

Descent Equivalence

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Abstract

For a \mathbf{C} -indexed category \mathbb{A} , an \mathbb{A} -descent equivalence is a morphism of bundles in \mathbf{C} which induces an equivalence between the \mathbb{A} -descent categories of its domain and codomain. In this note, properties of such morphisms are studied, and those morphisms which are \mathbb{A} -descent equivalences for all \mathbf{C} -indexed categories \mathbb{A} are fully characterized.

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0. Introduction. Descent Theory was developed by Grothendieck [1], [2] in the context of fibred categories. If the category \mathbf{E} is fibred over the category \mathbf{C} with pullbacks, then each morphism $p : E \rightarrow B$ of \mathbf{C} is associated with its descent category $\text{Des}_{\mathbf{E}}(p)$ (see, for example, [4], for details). Having defined descent structures, it seems natural to us to compare two bundles (E, p) and (X, φ) over B in the descent sense and to ask:

When do two bundles (E, p) and (X, φ) over B have the “same descent behavior”?

More clearly, we would like to know *under which conditions a morphism of the two bundles (E, p) and (X, φ) over B would render equivalent descent categories.* To this end, we shall examine here for morphisms of bundles the notion of descent equivalence, which was introduced in the first author’s Ph.D. thesis [3], and study its properties.

We formulate this notion in the (essentially equivalent) language of internal categories and of indexed categories (see [5,6,7]), rather than that of fibrations, making extensive use of some of the results of [5], which we recall here in sufficient detail.

After some preliminary observations concerning descent equivalences and their comparison with effective descent morphisms, in Theorem 1 we give a somewhat surprising necessary and sufficient condition for a morphism of bundles to be a descent equivalence (with respect to *all* indexed categories): one just needs the existence of *any* morphism of bundles in the opposite direction. In Theorem 2, we characterize those descent equivalences whose domain or codomain is given by an effective descent morphism.

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1. Internal categories. Recall that *an internal category* D (cf. [6]) in \mathbf{C} is given by a diagram

$$\begin{array}{ccccc}
 & \xrightarrow{\pi_2} & & \xrightarrow{d} & \\
 D_2 & \xrightarrow{m} & D_1 & \xleftarrow{e} & D_0 \\
 & \xrightarrow{\pi_1} & & \xleftarrow{c} & \\
 & & & &
 \end{array}$$

in \mathbf{C} , which satisfies

- I1. $de = 1_{D_0} = ce$,
- I2. $dm = d\pi_2$, $cm = c\pi_1$,
- I3. $m(1_{D_1} \times m) = m(m \times 1_{D_1})$,
- I4. $m \langle 1_{D_1}, ed \rangle = 1_{D_1} = m \langle ec, 1_{D_1} \rangle$,

where D_2 , π_1 , π_2 are given by the following pullback diagram in \mathbf{C} :

$$\begin{array}{ccc}
D_2 & \xrightarrow{\pi_2} & D_1 \\
\pi_1 \downarrow & & \downarrow c \\
D_1 & \xrightarrow{d} & D_0
\end{array}$$

An *internal functor* $f : D \rightarrow D'$ between two internal categories D, D' in \mathbf{C} is given by two morphisms $f_0 : D_0 \rightarrow D'_0, f_1 : D_1 \rightarrow D'_1$ of \mathbf{C} such that

F1. $f_0 d = d' f_1, f_0 c = c' f_1,$

F2. $f_1 e = e' f_0, f_1 m = m' f_2,$

where $f_2 = f_1 \times_{D_0} f_1 : D_1 \times_{D_0} D_1 \rightarrow D'_1 \times_{D'_0} D'_1.$

Composition of internal functors is defined in the obvious way. Hence one obtains $\mathbf{cat}(\mathbf{C}),$ the category of all internal categories and internal functors in $\mathbf{C}.$ It is actually a 2-category (see [5]) since one can define the notion of *internal natural transformation* $\alpha : f \rightarrow g$ of internal functors $f, g : D \rightarrow D',$ given by a morphism $\alpha : D_0 \rightarrow D'_1$ in \mathbf{C} such that

