Macro- and Micro-Aspects of Categorial Topology

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“Categorical Topology”

- Topos and Sheaf Theory: Grothendieck, Lawvere, Moerdijk, ...
- Locale (and Frame) Theory: Isbell, Johnstone, Banaschewski, Pultr, ...
- Topological Functors: Wyler, Brümmer, Herrlich, Hušek, Bentley, ...
- Closure Operators: Dikranjan, Giuli, T, Castellini ...
- Generalized metric spaces: Lawvere, Lowen, Clementino, Hofmann, T, ...
- Convergence and monads: Manes, Barr, Day, Gähler, Wyler, Mynard, Clementino, Hofmann, T, ...
- Algebraic Topology, Homotopy Theory, Groupoids, ...

A relatively complicated theorem:

$$\left.\begin{array}{r} f : X \to Y \text{ continuous} \\ X \text{ exponentiable} \\ Y \text{ locally Hausdorff} \end{array}\right\} \implies (X, f) \text{ fiberwise exponentiable}$$

$$\begin{array}{rcl} X \text{ exponentiable} & :\iff & \forall Z \; \exists Z^X \; \forall T : \; \dfrac{T \to Z^X}{X \times T \to Z} \\ & \iff & X \text{ core-compact (Day-Kelly 1970)} \\ (X, f) \text{ fw expble} & \iff & (X, f) \text{ fw c-c (Niefield 1982, Richter 2004)} \end{array}$$

A relatively easy theorem:

$$\left.\begin{array}{r} f : X \to Y \text{ continuous} \\ X \text{ compact} \\ Y \text{ Hausdorff} \end{array}\right\} \Longrightarrow f \text{ proper (=closed with compact fibers)}$$

less easy:

$$\left.\begin{array}{r} f : X \to Y, g : Y \to Z \text{ continuous} \\ g \circ f \text{ proper/perfect} \\ g \text{ separated} \end{array}\right\} \Longrightarrow f \text{ proper/perfect}$$

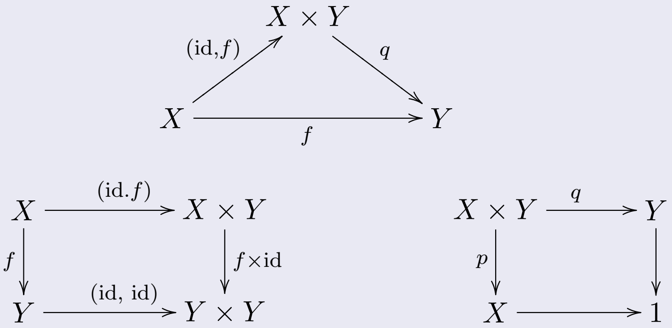
Two more theorems of the latter type:

$$f : X \to Y, g : Y \to Z \text{ continuous}$$

$$\left.\begin{array}{r} g \circ f \text{ local homeomorphism} \\ g \text{ locally injective} \end{array}\right\} \Longrightarrow f \text{ local homeomorphism}$$

$$\left.\begin{array}{r} g \circ f \text{ exponentiable} \\ g \text{ locally separated} \end{array}\right\} \Longrightarrow f \text{ exponentiable}$$

Proof of the eaasy theorem



Y Hausdorff *X compact*

f *proper* $\iff$ f *stably closed*

$\mathcal{X}$ finetely complete category

$\mathcal{P} \subseteq \mathrm{mor}\mathcal{X}$ *topology* on $\mathcal{X}$

$\iff$

1. $\mathrm{Iso}\mathcal{X} \subseteq \mathcal{P}$
2. $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$
3. $\mathcal{P}$ *stable under pullback* in $\mathcal{X}$

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array} \qquad f \in \mathcal{P} \Longrightarrow f' \in \mathcal{P}$$

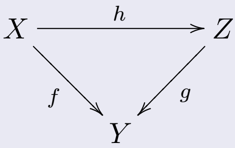
Some topologies on Top

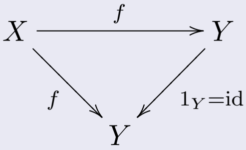
- proper maps
- open maps
- exponentiable maps
- $\mathcal{P}(\mathcal{F})$, with $\mathcal{F}$ satisfying 1,2
- $\mathcal{P}(\mathcal{F})$, with $\mathcal{F} = \{f : X \to Y \mid f \ c\text{-closed}\}$, c closure operator.

