

- Alternative presentation of $\text{Ha}(A, B)$ when U is ccd (constructively completely distributive):

$$\text{Ha}(A, B) = \bigvee \{v \in U \mid \forall x \in A \exists y \in B : v \leq a(x, y)\}$$

© $= \bigvee \{v \in U \mid \exists f: A \rightarrow B \text{ map } \forall x \in A : v \leq a(x, f(x))\}$

(Clementino - Hofmann, TAC 2004)

- The powerset monad \mathbb{P} of $\underline{\text{Set}}$ lifts to the Handorf monad \mathbb{H} of $U\text{-}\underline{\text{Cat}}$, and \mathbb{H} is Kock-Zöberlein.

- $(U\text{-}\underline{\text{Cat}})^{\mathbb{H}} \ni (X, a, \alpha: \mathbb{H}X \rightarrow X) : \alpha \text{ in } U\text{-}\underline{\text{Cat}}, 1 \leq e_X, a \cdot e_X = \alpha$
 α is necessarily the sup map of X in its induced order:

$(U\text{-}\underline{\text{Cat}})^{\mathbb{H}}$ is the cat. of U -cats that are complete in its induced order & satisfy $a(\bigvee A, y) = \bigwedge_{x \in A} a(x, y)$, with sup-preserving U -functors.

- Always $U \in (U\text{-}\underline{\text{Cat}})^{\mathbb{H}}$, with its given order.

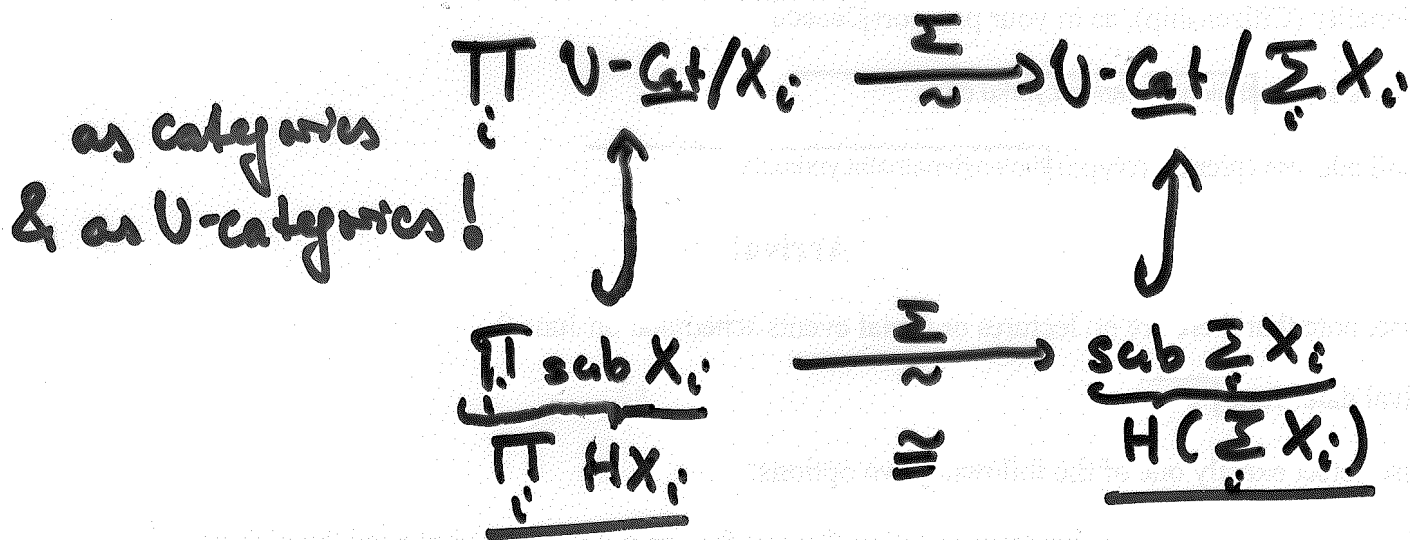
For $U=2$, $(U\text{-}\underline{\text{Cat}})^{\mathbb{H}} = \underline{\text{Sup}}$.

The extensive view of the Hausdorff metric:

- For $(X, a) \in \underline{U}\text{-Cat}$, any $f: A \rightarrow X, g: B \rightarrow X$, let

$$\begin{aligned}
 Ha(f, g) &= \bigwedge_{x \in A} \bigvee_{y \in B} a(f(x), g(y)) \\
 &= Ha(f(A) \hookrightarrow X, g(B) \hookrightarrow X)
 \end{aligned}$$

- $ob(\underline{U}\text{-Cat}/X)$ becomes a large \underline{U} -category
- Since $\underline{U}\text{-Cat}$ is extensive:



Easy to verify ad hoc, but it's more important to understand the categorical reason behind this isomorphism of \underline{U} -categories!

