Categorical and topological aspects of semi-abelian theories

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Introduction

In his "princeps" article about what was going to become the notion of abelian category, S. Mac Lane in *Duality for groups*, Bull. Am. Soc. (1950), emphasized the necessity to go beyong the abelian/additive case and to find some simple axioms which allow to conceptualize in a uniform way the properties of the categories of groups, rings, K-algebras etc. He indicated the achievement of the Noether isomorphisms as a good test for this kind of conceptualization.

The aim of these lectures is to introduce the notion of pointed protomodular [8] category which, together with the notion of regular category in the sense of Barr [2], allows to enter into the Mac-Lane project since, not only the combination of these two concepts (under the name of homological category [4]) allows to get the Noether isomorphisms, but also to get all the homological lemmas. Further, the notion of semi-abelian category [30], i.e. pointed protomodular, Barr exact category with sums, allows to get an intrinsic notion of semi-direct product.

Given an algebraic theory $\mathbb{T}$ whose category of models is pointed protomodular or, equivalently, semi-abelian, we then study the category $Top^{\mathbb{T}}$ of topological models of $\mathbb{T}$ (i.e. topological semi-abelian algebras [5]), we generalize in this way the most classical results on $GpTop$ the category of topological groups and we show that these categories $Top^{\mathbb{T}}$ are homological.

Both authors deeply thank F.Borceux for his strong help in the preparation of this document.

1 Preliminaries

First, let us recall that "elementwise arguments" are perfectly valid – under some precise conditions – for proving results in abstract categories.

Metatheorem 1.1 *Let $\mathcal{P}$ be a statement of the form $\varphi \Rightarrow \psi$, where φ and ψ can be expressed as conjunctions of properties in the following list:*

1. *some finite diagram is commutative;*
2. *some morphism is a monomorphism;*
3. *some morphism is an isomorphism;*
4. *some finite diagram is a limit diagram;*
5. *some arrow $f\colon A \longrightarrow B$ factors (of course, uniquely) through some specified monomorphism $s\colon S \rightarrowtail B$.*

If this statement $\mathcal{P}$ is valid in the category of sets, it is valid in every category.

Proof Let $\mathcal{E}$ be a category. The Yoneda embedding $Y_{\mathcal{E}}$ is full and faithful and preserves all existing limits. Thus $Y_{\mathcal{E}}$ preserves and reflects composition, commutativity of diagrams, isomorphisms, limits, monomorphisms and the existence of a factorization through a given subobject. (Let us recall that $f\colon A \rightarrowtail B$ being a monomorphism is a limit property: it means that the pullback of f along itself is – up to an isomorphism – the identity on A). This shows already that to prove the statement $\mathcal{P}$ in $\mathcal{E}$, it suffices to prove it in $[\mathcal{E}^{\mathsf{op}}, \mathsf{Set}]$, since the functor $Y_{\mathcal{E}}$ preserves and reflects all ingredients which appear in $\mathcal{P}$.

Now all the properties listed in our statement are valid in $[\mathcal{E}^{\mathsf{op}}, \mathsf{Set}]$ precisely when they are valid pointwise in Set. This is clear for the first four conditions. For condition 5, consider in $[\mathcal{E}^{\mathsf{op}}, \mathsf{Set}]$ the situation of Diagram 1 and assume that for every object $X \in \mathcal{E}$, the morphism f_X factors in Set through s_X via a morphism g_X. We must make sure that these mappings g_X define a morphism g in $[\mathcal{E}^{\mathsf{op}}, \mathsf{Set}]$, that is a natural transformation. If $x\colon X \longrightarrow Y$ is a morphism of $\mathcal{E}$, the outer part and the right hand square of Diagram 2 are commutative, because f and s are morphisms in $[\mathcal{E}^{\mathsf{op}}, \mathsf{Set}]$. Since s_X is a monomorphism, the left hand square is commutative as well, proving the naturality of g.

Thus to prove $\mathcal{P}$ in $\mathcal{E}$, it suffices to prove it in $[\mathcal{E}^{\mathsf{op}}, \mathsf{Set}]$ and this reduces to proving it pointwise in Set. □

Definition 1.2 *By a* pointed *category is meant a category with a zero object (an object which is both initial and terminal).*

Proposition 1.3 *When $\mathcal{E}$ is a pointed category with finite limits, the property*

6. *Some specified arrow is the zero arrow.*

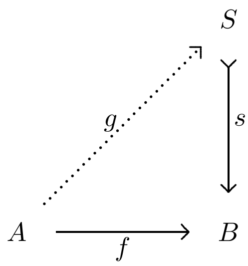


Diagram 1

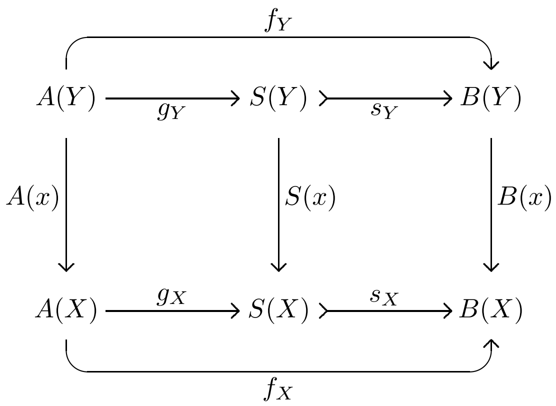


Diagram 2

can be added to the list of properties in Metatheorem 1.1.

Proof In a pointed category $\mathcal{E}$, every set $\mathcal{E}(A,B)$ of morphisms is pointed by the zero morphism, so that the Yoneda embeeding factors through pointed sets. This allows at once transposing the Metatheorem 1.1 to the pointed case, replacing the category of sets by that of pointed sets. □

Let us also recall a famous metatheorem due to M. Barr (see [2]):

Theorem 1.4 *When $\mathcal{E}$ is a regular category with finite limits, the property*

7. Some specified arrow is a regular epimorphism.

can be added to the list of properties in Metatheorem 1.1.

We shall freely use the three metatheorems above by simply saying that we develop a proof "elementwise". The following lemma will also be quite useful.

