c is discrete, indiscrete, wild or tame on S^F. By Theorem 5.2 these correspond to the closure operators μ, w, k^* and $k^\oplus$. Since neither of $k^*, w, w^\oplus$ and μ has the Tychonoff property by Example 5.13, we are left only with the possibility that c coincides with k on all finite powers S^F. Therefore, since both c and k have the Tychonoff property, but $k^\oplus$ does not, c and k must coincide on X as well. QED

Of the four characteristic properties in Theorem 5.15, certainly neither properness nor the Tychonoff property can be dropped (consider g and σ; note that both closure operators are productive, Čech, and hereditary, but σ does not have the Tychonoff property according to 5.13). But also hereditariness is essential : Velichko's θ-closure ([14]), defined by

$$\theta_X(M) = \{x \in X : U \cap M \neq \emptyset \text{ for every closed neighbourhood } U \text{ of } x \text{ in } X\},$$

is a proper, productive Čech closure operator with the Tychonoff property, which is neither hereditary nor idempotent. We do not have an example showing the essentiality of finite additivity for 5.15.

6 Testing the reals

Now we define further closure operators of **Top** which will be essential in this section. Each of them depends on a triple (P, Q, v) consisting of "pattern space" P, a subspace $Q \subseteq P$ and a point $v \in k_P(Q)$. For a subset M of a space X and $x \in X$ let $x \in c_X^{P,Q,v}(M)$ when there exists a continuous function $f : P \to X$ such that $f(v) = x$ and $f(Q) \subseteq M$. It is easy to see that $c^{P,Q,v}$ is a productive closure operator of **Top** that in general fails to be additive, idempotent or hereditary. Several particular instances are of major interest.

(i) One obtains the sequential closure operator as $\sigma = c^{P,Q,v}$ with P the one-point compactification $\mathbb{N}_\infty$ of the discrete space of natural numbers $\mathbb{N}$, $Q = \mathbb{N}$ and $v = \infty$.

(ii) For the sake of brevity we shall denote $c^{[0,1],\{1/n:n\in\mathbb{N}\},0}$ by α (i.e., $x \in \alpha_X(M)$ for $M \subseteq X$ if there exists a continuous function $f : [0,1] \to X$ with $f(0) = x$ and $f(1/n) \in M$ for each $n \in \mathbb{N}$). Then α is additive, but not hereditary.

(iii) We abbreviate $c^{[0,1],(0,1],0}$ by λ (i.e., $x \in \lambda_X(M)$ for $M \subseteq X$ if there exists a continuous function $f : [0,1] \to X$ with $f(0) = x$ and $f((0,1]) \subseteq M$). Then λ is hereditary but non-additive.

(iv) The closure operator $\delta = c^{[0,1],Q,0}$ with $Q = (0,1] \cap \mathbb{Q}$ satisfies $\delta \leq a * \sigma$. It is neither hereditary nor additive.

(v) More generally, let $C \subseteq (0,1]$ be a subset such that $0 \in k_P(C)$ and $0 \in k_P(Q)$, where $Q := (0,1] \setminus C$. Set $\lambda^{(C)} := c^{[0,1],Q,0}$ with the following relevant example $\lambda^{(1)} := \lambda^{(C)}$ with $C = \{1/n : n \in \mathbb{N}\}$. This closure operator is neither hereditary nor additive.

Obviously, $\mu \le k^* \le \lambda \le \lambda^{(1)} \le \delta \le \alpha \le a \le q$ and $\alpha \le \sigma$. Hence, for every space X

$$\alpha_X \le \sigma_X \wedge a_X. \tag{6}$$

The following easy example shows that in general (6) is not an equality even for metrizable separable spaces X.

Example 6.1 Actually, one can refine (6) as follows: $\alpha \le \sigma * a \le \sigma \wedge a$. To prove that even $\sigma * a = \sigma \wedge a$ may fail take the subspace

$$X = [0,1] \times \{0\} \cup \left\{ \left(0, \frac{1}{n}\right) : n \in \mathbb{N} \right\}$$

of the plane and $M = \{(1,0)\} \cup (X \setminus [0,1] \times \{0\})$. To see that (5) fails it suffices to check that M is $a * \sigma$-closed, while $(0,0) \in \sigma_X(M) \wedge a_X(M)$.

