Descent Equivalence

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Abstract

For a $\mathbf{C}$-indexed category $\mathbb{A}$, an $\mathbb{A}$-descent equivalence is a morphism of bundles in $\mathbf{C}$ which induces an equivalence between the $\mathbb{A}$-descent categories of its domain and codomain. In this note, properties of such morphisms are studied, and those morphisms which are $\mathbb{A}$-descent equivalences for all $\mathbf{C}$-indexed categories $\mathbb{A}$ are fully characterized.

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0. Introduction. Descent Theory was developed by Grothendieck [1], [2] in the context of fibred categories. If the category $\mathbf{E}$ is fibred over the category $\mathbf{C}$ with pullbacks, then each morphism $p : E \to B$ of $\mathbf{C}$ is associated with its descent category $\mathrm{Des}_{\mathbf{E}}(p)$ (see, for example, [4], for details). Having defined descent structures, it seems natural to us to compare two bundles (E, p) and (X, φ) over B in the descent sense and to ask:

When do two bundles (E, p) and (X, φ) over B have the "same descent behavior"?

More clearly, we would like to know *under which conditions a morphism of the two bundles (E, p) and (X, φ) over B would render equivalent descent categories.* To this end, we shall examine here for morphisms of bundles the notion of descent equivalence, which was introduced in the first author's Ph.D. thesis [3], and study its properties.

We formulate this notion in the (essentially equivalent) language of internal categories and of indexed categories (see [5,6,7]), rather than that of fibrations, making extensive use of some of the results of [5], which we recall here in sufficient detail.

After some preliminary observations concerning descent equivalences and their comparison with effective descent morphisms, in Theorem 1 we give a somewhat surprising necessary and sufficient condition for a morphism of bundles to be a descent equivalence (with respect to *all* indexed categories): one just needs the existence of *any* morphism of bundles in the opposite direction. In Theorem 2, we characterize those descent equivalences whose domain or codomain is given by an effective descent morphism.

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1. Internal categories. Recall that *an internal category D* (cf. [6]) in $\mathbf{C}$ is given by a diagram

$$\begin{tikzcd} D_2 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_1"'] & D_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & D_0 \arrow[l, "e" description] \end{tikzcd}$$

in $\mathbf{C}$, which satisfies

I1. $de = 1_{D_0} = ce$,

I2. $dm = d\pi_2,\ cm = c\pi_1$,

I3. $m(1_{D_1} \times m) = m(m \times 1_{D_1})$,

I4. $m < 1_{D_1}, ed > = 1_{D_1} = m < ec, 1_{D_1} >$,

where $D_2,\ \pi_1,\ \pi_2$ are given by the following pullback diagram in $\mathbf{C}$:

$$\begin{array}{ccc} D_2 & \xrightarrow{\pi_2} & D_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle c} \\ D_1 & \xrightarrow{d} & D_0 \end{array}$$

An internal functor $f : D \to D'$ between two internal categories D, D' in $\mathbf{C}$ is given by two morphisms $f_0 : D_0 \to D'_0, f_1 : D_1 \to D'_1$ of $\mathbf{C}$ such that

F1. $f_0 d = d' f_1,\ f_0 c = c' f_1,$

F2. $f_1 e = e' f_0,\ f_1 m = m' f_2,$

where $f_2 = f_1 \times f_1 : D_1 \times_{D_0} D_1 \to D'_1 \times_{D'_0} D'_1$.

Composition of internal functors is defined in the obvious way. Hence one obtains $\mathbf{cat}(\mathbf{C})$, the category of all internal categories and internal functors in $\mathbf{C}$. It is actually a 2-category (see [5]) since one can define the notion of *internal natural transformation* $\alpha : f \to g$ of internal functors $f,\ g : D \to D'$, given by a morphism $\alpha : D_0 \to D'_1$ in $\mathbf{C}$ such that

T1. $d'\alpha = f_0,\ c'\alpha = g_0,$

T2. $m' < \alpha c, f_1 > = m' < g_1, \alpha d >$.