Lemma 1.5 *Consider two finitely complete categories $\mathcal{A}$, $\mathcal{B}$ and a functor $F\colon \mathcal{A}\longrightarrow\mathcal{B}$ preserving finite limits. The functor F is conservative as soon as it is conservative on monomorphisms.*

Proof By "conservative on monomorphisms" we clearly mean that when $F(f)$ is an isomorphism with f a monomorphism, then f is an isomorphism. We have to infer the same property for an arbitrary arrow f.

First, observe that the functor F reflects monomorphisms. A morphism $f \in \mathcal{A}$ is a monomorphism when the equalizer k of its kernel pair (u,v) is an isomorphism. But then $F(k)$ is the equalizer of the kernel pair of $F(f)$. When $F(f)$ is a monomorphism, $F(k)$ is then an isomorphism and by assumption, k is an isomorphism. Thus f is a monomorphism.

In particular all morphisms f such that $F(f)$ is an isomorphism are monomorphisms, and therefore by assumption, are isomorphisms. □

$$\begin{array}{ccc} K & \xrightarrow{\;k\;} & X \\ {\scriptstyle \omega_K}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbf{1} & \xrightarrow[\alpha_K]{} & Y \end{array}$$

Diagram 3

2 Abelian groups versus groups

Let us write $\mathbf{1}$ for the terminal object. In a category with finite limits and a zero object, the pullback of Diagram 3 yields $k = \mathsf{ker}\, f$.

In the category of abelian groups, when (f, s) is a split epimorphism (that is, $f \circ s = \mathsf{id}_Y$), we obtain then the isomorphism of Diagram 4 between X and the direct sum of Y and K.

$$\begin{array}{ccccc} K & \xrightarrow{\;k\;} & X & \xleftarrow{\;\cong\;} & K \oplus Y \\ {\scriptstyle \omega_K}\big\downarrow & & \big\downarrow{\scriptstyle f} & & \\ \mathbf{1} & \xrightarrow[\alpha_K]{} & Y & & \end{array}$$

Diagram 4

$$\mathbf{1} \longrightarrow K \xrightarrow{\;k\;} X \underset{f}{\overset{s}{\leftrightarrows}} Y \longrightarrow \mathbf{1}.$$

In the category of groups, we obtain an isomorphism as well, as in Diagram 5 between X and only the

$$\begin{array}{ccccc} K & \xrightarrow{\;k\;} & X & \xleftarrow{\;\cong\;} & K \vee Y \\ {\scriptstyle \omega_K}\big\downarrow & & \big\downarrow{\scriptstyle f} & & \\ \mathbf{1} & \xrightarrow[\alpha_K]{} & Y & & \end{array}$$

Diagram 5

supremum $K \vee Y$ of K and Y in the lattice of subgroups of X.

Indeed, just as in the abelian case, given $x \in X$ we have

$$f\big(x - sf(x)\big) = f(x) - fsf(x) = f(x) - f(x) = 0$$

from which $x - sf(x) \in K$ and

$$x = \big(x - sf(x)\big) + sf(x)$$

with the first term in K and the second in Y.

The striking difference with the abelian case is that the mapping

$$X \longrightarrow K, \quad x \mapsto x - sf(x)$$

is no longer in general a group homomorphism, because the category Gp of groups is not additive. But we shall show that this absence is actually not too much inhibiting.

3 Pointed protomodular categories

The classical short five lemma in an abelian category $\mathcal{E}$ says the following. Given a the commutative Diagram 6 where p and q are (of course regular) epimorphisms and $u = \ker p$, $v = \ker q$, if a and c are isomorphisms, b is

$$\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & A & \overset{u}{\rightarrowtail} & B & \overset{p}{\twoheadrightarrow} & C & \longrightarrow & \mathbf{1} \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
\mathbf{1} & \longrightarrow & A' & \overset{v}{\rightarrowtail} & B' & \overset{q}{\twoheadrightarrow} & C' & \longrightarrow & \mathbf{1}
\end{array}$$

Diagram 6

an isomorphism as well.

It is well-known that the short five lemma holds as well in the category of groups, thus certainly, it is not characteristic of the abelian context.

The *split short five lemma* is a special instance of the short five lemma which has the particularity of being a purely "left exact statement": a statement involving only commutative diagrams and finite limits: nothing like "epimorphism" or "regular epimorphism". But in the situations studied in these notes, it will imply the general form of the short five lemma ... and much more!

Definition 3.1 *Let $\mathcal{E}$ be a pointed category. The* split short five lemma *holds in $\mathcal{E}$ when given the commutative Diagram 7*

$$b \circ u = v \circ a, \quad c \circ p = q \circ b, \quad b \circ s = t \circ c,$$

with p, q split epimorphisms

$$\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & A & \overset{u}{\rightarrowtail} & B & \underset{p}{\overset{s}{\leftrightarrows}} & C & \longrightarrow & \mathbf{1} \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
\mathbf{1} & \longrightarrow & A' & \overset{v}{\rightarrowtail} & B' & \underset{q}{\overset{t}{\leftrightarrows}} & C' & \longrightarrow & \mathbf{1}
\end{array}$$

Diagram 7

$$p \circ s = \mathsf{id}_C, \quad q \circ t = \mathsf{id}_{C'}$$

with respective kernels u and v

$$u = \ker p, \quad v = \ker q,$$

if a and c are isomorphisms, b is an isomorphism as well.

We introduce so one of the main definitions of these notes:

Definition 3.2 *A pointed category $\mathcal{E}$ is* protomodular *when*

1. *$\mathcal{E}$ has kernels of split epimorphisms;*
2. *the split short five lemma holds.*

The example on which we want to put full emphasize is the case of groups.

Example 3.3 *The category* Gp *of groups is pointed protomodular.*

It is well-known that the short five lemma holds in the category of groups; it implies obviously the split short five lemma. □

Of course one has also:

Example 3.4 *Every abelian category is protomodular.*

Again the short five lemma holds in an abelian category. □

Here is now a characterization of protomodular varieties (where "variety" means the category of models of an algebraic theory in the sense of Lawvere).