f $\mathcal{P}$-*proper* $\iff f \in \mathcal{P}$
X $\mathcal{P}$-*compact* $\iff (X \to 1) \in \mathcal{P} \iff \forall Y : (X \times Y \to Y) \in \mathcal{P}$

f $\mathcal{P}$-*separated* $\iff (X \to X \times_Y X) \in \mathcal{P}$
X $\mathcal{P}$-*Hausdorff* $\iff (X \to 1)$ $\mathcal{P}$-separated $\iff (X \to X \times X) \in \mathcal{P}$

$\mathcal{X}/Y$:



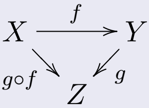


proper map in $\mathcal{X}$ = compact object in $\mathcal{X}/Y$
separated map in $\mathcal{X}$ = Hausdorff object in $\mathcal{X}/Y$

More proofs:

Complicated = easy: $\left.\begin{array}{l} X \text{ comp/ expble} \\ Y \text{ Hausd./ loc. Hausd.} \end{array}\right\} \Longrightarrow f \;\; \begin{array}{l}\text{comp/} \\ \text{expble}\end{array}$

map = object:



$\left.\begin{array}{l} g \circ f \in \mathcal{P} \\ g \in \mathcal{P}' \end{array}\right\} \Longrightarrow f \in \mathcal{P}$

$\left.\begin{array}{l} g \circ f \in \mathcal{P} \cap \mathcal{P}' \\ g \in \mathcal{P}' \end{array}\right\} \Longrightarrow f \in \mathcal{P} \cap \mathcal{P}'$

$\mathcal{P}$-open?

subobject $\iff$ regular monomorphism ("$\rightarrowtail$")

$\mathcal{P}_0 = \mathcal{P} \cap \mathrm{RegMono}$: *closed embeddings*

$m : D \rightarrowtail X$ $\mathcal{P}$*-dense* :

$$\begin{array}{ccccc} & & C & & \\ & \nearrow & & \searrow^{n \in \mathcal{P}_0} & \\ D & & \xrightarrow[m]{} & & X \end{array} \qquad \Longrightarrow n \text{ iso}$$

$f : X \to Y$ $\mathcal{P}$*-open* :

$$\begin{array}{ccc} D & \longrightarrow & E \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \qquad n \ \mathcal{P}\text{-dense} \Longrightarrow m \ \mathcal{P}\text{-dense}$$

$\big($Top: f open $\iff \forall\, S \subseteq Y : \ f^{-1}(\overline{S}) = \overline{f^{-1}(S)}\big)$

$$\mathcal{P} \mapsto \rightsquigarrow \mathcal{P} \cap \mathcal{P}'$$

$$\{\mathcal{P} - \text{open}\}$$

$$\vdots$$

local properties: local compactness
local connectedness

M.M Clementino, E.Giuli, W. Tholen
"A Functional Approach to General Topology", in:
Categorical Foundations, Cambridge University Press 2004

Lawvere 1973

Cat

X: "objects"

hom:$X \times X \longrightarrow \{\text{sets}\}$

$1 = \{*\} \longrightarrow \hom(x, x)$

$\hom(x, y) \times \hom(y, z) \longrightarrow \hom(x, z)$

$(\mathsf{Set}, \times, 1)$

Met

X: "points"

$d : X \times X \longrightarrow [0, \infty]$

$0 \geq d(x, x)$

$d(x, y) + d(y, z) \geq d(x, z)$

$([0, \infty]^{\mathrm{op}}, +, 0)$

Ord

X: "elements"

$\leq : X \times X \longrightarrow \{\text{true}, \text{false}\}$

$\text{true} \vDash x \leq x$

$x \leq y \wedge y \leq z \vDash x \leq z$

$(2, \wedge, \text{true})$

$\mathcal{V}$ commutative, unital quantale
$=$ complete lattice, $\otimes, k(\neq \bot)$
$a \otimes (\bigvee b_i) = \bigvee (a \otimes b_i)$