The *composite* $\beta\alpha : f \to h$ of internal natural transformations $\alpha : f \to g$ and $\beta : g \to h$ is the morphism

$$m' < \beta, \alpha >: D_0 \to D'_1,$$

and the *identity internal natural transformation* $1_f : f \to f$ is the morphism

$$e' f_0 : D_0 \to D'_1.$$

An internal functor $f : D \to D'$ of $\mathbf{C}$ is an *internal category equivalence* if there is an internal functor $g : D' \to D$ such that

$$gf \cong 1_D \ \text{ and } \ fg \cong 1_{D'}.$$

For example, if $p : E \to B$ is a morphism in $\mathbf{C}$, then

$$(E \times_B E) \times_E (E \times_B E) \cong E \times_B E \times_B E \quad \begin{array}{c} \xrightarrow{\pi_{23}} \\ \xrightarrow{\pi_{13}} \\ \xrightarrow[\pi_{12}]{} \end{array} E \times_B E \quad \begin{array}{c} \xrightarrow{\pi_2} \\ \xleftarrow{e} \\ \xrightarrow[\pi_1]{} \end{array} \quad E$$

is an internal category in $\mathbf{C}$, where $e =< 1_E, 1_E >$, (π_1, π_2) is the kernel pair of p, π_{12} and π_{23} are such that $\pi_1\pi_{23} = \pi_2\pi_{12}$ (pullback square) and $\pi_{13} =< \pi_1\pi_{12}, \pi_2\pi_{23} >$. This internal category is denoted by $\mathrm{Eq}(p)$. Every object B in $\mathbf{C}$ can be viewed as a discrete internal category B of $\mathbf{C}$:

$$
\begin{tikzcd}
B \arrow[r, shift left=2, "1_B"] \arrow[r, "1_B" description] \arrow[r, shift right=2, "1_B"'] & B \arrow[r, shift left=2, "1_B"] \arrow[r, shift right=2, "1_B"'] & B \arrow[l, "1_B" description]
\end{tikzcd}
$$

Clearly, $\mathrm{Eq}(1_B)$ is isomorphic to the above discrete internal category B.

For any morphism $q : (E, p) \to (X, \varphi)$ in $\mathbf{C}/B$, as in [5] one constructs the internal functor

$$\tilde{q}\colon \mathrm{Eq}(p) \to \mathrm{Eq}(\varphi),$$

where $\tilde{q}_0 = q$, $\tilde{q}_1 = q \times_B q$. Then, for a fixed object B of $\mathbf{C}$, the assignments:

$$(E, p) \mapsto \mathrm{Eq}(p) \;\text{ and }\; q \mapsto \tilde{q},$$

define the functor

$$\mathrm{Eq}_B : \; \mathbf{C}/B \to \mathbf{cat}(\mathbf{C}).$$

2. Indexed categories. A $\mathbf{C}$-*indexed category* $\mathbb{A}$ or a pseudo-functor $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ (cf. [5,7,8]) consists of the following data:

- for every object E of $\mathbf{C}$ a category $\mathbb{A}^E$
- for every morphism $f : E \to D$ of $\mathbf{C}$ a functor $f^* : \mathbb{A}^D \to \mathbb{A}^E$,
- for every $f : E \to D,\ g : D \to B$ in $\mathbf{C}$, two natural isomorphisms:

$$i^D : 1_{\mathbb{A}^D} \to (1_D)^*, \quad j^{f,g} : f^* g^* \to (gf)^*$$

which make the diagrams

$$
\begin{tikzcd}
f^* \arrow[r, "f^* i^D"] \arrow[d, "i^E f^*"'] \arrow[dr, "1_{f^*}"] & f^*(1_D)^* \arrow[d, "j^{f,1_D}"] \\
(1_E)^* f^* \arrow[r, "j^{1_E,f}"'] & f^*
\end{tikzcd}
$$

and

$$\begin{tikzcd}
f^*g^*h^* \arrow[r, "f^*j^{g,h}"] \arrow[d, "j^{f,g}h^*"'] & f^*(hg)^* \arrow[d, "j^{f,hg}"] \\
(gf)^*h^* \arrow[r, "j^{gf,h}"'] & (hgf)^*
\end{tikzcd}$$

commute.

For example,

$$\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$$

given by $B \longmapsto \mathbf{C}/B$ and $(f : E \to B) \longmapsto f^* : \mathbf{C}/B \to \mathbf{C}/E$, the pullback functor along f, is a $\mathbf{C}$-indexed category, also called the *basic* $\mathbf{C}$*-indexed category*.