T1. $d' \alpha = f_0, c' \alpha = g_0,$

T2. $m' \langle \alpha c, f_1 \rangle = m' \langle g_1, \alpha d \rangle .$

The *composite* $\beta \alpha : f \rightarrow h$ of internal natural transformations $\alpha : f \rightarrow g$ and $\beta : g \rightarrow h$ is the morphism

$$m' \langle \beta, \alpha \rangle : D_0 \rightarrow D'_1,$$

and the *identity internal natural transformation* $1_f : f \rightarrow f$ is the morphism

$$e' f_0 : D_0 \rightarrow D'_1.$$

An internal functor $f : D \rightarrow D'$ of \mathbf{C} is an *internal category equivalence* if there is an internal functor $g : D' \rightarrow D$ such that

$$gf \cong 1_D \text{ and } fg \cong 1_{D'}.$$

For example, if $p : E \rightarrow B$ is a morphism in $\mathbf{C},$ then

$$\begin{array}{ccc}
(E \times_B E) \times_E (E \times_B E) & \xrightarrow{\pi_{23}} & E \times_B E \\
\cong E \times_B E \times_B E & \xrightarrow{\pi_{13}} & E \times_B E \\
& \xrightarrow{\pi_{12}} & E \\
& & \xleftarrow{e} E \\
& & \xrightarrow{\pi_1}
\end{array}$$

is an internal category in $\mathbf{C},$ where $e = \langle 1_E, 1_E \rangle, (\pi_1, \pi_2)$ is the kernel pair of p, π_{12} and π_{23} are such that $\pi_1 \pi_{23} = \pi_2 \pi_{12}$ (pullback square) and $\pi_{13} = \langle \pi_1 \pi_{12}, \pi_2 \pi_{23} \rangle.$ This internal category is denoted by $\text{Eq}(p).$ Every object B in \mathbf{C} can be viewed as a discrete internal category B of $\mathbf{C}:$

$$\begin{array}{ccc}
B & \begin{array}{c} \xrightarrow{1_B} \\ \xrightarrow{1_B} \\ \xrightarrow{1_B} \end{array} & B \\
& & \begin{array}{c} \xleftarrow{1_B} \\ \xleftarrow{1_B} \\ \xleftarrow{1_B} \end{array} & B
\end{array}$$

Clearly, $\text{Eq}(1_B)$ is isomorphic to the above discrete internal category B .

For any morphism $q : (E, p) \rightarrow (X, \varphi)$ in \mathbf{C}/B , as in [5] one constructs the internal functor

$$\tilde{q} : \text{Eq}(p) \rightarrow \text{Eq}(\varphi),$$

where $\tilde{q}_0 = q$, $\tilde{q}_1 = q \times_B q$. Then, for a fixed object B of \mathbf{C} , the assignments:

$$(E, p) \mapsto \text{Eq}(p) \text{ and } q \mapsto \tilde{q},$$

define the functor

$$\text{Eq}_B : \mathbf{C}/B \rightarrow \mathbf{cat}(\mathbf{C}).$$

2. Indexed categories. A \mathbf{C} -indexed category \mathbb{A} or a pseudo-functor $\mathbb{A} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{CAT}$ (cf. [5,7,8]) consists of the following data:

- for every object E of \mathbf{C} a category \mathbb{A}^E
- for every morphism $f : E \rightarrow D$ of \mathbf{C} a functor $f^* : \mathbb{A}^D \rightarrow \mathbb{A}^E$,
- for every $f : E \rightarrow D$, $g : D \rightarrow B$ in \mathbf{C} , two natural isomorphisms:

$$i^D : 1_{\mathbb{A}^D} \rightarrow (1_D)^*, \quad j^{f,g} : f^* g^* \rightarrow (gf)^*$$

which make the diagrams

$$\begin{array}{ccc}
f^* & \xrightarrow{f^* i^D} & f^* (1_D)^* \\
\downarrow i^E f^* & \searrow 1_{f^*} & \downarrow j^{f, 1_D} \\
(1_E)^* f^* & \xrightarrow{j^{1_E, f}} & f^*
\end{array}$$

and

$$\begin{array}{ccc}
f^*g^*h^* & \xrightarrow{f^*j^{g,h}} & f^*(hg)^* \\
\downarrow j^{f,gh^*} & & \downarrow j^{f,hg} \\
(gf)^*h^* & \xrightarrow{j^{gf,h}} & (hgf)^*
\end{array}$$

commute.