Proposition 6.2 *Let X be a locally arcwise connected space. Then $a_X = q_X$, so that* (6) *becomes an equality for X if and only if $\alpha_X = \sigma_X$. This occurs when X is also first countable.*

Proof. We note that now each component $a_X(x)$ is a clopen set, thus coincides with $q_X(x)$. Then arbitrary unions of quasi-components $q_X(x)$ will be clopen as well. Hence $k_X \le a_X = q_X$. Now equality in (2) is equivalent to $\alpha_X = \sigma_X$ since always $\sigma_X \le q_X$.

Assume that X is first countable. Let $x \in \sigma_X(M) \wedge a_X(M)$ for some subset M of X. Then this must be witnessed by a sequence $x_n \to x$ with $x_n \in M$ and by some $m \in M$ such that $x \in a_X(\{m\})$. As the latter set is clopen, we can assume without loss of generality that all $x_n \in a_X(\{m\})$ and consequently $x_n \in a_X(\{x\})$. Since $x = \lim_n x_n$, we can choose a subsequence $y_k = x_{n_k}$ and a decreasing sequence of neighbourhoods $U_1 \supseteq U_2 \supseteq \dots \supseteq U_n \supseteq \dots$ of x forming a base of the neighbourhood filter of x and such that there exists a continuous function $f_k : [0,1] \to X$ with image contained in U_k such that $f_k(0) = x$ and $f_k(1) = y_k$ for every $k \in \mathbb{N}$. Making use of the functions f_k one can define a single continuous function $f : [0,1] \to X$ with $f(0) = x$ and $f(1/k) = y_k \in M$ for each $k \in \mathbb{N}$. This proves that $x \in \alpha_X(M)$. The equality (5) is proved. QED

In the following lemma we compute the hereditary hulls of these closure operators.

Lemma 6.3 *$\alpha^{he} = \sigma$, $a^{he} = e^{he} = q^{he} = g$ hold both in* **Top** *and* **Tych**.

Proof. Easy extension spaces (which remain in the respective subcategory) provide the necessary inclusions. QED

Let us recall that a closure operator c is $\mathbb{R}$-*discrete* (resp. $\mathbb{R}$-*tame*, $\mathbb{R}$-*indiscrete*) if $c_{\mathbb{R}} = s_{\mathbb{R}}$ (resp. if $c_{\mathbb{R}} = k_{\mathbb{R}}$, resp. if $c_{\mathbb{R}} = g_{\mathbb{R}}$). We prove below that each Čech closure operator in **Top** belongs to one of these three types. To this end we need the following lemma.

Lemma 6.4 *Let c be a closure operator of* **Top**.

(i) c is $\mathbb{R}$-indiscrete (i. e., $c_{\mathbb{R}} = g_{\mathbb{R}} (= q_{\mathbb{R}})$) iff $a \le c$ (and consequently, $a \le c^{\oplus}$).

(ii) $c_{\mathbb{R}} < g_{\mathbb{R}}$ iff $c \le z$.

(iii) $c_{\mathbb{R}} \ge k_{\mathbb{R}}$ iff $c \ge \alpha$.

(iv) $c_{\mathbb{R}} = s_{\mathbb{R}}$ (i. e. c is $\mathbb{R}$-discrete) iff $c \le \varrho$.

Proof. Let X be an arbitrary topological space.

(i) Assume that c is $\mathbb{R}$-indiscrete and let $x \in a_X(\{y\})$. Then there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. One can easily extend f to a continuous function $\tilde{f} : \mathbb{R} \to X$. Then $0 \in c_{\mathbb{R}}(1)$ yields $x = \tilde{f}(0) \in c_X(\{y\})$, hence $a_X(\{y\}) \subseteq c_X(\{y\}) = c_X^{\oplus}(\{y\})$. This proves that $a_X \leq c_X^{\oplus}$. Since X be an arbitrary this implies $a \leq c$. Clearly, $a \leq c$ implies that c is $\mathbb{R}$-indiscrete.

(ii) Let $x \in X$ be such that $x \notin z_X(A)$. Then there is a continuous function $g\colon X \to \mathbb{R}$ such that $g(A) = \{0\}$ and $g(x) = \{1\}$. Since $c_{\mathbb{R}}$ is not indiscrete, $g(x) = 1 \notin c(\{0\}) = c(g(A))$. Since g is "c-continuous" this yields $x \notin c_X(A)$. The inverse implication is again trivial since $z_{\mathbb{R}} = k_{\mathbb{R}}$.