Let D be an internal category in $\mathbf{C}$. One defines $\mathbb{A}^D$ (cf. [5]) to be the category with

- objects all pairs of (C, ξ), where $C \in \mathrm{ob}\mathbb{A}^{D_0}$ and $\xi : d^*C \to c^*C$ is a morphism in $\mathbb{A}^{D_1}$ such that

$$\begin{tikzcd}
e^*d^*C \arrow[rr, "e^*\xi"] \arrow[dr, "\cong"'] & & e^*c^*C \arrow[dl, "\cong"] \\
& C &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& (\pi_2)^*c^*C \arrow[r, "\cong"] & (\pi_1)^*d^*C \arrow[dr, "(\pi_1)^*\xi"] & \\
(\pi_2)^*d^*C \arrow[ur, "(\pi_2)^*\xi"] \arrow[dr, "\cong"'] & & & (\pi_1)^*c^*C \\
& m^*d^*C \arrow[r, "m^*\xi"] & m^*c^*C \arrow[ur, "\cong"'] &
\end{tikzcd}$$

commute, in $\mathbb{A}^{D_0}$ and $\mathbb{A}^{D_2}$, respectively, with the above natural isomorphisms arising from I1 and I2,

- morphisms $h : (C, \xi) \to (C', \xi')$ of $\mathbb{A}^D$ given by morphisms $h : C \to C'$ of $\mathbb{A}^{D_0}$ such that

$$\begin{tikzcd} d^*C \arrow[r, "d^*h"] \arrow[d, "\xi"'] & d^*C' \arrow[d, "\xi'"] \\ c^*C \arrow[r, "c^*h"] & c^*C' \end{tikzcd}$$

commutes in $\mathbb{A}^{D_1}$.

In [5], it was proved that, *for every* $\mathbf{C}$*-indexed category* $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$*, the extension*

$$\mathbb{A} : \mathbf{cat}(\mathbf{C})^{\mathrm{op}} \to \mathbf{CAT}$$

given by the assignment $D \to \mathbb{A}^D$*, is a pseudo-functor of* 2*-categories.* As a consequence, one obtains that *for every internal category equivalence* $f : D \to D'$ *of* $\mathbf{C}$*, the functor* $f^* : \mathbb{A}^{D'} \to \mathbb{A}^D$ *is an equivalence of categories.*

3. Effective descent, descent equivalence. Now, let $\mathrm{Des}_{\mathbb{A}}$ be the pseudo-functor $\mathbb{A} \circ \mathrm{Eq}_B$:

$$\begin{tikzcd} (\mathbf{C}/B)^{\mathrm{op}} \arrow[rr, "\mathrm{Des}_{\mathbb{A}}(\,)"] \arrow[dr, "\mathrm{Eq}_B"'] & & (\mathbf{C}/B)\backslash\mathbf{CAT} \\ & \mathbf{cat}(\mathbf{C})^{\mathrm{op}} \arrow[ur, "\mathbb{A}"'] & \end{tikzcd}$$

The discrete functor $p : E \to B$ can be factored as

$$\begin{tikzcd} B & & \mathrm{Eq}(p) \arrow[ll, "\overline{p}"'] \\ & E \arrow[ul, "p"] \arrow[ur, "\delta"'] & \end{tikzcd}$$

where $\overline{p}_0 = p$, $\overline{p}_1 = p\pi_1 = p\pi_2$, $\delta_0 = 1_E$, $\delta_1 = e =< 1_E, 1_E >$, with (π_1, π_2) the kernel pair of p. Applying $\mathbb{A}$ to the last diagram, one has a commutative diagram (up to natural isomorphism) in $\mathbf{CAT}$:

$$\begin{tikzcd} \mathbb{A}^B \arrow[rr, "\Phi^p = \overline{p}^*"] \arrow[dr, "p^*"'] & & \mathrm{Des}_{\mathbb{A}}(p) \arrow[dl, "\delta^*"] \\ & \mathbb{A}^E & \end{tikzcd}$$

p is called *an $\mathbb{A}$-descent morphism (effective $\mathbb{A}$-descent morphism)* if the comparison functor Φ^p is full and faithful (an equivalence of categories). p is called *an absolute (effective) descent morphism* if it is an (effective) $\mathbb{A}$-descent morphism for every $\mathbf{C}$-indexed category $\mathbb{A}$.

