

# Macro- and Micro-Aspects of Categorical Topology

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# “Categorical Topology”

- Topos and Sheaf Theory: Grothendieck, Lawvere, Moerdijk, ...
- Locale (and Frame) Theory: Isbell, Johnstone, Banaschewski, Pultr, ...
- Topological Functors: Wyler, Brümmer, Herrlich, Hušek, Bentley, ...
- Closure Operators: Dikranjan, Giuli, T, Castellini ...
- Generalized metric spaces: Lawvere, Lowen, Clementino, Hofmann, T, ...
- Convergence and monads: Manes, Barr, Day, Gähler, Wyler, Mynard, Clementino, Hofmann, T, ...
- Algebraic Topology, Homotopy Theory, Groupoids, ...

# A relatively complicated theorem:

$$\left. \begin{array}{l} f : X \rightarrow Y \text{ continuous} \\ X \text{ exponentiable} \\ Y \text{ locally Hausdorff} \end{array} \right\} \implies (X, f) \text{ fiberwise exponentiable}$$

$$X \text{ exponentiable} : \iff \forall Z \exists Z^X \forall T : \frac{T \rightarrow Z^X}{X \times T \rightarrow Z}$$

$$\iff X \text{ core-compact (Day-Kelly 1970)}$$

$$(X, f) \text{ fw expble} \iff (X, f) \text{ fw c-c (Niefield 1982, Richter 2004)}$$

A relatively easy theorem:

$$\left. \begin{array}{l} f : X \rightarrow Y \text{ continuous} \\ X \text{ compact} \\ Y \text{ Hausdorff} \end{array} \right\} \implies f \text{ proper (=closed with compact fibers)}$$

less easy:

$$\left. \begin{array}{l} f : X \rightarrow Y, g : Y \rightarrow Z \text{ continuous} \\ g \circ f \text{ proper/perfect} \\ g \text{ separated} \end{array} \right\} \implies f \text{ proper/perfect}$$

## Two more theorems of the latter type:

$f : X \rightarrow Y, g : Y \rightarrow Z$  continuous

$\left. \begin{array}{l} g \circ f \text{ local homeomorphism} \\ g \text{ locally injective} \end{array} \right\} \implies f \text{ local homeomorphism}$

$\left. \begin{array}{l} g \circ f \text{ exponentiable} \\ g \text{ locally separated} \end{array} \right\} \implies f \text{ exponentiable}$

# Proof of the easy theorem

$$\begin{array}{ccc}
 & X \times Y & \\
 (\text{id}, f) \nearrow & & \searrow q \\
 X & \xrightarrow{f} & Y
 \end{array}$$

$$\begin{array}{ccc}
 X & \xrightarrow{(\text{id}, f)} & X \times Y \\
 f \downarrow & & \downarrow f \times \text{id} \\
 Y & \xrightarrow{(\text{id}, \text{id})} & Y \times Y
 \end{array}$$

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{q} & Y \\
 p \downarrow & & \downarrow \\
 X & \longrightarrow & 1
 \end{array}$$

$Y$  Hausdorff

$X$  compact

$f$  proper  $\iff$   $f$  stably closed

$\mathcal{X}$  finitely complete category

$\mathcal{P} \subseteq \text{mor } \mathcal{X}$  topology on  $\mathcal{X}$

$\iff$

1.  $\text{Iso } \mathcal{X} \subseteq \mathcal{P}$

2.  $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$

3.  $\mathcal{P}$  stable under pullback in  $\mathcal{X}$

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{f'} & Y \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array} \quad f \in \mathcal{P} \implies f' \in \mathcal{P}$$

# Some topologies on Top

- proper maps
- open maps
- exponentiable maps
- $\mathcal{P}(\mathcal{F})$ , with  $\mathcal{F}$  satisfying 1,2
- $\mathcal{P}(\mathcal{F})$ , with  $\mathcal{F} = \{f : X \rightarrow Y \mid f \text{ } c\text{-closed}\}$ ,  $c$  closure operator.