For example,

$$\mathbb{A} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{CAT}$$

given by $B \mapsto \mathbf{C}/B$ and $(f : E \rightarrow B) \mapsto f^* : \mathbf{C}/B \rightarrow \mathbf{C}/E$, the pullback functor along f , is a \mathbf{C} -indexed category, also called the *basic \mathbf{C} -indexed category*.

Let D be an internal category in \mathbf{C} . One defines \mathbb{A}^D (cf. [5]) to be the category with
- objects all pairs of (C, ξ) , where $C \in \text{ob}\mathbb{A}^{D_0}$ and $\xi : d^*C \rightarrow c^*C$ is a morphism in \mathbb{A}^{D_1} such that

$$\begin{array}{ccc}
e^*d^*C & \xrightarrow{e^*\xi} & e^*c^*C \\
\cong \searrow & & \swarrow \cong \\
& C &
\end{array}$$

and

$$\begin{array}{ccccc}
& & \cong & & \\
& & (\pi_2)^*c^*C & \xrightarrow{\quad} & (\pi_1)^*d^*C \\
(\pi_2)^*\xi \nearrow & & & & \searrow (\pi_1)^*\xi \\
(\pi_2)^*d^*C & & & & (\pi_1)^*c^*C \\
\cong \searrow & & & & \swarrow \cong \\
& m^*d^*C & \xrightarrow{m^*\xi} & m^*c^*C &
\end{array}$$

commute, in \mathbb{A}^{D_0} and \mathbb{A}^{D_2} , respectively, with the above natural isomorphisms arising from I1 and I2,

- morphisms $h : (C, \xi) \rightarrow (C', \xi')$ of \mathbb{A}^D given by morphisms $h : C \rightarrow C'$ of \mathbb{A}^{D_0} such that

$$\begin{array}{ccc}
d^*C & \xrightarrow{d^*h} & d^*C' \\
\xi \downarrow & & \downarrow \xi' \\
c^*C & \xrightarrow{c^*h} & c^*C'
\end{array}$$

commutes in \mathbb{A}^{D_1} .

In [5], it was proved that, for every \mathbf{C} -indexed category $\mathbb{A} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{CAT}$, the extension

$$\mathbb{A} : \mathbf{cat}(\mathbf{C})^{\text{op}} \rightarrow \mathbf{CAT}$$

given by the assignment $D \rightarrow \mathbb{A}^D$, is a pseudo-functor of 2-categories. As a consequence, one obtains that for every internal category equivalence $f : D \rightarrow D'$ of \mathbf{C} , the functor $f^* : \mathbb{A}^{D'} \rightarrow \mathbb{A}^D$ is an equivalence of categories.

3. Effective descent, descent equivalence. Now, let $\text{Des}_{\mathbb{A}}$ be the pseudo-functor $\mathbb{A} \circ \text{Eq}_B$:

$$\begin{array}{ccc}
(\mathbf{C}/B)^{\text{op}} & \xrightarrow{\text{Des}_{\mathbb{A}}(\)} & (\mathbf{C}/B) \setminus \mathbf{CAT} \\
& \searrow \text{Eq}_B & \nearrow \mathbb{A} \\
& & \mathbf{cat}(\mathbf{C})^{\text{op}}
\end{array}$$

The discrete functor $p : E \rightarrow B$ can be factored as

$$\begin{array}{ccc}
B & \xleftarrow{\bar{p}} & \text{Eq}(p) \\
& \swarrow p & \nearrow \delta \\
& & E
\end{array}$$

where $\bar{p}_0 = p$, $\bar{p}_1 = p\pi_1 = p\pi_2$, $\delta_0 = 1_E$, $\delta_1 = e = \langle 1_E, 1_E \rangle$, with (π_1, π_2) the kernel pair of p . Applying \mathbb{A} to the last diagram, one has a commutative diagram (up to natural isomorphism) in \mathbf{CAT} :

$$\begin{array}{ccc}
\mathbb{A}^B & \xrightarrow{\Phi^p = \bar{p}^*} & \text{Des}_{\mathbb{A}}(p) \\
& \searrow p^* & \nearrow \delta^* \\
& & \mathbb{A}^E
\end{array}$$

p is called an \mathbb{A} -descent morphism (effective \mathbb{A} -descent morphism) if the comparison functor Φ^p is full and faithful (an equivalence of categories). p is called an absolute (effective) descent morphism if it is an (effective) \mathbb{A} -descent morphism for every \mathbf{C} -indexed category \mathbb{A} .