$\mathcal{V} \circ \mathsf{Set}$

objects: sets
morphisms $=$ $\mathcal{V}$-relations: $(r : X \relbar\!\mapstochar\!\rightarrow Y) = (r : X \times Y \to \mathcal{V})$
composition: $s : Y \relbar\!\mapstochar\!\rightarrow Z$

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

$\mathsf{Set} \to \mathcal{V} \circ \mathsf{Set}$

$X \mapsto X \qquad f \mapsto f$ with $f(x, y) = \begin{cases} k & \text{if } f(x) = y \\ \bot & \text{else} \end{cases}$

$\mathcal{V}$-*category*: set X with $a : X \nrightarrow X$ $(a : X \times X \to \mathcal{V})$

$$1_X \leq a \qquad k \leq a(x,x)$$

$$a \cdot a \leq a \quad a(x,y) \otimes a(y,z) \leq a(x,z)$$

$\mathcal{V}$-*functor* :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow\!\!\!\!\!\!\!\; \shortmid & & \shortmid\!\!\!\!\!\!\downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array} \qquad a(x_1,x_2) \leq b(f(x_1), f(x_2))$$

$\rightsquigarrow$ $\mathcal{V}$-Cat : $\quad$ 2-$\mathsf{Cat} = \mathsf{Ord}$ $\quad$ $\mathbb{P}_+$-$\mathsf{Cat} = \mathsf{Met}$

$(\mathsf{Set}$-$\mathsf{Cat} = \mathsf{Cat}$ $\quad$ AbGrp-$\mathsf{Cat} = \mathsf{AddCat})$

Yoneda embedding

Category $\mathcal{X} \xrightarrow{y} \mathsf{Set}^{\mathcal{X}^{\mathrm{op}}}$

$X \longmapsto \mathcal{X}(-, X)$

$\mathcal{V}$-Category $X \xrightarrow{y} \mathcal{V}^{X^{\mathrm{op}}} = \{f \mid f : X^{\mathrm{op}} \to \mathcal{V}\}$

$x \longmapsto a(-, x)$

Need structure on $\mathcal{V}$ and $\mathcal{V}^{X^{\mathrm{op}}}$!

$\mathcal{V}$ as a $\mathcal{V}$-category:

$$z \leq v \ominus u \iff u \otimes z \leq v$$

$$v \ominus u = \bigvee_{u \otimes z \leq v} z$$

Y^X as a $\mathcal{V}$-category:

$$e(f,g) = \bigwedge_{x \in X} b(f(x), g(x))$$

Ord: pointwise order   Met: sup-metric

Closure operator in $\mathcal{V}$-Cat:

$M \subseteq X \quad \overline{M} = \{x \in X \mid k \leq \bigvee_{z \in M} a(x,z) \otimes a(z,x)\}$

Ord : $\quad \overline{M} = \{x \in X \mid \exists z \in M (x \leq z \ \& \ z \leq x)\}$

Met : $\quad \overline{M} = \{x \in X \mid \inf\{d(x,z) + d(z,x) \mid z \in M\} = 0\}$

$\overline{(\ )}$ extensive, monotone, idempotent

$\overline{(\ )}$ additive if: $\quad \forall u, v, \in \mathcal{V} (k \leq u \vee v \implies k \leq u \text{ or } k \leq v)$

$\mathcal{V}$-Cat $\rightarrow$ Top "L-topology"

$$
\begin{aligned}
X \;\; L\textit{-separated} \quad &\iff \quad \triangle_X \subseteq X \times X \;\; L\text{-closed} \\
&\iff \quad \forall\, x, y \in X (a(x,y) \wedge a(y,x) \leq k \implies x = y) \\
&\iff \quad \mathrm{y} : X \to \mathcal{V}^{X^{\mathrm{op}}} \text{ injective}
\end{aligned}
$$

$\mathsf{Ord}_{\mathrm{sep}}$: posets

$\mathsf{Met}_{\mathrm{sep}}$: $d(x,y) = 0 = d(y,x) \implies x = y$

$$\widetilde{X} := \overline{y(X)} \subseteq \mathcal{V}^{X^{\mathrm{op}}}$$

$$(h : X^{\mathrm{op}} \to \mathcal{V}) \in \widetilde{X} \iff h \text{ tight}$$

$$\iff k \leq \bigvee_y \Big(h(y) \otimes \bigwedge_x \big(a(x,y) \ominus h(x) \big) \Big)$$