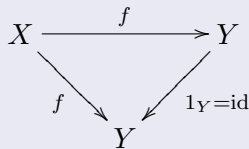
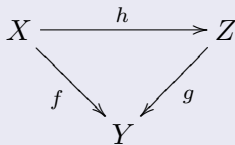
$f$   $\mathcal{P}$ -proper  $\iff f \in \mathcal{P}$

$X$   $\mathcal{P}$ -compact  $\iff (X \rightarrow 1) \in \mathcal{P} \iff \forall Y : (X \times Y \rightarrow Y) \in \mathcal{P}$

$f$   $\mathcal{P}$ -separated  $\iff (X \rightarrow X \times_Y X) \in \mathcal{P}$

$X$   $\mathcal{P}$ -Hausdorff  $\iff (X \rightarrow 1)$   $\mathcal{P}$ -separated  $\iff (X \rightarrow X \times X) \in \mathcal{P}$

$\mathcal{X}/Y :$



proper map in  $\mathcal{X}$  = compact object in  $\mathcal{X}/Y$

separated map in  $\mathcal{X}$  = Hausdorff object in  $\mathcal{X}/Y$

## More proofs:

Complicated = easy:  $\left. \begin{array}{l} X \text{ comp/ expble} \\ Y \text{ Hausd./ loc. Hausd.} \end{array} \right\} \implies f \text{ comp/ expble}$

map = object:  $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{g \circ f} & \swarrow_{g} \\ & Z & \end{array} \quad \left. \begin{array}{l} g \circ f \in \mathcal{P} \\ g \in \mathcal{P}' \end{array} \right\} \implies f \in \mathcal{P}$

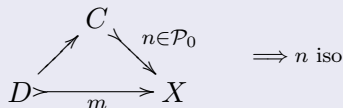
$\left. \begin{array}{l} g \circ f \in \mathcal{P} \cap \mathcal{P}' \\ g \in \mathcal{P}' \end{array} \right\} \implies f \in \mathcal{P} \cap \mathcal{P}'$

# $\mathcal{P}$ -open?

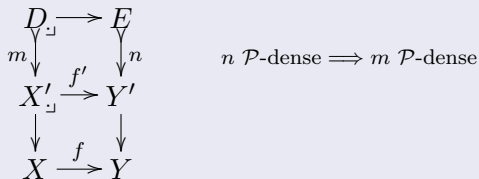
subobject  $\iff$  regular monomorphism (" $\hookrightarrow$ ")

$\mathcal{P}_0 = \mathcal{P} \cap \text{RegMono}$ : *closed embeddings*

$m : D \hookrightarrow X$   $\mathcal{P}$ -dense :



$f : X \rightarrow Y$   $\mathcal{P}$ -open :



(Top:  $f$  open  $\iff \forall S \subseteq Y : f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$ )

$$\mathcal{P} \mid \rightsquigarrow \mathcal{P} \cap \mathcal{P}'$$

$\{\mathcal{P} - \text{open}\}$

$\vdots$

local properties:    local compactness  
                          local connectedness

M.M Clementino, E.Giuli, W. Tholen

“A Functional Approach to General Topology”, in:

*Categorical Foundations*, Cambridge University Press 2004

## Cat

$X$ : “objects”

$\text{hom}: X \times X \rightarrow \{\text{sets}\}$

$1 = \{*\} \rightarrow \text{hom}(x, x)$

$\text{hom}(x, y) \times \text{hom}(y, z) \rightarrow \text{hom}(x, z)$

$(\text{Set}, \times, 1)$

## Met

$X$ : “points”

$d: X \times X \rightarrow [0, \infty]$

$0 \geq d(x, x)$

$d(x, y) + d(y, z) \geq d(x, z)$

$([0, \infty]^{\text{op}}, +, 0)$

## Ord

$X$ : “elements”

$\leq: X \times X \rightarrow \{\text{true}, \text{false}\}$

$\text{true} \vDash x \leq x$

$x \leq y \wedge y \leq z \vDash x \leq z$

$(2, \wedge, \text{true})$

$\mathcal{V}$  commutative, unital quantale  
= complete lattice,  $\otimes, k (\neq \perp)$   
 $a \otimes (\bigvee b_i) = \bigvee (a \otimes b_i)$