For a morphism $q : (E, p) \rightarrow (X, \varphi)$ in \mathbf{C}/B , the authors of [9] considered the following diagram in $\mathbf{cat}(\mathbf{C})$:

$$\begin{array}{ccc}
 & \text{Eq}(1_B) & \\
 \bar{\varphi} \nearrow & & \nwarrow \bar{p} \\
 \text{Eq}(\varphi) & \xleftarrow{\tilde{q}} & \text{Eq}(p) \\
 \delta_X \uparrow & & \uparrow i_\varphi \\
 \text{Eq}(1_X) & \xleftarrow{\bar{q}} & \text{Eq}(q)
 \end{array} \tag{1}$$

where $(i_\varphi)_0 = 1_E$, $(i_\varphi)_1 = 1_E \times_\varphi 1_E$, $\tilde{q}_0 = q$, $\tilde{q}_1 = q \times_B q$, $(\delta_X)_0 = 1_X$, $(\delta_X)_1 = \Delta_X$, $\bar{p}_0 = p$, $\bar{p}_1 = p\pi_1 = p\pi_2$, and where (π_1, π_2) is the kernel pair of p .

Applying \mathbb{A} to diagram (1), one obtains the following commutative diagram (up to natural isomorphisms) in \mathbf{CAT} :

$$\begin{array}{ccc}
 & \mathbb{A}^B & \\
 \Phi^\varphi \nearrow & & \nwarrow \Phi^p \\
 \text{Des}_\mathbb{A}(X, \varphi) & \xrightarrow{\text{Des}_\mathbb{A}(q)} & \text{Des}_\mathbb{A}(E, p) \\
 U^\varphi \downarrow & & \downarrow V^\varphi \\
 \mathbb{A}^X & \xrightarrow{\Phi^q} & \text{Des}_\mathbb{A}(E, q)
 \end{array} \tag{2}$$

where $U^\varphi = \delta_X^*$, $V^\varphi = i_\varphi^*$, $\Phi^p = \bar{p}^*$, and $\text{Des}_\mathbb{A}(q) = \tilde{q}^*$.

Definition. Let $q : (E, p) \rightarrow (X, \varphi)$ be a morphism in \mathbf{C}/B . We call q an \mathbb{A} -descent equivalence (\mathbb{A} -descent pre-equivalence) if $\text{Des}_\mathbb{A}(q)$ is an equivalence of categories (full and faithful). We call

q an absolute descent equivalence (absolute descent pre-equivalence) if $\text{Des}_{\mathbb{A}}(q)$ is an equivalence of categories (full and faithful) for every \mathbf{C} -indexed category \mathbb{A} .

4. Properties of descent equivalences. Functoriality of $\text{Des}_{\mathbb{A}}(\)$ leads immediately to a number of consequences.

Proposition 1. *The morphism $p : (E, p) \rightarrow (B, 1_B)$ in \mathbf{C}/B is an \mathbb{A} -descent pre-equivalence (\mathbb{A} -descent equivalence) if and only if p is an \mathbb{A} -descent (effective \mathbb{A} -descent) morphism.*

Proof. Applying Eq to the following commutative diagram:

$$\begin{array}{ccc}
 E & \xrightarrow{p} & B \\
 & \searrow p & \swarrow 1_B \\
 & & B
 \end{array}$$

we obtain the commutative diagram:

$$\begin{array}{ccc}
 & \text{Eq}(1_B) & \\
 \bar{p} \nearrow & & \nwarrow \bar{1}_B \\
 \text{Eq}(p) & \xrightarrow{\tilde{p}} & \text{Eq}(1_B)
 \end{array}$$

Clearly, with the notation of the previous section, $\bar{1}_B = 1_{\text{Eq}(1_B)}$, $\tilde{p} = \bar{p}$. Hence, \tilde{p}^* is an equivalence of categories if and only if \bar{p}^* is an equivalence of categories, as desired. \square