Ord: tight h corresponds to some $\downarrow y = \{x \in X \mid x \leq y\}$

Met: h tight $\iff$ $h(x) \leq d(x,y) + h(y)$

$\& \ \inf_y \Big(h(y) + \sup_x \big(d(x,y) \ominus h(x) \big) \Big) = 0$

$$
\begin{aligned}
X \ L\text{-complete} \quad &\iff \quad \mathsf{y} : X \to \widetilde{X} \text{ surjective} \\
&\iff \quad \forall h \text{ tight } \exists y \in X : h = \mathsf{y}(y)
\end{aligned}
$$

$\mathsf{Ord}_{cpl} = \mathsf{Ord}$
$\mathsf{Met}_{cpl} =$ Cauchy-complete metric spaces

$\mathcal{V}$ is L-complete and L-separated
$\widetilde{X}$ is L-complete and L-separated (any X)

X is L-complete and L-separated
$\iff$ X is injective wrt { full & faithful L-dense $\mathcal{V}$-functors} $= \mathcal{D}$

$f : Y \to Z$ fff $\iff$ $b(y_1, y_2) = c\big(f(y_1), f(y_2)\big)$ for all $y_1, y_2 \in Y$
$f L$-dense $\iff$ $\overline{f(Y)} = Z$

$(\mathcal{V}\text{-}\mathsf{Cat})_{\mathrm{cpl\&sep}}$ is firmly $\mathcal{D}$-reflective in $\mathcal{V}$-Cat

($\to$ Brümmer, Giuli, Herrlich, Holgate)

Manes 1969

X compact Hausdorff space $\iff$ X carries $UX \xrightarrow{c} X$ with

$$\begin{array}{ccc} X & \xrightarrow{e} & UX \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle c} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} UUX & \xrightarrow{Uc} & UX \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle c} \\ UX & \xrightarrow[c]{} & X \end{array}$$

$e(x) = \dot{x} =$ principal ultrafilter

$m(\mathcal{X}) = \sum \mathcal{X} =$ Kowalsky sum of $\mathcal{X}$:

$$A \in \sum \mathcal{X} \iff \{\mathfrak{x} \in \mathcal{X} \mid A \in \mathfrak{x}\} \in \mathcal{X}$$

$Uc(\mathcal{X}) \ni A \iff c^{-1}[A] \in \mathcal{X}$

Barr 1970

X topological space $\iff$ X carries $c : UX \nrightarrow X$ with

$$\begin{array}{ccc} X & \xrightarrow{e} & UX \\ & {}_{1_X}\searrow \;\; \leq & \downarrow c \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} UUX & \overset{Uc}{\nrightarrow} & UX \\ m\downarrow & \geq & \downarrow c \\ UX & \underset{c}{\nrightarrow} & X \end{array}$$

$$\text{true} \vDash \; \dot{x} \xrightarrow{\;c\;} x$$

$$\mathcal{X} \xrightarrow{\;Uc\;} y \;\;\&\;\; y \xrightarrow{\;c\;} z \;\vDash \sum \mathcal{X} \to z$$

$f : X \to Y$ cont. $\iff$ $\begin{array}{ccc} UX & \overset{Uf}{\to} & UY \\ \downarrow & \leq & \downarrow \\ X & \underset{f}{\longrightarrow} & Y \end{array}$ $\iff$ $\big(\chi \to x \implies Uf(\chi) \to f(x)\big)$

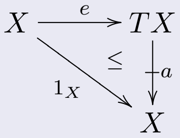
Clementino–Hofmann, Clementino–T 2004, CHT 2004:

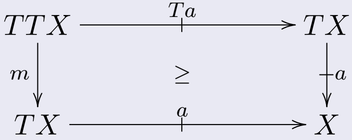
(T, e, m) "suitable" monad on Set:

$$\begin{array}{ccccc} T & \xrightarrow{eT} & TT & \xleftarrow{Te} & T \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle 1} & \\ & & T & & \end{array} \qquad \begin{array}{ccc} TTT & \xrightarrow{Tm} & TT \\ {\scriptstyle mT}\downarrow & & \downarrow{\scriptstyle m} \\ TT & \xrightarrow{m} & T \end{array}$$