For a morphism $q : (E, p) \to (X, \varphi)$ in $\mathbf{C}/B$, the authors of [9] considered the following diagram in $\mathbf{cat}(\mathbf{C})$:

$$\begin{tikzcd}
 & \mathrm{Eq}(1_B) & \\
\mathrm{Eq}(\varphi) \arrow[ur, "\overline{\varphi}"] & & \mathrm{Eq}(p) \arrow[ul, "\overline{p}"'] \arrow[ll, "\tilde{q}"] \\
\mathrm{Eq}(1_X) \arrow[u, "\delta_X"] & & \mathrm{Eq}(q) \arrow[u, "i_\varphi"'] \arrow[ll, "\overline{q}"']
\end{tikzcd} \tag{1}$$

where $(i_\varphi)_0 = 1_E,\ (i_\varphi)_1 = 1_E \times_\varphi 1_E,\ \tilde{q}_0 = q,\ \tilde{q}_1 = q \times_B q,\ (\delta_X)_0 = 1_X,\ (\delta_X)_1 = \Delta_X,\ \overline{p}_0 = p,\ \overline{p}_1 = p\pi_1 = p\pi_2$, and where (π_1, π_2) is the kernel pair of p.

Applying $\mathbb{A}$ to diagram (1), one obtains the following commutative diagram (up to natural isomorphisms) in $\mathbf{CAT}$:

$$\begin{tikzcd}
 & \mathbb{A}^B \arrow[dl, "\Phi^\varphi"'] \arrow[dr, "\Phi^p"] & \\
\mathrm{Des}_{\mathbb{A}}(X, \varphi) \arrow[rr, "\mathrm{Des}_{\mathbb{A}}(q)"'] \arrow[d, "U^\varphi"'] & & \mathrm{Des}_{\mathbb{A}}(E, p) \arrow[d, "V^\varphi"] \\
\mathbb{A}^X \arrow[rr, "\Phi^q"'] & & \mathrm{Des}_{\mathbb{A}}(E, q)
\end{tikzcd} \tag{2}$$

where $U^\varphi = \delta_X^*,\ \ V^\varphi = i_\varphi^*,\ \ \Phi^p = \overline{p}^*$, and $\mathrm{Des}_{\mathbb{A}}(q) = \tilde{q}^*$.

Definition. *Let $q : (E, p) \to (X, \varphi)$ be a morphism in $\mathbf{C}/B$. We call q* an $\mathbb{A}$-descent equivalence ($\mathbb{A}$-descent pre-equivalence) *if $\mathrm{Des}_{\mathbb{A}}(q)$ is an equivalence of categories (full and faithful). We call*

q an absolute descent equivalence (absolute descent pre-equivalence) *if* $\mathrm{Des}_{\mathbb{A}}(q)$ *is an equivalence of categories (full and faithful) for every* $\mathbf{C}$*-indexed category* $\mathbb{A}$.

4. Properties of descent equivalences. Functoriality of $\mathrm{Des}_{\mathbb{A}}(\ )$ leads immediately to a number of consequences.

Proposition 1. *The morphism* $p : (E,p) \to (B,1_B)$ *in* $\mathbf{C}/B$ *is an* $\mathbb{A}$*-descent pre-equivalence (*$\mathbb{A}$*-descent equivalence) if and only if* p *is an* $\mathbb{A}$*-descent (effective* $\mathbb{A}$*-descent) morphism.*

Proof. Applying Eq to the following commutative diagram:

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "p"'] & & B \arrow[dl, "1_B"] \\ & B & \end{tikzcd}$$

we obtain the commutative diagram:

$$\begin{tikzcd} & \mathrm{Eq}(1_B) & \\ \mathrm{Eq}(p) \arrow[ur, "\overline{p}"] \arrow[rr, "\tilde{p}"'] & & \mathrm{Eq}(1_B) \arrow[ul, "\overline{1}_B"'] \end{tikzcd}$$

Clearly, with the notation of the previous section, $\overline{1}_B = 1_{\mathrm{Eq}(1_B)}$, $\tilde{p} = \overline{p}$. Hence, $\tilde{p}^*$ is an equivalence of categories if and only if $\overline{p}^*$ is an equivalence of categories, as desired. □

One also easily obtains:

Proposition 2. *Let* $q : (E,p) \to (X,\varphi)$, $r : (X,\varphi) \to (Y,\xi)$ *be morphisms in* $\mathbf{C}/B$.

