- Alternative presentation of Ha(A,B) when $\mathcal{V}$ is ccd (constructively completely distributive):

$$Ha(A,B) = \bigvee\{v \in \mathcal{V} \mid \forall x \in A\ \exists y \in B : v \le a(x,y)\}$$

© $$= \bigvee\{v \in \mathcal{V} \mid \exists f : A \xrightarrow[\text{map}]{} B\ \forall x \in A : v \le a(x, f(x))\}$$

(Clementino - Hofmann, TAC 2004)

- The powerset monad $\mathbb{P}$ of Set lifts to the Hausdorff monad $\mathbb{H}$ of $\mathcal{V}$-Cat, and $\mathbb{H}$ is Kock-Zöberlein.

- $(\mathcal{V}\text{-}\underline{Cat})^{\mathbb{H}} \ni (X, a, \alpha : HX \to X) : \alpha$ in $\mathcal{V}$-Cat, $1 \le e_X \cdot a$, $a \cdot e_X =$

α is necessarily the sup map of X in its induced order:

$(\mathcal{V}\text{-}\underline{Cat})^{\mathbb{H}}$ is the cat. of $\mathcal{V}$-cats that are complete in its induced order & satisfy $a(\bigvee A, y) = \bigwedge_{x \in A} a(x,y)$, with sup-preserving $\mathcal{V}$-functors.

- Always $\mathcal{V} \in (\mathcal{V}\text{-}\underline{Cat})^{\mathbb{H}}$, with its given order.

For $\mathcal{V} = 2$, $(\mathcal{V}\text{-}\underline{Cat})^{\mathbb{H}} = \underline{Sup}$.

The extensive view of the Hausdorff metric:

- For $(X,a) \in \mathcal{V}\text{-}\underline{\mathrm{Cat}}$, any $f: A \to X$, $g: B \to X$, let

$$Ha(f,g) = \bigwedge_{x \in A} \bigvee_{y \in B} a(f(x), g(y))$$
$$= Ha(f(A) \hookrightarrow X, g(B) \hookrightarrow X)$$

- $\mathrm{ob}(\mathcal{V}\text{-}\underline{\mathrm{Cat}}/X)$ becomes a large $\mathcal{V}$-category

- Since $\mathcal{V}\text{-}\underline{\mathrm{Cat}}$ is extensive:

as categories & as $\mathcal{V}$-categories!

$$\begin{tikzcd}
\prod_i \mathcal{V}\text{-}\underline{\mathrm{Cat}}/X_i \arrow[r, "\Sigma", "\sim"'] & \mathcal{V}\text{-}\underline{\mathrm{Cat}}/\sum_i X_i \\
\underbrace{\prod_i \mathrm{sub}\, X_i}_{\prod_i HX_i} \arrow[u, hook] \arrow[r, "\Sigma", "\sim"'] & \underbrace{\mathrm{sub} \sum_i X_i}_{H(\sum_i X_i)} \arrow[u, hook]
\end{tikzcd}$$

$$\prod_i HX_i \cong H\left(\sum_i X_i\right)$$

Easy to verify ad hoc, but it's more important to understand the categorical reason behind this isomorphism of $\mathcal{V}$-categories!

4. Gromov

$$\operatorname{colim}\left(\mathcal{V}\text{-}\underline{Cat}_{emb} \to \mathcal{V}\text{-}\underline{Cat} \xrightarrow{H} \mathcal{V}\text{-}\underline{Cat} \hookrightarrow \mathcal{V}\text{-}\underline{CAT}\right)$$

$$=: (\mathcal{G}, GH) = \left(ob(\mathcal{V}\text{-}\underline{Cat})/\cong\,,\ GH\right)$$

$$\underline{GH(X,Y) = \bigvee_{X \hookrightarrow Z \hookleftarrow Y} h_Z(X,Y)}$$

GH is a structure on $\mathcal{G}$: 1) $k \leq h_X(X,X) \leq GH(X,X)$

$$\begin{tikzcd}
 & & W & & \\
 & U \arrow[ur, hook] & & V \arrow[ul, hook'] & \\
X \arrow[ur, hook] & & Y \arrow[ul, hook'] \arrow[ur, hook] & & Z \arrow[ul, hook']
\end{tikzcd}$$

amalgamation

$$2)\ \left.\begin{aligned} h_U(X,Y) &\leq h_W(X,Y) \\ \otimes \quad & \quad\ \otimes \\ h_V(Y,Z) &\leq h_W(Y,Z) \end{aligned}\right\} \leq h_W(X,Z)$$

$$A \longmapsto [A]$$

Colimit property:

$$\begin{tikzcd}
HX \arrow[r, "\lambda_X"] \arrow[dr, "\alpha_X"'] & \mathcal{G} \arrow[d, dashed, "F"] & {[X]} \arrow[d, mapsto] \\
 & \mathcal{J} & \alpha_X(X)
\end{tikzcd}$$

This works for arbitrary $K : \mathcal{V}\text{-}\underline{Cat} \to \mathcal{V}\text{-}\underline{Cat}$ in lieu of H!

$$\begin{tikzcd}
\mathcal{V}\text{-}\underline{Cat} \arrow[r, "K"] \arrow[d] & \mathcal{V}\text{-}\underline{Cat} \arrow[d] \\
\underline{Set} \arrow[r, "P"] & \underline{Set}
\end{tikzcd}$$

Now $H : \mathcal{G} \to \mathcal{G}$ is a $\mathcal{V}$-functor!

Also: $\mathcal{G} \otimes \mathcal{G} \xrightarrow{\otimes} \mathcal{G}$, $(X,Y) \mapsto X \otimes Y$,

$(k=T)$ $\mathcal{G} \otimes \mathcal{G} \xrightarrow{\times} \mathcal{G}$, $\mapsto X \times Y$,

$(k=T)$ $\mathcal{G} \otimes \mathcal{G} \xrightarrow{+} \mathcal{G}$, $\mapsto X + Y$,

are all $\mathcal{V}$-functors. Hence:

$(\mathcal{G}, \otimes)$, $(\mathcal{G}, \times)$, $(\mathcal{G}, +)$ are monoids of the monoidal category $\mathcal{V}$-$\underline{CAT}$, and

$$H : (\mathcal{G}, +) \longrightarrow (\mathcal{G}, \times)$$

is a homomorphism of monoids in $\mathcal{V}$-$\underline{CAT}$.

Note: one has always (<u>without</u> $k=T$) that

$$\mathcal{G} \times \mathcal{G} \xrightarrow{+} \mathcal{G}$$

is a monoid (w.r.t. the cartesian structure of $\mathcal{V}$-$\underline{CAT}$),

5. Problems

compact := $\underset{\text{Lawvere}}{\underline{\text{complete}}}$ + $\underset{\text{(straightforward)}}{\underline{\text{totally bounded}}}$

- X compact + separated $\overset{?}{\Longrightarrow}$ $\overset{s}{H}X$ complete
- $(\mathcal{G} \cap \{\text{comp} + \text{sep}\}, d_{H^s})$ complete ?