$(\mathcal{V}, \otimes, k)$ quantale (or a symmetric monoidal-closed category)

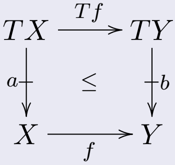
$(T,\mathcal{V})$-*algebra*: set X with $a : TX \nrightarrow X$ s.th. Barr's axioms hold mut-mut:





$$k \le a(x,x) \qquad\qquad Ta(\mathcal{X},y)\otimes a(y,z) \le a(m(\mathcal{X}),z)$$

$f : X \to Y$ *lax homomorphism:*



$$a(\chi,x) \le b(Tf(\chi),f(x))$$

$\mathsf{Alg}(T, \mathcal{V})$

$$\begin{array}{lll} \mathsf{Alg}(\mathrm{Id}, \mathcal{V}) = \mathcal{V}\text{-}\mathsf{Cat}: & \mathsf{Ord} & \mathsf{Met} \\ \mathsf{Alg}(U, \mathcal{V}): & \mathsf{Top} & \mathsf{App} \end{array}$$

(Lowen 1997, CH 2003)

What is L-closure, Yoneda embedding ...?

"Topologize" $\mathsf{Alg}(T, \mathcal{V})$

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\big\downarrow\!\!\!+ & \leq & +\!\!\!\big\downarrow{\scriptstyle b} \\ X & \xrightarrow[]{f} & Y \end{array} \qquad \begin{array}{ccc} TX & \xleftarrow{(Tf)^\circ}\!\!\!\!\!\!\!+\;\;\; & TY \\ {\scriptstyle a}\big\downarrow\!\!\!+ & \leq & +\!\!\!\big\downarrow{\scriptstyle b} \\ X & \xleftarrow{f^\circ}\!\!\!\!\!+\;\;\; & Y \end{array}$$

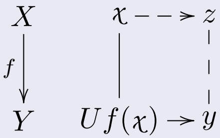
$$f \cdot a \leq b \cdot (Tf) \iff a \cdot (Tf)^\circ \leq f^\circ \cdot b$$

$$f \;\; strict \iff f \cdot a \geq b \cdot (Tf) \qquad a \cdot (Tf)^\circ \geq f^\circ \cdot b \iff f \;\; costrict$$

f strict

Ord : $\forall\, x, y\big(f(x \leq y \implies \exists z(x \leq z \,\&\, f(z) = y)\big)$

Top : $\forall\, x, y\big(Uf(x) \to y \implies \exists z(x \to z \,\&\, f(z) = y)\big)$

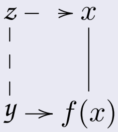


f proper

f costrict

$\forall\, x, y\big(y \leq f(x) \implies \exists z(z \leq x \,\&\, f(z) = y)\big)$

$\forall\, x, y\big(y \to f(x) \implies \exists z(z \to x \,\&\, Uf(z) = y)\big)$



f open

$\mathcal{V}$ frame $\left(v \wedge (\bigvee w_i) = \bigvee (v \wedge w_i)\right)$
$\implies$ {strict morphisms} form a topology on $\mathsf{Alg}(T, \mathcal{V})$

$\mathcal{V}$ frame, T satisfies Beck-Chevalley property
(T sends pullbacks to weak pullbacks)
$\implies$ {costrict morphims} form a topology on $\mathsf{Alg}(T, \mathcal{V})$

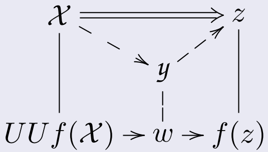
Schubert 2005: $\mathcal{V}$ completely distributive $\implies$ Tychonoff's Theorem holds

Pisani 1999

$$X \text{ exponentiable in } \mathsf{Top} \iff \Big(\sum_{\mathcal{X} \Longrightarrow z} \mathcal{X} \to z \implies \exists y(\mathcal{X} \to y \to z)\Big)$$

CHT 2003

$f : X \to Y$ expble in Top $\iff$



Hofmann 2006

T sat's BCP $\implies$ Pisani's charcaterization holds in $\mathsf{Alg}(T, \mathcal{V})$ (with $\times$ replaced by $\otimes$)