(1) *If two of* q, r, *and* rq *are* $\mathbb{A}$*-descent equivalences, so is the third one.*

(2) *If* r *is an* $\mathbb{A}$*-descent equivalence, then* q *is an* $\mathbb{A}$*-descent pre-equivalence if and only if* rq *is an* $\mathbb{A}$*-descent pre-equivalence.* □

It is also easy to show that $\mathbb{A}$-descent (pre-)equivalences have the intended invariance property:

Proposition 3. *Let $q : (E,p) \to (X,\varphi)$ be an $\mathbb{A}$-descent pre-equivalence ($\mathbb{A}$-descent equivalence) in $\mathbf{C}/B$. Then p is an $\mathbb{A}$-descent (effective $\mathbb{A}$-descent) morphism if and only if φ is an $\mathbb{A}$-descent (effective $\mathbb{A}$-descent) morphism.*

Proof. By Diagram (2), $\mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi} = \Phi^{p}$ (up to natural isomorphism). If q is an $\mathbb{A}$-descent equivalence, then $\mathrm{Des}_{\mathbb{A}}(q)$ is an equivalence of categories. Therefore, Φ^{φ} is an equivalence of categories if and only if Φ^{p} is an equivalence of categories. Hence p is an effective $\mathbb{A}$-descent morphism if and only if φ is an effective $\mathbb{A}$-descent morphism.

Suppose now that q is an $\mathbb{A}$-descent pre-equivalence. Then $\mathrm{Des}_{\mathbb{A}}(q)$ is full and faithful. If φ is $\mathbb{A}$-descent pre-quivalence morphism, then $\Phi^{p} = \mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi}$ (up to isomorphism) is full and faithful. Hence p is an $\mathbb{A}$-descent morphism. On the other hand, if p is $\mathbb{A}$-descent morphism, then $\mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi} = \Phi^{p}$ (up to isomorphism) is full and faithful, and so is Φ^{φ}. □

5. A necessary and sufficient condition for absolute descent equivalences. In any category, the absolutely effective descent morphisms are precisely the split epimorphisms [5]. A characterization of the absolute descent equivalences is given by the following:

Theorem 1. *Let $q : (E,p) \to (X,\varphi)$ be a morphism in $\mathbf{C}/B$. Then q is an absolute descent equivalence if and only if there is any morphism $s : (X,\varphi) \to (E,p)$ in $\mathbf{C}/B$.*

Proof. $\Longleftarrow$: By hypothesis, we have

$$p = \varphi q \text{ and } ps = \varphi.$$

So there exist two internal functors

$$\tilde{s}\colon \mathrm{Eq}(\varphi) \to \mathrm{Eq}(p) \text{ and } \tilde{q}\colon \mathrm{Eq}(p) \to \mathrm{Eq}(\varphi).$$

We claim that $\tilde{s}\tilde{q} \cong 1_{\mathrm{Eq}(p)}$ and $\tilde{q}\tilde{s} \cong 1_{\mathrm{Eq}(\varphi)}$. In order to prove this it suffices to construct natural transformations between the respective pairs of functors since all natural transformations between internal functors whose codomain is a groupoid are natural isomorphisms. To this end we define $\alpha : \tilde{s}\tilde{q} \to 1_{\mathrm{Eq}(p)}$ by

$$\alpha =< 1_E, sq >: E \to E \times_B E \text{ in } \mathbf{C}.$$

It is easy to check that

$$\pi_2\alpha = sq,\ \pi_1\alpha = 1_E,$$

and

$$\pi_{13} < \alpha\pi_1, (s \times_B s)(q \times_B q) >= \pi_{13} < 1_{E\times_B E}, \alpha\pi_2 > .$$

Hence α is an internal natural transformation.

Similarly one shows that $\beta : 1_{\mathrm{Eq}(\varphi)} \to: \tilde{q}\tilde{s}$, given by $\beta = < qs, 1_X >: X \to X \times_B X$, is an internal natural transformation. Therefore, $\mathrm{Des}_{\mathbb{A}}(q)$ is an equivalence of categories.