(iii) Let $x \in \alpha_X(M)$ be witnessed by the continuous function $f : [0,1] \to X$ with $f(0) = x$ and $f(A) \subseteq M$, where $A = \{1/n : n \in \mathbb{N}\}$. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(y) = y$ for all $y \in [0,1]$, $h(y) = 1$ for $y \geq 1$ and $h(y) = 0$ for $y \leq 0$. Then $g = f \circ h : [0,1] \to X$ satisfies $g(A) \subseteq M$. Moreover, $0 \in k_{\mathbb{R}}(A) \subseteq c_{\mathbb{R}}(A)$ by hypothesis, so

$$x = g(0) \in g(c_{\mathbb{R}}(A)) \subseteq c_X(g(A)) \subseteq c_X(M).$$

(iv) Let X be a Tychonoff space and assume that $M \subseteq X$ is ϱ-closed. We have to show that M is also c-closed. Since every ϱ-closed set of X is an intersection of inverse images of subsets of $\mathbb{R}$ under continuous maps, it suffices to observe that subsets of $\mathbb{R}$ are c-closed by hypothesis. This proves $c_X \leq \varrho_X$. The inverse implication follows trivially from the equality $\varrho_{\mathbb{R}} = j_{\mathbb{R}}$. QED

Remark 6.5 (1) Obviously, one cannot improve item (iii) by replacing α by a *coarser* closure operator d (like $\sigma \wedge a$, for example), since (iii) applied to $c = \alpha$ (this is possible as $\alpha_{\mathbb{R}} = k_{\mathbb{R}}$) would give $\alpha \geq d$.

(2) By (i) and Proposition 6.2, $c < g$ and c is $\mathbb{R}$-indiscrete if and only if $a \leq c \leq q$. This gives also $a_X \leq c_X^{\oplus} \leq q_X^{\oplus}$.

(3) Note that for a Tychonoff space X (iv) gives $c_X \leq p_X$ (in particular $c_X = s_X$ if X has countable pseudocharacter[3]).

This lemma gives the possibility to classify the closure operators in **Tych** with respect to the effect they produce on the reals $\mathbb{R}$. Let us start with some examples.

Example 6.6 (a) $\mathbb{R}$-indiscrete closure operators: g, q, e, a. More generally, any closure operator coarser than a is $\mathbb{R}$-indiscrete. It can be shown that an $\mathbb{R}$-indiscrete closure operator c satisfies $c_S = g_S$ (compare with the previous section).

(b) $\mathbb{R}$-tame closure operators: θ, k, κ, σ, α. More generally, any closure operator c satisfying $\alpha \leq c \leq z$ is $\mathbb{R}$-tame.

(c) $\mathbb{R}$-discrete closure operators: ϱ, p, k^*, b, μ. More generally, any closure operator finer than p or k^* is $\mathbb{R}$-discrete.

We will separately classify the $\mathbb{R}$-discrete, the $\mathbb{R}$-tame and the $\mathbb{R}$-indiscrete closure operators on spaces close to being metrizable. Let us first see that these are the sole possibilities for a Čech closure operator of **Top**:

Theorem 6.7 *Let c be a closure operator of* **Top**. *Then:*

(i) Either c is $\mathbb{R}$-indiscrete, or $c_{\mathbb{R}} \leq k_{\mathbb{R}}$.

[3]=each point is a G_δ set

(ii) Assume c is neither $\mathbb{R}$-indiscrete nor $\mathbb{R}$-tame. Then $c_\mathbb{R}$ is Čech if and only if c is $\mathbb{R}$-discrete.

Proof. (i) Follows from item (ii) of Lemma 6.4 with $X = \mathbb{R}$.

(ii) Under this assumption on the closure operator c there exists a converging sequence $a_n \to a$ in $\mathbb{R}$ such that

$$a \notin c_\mathbb{R}(A), \tag{6}$$

where $A = \{a_n : n \in \mathbb{N}\}$. It suffices to prove that for each point $x \in \mathbb{R}$ we have $x \notin c_\mathbb{R}(\mathbb{R} \setminus \{x\})$. To this end we "split" $\mathbb{R} \setminus \{x\}$ into a union $\mathbb{R} \setminus \{x\} = M \cup N$ and we prove that $x \notin c_\mathbb{R}(M)$ and $x \notin c_\mathbb{R}(N)$. Then additivity of $c_\mathbb{R}$ applies. For $n \in \mathbb{N}$ set