Theorem 3.5 *A variety is pointed protomodular when the corresponding theory contains, for some given natural number $n \in \mathbb{N}$:*

- *a unique constant* 0*;*
- n *binary operations* $\alpha_i(x, y)$ *such that* $\alpha_i(x, x) = 0$*;*
- *a* $(n+1)$*-ary operation* θ *such that* $\theta\big(\alpha_1(x, y), \ldots, \alpha_n(x, y), y\big) = x$*.*

Proof In a variety, the initial object is the algebra of constants of the theory. The initial object is thus also terminal (i.e. a singleton) if and only if the theory has exactly one constant. From now on we assume that this is the case and we write 0 for that constant.

Let us write further $F(x, y)$ for the free algebra on two generators, $F(y)$ for the free algebra on one generator and $\mathbf{1}$ for the zero algebra.

We assume first that the variety is protomodular. We consider the following situation

$$K \xrightarrow{\;k\;} F(x,y) \underset{p}{\overset{s}{\rightleftarrows}} F(y)$$

where the various ingredients of this diagram are defined by

$$p(x) = y = p(y), \quad s(y) = y, \quad K = \ker p.$$

Clearly, $p \circ s = \mathsf{id}$, thus (p, s) is a split epimorphism. This diagram presents also K and $F(y)$ as two subobjects of $F(x, y)$, with respective inclusions k and s. We consider their union $K \vee F(y)$ as subobject of $F(x, y)$ in the variety. This yields a subobject in the category of split epimorphisms over $F(y)$, simply because s factors through $F(y)$, thus through $K \vee F(y)$ (see Diagram 8) But then the kernel of $p \circ i$ is still K and by protomodularity, i

$$\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & K & \longrightarrow & K \vee F(Y) & \underset{s}{\overset{p \circ i}{\rightleftarrows}} & F(y) & \longrightarrow & \mathbf{1} \\
 & & \| & & \downarrow{\scriptstyle i} & & \| & & \\
\mathbf{1} & \longrightarrow & K & \longrightarrow & F(x,y) & \underset{p}{\overset{s}{\rightleftarrows}} & F(y) & \longrightarrow & \mathbf{0}
\end{array}$$

Diagram 8

is an isomorphism.

We have already proved that $F(x, y) = K \vee F(y)$, thus in particular $x \in K \vee F(y)$. This proves the existence of finitely many (let us say, n) operations $\alpha_i(x, y) \in K$ and a $(n + 1)$-ary operation θ and such that

$$x = \theta\big(\alpha_1(x, y), \ldots, \alpha_n(x, y), y\big).$$

It remains to notice that by definition of K, $\alpha_i(x, x) = 0$ for each index i.

Conversely, let us assume the existence of a natural number n and operations α_i, θ as in the statement. Let us observe first that given two elements $a, b \in A$ in an algebra

$$a = b \;\text{ if and only if }\; \forall i \;\alpha_i(a, b) = 0.$$

If $a = b$, $\alpha_i(a, b) = 0$ by assumption. Conversely, if each $\alpha_i(a, b) = 0$, then

$$b = \theta(\alpha_1(b, b), \ldots, \alpha_n(b, b), b\big) = \theta(0, \ldots, 0, b\big) = \theta(\alpha_1(a, b), \ldots, \alpha_n(a, b), b\big) = a.$$

Consider now the situation of the split short five lemma as in Definition 3.1, with a and c isomorphisms. We must prove that b is an isomorphism as well. For simplicity, we view u, v as canonical inclusions.

Given $x, y \in B$ such that $b(x) = b(y)$, we have

$$cp(x) = qb(x) = qb(y) = cp(y)$$

from which $p(x) = p(y)$, since c is an isomorphism. Therefore

$$p\big(\alpha_i(x,y)\big) = \alpha_i\big(p(x), p(y)\big) = 0$$

proving that $\alpha_i(x,y) \in A$ for each index i. But

$$b\big(\alpha_i(x,y)\big) = \alpha_i\big(b(x), b(y)\big) = 0$$

since $b(x) = b(y)$. Therefore $a\big(\alpha_i(x,y)\big) = 0$ and thus $\alpha_i(x,y) = 0$ since a is an isomorphism. This implies $x = y$, as we have seen. This proves already that b is injective.

Next given $z \in B'$, we can write $q(z) = cp(x)$ for some $x \in B$, since c and p are surjective. We have then, for each index i,

$$q\big(\alpha_i(z, b(x))\big) = \alpha_i\big(q(z), qb(x)\big) = \alpha_i\big(q(z), cp(x)\big) = 0$$

proving that $\alpha_i\big(z, b(x)\big) \in A'$. Since a is surjective, let us write $\alpha_i\big(z, b(x)\big) \in A' = a(y_i)$ for some $y_i \in A$. It remains to consider the element

$$y = \theta(y_1, \ldots, y_n, x) \in B$$

and observe that

$$b(y) = \theta\big(b(y_1), \ldots, b(y_n), b(x)\big) = \theta\big(a(y_1), \ldots, a(y_n), b(x)\big) = \theta\big(\alpha_1(z, b(x)), \ldots, \alpha_n(z, b(x)), b(x)\big) = z.$$

This proves the surjectivity of b. □

In the proof, we have observed that

Corollary 3.6 *In the conditions of Theorem 3.5, for elements $a, b \in A$*

$$a = b \quad \textit{iff} \quad \forall i \;\; \alpha_i(a) = \alpha_i(b).$$ □

Here are now various situations and constructions which produce new protomodular categories from given ones.