One also easily obtains:

Proposition 2. *Let $q : (E, p) \rightarrow (X, \varphi)$, $r : (X, \varphi) \rightarrow (Y, \xi)$ be morphisms in \mathbf{C}/B .*

- (1) *If two of q , r , and rq are \mathbb{A} -descent equivalences, so is the third one.*
- (2) *If r is an \mathbb{A} -descent equivalence, then q is an \mathbb{A} -descent pre-equivalence if and only if rq is an \mathbb{A} -descent pre-equivalence.* \square

It is also easy to show that \mathbb{A} -descent (pre-)equivalences have the intended invariance property:

Proposition 3. *Let $q : (E, p) \rightarrow (X, \varphi)$ be an \mathbb{A} -descent pre-equivalence (\mathbb{A} -descent equivalence) in \mathbf{C}/B . Then p is an \mathbb{A} -descent (effective \mathbb{A} -descent) morphism if and only if φ is an \mathbb{A} -descent (effective \mathbb{A} -descent) morphism.*

Proof. By Diagram (2), $\text{Des}_{\mathbb{A}}(q)\Phi^\varphi = \Phi^p$ (up to natural isomorphism). If q is an \mathbb{A} -descent equivalence, then $\text{Des}_{\mathbb{A}}(q)$ is an equivalence of categories. Therefore, Φ^φ is an equivalence of categories if and only if Φ^p is an equivalence of categories. Hence p is an effective \mathbb{A} -descent morphism if and only if φ is an effective \mathbb{A} -descent morphism.

Suppose now that q is an \mathbb{A} -descent pre-equivalence. Then $\text{Des}_{\mathbb{A}}(q)$ is full and faithful. If φ is \mathbb{A} -descent pre-equivalence morphism, then $\Phi^p = \text{Des}_{\mathbb{A}}(q)\Phi^\varphi$ (up to isomorphism) is full and faithful. Hence p is an \mathbb{A} -descent morphism. On the other hand, if p is \mathbb{A} -descent morphism, then $\text{Des}_{\mathbb{A}}(q)\Phi^\varphi = \Phi^p$ (up to isomorphism) is full and faithful, and so is Φ^φ . \square

5. A necessary and sufficient condition for absolute descent equivalences. In any category, the absolutely effective descent morphisms are precisely the split epimorphisms [5]. A characterization of the absolute descent equivalences is given by the following:

Theorem 1. *Let $q : (E, p) \rightarrow (X, \varphi)$ be a morphism in \mathbf{C}/B . Then q is an absolute descent equivalence if and only if there is any morphism $s : (X, \varphi) \rightarrow (E, p)$ in \mathbf{C}/B .*

Proof. \Leftarrow : By hypothesis, we have

$$p = \varphi q \text{ and } ps = \varphi.$$

So there exist two internal functors

$$\tilde{s} : \text{Eq}(\varphi) \rightarrow \text{Eq}(p) \text{ and } \tilde{q} : \text{Eq}(p) \rightarrow \text{Eq}(\varphi).$$

We claim that $\tilde{s}\tilde{q} \cong 1_{\text{Eq}(p)}$ and $\tilde{q}\tilde{s} \cong 1_{\text{Eq}(\varphi)}$. In order to prove this it suffices to construct natural transformations between the respective pairs of functors since all natural transformations between internal functors whose codomain is a groupoid are natural isomorphisms. To this end we define $\alpha : \tilde{s}\tilde{q} \rightarrow 1_{\text{Eq}(p)}$ by

$$\alpha = \langle 1_E, sq \rangle : E \rightarrow E \times_B E \text{ in } \mathbf{C}.$$

It is easy to check that

$$\pi_2\alpha = sq, \pi_1\alpha = 1_E,$$

and

$$\pi_{13} \langle \alpha\pi_1, (s \times_B s)(q \times_B q) \rangle = \pi_{13} \langle 1_{E \times_B E}, \alpha\pi_2 \rangle.$$

Hence α is an internal natural transformation.