## $\mathcal{V} \circ \text{Set}$

objects: sets

morphisms =  $\mathcal{V}$ -relations:  $(r : X \dashrightarrow Y) = (r : X \times Y \rightarrow \mathcal{V})$

composition:  $s : Y \dashrightarrow Z$

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

$\text{Set} \rightarrow \mathcal{V} \circ \text{Set}$

$$X \mapsto X \quad f \mapsto f \text{ with } f(x, y) = \begin{cases} k & \text{if } f(x) = y \\ \perp & \text{else} \end{cases}$$

$\mathcal{V}$ -category: set  $X$  with  $a : X \dashrightarrow X$  ( $a : X \times X \rightarrow \mathcal{V}$ )

$$1_X \leq a \quad k \leq a(x, x)$$

$$a \cdot a \leq a \quad a(x, y) \otimes a(y, z) \leq a(x, z)$$

$$\mathcal{V}\text{-functor} : \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow a & & \downarrow b \\ X & \xrightarrow{f} & Y \end{array} \quad a(x_1, x_2) \leq b(f(x_1), f(x_2))$$

$$\rightsquigarrow \mathcal{V}\text{-Cat} : \quad \begin{array}{ll} 2\text{-Cat} = \text{Ord} & \mathbb{P}_+\text{-Cat} = \text{Met} \\ (\text{Set-Cat} = \text{Cat}) & \text{AbGrp-Cat} = \text{AddCat} \end{array}$$

# Yoneda embedding

$$\begin{aligned} \text{Category } \mathcal{X} &\xrightarrow{y} \text{Set}^{\mathcal{X}^{\text{op}}} \\ X &\longmapsto \mathcal{X}(-, X) \end{aligned}$$

$$\begin{aligned} \mathcal{V}\text{-Category } X &\xrightarrow{y} \mathcal{V}^{X^{\text{op}}} = \{f \mid f : X^{\text{op}} \rightarrow \mathcal{V}\} \\ x &\longmapsto a(-, x) \end{aligned}$$

Need structure on  $\mathcal{V}$  and  $\mathcal{V}^{X^{\text{op}}}$ !

$\mathcal{V}$  as a  $\mathcal{V}$ -category:

$$z \leq v \ominus u \iff u \otimes z \leq v$$

$$v \ominus u = \bigvee_{u \otimes z \leq v} z$$

$Y^X$  as a  $\mathcal{V}$ -category:

$$e(f, g) = \bigwedge_{x \in X} b(f(x), g(x))$$

Ord: pointwise order    Met: sup-metric

## Closure operator in $\mathcal{V}$ -Cat:

$$M \subseteq X \quad \overline{M} = \{x \in X \mid k \leq \bigvee_{z \in M} a(x, z) \otimes a(z, x)\}$$

$$\text{Ord} : \quad \overline{M} = \{x \in X \mid \exists z \in M (x \leq z \ \& \ z \leq x)\}$$

$$\text{Met} : \quad \overline{M} = \{x \in X \mid \inf\{d(x, z) + d(z, x) \mid z \in M\} = 0\}$$

$\overline{(\quad)}$  extensive, monotone, idempotent

$\overline{(\quad)}$  additive if:  $\forall u, v \in \mathcal{V} (k \leq u \vee v \implies k \leq u \text{ or } k \leq v)$   
 $\mathcal{V}\text{-Cat} \rightarrow \text{Top}$  “L-topology”

$$\begin{aligned}
X \text{ } L\text{-separated} &\iff \Delta_X \subseteq X \times X \text{ } L\text{-closed} \\
&\iff \forall x, y \in X (a(x, y) \wedge a(y, x) \leq k \implies x = y) \\
&\iff y : X \rightarrow \mathcal{V}^{X^{\text{op}}} \text{ injective}
\end{aligned}$$

$\text{Ord}_{\text{sep}}$ : posets

$\text{Met}_{\text{sep}}$ :  $d(x, y) = 0 = d(y, x) \implies x = y$

$$\tilde{X} := \overline{y(X)} \subseteq \mathcal{V}^{X^{\text{op}}}$$

$$(h : X^{\text{op}} \rightarrow \mathcal{V}) \in \tilde{X} \iff h \text{ tight}$$

$$\iff k \leq \bigvee_y \left( h(y) \otimes \bigwedge_x (a(x, y) \ominus h(x)) \right)$$