Example 3.7 *Consider two pointed categories $\mathcal{E}$, $\mathcal{E}'$ with kernels of split epimorphisms. Consider next a functor $U\colon \mathcal{E} \longrightarrow \mathcal{E}'$ which preserves kernels of split epimorphisms and reflects isomorphisms. If $\mathcal{E}'$ is protomodular, $\mathcal{E}$ is protomodular as well.*

Immediate from the definitions. □

Example 3.8 *Every theory $\mathbb{T}$ containing a unique constant and a group operation gives rise to a pointed protomodular variety of $\mathbb{T}$-algebras. This is in particular the case for groups, rings without unit, modules on a ring, algebras without unit, and so on.*

Writing the group operation additively, in Theorem 3.5 it suffices to choose $n = 1$ and

$$\alpha_1(x,y) = x - y, \quad \theta(x,y) = x + y.$$

Indeed, $x - x = 0$ and $(x - y) + y = x$. □

Example 3.9 *Let $\mathcal{E}'$ be a category which is monadic over the pointed protomodular category $\mathcal{E}$. Then $\mathcal{E}'$ is pointed protomodular as well.*

By 3.7 again. □

Example 3.10 *Every additive category with finite limits is pointed protomodular.*

If $\mathcal{C}$ is an additive category with finite limits, the category $\mathsf{Add}[\mathcal{C}, \mathsf{Ab}]$ of additive functors to the category of abelian groups is well-known to be abelian; thus it satisfies the short five lemma and is protomodular. The Yoneda embedding

$$Y \colon \mathcal{C} \longrightarrow \mathsf{Add}[\mathcal{C}^{\mathsf{op}}, \mathsf{Ab}], \quad C \mapsto \mathcal{C}(-, C)$$

satisfies the conditions of 3.7, thus $\mathcal{C}$ is protomodular. □

Example 3.11 *Let $\mathcal{E}$ be a category with finite limits and $\mathbb{T}$ an algebraic theory yielding a pointed protomodular category of $\mathbb{T}$-algebras. The category $\mathcal{E}^{\mathbb{T}}$ of $\mathbb{T}$-algebras in $\mathcal{E}$ is pointed protomodular.*

The composite functor

$$\mathcal{E}^{\mathbb{T}} \xrightarrow{\ U\ } \mathcal{E} \xrightarrow{\ Y_{\mathcal{E}}\ } [\mathcal{E}^{\mathsf{op}}, \mathsf{Set}], \quad A \mapsto \mathcal{E}(-, A)$$

where U is the forgetful functor and $Y_{\mathcal{E}}$ is the Yoneda embedding, factors through the category $[\mathcal{E}^{\mathsf{op}}, \mathsf{Set}^{\mathbb{T}}]$, where $\mathsf{Set}^{\mathbb{T}}$ is now the usual category of $\mathbb{T}$-algebras in Set. This factorization

$$\overline{Y} \colon \mathcal{E}^{\mathbb{T}} \longrightarrow [\mathcal{E}^{\mathsf{op}}, \mathsf{Set}^{\mathbb{T}}]$$

preserves finite limits because U and $Y_{\mathcal{E}}$ do; it is also full and faithful because so is $Y_{\mathcal{E}}$ and therefore it reflects isomorphisms. Thus we can apply 3.7 and it suffices to prove that $[\mathcal{E}^{\mathsf{op}}, \mathsf{Set}^{\mathbb{T}}]$ is protomodular.

In the category $[\mathcal{E}^{\mathsf{op}}, \mathsf{Set}^{\mathbb{T}}]$, the notions of split epimorphism, pullback and isomorphism are defined pointwise. Thus the protomodularity of $\mathsf{Set}^{\mathbb{T}}$ implies that of $[\mathcal{E}^{\mathsf{op}}, \mathsf{Set}^{\mathbb{T}}]$. □

Let us emphasize the following case, which will be studied more intensively in subsequent sections:

Example 3.12 *Let $\mathcal{E}$ be a category with finite limits. Then the category $\mathsf{Gp}(\mathcal{E})$ of internal groups in $\mathcal{E}$ is pointed protomodular. This is thus in particular the case for the category $\mathsf{Gp}(\mathsf{Top})$ of topological groups.* □

Example 3.13 *The dual category of an elementary topos $\mathcal{E}$ is a protomodular category*

See [19] for a proof. □

4 Some well-known algebraic properties

We are now going to show that, in pointed protomodular categories, kernels and cokernels behave in a manner which is analogous to the situation in abelian categories.

Proposition 4.1 *Consider a protomodular category $\mathcal{E}$ with finite limits. Pulling back in $\mathcal{E}$ reflects monomorphisms, that is, given a pullback as in Diagram 9, if f' is a monomorphism, f is a monomorphism as well.*

$$\begin{array}{ccc} X' & \xrightarrow{\ \beta\ } & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ \alpha\ ]{} & Y \end{array}$$

Diagram 9

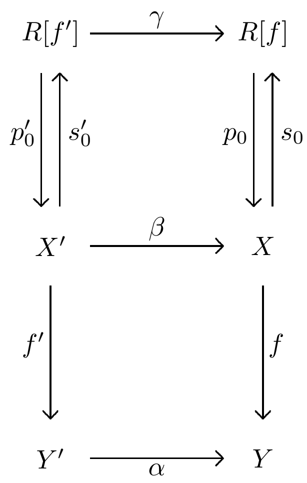


Diagram 10

Proof Let us consider Diagram 10, with $R[f']$ and $R[f]$ the kernel pair relations of f' and f. The first projections p_0' and p_0 admit the diagonals s_0' and s_0 as sections. And since the bottom square is a pullback, the square involving the projections is a pullback as well. Indeed, working elementwise,

$$\begin{aligned} R[f'] &= \{(u,v) \big| u,v \in X', \quad f'(u) = f'(v)\} \\ &= \{(x,a,y,b) \big| x,y \in Y', \quad a,b \in X, \quad \alpha(x) = f(a), \quad \alpha(y) = f(b), \quad x = y\} \\ &\cong \{((x,a),(a,b)) \big| (x,a) \in X', \quad (a,b) \in R[f]\} \\ &= X' \times_X R[f]. \end{aligned}$$

Since f' is a monomorphism, p_0' is an isomorphism. Pulling back further along $\mathbf{1} \longrightarrow X'$, we find both the kernel of p_0 and of p_0', which is thus $\mathbf{1}$ since p_0 is an isomorphism. Considering then Diagram 11 we conclude

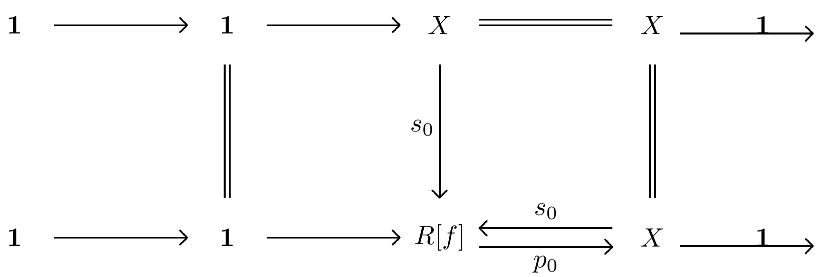


Diagram 11

by the split short five lemma that s_0 is an isomorphism. Thus its left inverse p_0 is an isomorphism as well and f is a monomorphism. $\square$