Similarly one shows that $\beta : 1_{\text{Eq}(\varphi)} \rightarrow: \tilde{q}\tilde{s}$, given by $\beta = \langle qs, 1_X \rangle : X \rightarrow X \times_B X$, is an internal natural transformation. Therefore, $\text{Des}_{\mathbb{A}}(q)$ is an equivalence of categories.

\implies : We show more precisely:

- (1) If $\text{Des}_{\mathbb{A}}(q)$ is essentially surjective on objects for every \mathbf{C} -indexed category \mathbb{A} , then there is a morphism $s : X \rightarrow E$ in \mathbf{C} with $psq = p$;
- (2) If, furthermore, $\text{Des}_{\mathbb{A}}(q)$ is full and faithful for every \mathbb{A} , then s of (1) yields a morphism $s : (X, \varphi) \rightarrow (E, p)$ in \mathbf{C}/B .

- (1) Consider the \mathbf{C} -indexed category \mathbb{A}_p of Theorem 3.5 [5]:

$$\begin{array}{ccc}
 \mathbf{C}^{\text{op}} & \xrightarrow{\mathbb{A}_p} & \mathbf{CAT} \\
 A & \mapsto & \mathbf{C}(A, E) \\
 \uparrow t & & \uparrow t^* \\
 B & \mapsto & \mathbf{C}(B, E)
 \end{array}$$

where $\mathbb{A}_p^A = \mathbf{C}(A, E)$ carries an equivalence relation given by

$$u \sim v \Leftrightarrow pu = pv,$$

making it a category (in fact, a groupoid), and where $t^* : \mathbf{C}(B, E) \rightarrow \mathbf{C}(A, E)$ is the composition functor with t . Since

$$p\pi_1 = p\pi_2, \pi_1^*(1_E) = \pi_1 \sim \pi_2 = \pi_2^*(1_E),$$

the object 1_E of \mathbb{A}_p^E has a descent structure $\xi : \pi_2^*(1_E) \rightarrow \pi_1^*(1_E)$, where (π_1, π_2) is the kernel pair of p . Hence, by diagram (2) and the proof of Theorem 2.5 [5],

$$V^\varphi(1_E, \xi) = (i_\varphi)^*(1_E, \xi) = ((1_E)^*(1_E), \xi_{i_\varphi}) = (1_E, \xi') \in \text{Des}_{\mathbb{A}_p}(E, q).$$

But $\text{Des}_{\mathbb{A}_p}(q)$ is essentially surjective, so there is $(s, \mu) \in \text{Des}_{\mathbb{A}_p}(X, \varphi)$ such that

$$\text{Des}_{\mathbb{A}_p}(q)(s, \mu) \cong (1_E, \xi),$$

and therefore

$$V^\varphi \text{Des}_{\mathbb{A}_p}(q)(s, \mu) \cong V^\varphi(1_E, \xi) = (1_E, \xi').$$

That is

$$\Phi^q U^\varphi(s, \mu) \cong (1_E, \xi^l).$$

But Φ^q is just a lifting of q^* ,

$$q^* U^\varphi(s, \mu) \cong \delta^* \Phi^q U^\varphi(s, \mu) \cong \delta^*(1_E, \xi^l).$$

Hence

$$q^* s \sim 1_E \text{ in } \mathbb{A}_p^E,$$

and therefore

$$psq = p.$$

(2) In order to prove that $ps = \varphi$, again, we consider the \mathbf{C} -indexed category $\mathbb{B} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{CAT}$ of Theorem 3.5 of [5] with $\mathbb{B}^A = \mathbf{C}(A, B)$ considered a discrete category, for every $A \in \mathbf{C}$, and with t^* the composition functor with t , for every $t : A \rightarrow B$ in \mathbf{C} . It is easy to check that $(ps, 1)$ and $(\varphi, 1)$ are objects of $\text{Des}_{\mathbb{B}}(X, \varphi)$ and that

$$\text{Des}_{\mathbb{B}}(q)(ps, 1) = \text{Des}_{\mathbb{B}}(q)(\varphi, 1) = (p, 1),$$

by the fact that $psq = p$. Since $\text{Des}_{\mathbb{B}}(q)$ is full and faithful, $(ps, 1)$ is isomorphic to $(\varphi, 1)$, which yields

$$ps = \varphi.$$

□

From Theorem 1 one obtains:

Corollary 1. *Let $q : E \rightarrow X$ and $\varphi : X \rightarrow B$ be two morphisms of \mathbf{C} . Then $q : (E, \varphi q) \rightarrow (X, \varphi)$ is an absolute descent equivalence if and only if there is a morphism $s : X \rightarrow E$ in \mathbf{C} such that $\varphi q s = \varphi$*

Corollary 2. *Let $q : (E, p) \rightarrow (X, \varphi)$ be a morphism in \mathbf{C}/B . Then q is an absolute descent equivalence if either q is a split epimorphism in \mathbf{C} or q is a split monomorphism in \mathbf{C}/B .*

Remark. Corollary 1 implies in particular that split epimorphisms are the absolutely effective descent morphisms (see Thm. 3.5 of [5]). In fact, if $p : E \rightarrow B$ has a splitting s with $ps = 1_B$, then we may apply Corollary 1 to $p : (E, p) \rightarrow (B, 1_B)$, so that with 1_B also p is an absolute effective descent morphisms (i.e., effective descent w.r.t. every \mathbf{C} -indexed category \mathbb{A}), by Proposition 3.

6. Descent equivalences whose domain or codomain is effective descent. With the help of Corollary 2, Proposition 3 can be refined, as follows. Given any morphism $q : (E, p) \rightarrow (X, \varphi)$ in \mathbf{C}/B , we form the pullback diagram

$$\begin{array}{ccc}
 & E \times_B X & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 E & & X \\
 p \searrow & & \swarrow \varphi \\
 & B &
 \end{array} \tag{3}$$

in which π_1 is a split epimorphism. Hence $\pi_1 : (E \times_B X, p\pi_1) \rightarrow (E, p)$ is an absolute descent equivalence, by Corollary 2.

Theorem 2. *The following conditions are equivalent:*

- (i) p is an effective \mathbb{A} -descent morphism and $\pi_2 : (E \times_B X, p\pi_1) \rightarrow (X, \varphi)$ is an \mathbb{A} -descent equivalence,
- (ii) φ is an effective \mathbb{A} -descent morphism, and $q : (E, p) \rightarrow (X, \varphi)$ is an \mathbb{A} -descent equivalence.

Proof. (i) \implies (ii): By Prop.1, $p : (E, p) \rightarrow (B, 1_B)$ is an \mathbb{A} -descent equivalence. Since π_1 is an \mathbb{A} -descent equivalence, also $p\pi_1 = \varphi\pi_2 : (E \times_B X, p\pi_1) \rightarrow (B, 1_B)$ is an \mathbb{A} -descent equivalence, and so is $\varphi : (X, \varphi) \rightarrow (B, 1_B)$, by Prop.2 and the hypothesis on π_2 . Then, another application of Propositions 1 and 2 gives (ii).

(ii) \implies (i): By Prop.3, p is an effective \mathbb{A} -descent morphism. As before then, $p\pi_1 = \varphi\pi_2$ is an \mathbb{A} -descent equivalence, and so are q (by hypothesis), p , φ , and then π_2 , by repeated application of Propositions 1 and 2. \square

Remark. We note that in (i) it is enough to assume that $\text{Des}_{\mathbb{A}}(\pi_2)$ be full and faithful, rather than an equivalence of categories. Indeed, since π_1 is an \mathbb{A} -descent equivalence, also $p\pi_1 = \varphi\pi_2$ is an \mathbb{A} -descent equivalence when p is an effective \mathbb{A} -descent morphism, which implies $\text{Des}_{\mathbb{A}}(\pi_2)$ is essentially surjective on objects.

If \mathbb{A} is the basic fibration, Theorem 2 may be simplified, as follows:

Corollary 3. *For any morphism $q : (E, p) \rightarrow (X, \varphi)$ in \mathbf{C}/B , p is an effective descent morphism if and only if φ is an effective descent morphism and $q : (E, p) \rightarrow (X, \varphi)$ is a descent equivalence.*

Proof. Using pullback-stability of effective descent morphisms (see [10]) and the composition-cancellation rule of [9], for “only if” one can argue as in (i) \implies (ii) of Theorem 3. Likewise for “if”. \square

References

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