Two Gromov Theorems

- GH^3 is a complete metric on the totality of isometry classes of compact metric spaces.

- G finitely generated group of polynomial growth
  $\Rightarrow$ G has nilpotent subgroup of finite index

$$B_r(e_G) = \{x \in G \mid \underbrace{\|x\|}_{\text{length of shortest word representing } x} \le r\}, \quad r > 0$$

$$\exists C, d : |B_r(e_G)| \le C r^d$$

(Being of polynomial growth does not depend on the choice of the finite generating system.)

2. Quantale-enriched Categories

$\mathcal{V} = (\mathcal{V}, \otimes, k)$ comm. unital quantale, $\bot < k$

$$u \otimes \bigvee_i v_i = \bigvee_i u \otimes v_i$$

Examples: $2 = (\{\bot < \top\}, \wedge, \top)$, $\mathbb{P}_+ = (([0,\infty], \geq), +, 0)$

'73 Lawvere: logic metric

$$\begin{tikzcd}
\mathcal{V}\text{-}\underline{\mathrm{Rel}} \\
\underline{\mathrm{Set}} \arrow[u, "\text{graph}"'] \\
\mathcal{V}\text{-}\underline{\mathrm{Cat}} \arrow[u, "\text{forget: topological}"']
\end{tikzcd}$$

$\mathcal{V}$-$\underline{\mathrm{Rel}}$: $X \xrightarrow{r} Y$: $r : X \times Y \to \mathcal{V}$

$$(s \cdot r)(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z)$$

$\mathcal{V}$-$\underline{\mathrm{Cat}}$:

$$\begin{tikzcd}
(X,a) \arrow[d, "f"'] \\
(Y,b)
\end{tikzcd}$$

$1_X \leq a$, $a \cdot a \leq a$

$k \leq a(x,x)$, $a(x,y) \otimes a(y,z) \leq a(x,z)$

$f \cdot a \leq b \cdot f$

$a(x,y) \leq b(f(x), f(y))$

$2\text{-}\underline{\mathrm{Cat}} = \underline{\mathrm{Ord}}$, $\mathbb{P}_+\text{-}\underline{\mathrm{Cat}} = \underline{\mathrm{Met}}$

- V is a V-category: $z \leq \underbrace{x \multimap y}$ $\Leftrightarrow$ $z \otimes x \leq y$
- V-$\underline{Cat}$ is <u>monoidal closed</u>:

$$(X \otimes Y, a \otimes b)\,, \quad a \otimes b((x,y),(x',y')) = a(x,x') \otimes b(y,y')$$

$$X \multimap Y = (V\text{-}\underline{Cat}(X,Y), c)\,, \quad c(f,g) = \bigwedge_{x \in X} b(f(x),g(x))$$

- $V \to 2$ $\quad \rightsquigarrow \quad V\text{-}\underline{Cat} \longrightarrow \underline{Ord}$
  $(v \mapsto \top)$ iff $v \geq k$

$$\underline{x \leq y \Leftrightarrow k \leq a(x,y)}$$

- $V\text{-}\underline{Rel}(X,Y) = V^{X \times Y}$: V-$\underline{Rel}$ is V-$\underline{Cat}$ enriched
- $V\text{-}\underline{Mod}(X,Y) = (X^{op} \otimes Y \multimap V)$: <u>V-$\underline{Mod}$ is V-$\underline{Cat}$ enriched</u>
- $V\text{-}\underline{Cat} \longrightarrow V\text{-}\underline{Mod}$, $\quad V\text{-}\underline{Cat}^{op} \longrightarrow V\text{-}\underline{Mod}$

$$(X \xrightarrow{f} Y) \longmapsto f_* : X \to Y \qquad f \longmapsto f^* : Y \to X$$

$$f_* = b \cdot f \qquad\qquad f^* = f^{\circ} \cdot b$$

- <u>V-$\underline{Cat}$ is ordered</u> : $f \leq g \Leftrightarrow f^* \leq g^* \Leftrightarrow \forall x : f(x) \leq g(x)$

  in particular: $\underline{x \leq y \Leftrightarrow x^* \leq y^*}$: same as above

3. Hausdorff

free sup lattice

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "y_X"', "\text{Yoneda}"] & PX \arrow[d, dashed, "\mathcal{Y}_X"] \\ & (X^{op} \multimap V) = P_V X \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, dashed, maps to] \\ a_B(x) = \bigvee_{y\in B} a(x,y) \end{tikzcd}$$

a sup lattice

V-$\underline{\mathrm{Cat}} \to \underline{\mathrm{Set}}$ topological: PX inherits V-$\underline{\mathrm{Cat}}$-structure from $P_V X$:

$$h(A,B) := \bigwedge_{x\in X} a_A(x) \multimap a_B(x) = \bigwedge_{x\in A} a_B(x)$$

$$= \bigwedge_{x\in A} \bigvee_{y\in B} a(x,y) \underline{=: Ha(A,B)}$$

This could be done for every $\varphi : X \nrightarrow Y$ in lieu of $a : X \nrightarrow X$. Get

$$\begin{tikzcd} V\text{-}\underline{\mathrm{Mod}} \arrow[r, "H"] & V\text{-}\underline{\mathrm{Mod}} \\ V\text{-}\underline{\mathrm{Cat}} \arrow[u, "(-)_*"] \arrow[r, "H"] \arrow[d] \arrow[ur, phantom, "(X,a) \mapsto (PX, Ha)"] & V\text{-}\underline{\mathrm{Cat}} \arrow[u, "(-)_*"'] \arrow[d] \\ \underline{\mathrm{Set}} \arrow[r, "P"] & \underline{\mathrm{Set}} \end{tikzcd}$$