Theorem 4.2 *In a pointed protomodular category with finite limits, the following conditions are equivalent:*

1. *the morphism $f\colon X \longrightarrow Y$ is a monomorphism;*
2. *the kernel $K[f]$ of f is the zero object.*

Proof The first condition implies obviously the second one in every pointed category. Conversely, the kernel of f is given by the pullback of Diagram 12, where the left vertical arrow is an isomorphism by assumption. By 4.1, f is a monomorphism. $\square$

Lemma 4.3 *In a pointed protomodular category with finite limits, consider the pullback of Diagram 13 where p is a split epimorphism with section s. Then the pair (g, s) is strongly epimorphic.*

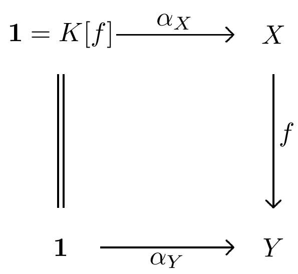


Diagram 12

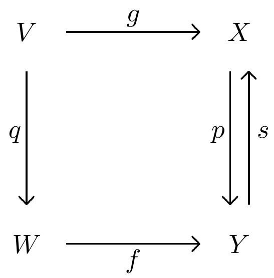


Diagram 13

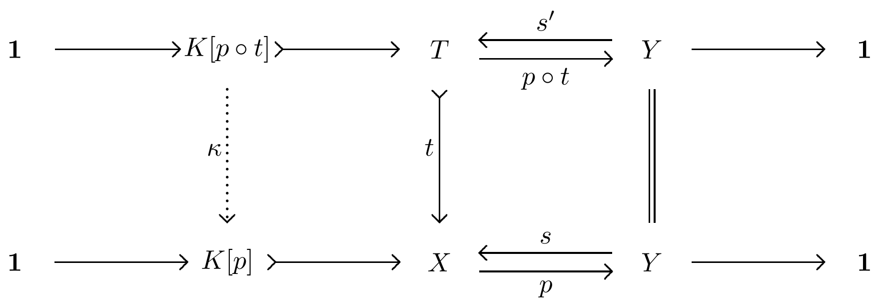


Diagram 14

Proof If g and s factor through some subobject $t\colon T \rightarrowtail X$ – let us say: $g = t \circ g'$ and $s = t \circ s'$ – we obtain the following situation of Diagram 14. The factorization κ through the kernels is thus a monomorphism. But the factorization $g = t \circ g'$ induces corresponding factorizations

$$K[q] \longrightarrow K[p \circ t] \xrightarrow{\kappa} K[p].$$

Since the original square is a pullback, $\ker q \cong \ker p$ proving finally that the monomorphism κ is in fact an isomorphism. By the split short five lemma, t is now an isomorphism. □

It should be mentioned that for a category $\mathcal{E}$ with finite limits, the property in Lemma 4.3 is in fact equivalent to the protomodularity of $\mathcal{E}$.

Corollary 4.4 *In a pointed protomodular category with finite limits, the pair of morphisms*

$$A \xrightarrow{(\mathsf{id}_A, 0)} A \times B \xleftarrow{(0, \mathsf{id}_B)} B$$

is strongly epimorphic.

Proof The projection $p_B\colon A \times B \longrightarrow B$ admits $(0, \mathsf{id}_B)$ as section and $(\mathsf{id}_A, 0)$ as kernel, that is, as pullback along $\mathbf{1} \longrightarrow B$. □

A category $\mathcal{E}$ with finite limits satisfying the condition of Corollary 4.4 is called *unital* [15].

Theorem 4.5 *Let $\mathcal{E}$ be a pointed protomodular category with finite limits. For a morphism $f\colon X \longrightarrow Y$, the following conditions are equivalent:*

1. *f is a regular epimorphism;*
2. *$f = \mathsf{coker}\,(\mathsf{ker}\, f)$.*

Proof Condition 2 implies obviously condition 1. Conversely, let us consider Diagram 15, with $R[f]$ the kernel relation of f. The upper squares are pullbacks, as follows at once from an elementwise argument. Indeed, for

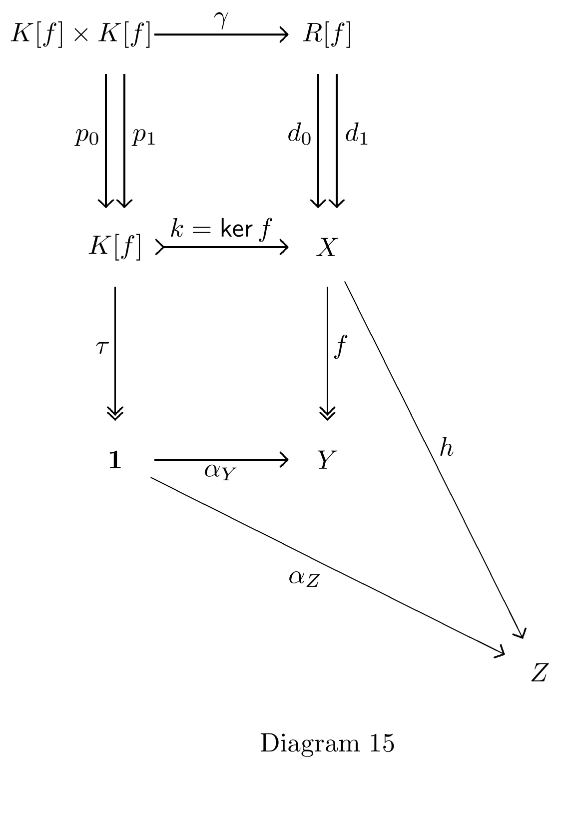


Diagram 15

the index 0,

$$\begin{aligned} K[f] \times K[f] &= \big\{(a,b) \in X \times X \big| f(a) = 1, \quad f(b) = 1\big\} \\ &\cong \big\{\big(a, (a', b)\big) \big| a = a', \quad f(a') = f(b)\big\}. \end{aligned}$$