$$F_n = \{y \in \mathbb{R} : 1/(n+1) \leq \min\{|x-y|, 1\} \leq 1/n\}.$$

Now define $f : \mathbb{R} \to \mathbb{R}$ by $f(y) = \min\{|x-y|, 1\}$. This is a Lipschitz function, so in particular (uniformly) continuous. Its range is in fact $[0,1]$. Next we compose f with the piece-wise linear continuous function $h : \mathbb{R} \to [0,1]$, such that $h((-\infty, 0] = a$, $h([1/(2n+1), 1/(2n)] = a_n$ $(n \in \mathbb{N})$ and $h([1, +\infty) = a_1$. Then the function h will be continuous. The composition $g = hf : \mathbb{R} \to \mathbb{R}$ sends the set $M = \bigcup_{n \in \mathbb{N}} F_{2n}$ into the set A. Since $g(x) = a$ and $f(M) \subseteq A$, (6) yields (by the continuity property of closure operators, cf. Definition 2.1) that $x \notin c_\mathbb{R}(M)$. Analogously one proves that $x \notin c_\mathbb{R}(N)$, where $N = \bigcup_{n \in \mathbb{N}} F_{2n-1}$. Since obviously $\mathbb{R} \setminus \{x\} = M \cup N$, we are done. QED

Remark 6.8 One can show that there exist infinitely many non-additive closure operators that are distinct on $\mathbb{R}$. More precisely, for $c \in CL(\mathbf{Top})$ $c_\mathbb{R}$ is not Čech if and only if $c_\mathbb{R} = \lambda_\mathbb{R}$ or $\lambda_\mathbb{R}^{(1)} \leq c_\mathbb{R} \leq \delta_\mathbb{R}$. Appropriate choices of C give $\mathfrak{c}$-many closure operators of the form $\lambda_\mathbb{R}^{(C)}$ that are distinct on $\mathbb{R}$ (see [7]).

7 Characterization of the Kuratowski closure in smaller subcategories

Here we characterize the Kuratowski closure as a Čech operator in smaller subcategories of **Top**.

First we classify Tychonoff spaces of countable pseudocharacter by means of $\mathbb{R}$-discrete closure operators.

Theorem 7.1 *For a Tychonoff space X the following are equivalent:*

(i) every $\mathbb{R}$-discrete closure operator of **Top** *is discrete on X, i. e., $\mathbb{R}$-discrete yields X-discrete;*

(ii) X has countable pseudocharacter.

Proof. Apply Lemma 6.4. QED

Now we give the same result from the point of view of the closure operator.

Corollary 7.2 *A closure operator c on* **Tych** *is $\mathbb{R}$-discrete if and only if it satisfies $c \leq p$. In particular, c is discrete on every Tychonoff space of countable pseudocharacter.*

A space X is submetrizable if X admits a coarser metrizable topology.

Corollary 7.3 *Let c be an $\mathbb{R}$-discrete closure operator of* **Tych**. *Then it is X-discrete for every Tychonoff space which is either first countable or submetrizable.*

One can be rather satisfied with this classification of $\mathbb{R}$-discrete closure operators since p is the ω_1-additive core of k, so that we have again a kind of characterization of k.

Now we classify $\mathbb{R}$-indiscrete closure operators.

Corollary 7.4 *Let X be a locally arcwise connected space and c be a proper $\mathbb{R}$-indiscrete closure operator of* **Top**. *Then $k_X \leq a_X = c_X = q_X$. Consequently, $c_X = k_X$ iff X is a symmetric Alexandroff-Tucker space.*

Proof. By Proposition 6.2 $a_X = q_X$. Now Lemma 6.4 applies to get $k_X \leq a_X = c_X = q_X$. The conclusion follows from Proposition 4.7 since the hypothesis implies that k_X is fully additive and symmetric (as $c_X = q_X^{\oplus}$). Hence X is a symmetric Alexandroff-Tucker space. QED

This corollary shows that for a proper closure operator c with $c_{\mathbb{R}} = g_{\mathbb{R}}$ and a locally arcwise connected space X the equality $c_X = q_X$ holds.

Now we classify $\mathbb{R}$-tame closure operators.

Theorem 7.5 *Let c be a $\mathbb{R}$-tame closure operator of* **Top**. *Then $\alpha \leq c \leq z$ in* **Top** *and $\alpha \leq c \leq k$ in* **Tych**.