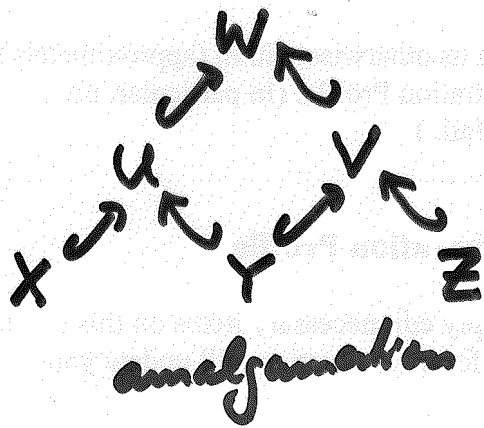
4. Gromov

$$\text{colim} (U\text{-Cat}_{\text{emb}} \rightarrow U\text{-Cat} \xrightarrow{H} U\text{-Cat} \hookrightarrow U\text{-CAT})$$

$$=: (\mathcal{G}, GH) = (\text{ob}(U\text{-Cat}) / \cong, GH)$$

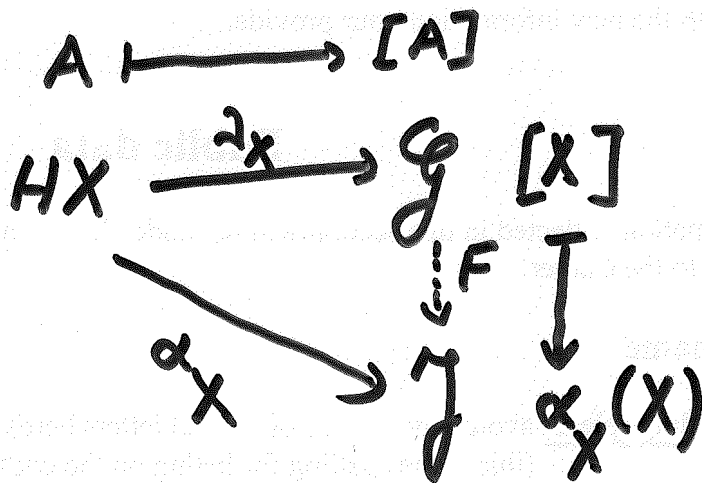
$$\underline{GH(X, Y)} = \bigvee_{X \hookrightarrow Z \hookrightarrow Y} \underline{h_Z(X, Y)}$$

GH is a structure on \mathcal{G} : 1) $k \subseteq h_X(X, X) \subseteq GH(X, X)$



$$2) \left. \begin{array}{l} h_u(X, Y) \subseteq h_w(X, Y) \\ h_v(Y, Z) \subseteq h_w(Y, Z) \end{array} \right\} \subseteq h_w(X, Z)$$

Colimit property:



This works for arbitrary $K: U\text{-Cat} \rightarrow U\text{-Cat}$ in lieu of H !

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \text{Set} & \xrightarrow{P} & \text{Set} \end{array}$$

Now $H: \mathcal{C} \rightarrow \mathcal{C}$ is a \mathcal{U} -functor!

Also: $\mathcal{C} \otimes \mathcal{C} \xrightarrow{\otimes} \mathcal{C}, (X, Y) \mapsto X \otimes Y,$

$(R=T) \quad \mathcal{C} \otimes \mathcal{C} \xrightarrow{\times} \mathcal{C}, \quad \mapsto X+Y,$

$(R=T) \quad \mathcal{C} \otimes \mathcal{C} \xrightarrow{+} \mathcal{C}, \quad \mapsto X+Y,$

are all \mathcal{U} -functors. Hence:

$(\mathcal{C}, \otimes), (\mathcal{C}, \times), (\mathcal{C}, +)$ are monoids of the monoidal category $\mathcal{U}\text{-CAT}$, and

$$H: (\mathcal{C}, +) \rightarrow (\mathcal{C}, \times)$$

is a homomorphism of monoids in $\mathcal{U}\text{-CAT}$.

Note: one has always (without $R=T$) that

$$\mathcal{C} \times \mathcal{C} \xrightarrow{+} \mathcal{C}$$

is a monoid (w.r.t. the cartesian structure of $\mathcal{U}\text{-CAT}$).

5. Problems

compact := complete + totally bounded
Lauvere (straightforward)

- X compact + separated $\stackrel{?}{\Rightarrow} H^s X$ complete
- $(\mathcal{C}_b \cap \{\text{comp} + \text{sep}\}, \in H^s)$ complete?