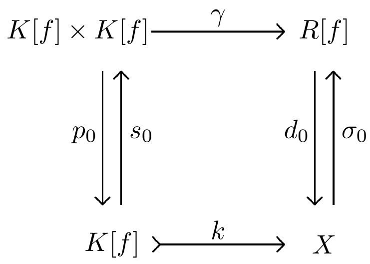


Diagram 16

So Diagram 16 is commutative, where s_0 and σ_0 indicate the diagonals and the downward directed square is a pullback. By Lemma 4.3, the pair (γ, σ_0) is epimorphic. Choose now h such that $h \circ k = \omega_{K[f],Z}$; we must prove that h factors uniquely through f. Since $f = \mathsf{Coeq}(d_0, d_1)$, it suffices to prove that $h \circ d_0 = h \circ d_1$. This reduces further to proving

$$h \circ d_0 \circ \gamma = h \circ d_1 \circ \gamma, \quad h \circ d_0 \circ \sigma_0 = h \circ d_1 \circ \sigma_0.$$

We have indeed

$$h \circ d_0 \circ \gamma = h \circ k \circ p_0 = \omega_{K[f]\times K[f]} = h \circ k \circ p_1 = h \circ d_1 \circ \gamma;$$
$$h \circ d_0 \circ \sigma_0 = h = h \circ d_1 \circ \sigma_0.$$

□

5 Protomodularity and categories of points

It is well-known that given a set I, the category Set/I is equivalent to the category of I-families of sets. Indeed given a mapping $p\colon X \longrightarrow I$, one gets at once the corresponding family $\big(p^{-1}(i)\big)_{i\in I}$ of sets, while given a family $(X_i)_{i\in I}$ it suffices to define $X = \amalg_{i\in I} X_i$, with $p(x) = i$ when $x \in X_i$.

Observe now that giving a family $(X_i, x_i)_{i\in I}$ of pointed sets requires to give the choice of a base point for every index i, that is a mapping

$$s\colon I \longrightarrow \coprod X_i, \quad i \mapsto x_i \in X_i.$$

This is equivalent to giving

$$s\colon I \longrightarrow X, \quad p \circ s = \mathsf{id}_I.$$

This justifies the terminology in the following definition:

Definition 5.1 *Let $\mathcal{E}$ be an arbitrary category. The category $\mathsf{Pt}_I(\mathcal{E})$ of points over $I \in \mathcal{E}$ is defined as follows.*

1. *An object is a pair $\big(p, s\colon X \leftrightarrows I\big)$ of morphisms of $\mathcal{E}$ with $p \circ s = \mathsf{id}_I$;*
2. *a morphism*
$$u\colon \big(q, t\colon Y \leftrightarrows I\big) \longrightarrow \big(p, s\colon X \leftrightarrows I\big)$$
in $\mathsf{Pt}_I(\mathcal{E})$ is a morphism $u\colon Y \longrightarrow X$ in $\mathcal{E}$ such that $p \circ u = q$ and $u \circ t = s$ (see Diagram 17).

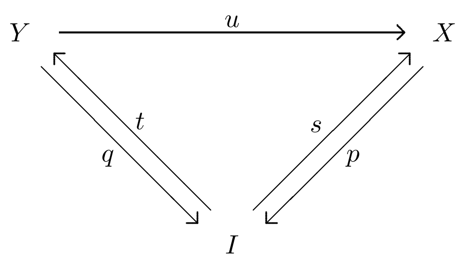


Diagram 17

Proposition 5.2 *Let $\mathcal{E}$ be a category with pullbacks of split epimorphisms. Pulling back along a morphism $v\colon J \longrightarrow I$ in $\mathcal{E}$ induces a functor*

$$v^*\colon \mathsf{Pt}_I(\mathcal{E}) \longrightarrow \mathsf{Pt}_J(\mathcal{E}).$$

called "inverse image along v".

Proof Trivial. □

To emphasize an important property of the categories of points, let us recall that given a functor $F\colon \mathcal{F} \longrightarrow \mathcal{E}$ and an object $I \in \mathcal{E}$, the fibre of F at I is the (non full) subcategory $\mathcal{F}_I \subseteq \mathcal{F}$ of those objects $X \in \mathcal{F}$ such that $F(X) = I$ and of those morphisms $f \in \mathcal{F}$ such that $F(f) = \mathsf{id}_I$. The functor F is a *fibration* when, for every arrow $\alpha\colon J \longrightarrow I$ in $\mathcal{E}$ and every object $X \in \mathcal{F}_I$, there is a universal morphism $f\colon Y \longrightarrow X$ in $\mathcal{F}$ such that $F(f) = \alpha$. This universal map f is called the "cartesian map above α".

Theorem 5.3 *Let $\mathcal{E}$ be a category with pullbacks of split epimorphisms.*

1. *The codomain functor*

$$\pi\colon \mathsf{Pt}(\mathcal{E}) \longrightarrow \mathcal{E}, \quad \pi\big(p, s\colon X \leftrightarrows I\big) \mapsto I, \quad (u,v) \mapsto v$$

   is a fibration, called the "fibration of points" of $\mathcal{E}$.

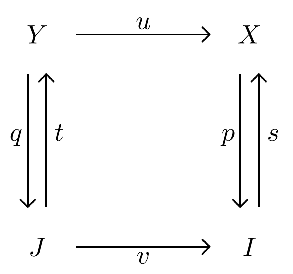
2. *The fibre of this fibration at some object $I \in \mathcal{E}$ is the category $\mathsf{Pt}_I(\mathcal{E})$ of points over I.*

3. *The inverse image functors v^* of this fibration are those described in Proposition 5.2.*

4. *A morphism*

$$(u,v)\colon \big(q, t\colon Y \leftrightarrows J\big) \longrightarrow \big(p, s\colon X \leftrightarrows I\big)$$

   in $\mathsf{Pt}(\mathcal{E})$ is cartesian precisely when the downward directed square in Diagram 18 is a pullback.