Ord: tight  $h$  corresponds to some  $\downarrow y = \{x \in X \mid x \leq y\}$

Met:  $h$  tight  $\iff h(x) \leq d(x, y) + h(y)$   
 $\& \inf_y \left( h(y) + \sup_x (d(x, y) \ominus h(x)) \right) = 0$

$$\begin{aligned} X \text{ } L\text{-complete} &\iff y : X \rightarrow \tilde{X} \text{ surjective} \\ &\iff \forall h \text{ tight } \exists y \in X : h = y(y) \end{aligned}$$

$\text{Ord}_{cpl} = \text{Ord}$

$\text{Met}_{cpl} = \text{Cauchy-complete metric spaces}$

$\mathcal{V}$  is  $L$ -complete and  $L$ -separated

$\tilde{X}$  is  $L$ -complete and  $L$ -separated (any  $X$ )

$X$  is  $L$ -complete and  $L$ -separated

$\iff X$  is injective wrt  $\{ \text{full \& faithful } L\text{-dense } \mathcal{V}\text{-functors} \} = \mathcal{D}$

$f : Y \rightarrow Z$   $f f f \iff b(y_1, y_2) = c(f(y_1), f(y_2))$  for all  $y_1, y_2 \in Y$   
 $f L\text{-dense} \iff \overline{f(Y)} = Z$

$(\mathcal{V}\text{-Cat})_{\text{cpl\&sep}}$  is firmly  $\mathcal{D}$ -reflective in  $\mathcal{V}\text{-Cat}$

( $\rightarrow$  Brümmer, Giuli, Herrlich, Holgate)

$X$  compact Hausdorff space  $\iff X$  carries  $UX \xrightarrow{c} X$  with

$$\begin{array}{ccc}
 X & \xrightarrow{e} & UX \\
 & \searrow 1_X & \downarrow c \\
 & & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 UUX & \xrightarrow{Uc} & UX \\
 m \downarrow & & \downarrow c \\
 UX & \xrightarrow{c} & X
 \end{array}$$

$e(x) = \dot{x} =$  principal ultrafilter

$m(\mathcal{X}) = \sum \mathcal{X} =$  Kowalsky sum of  $\mathcal{X}$  :  
 $A \in \sum \mathcal{X} \iff \{\chi \in \mathcal{X} \mid A \in \chi\} \in \mathcal{X}$

$Uc(\mathcal{X}) \ni A \iff c^{-1}[A] \in \mathcal{X}$

$X$  topological space  $\iff X$  carries  $c : UX \dashrightarrow X$  with

$$\begin{array}{ccc}
 X & \xrightarrow{e} & UX \\
 & \searrow \leq \downarrow c & \\
 & 1_X & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 UUX & \xrightarrow{Uc} & UX \\
 m \downarrow & \geq \downarrow c & \\
 UX & \xrightarrow{c} & X
 \end{array}$$

$$\text{true} \models \dot{x} \xrightarrow{c} x$$

$$\mathcal{X} \xrightarrow{Uc} y \ \& \ y \xrightarrow{c} z \models \sum \mathcal{X} \rightarrow z$$

$$\begin{array}{ccc}
 f : X \rightarrow Y \text{ cont.} & \iff & UX \xrightarrow{Uf} UY \\
 & & \downarrow \leq \downarrow \\
 & & X \xrightarrow{f} Y
 \end{array}
 \iff (\dot{x} \rightarrow x \implies Uf(\dot{x}) \rightarrow f(x))$$

$(T, e, m)$  “suitable” monad on  $\text{Set}$ :

$$\begin{array}{ccc}
 T & \xrightarrow{eT} TT & \xleftarrow{Te} T \\
 \searrow & \downarrow m & \swarrow \\
 & T & \\
 1 & & 1
 \end{array}
 \qquad
 \begin{array}{ccc}
 TTT & \xrightarrow{Tm} TT \\
 mT \downarrow & & \downarrow m \\
 TT & \xrightarrow{m} T
 \end{array}$$

$(\mathcal{V}, \otimes, k)$  quantale (or a symmetric monoidal-closed category)