Proof. If $c_{\mathbb{R}} = k_{\mathbb{R}}$, then by Lemma 6.4 we get $\alpha_X \leq c_X \leq z_X$. QED

It would be interesting to characterize the class **A** (**B**) of Tychonoff spaces (Fréchet-Urysohn Tychonoff spaces, respectively) X such that $\alpha_X = \sigma_X$. Since $\alpha_X \leq c_X \leq k_X$, for Tychonoff spaces, then obviously $\sigma_X \leq c_X \leq k_X$ will hold for each space $X \in \mathbf{A}$, and so $c_X = k_X$ for spaces $X \in \mathbf{B}$.

The next theorem characterizes k as the only proper, hereditary, Čech closure operator on Fréchet-Urysohn spaces.

Theorem 7.6 *Let c be an $\mathbb{R}$-Čech closure operator of* **Top**, *such that the restriction of c on the subcategory of Tychonoff spaces of countable pseudocharacter is proper. Then $c|_{\mathbf{FU}} = k|_{\mathbf{FU}}$* [4], *iff the restriction $c|_{\mathbf{FU}}$ is hereditary*

Proof. According to Theorem 6.7 there are three possibilities for c. Our first aim is to rule out the two of them.

Assume that c is $\mathbb{R}$-indiscrete, i. e. $c_{\mathbb{R}} = g_{\mathbb{R}}$, then $a \leq c$ holds by item i) of Lemma 6.4. Now hereditariness and properness of c give $c = c_{he} \leq q_{he} = w$, so that $c_{\mathbb{R}} \leq w_{\mathbb{R}} = k_{\mathbb{R}} < g_{\mathbb{R}}$, a contradiction.

Assume now c is $\mathbb{R}$-discrete. Then by Corollary 7.2 c is discrete for each Tychonoff space of countable pseudocharacter, this contradicts our hypothesis. Therefore c is $\mathbb{R}$-tame according to Theorem 6.7. Now we get also $\alpha_X \leq c_X \leq z$ by Theorem 7.5. So again by hereditariness of c and Lemma 6.3 we get $\sigma_X = \alpha_X^{he} \leq c_X \leq k_X$ for every Tychonoff space X. Hence $c_X = k_X$ for each Fréchet-Urysohn space X. QED

One may ask how strong is the condition of hereditariness. The following example shows that the conclusion of the above theorem, i. e. $c_X = k_X$, does not remain valid even for separable metric spaces if the closure operator is not hereditary.

Example 7.7 Let X be the disjoint sum of an interval and a converging sequence. Then one has $j_X < \alpha_X < k_X$.

[4] i. e. $c_X = k_X$ for each Fréchet-Urysohn space X.

Theorem 7.8 *Let c be a Cech closure operator of* **Top**. *If c_X is proper on X, then c_X coincides with the Kuratowski closure operator k_X for every first countable Tychonoff space X which is either connected and locally arcwise connected or zero-dimensional.*

Proof. In case X is connected and locally arcwise connected, $c_X = q_X$ by Corollary 7.4 when c is $\mathbb{R}$-indiscrete. Since X is connected, $q_X = g_X$, a contradiction. If c were $\mathbb{R}$-discrete, then $c_X = j_X$ by Corollary 7.2. Therefore c must be $\mathbb{R}$-tame. Then by Theorem 7.5 $\alpha_X \leq c_X \leq k_X$. Since $\alpha = \sigma$ by Proposition 6.2, we conclude that $\sigma_X = c_X = k_X$ as the space X is first countable.

Now we prove that any proper additive closure operator c on a first countable space zero-dimensional space X coincides with the Kuratowski closure k_X. By Proposition 6.2 $c_X \leq q_X$. Since X is zero-dimensional, $q_X = k_X$ by Proposition 2.2. Thus $c_X \leq k_X$. Assume that $c_X < k_X$. Then there exists a converging sequence $a_n \to a$ in X such that

$$a \notin c_X(A), \tag{7}$$

where $A = \{a_n : n \in \mathbb{N}\}$. Now we are ready to prove that under this assumption the closure operator c_X is discrete. It suffices to prove that for each point $x \in X$ we have $x \notin c_X(X \setminus \{x\})$. To this end we "split" $X \setminus \{x\}$ into a union $X \setminus \{x\} = M \cup N$ and we prove that $x \notin c_X(M)$ and $x \notin c_X(N)$. Then additivity of c applies. For every $n \in \mathbb{N}$ choose a clopen neighbourhood F_n of x such that the family $\{F_n : n \in \mathbb{N}\}$ is a proper descending chain and form a base of the neighbourhood system of x. Set $M = \bigcup_{n\in\mathbb{N}} F_{2n} \setminus F_{2n+1}$ and $M' = M \cup \{x\}$. Now define $f : X \to X$ by $f(x) = a = f[X \setminus M']$, $f[F_{2n} \setminus F_{2n+1}] = a_n$ for $n \in \mathbb{N}$. This is a continuous function which sends M into the set A. Since $f(x) = a$ and $f(M) \subseteq A$, (7) yields (by continuity of closure operators, cf. Definition 2.1) that $x \notin c_X(M)$. Analogously one proves that $x \notin c_X(N)$, where $N = \bigcup_{n\in\mathbb{N}} F_{2n-1} \setminus F_{2n}$. Since obviously $X \setminus \{x\} = M \cup N$, we are done. QED