$\Longrightarrow$: We show more precisely:

(1) *If* $\mathrm{Des}_{\mathbb{A}}(q)$ *is essentially surjective on objects for every* $\mathbf{C}$*-indexed category* $\mathbb{A}$*, then there is a morphism* $s : X \to E$ *in* $\mathbf{C}$ *with* $psq = p$*;*

(2) *If, furthermore,* $\mathrm{Des}_{\mathbb{A}}(q)$ *is full and faithful for every* $\mathbb{A}$*, then* s *of* (1) *yields a morphism* $s : (X, \varphi) \to (E, p)$ *in* $\mathbf{C}/B$*.*

(1) Consider the $\mathbf{C}$-indexed category $\mathbb{A}_p$ of Theorem 3.5 [5]:

$$\mathbf{C}^{\mathrm{op}} \xrightarrow{\mathbb{A}_p} \mathbf{CAT}$$

$$\begin{array}{ccc} A & \longmapsto & \mathbf{C}(A,\ E) \\ {\scriptstyle t}\big\uparrow & & \big\uparrow{\scriptstyle t^*} \\ B & \longmapsto & \mathbf{C}(B,\ E) \end{array}$$

where $\mathbb{A}_p^A = \mathbf{C}(A,\ E)$ carries an equivalence relation given by

$$u \sim v \Leftrightarrow pu = pv,$$

making it a category (in fact, a groupoid), and where $t^* : \mathbf{C}(B,\ E) \to \mathbf{C}(A,\ E)$ is the composition functor with t. Since

$$p\pi_1 = p\pi_2, \pi_1^*(1_E) = \pi_1 \sim \pi_2 = \pi_2^*(1_E),$$

the object 1_E of $\mathbb{A}_p^E$ has a descent structure $\xi : \pi_2^*(1_E) \to \pi_1^*(1_E)$, where (π_1, π_2) is the kernel pair of p. Hence, by diagram (2) and the proof of Theorem 2.5 [5],

$$V^{\varphi}(1_E, \xi) = (i_{\varphi})^*(1_E, \xi) = ((1_E)^*(1_E), \xi_{i_{\varphi}}) = (1_E, \xi') \in \mathrm{Des}_{\mathbb{A}_p}(E, q).$$

But $\mathrm{Des}_{\mathbb{A}_p}(q)$ is essentially surjective, so there is $(s, \mu) \in \mathrm{Des}_{\mathbb{A}_p}(X, \varphi)$ such that

$$\mathrm{Des}_{\mathbb{A}_p}(q)(s, \mu) \cong (1_E, \xi),$$

and therefore

$$V^{\varphi}\mathrm{Des}_{\mathbb{A}_p}(q)(s, \mu) \cong V^{\varphi}(1_E, \xi) = (1_E, \xi').$$

That is

$$\Phi^q U^\varphi(s,\mu) \cong (1_E, \xi').$$

But Φ^q is just a lifting of q^*,

$$q^* U^\varphi(s,\mu) \cong \delta^* \Phi^q U^\varphi(s,\mu) \cong \delta^*(1_E, \xi').$$

Hence

$$q^* s \sim 1_E \text{ in } \mathbb{A}_p^E,$$

and therefore

$$psq = p.$$

(2) In order to prove that $ps = \varphi$, again, we consider the C-indexed category $\mathbb{B} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ of Theorem 3.5 of [5] with $\mathbb{B}^A = \mathbf{C}(A, B)$ considered a discrete category, for every $A \in \mathbf{C}$, and with t^* the composition functor with t, for every $t : A \to B$ in $\mathbf{C}$. It is easy to check that $(ps, 1)$ and $(\varphi, 1)$ are objects of $\mathrm{Des}_{\mathbb{B}}(X, \varphi)$ and that

$$\mathrm{Des}_{\mathbb{B}}(q)(ps, 1) = \mathrm{Des}_{\mathbb{B}}(q)(\varphi, 1) = (p, 1),$$

by the fact that $psq = p$. Since $\mathrm{Des}_{\mathbb{B}}(q)$ is full and faithful, $(ps, 1)$ is isomorphic to $(\varphi, 1)$, which yields

$$ps = \varphi.$$

□

From Theorem 1 one obtains:

Corollary 1. *Let $q : E \to X$ and $\varphi : X \to B$ be two morphisms of $\mathbf{C}$. Then $q : (E, \varphi q) \to (X, \varphi)$ is an absolute descent equivalence if and only if there is a morphism $s : X \to E$ in $\mathbf{C}$ such that $\varphi q s = \varphi$*

Corollary 2. *Let $q : (E, p) \to (X, \varphi)$ be a morphism in $\mathbf{C}/B$. Then q is an absolute descent equivalence if either q is a split epimorphism in $\mathbf{C}$ or q is a split monomorphism in $\mathbf{C}/B$.*

Remark. Corollary 1 implies in particular that split epimorphisms are the absolutely effective descent morphisms (see Thm. 3.5 of [5]). In fact, if $p : E \to B$ has a splitting s with $ps = 1_B$, then we may apply Corollary 1 to $p : (E, p) \to (B, 1_B)$, so that with 1_B also p is an absolute effective descent morphisms (i.e., effective descent w.r.t. every $\mathbf{C}$-indexed category $\mathbb{A}$), by Proposition 3.

6. Descent equivalences whose domain or codomain is effective descent. With the help of Corollary 2, Proposition 3 can be refined, as follows. Given any morphism $q : (E, p) \to (X, \varphi)$ in $\mathbf{C}/B$, we form the pullback diagram

$$\begin{tikzcd}
& E \times_B X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
E \arrow[dr, "p"'] & & X \arrow[dl, "\varphi"] \\
& B &
\end{tikzcd} \tag{3}$$

in which π_1 is a split epimorphism. Hence $\pi_1 : (E \times_B X, p\pi_1) \to (E, p)$ is an absolute descent equivalence, by Corollary 2.

Theorem 2. *The following conditions are equivalent:*

(i) *p is an effective $\mathbb{A}$-descent morphism and $\pi_2 : (E \times_B X, p\pi_1) \to (X, \varphi)$ is an $\mathbb{A}$-descent equivalence,*

(ii) *φ is an effective $\mathbb{A}$-descent morphism, and $q : (E, p) \to (X, \varphi)$ is an $\mathbb{A}$-descent equivalence.*

Proof. (i)$\Longrightarrow$(ii): By Prop.1, $p : (E, p) \to (B, 1_B)$ is an $\mathbb{A}$-descent equivalence. Since π_1 is an $\mathbb{A}$-descent equivalence, also $p\pi_1 = \varphi\pi_2 : (E \times_B X, p\pi_1) \to (B, 1_B)$ is an $\mathbb{A}$-descent equivalence, and so is $\varphi : (X, \varphi) \to (B, 1_B)$, by Prop.2 and the hypothesis on π_2. Then, another application of Propositions 1 and 2 gives (ii).

(ii)$\Longrightarrow$(i): By Prop.3, p is an effective $\mathbb{A}$-descent morphism. As before then, $p\pi_1 = \varphi\pi_2$ is an $\mathbb{A}$-descent equivalence, and so are q (by hypothesis), p, φ, and then π_2, by repeated application of Propositions 1 and 2. $\square$

Remark. We note that in (i) it is enough to assume that $\mathrm{Des}_{\mathbb{A}}(\pi_2)$ be full and faithful, rather than an equivalence of categories. Indeed, since π_1 is an $\mathbb{A}$-descent equivalence, also $p\pi_1 = \varphi\pi_2$ is an $\mathbb{A}$-descent equivalence when p is an effective $\mathbb{A}$-descent morphism, which implies $\mathrm{Des}_{\mathbb{A}}(\pi_2)$ is essentially surjective on objects.

If $\mathbb{A}$ *is the basic fibration*, Theorem 2 may be simplified, as follows:

Corollary 3. *For any morphism* $q : (E, p) \to (X, \varphi)$ *in* $\mathbf{C}/B$, *p is an effective descent morphism if and only if φ is an effective descent morphism and* $q : (E, p) \to (X, \varphi)$ *is a descent equivalence.*

Proof. Using pullback-stability of effective descent morphisms (see [10]) and the composition-cancellation rule of [9], for "only if" one can argue as in (i)$\Longrightarrow$(ii) of Theorem 3. Likewise for "if". $\square$

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