$(T, \mathcal{V})$ -algebra: set  $X$  with  $a : TX \rightarrow X$  s.th. Barr's axioms hold  
mut-mut:

$$\begin{array}{ccc}
 X & \xrightarrow{e} & TX \\
 & \searrow^{1_X} & \downarrow a \\
 & & X
 \end{array}
 \quad \leq$$

$$\begin{array}{ccc}
 TT X & \xrightarrow{Ta} & TX \\
 m \downarrow & \geq & \downarrow a \\
 TX & \xrightarrow{a} & X
 \end{array}$$

$$k \leq a(x, x)$$

$$Ta(\mathcal{X}, y) \otimes a(y, z) \leq a(m(\mathcal{X}), z)$$

$f : X \rightarrow Y$  lax homomorphism:

$$\begin{array}{ccc}
 TX & \xrightarrow{Tf} & TY \\
 a \downarrow & \leq & \downarrow b \\
 X & \xrightarrow{f} & Y
 \end{array}$$

$$a(\mathcal{X}, x) \leq b(Tf(\mathcal{X}), f(x))$$

$\text{Alg}(\text{Id}, \mathcal{V}) = \mathcal{V}\text{-Cat} : \text{Ord} \quad \text{Met}$   
 $\text{Alg}(U, \mathcal{V}) : \quad \quad \quad \text{Top} \quad \text{App}$

(Lowen 1997, CH 2003)

What is  $L$ -closure, Yoneda embedding ...?

# “Topologize” $\text{Alg}(T, \mathcal{V})$

$$\begin{array}{ccc}
 TX & \xrightarrow{Tf} & TY \\
 a \downarrow & \leq & \downarrow b \\
 X & \xrightarrow{f} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 TX & \xleftarrow{(Tf)^\circ} & TY \\
 a \downarrow & \leq & \downarrow b \\
 X & \xleftarrow{f^\circ} & Y
 \end{array}$$

$$f \cdot a \leq b \cdot (Tf) \iff a \cdot (Tf)^\circ \leq f^\circ \cdot b$$

$$f \text{ strict} \iff f \cdot a \geq b \cdot (Tf) \iff a \cdot (Tf)^\circ \geq f^\circ \cdot b \iff f \text{ costrict}$$

## $f$ strict

$$\text{Ord : } \forall x, y (f(x \leq y \implies \exists z (x \leq z \& f(z) = y)))$$

$$\text{Top : } \forall x, y (Uf(x) \rightarrow y \implies \exists z (x \rightarrow z \& f(z) = y))$$

$$\begin{array}{ccc} X & & x \dashrightarrow z \\ \downarrow f & & \downarrow \\ Y & & Uf(x) \rightarrow y \end{array}$$

$f$  proper

## $f$ costrict

$$\forall x, y (y \leq f(x) \implies \exists z (z \leq x \& f(z) = y))$$

$$\forall x, y (y \rightarrow f(x) \implies \exists z (z \rightarrow x \& Uf(z) = y))$$

$$\begin{array}{ccc} z \dashrightarrow x & & \\ \downarrow & & \downarrow \\ y \rightarrow f(x) & & \end{array}$$

$f$  open

$\mathcal{V}$  frame  $(v \wedge (\bigvee w_i) = \bigvee(v \wedge w_i))$   
 $\implies$  {strict morphisms} form a topology on  $\text{Alg}(T, \mathcal{V})$

$\mathcal{V}$  frame,  $T$  satisfies Beck-Chevalley property  
( $T$  sends pullbacks to weak pullbacks)  
 $\implies$  {costrict morphisms} form a topology on  $\text{Alg}(T, \mathcal{V})$

Schubert 2005:  $\mathcal{V}$  completely distributive  $\implies$  Tychonoff's Theorem holds

$$X \text{ exponentiable in Top} \iff \left( \sum_{\mathcal{X} \Rightarrow z} \mathcal{X} \rightarrow z \implies \exists y(\mathcal{X} \rightarrow y \rightarrow z) \right)$$

$$f : X \rightarrow Y \text{ expble in Top} \iff \begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad} & z \\ \downarrow & \dashrightarrow y & \downarrow \\ Uf(\mathcal{X}) & \rightarrow w \rightarrow & f(z) \end{array}$$

## Hofmann 2006

$T$  sat's BCP  $\implies$  Pisani's characterization holds in  $\text{Alg}(T, \mathcal{V})$  (with  $\times$  replaced by  $\otimes$ )