Remark 7.9 (a) By Example 7.7 one cannot omit local arcwise connectedness (zero-dimensionality, respectively) of the metrizable space X to claim $c_X = k_X$. On the other hand, one cannot omit first countability of the space X even when X is a connected and locally arcwise connected topological group, while the closure operator is even hereditary (take $X = (\mathbb{R}/\mathbb{Q})^{\omega_1}$, then $\sigma_X < k_X$.)

(b) To see that additivity is essential in the above theorem it suffices to take $c = c^{P,Q,v}$, where $P = [0,1) \cap \mathbb{Q}$, $v = 0$ and $Q = P \setminus \{v\}$. This is a hereditary non-additive closure operator strictly finer than k.

Corollary 7.10 *All proper Čech closure operators of* **Top** *coincide on a connected locally Hilbert space (= Q-manifold) with the Kuratowski closure.*

Corollary 7.11 *Let X be one of the spaces $\mathbb{R}^n$ or ℓ_2. If a Čech closure operator c of* **Top** *is proper on X, then c_X coincides with k_X.*

Roughly speaking, we have shown that a closure operator on a locally "good" space X must coincide with the Kuratowski closure.

References

[1] A. V. Arhangel'skii and S. Franklin, *Ordinal invariants for topological spaces*, *Mich. Math. J.* **15**, (1968) 313-320.

[2] S. Baron, *Note on epi in T_o*, Canad. Math. Bull., **11** (1968) 503-504.

[3] D. Dikranjan, *Semiregular closure operators and epimorphisms in topological categories*, Suppl. Rend. Circ. Mat. Palermo, Serie II, n. **29** (1992) 105-160.

[4] D. Dikranjan and E. Giuli, *Closure operators I*, Topology Appl., **27** (1987) 129-143.

[5] D. Dikranjan, E. Giuli and W. Tholen, *Closure operators II*, in: J. Adamek and S. McLane ed., *Proceedings of the International Conference on Categorical Topology*, Prague, 1988 (World Scientific, Singapore 1989) 297-335.

[6] D. Dikranjan and Walter Tholen, *Categorical Structure of Closure Operators with Applications to Topology, Algebra and Discrete Mathematics*, *Mathematics and its Applications*, vol. **346**, Kluwer Academic Publishers, Dordrecht-Boston-London 1995, pp. 358+xviii.

[7] D. Dikranjan, Walter Tholen and Stephen Watson, *Classification of closure operators for categories of topological spaces, II: non-additive closure operators, regular closure operators and weak hereditariness*, work in preparation.

[8] D. Dikranjan and Stephen Watson, *The Category of $S(\alpha)$-Spaces is not Cowellpowered*, Topology Appl., **61** (1995) 137-150.

[9] D. Dikranjan and Stephen Watson, *Finite topological spaces and their closure operators*, work in progress.

[10] R. Engelking, General Topology, Revised and Completed Edition (*Sigma Series in Pure Mathematics*, vol. **6**, Heldermann Verlag, Berlin, 1989).

[11] H. Herrlich, *Wann sind alle stetigen Abbildungen in Y konstant ?* Math. Zeitschr. **90** (1965), 152-154.

[12] L. Nel and R. G. Wilson, *Epireflections in the category of T_0-spaces*, Fund. Math. **75** (1972) 69-74.

[13] S. Salbany, *Reflective subcategories and closure operators*, Categorical topology (Proc. Conf., Mannheim, 1975), pp. 548–565. Lecture Notes in Math., Vol. **540**, Springer, Berlin, 1976.

[14] H. Velichko, H-closed topological spaces, *Mat. Sb.(N.S.)* **70**(112), (1966) 98-112 (Amer. Math. Soc. Transl. **78**, Ser. 2 (1969) 103-118.)