$u \circ t = s \circ v,$

$v \circ q = p \circ u;$

Diagram 18

Proof Notice at once that the fibre $\mathsf{Pt}_I(\mathcal{E})$ over an object $I \in \mathcal{E}$ is the category $\mathsf{Pt}_I(\mathcal{E})$ of points over I.

Given a morphism $v\colon J \longrightarrow I$ in $\mathcal{E}$ and a point $\big(p, s\colon X \leftrightarrows I\big)$ in $\mathsf{Pt}_I(\mathcal{E})$, we consider the commutative Diagram 19 where the square is a pullback.

This diagram shows in particular that the pullback of the split epimorphism p along v is a split epimorphism q, with section t. It follows also at once from 5.2 that

$$\big(q, t\colon Y \leftrightarrows J\big) = v^*\big(p, s\colon X \leftrightarrows I\big).$$

We shall prove that

$$(u,v)\colon \big(q, t\colon Y \leftrightarrows J\big) \longrightarrow \big(p, s\colon X \leftrightarrows I\big)$$

is the expected cartesian morphism.

Let us consider the situation of Diagram 20, where

$$(x,y)\colon \big(h, r\colon Z \leftrightarrows K\big) \longrightarrow \big(p, s\colon X \leftrightarrows I\big)$$

is a morphism of $\mathsf{Pt}(\mathcal{E})$ and $y = v \circ z$ in $\mathcal{E}$. This yields at once a factorization w through the pullback defining Y:

$$u \circ w = x, \quad q \circ w = z \circ h.$$

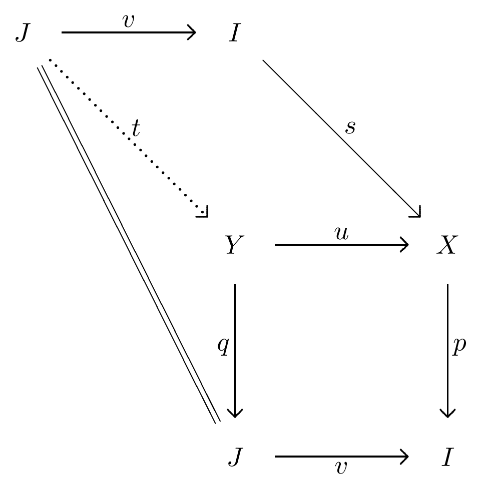


Diagram 19

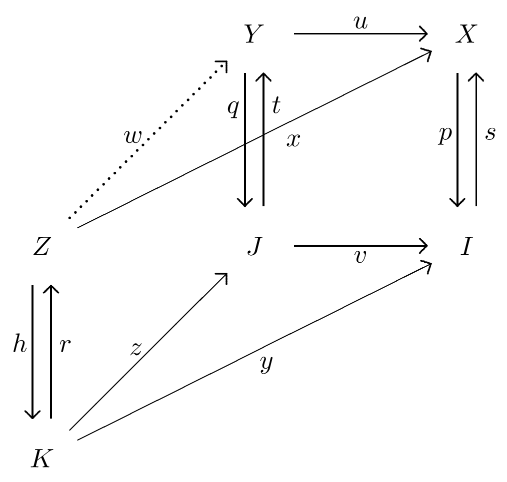


Diagram 20

It follows that

$$q \circ w \circ r = z \circ h \circ r = z = q \circ t \circ z,$$
$$u \circ w \circ r = x \circ r = s \circ y = s \circ v \circ z = u \circ t \circ z,$$

from which $w \circ r = t \circ z$, by uniqueness of the factorization through a pullback. This means precisely the existence of a unique morphism

$$(w, z)\colon \big(h, r\colon Z \leftrightarrows K\big) \longrightarrow \big(q, t\colon Y \leftrightarrows J\big)$$

in $\mathsf{Pt}(\mathcal{E})$ such that $(u, v) \circ (w, z) = (x, y)$. $\square$

The fibration of points provides an elegant way to define protomodular categories in the non-pointed context.

Definition 5.4 *A category $\mathcal{C}$ is* protomodular *when*

1. *it admits pullbacks of split epimorphisms along arbitrary morphisms;*
2. *for every morphism $v\colon J \longrightarrow I$ in $\mathcal{E}$, the inverse image functor*

$$v^*\colon \mathsf{Pt}_I(\mathcal{E}) \longrightarrow \mathsf{Pt}_J(\mathcal{E}).$$

of the fibration of points reflects isomorphisms.

And we observe immediately that:

Proposition 5.5 *Let $\mathcal{C}$ be a pointed category with pullbacks of split epimorphisms. The two notions of protomodularity, in Definitions 3.2 and 5.4 are equivalent.*

Proof Taking the kernel of a split epimorphism $p, s\colon A \overset{\longleftarrow}{\longrightarrow} I$ is computing the pullback along the morphism $\mathbf{1} \longrightarrow I$; thus Definition 5.5 implies certainly Definition 3.2.

Conversely, consider a morphism $v\colon J \longrightarrow I$ in $\mathcal{E}$ and the morphism $\alpha_J\colon \mathbf{1} \longrightarrow J$. Pulling back along $v \circ \alpha_J$ is the kernel functor, thus reflects isomorphisms by assumption. Therefore v^* reflects isomorphisms as well. $\square$

But we can even do a little bit more:

Proposition 5.6 *Let $\mathcal{C}$ be a finitely complete category. This category is protomodular if and only if the inverse image functors of the fibration of points are conservative on monomorphisms.*

Proof It is straightforward to observe that when $\mathcal{C}$ has finite limits, so do all the fibres of the fibration of points. And since in a category, pulling-back commutes with finite limits, one infers easily that the inverse image functors v^* preserve finite limits. One concludes by Lemma 1.5. $\square$

One should observe further that in the case of the category Gp of groups, saying that the inverse image functor of the fibration of points along the morphism $\mathbf{1} \longrightarrow X$ is conservative on monomorphisms is exactly rephrasing the condition $K \vee Y \cong X$ considered in Section 2. Consequently, in the pointed context, the protomodular condition is entirely contained in this isomorphism.

The following result will also be useful.

Proposition 5.7 *If a category $\mathcal{E}$ is protomodular, all the categories $\mathsf{Pt}_I(\mathcal{E})$ of points are pointed protomodular.*

Proof The category $\mathsf{Pt}_I(\mathcal{E})$ is pointed, with the identity on I as zero object. The result follows then at once from the fact that pullbacks in a category of points are computed as in $\mathcal{E}$. $\square$

Given Diagram 21 in any category, if the outer rectangle and the square (2) are pullbacks, the square (1) is a pullback as well. The other cancellation property, obtained by interchanging the roles of the squares (1) and (2), is not valid in general. A special case of it remains nevertheless valid in protomodular categories.

Proposition 5.8 *Let $\mathcal{E}$ be a protomodular category with pullbacks. Given Diagram 21 where $p \circ s = \mathsf{id}_Y$ and the downward directed squares are commutative, if the square* (1) *and the outer rectangle are pullbacks, the square* (2) *is a pullback as well.*

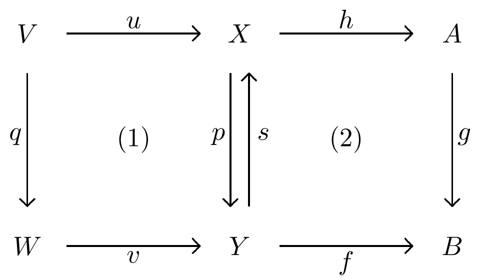


Diagram 21

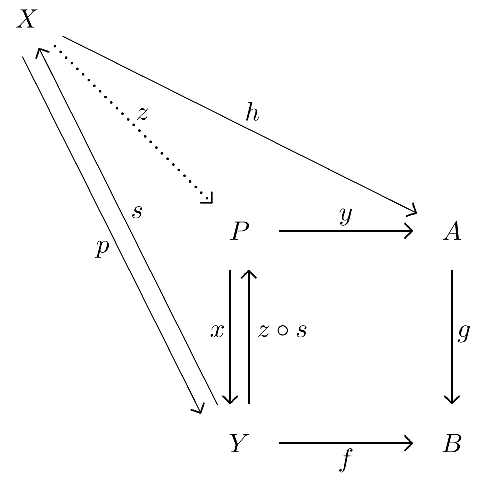


Diagram 22

Proof Pulling back the split epimorphism p along v yields a split epimorphism q; more precisely, when the square (1) is a pullback, there is a morphism $t\colon W \longrightarrow V$ such that $q \circ t = \mathsf{id}_W$ and $u \circ t = s \circ v$.

If $\mathcal{E}$ is protomodular and the situation of condition 2 is given, consider Diagram 22 where the downward directed square is a pullback and z is the unique factorization of the outer downward directed quadrilateral through this pullback. This yields a morphism

$$z\colon \big(p, s\colon X \leftrightarrows Y\big) \longrightarrow \big(x, z \circ s\colon P \leftrightarrows Y\big)$$

in the fibre $\mathsf{Pt}_Y(\mathcal{E})$. The pullback assumptions in condition 2 imply

$$v^*\big(p, s\colon X \leftrightarrows Y\big) \cong \big(q, t\colon V \leftrightarrows W\big) \cong v^*\big(x, z \circ s\colon P \leftrightarrows Y\big).$$

Thus $v^*(z)$ is an isomorphism and therefore z is an isomorphism, by protomodularity of $\mathcal{E}$ (see 3.2). By construction of P, this proves that the square (2) is a pullback. □

It should be mentioned that the "cancellation property" in Proposition 5.8 is in fact equivalent to the protomodularity of the category.

Let us conclude with an important property of protomodular categories.

Theorem 5.9 *In a finitely complete protomodular category, every reflexive relation is an equivalence relation.*

Proof We consider a reflexive relation $r\colon R \rightarrowtail Y \times Y$ in $\mathcal{E}$. We write d_0, d_1 for its two projections

$$d_0, d_1\colon R \rightrightarrows Y, \quad s_0\colon Y \longrightarrow R.$$

and s_0 for the factorization of the diagonal of $Y \times Y$ through R. We must prove that R is an equivalence relation.

For this, we construct by straightformard pullbacks the so-called "simplicial kernel" $K = K[d_0, d_1]$ of the relation R, which in the category of sets is given by

$$K = K[d_0, d_1] = \big\{(x, y, z) \big| xRy, \quad yRz, \quad xRz\big\}.$$

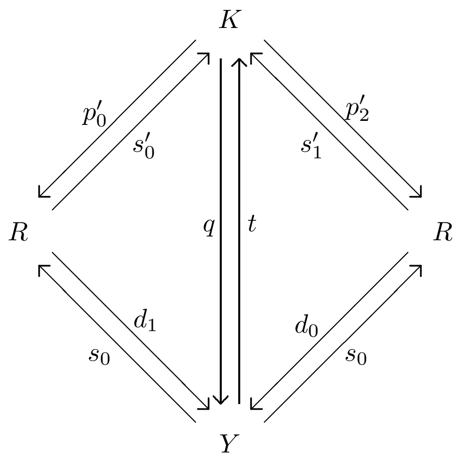


Diagram 23

We shall write $d'_i\colon K \longrightarrow Y$ $(i = 0, 1, 2)$ for the three projections of $K = K[d_0, d_1]$.

We consider now Diagram 23. The morphism s'_0 is defined via the equalities

$$d'_0 \circ s'_0 = d_0, \quad d'_1 \circ s'_0 = d_1, \quad d'_2 \circ s'_0 = d_1$$

and analogously for s_1

$$d'_0 \circ s'_1 = d_0, \quad d'_1 \circ s'_1 = d_0, \quad d'_2 \circ s'_1 = d_1.$$

In terms of elements, in the category of sets, the original morphisms are given by

$$d_0(x, y) = x, \quad d_1(x, y) = y, \quad s_0(x) = (x, x).$$

Our Diagram 23 has then the morphisms

$$p'_0(x, y, z) = (x, y), \quad p'_2(x, y, z) = (y, z), \quad s'_0(x, z) = (x, z, z), \quad s'_1(x, z) = (x, x, z).$$

Working elementwise, we observe at once that $d_0 \circ p'_2 = d_1 \circ p'_0$, which we choose as morphism q. In the same way $s'_0 \circ s_0 = s'_1 \circ s_0$ and we choose this composite as morphism t. Elementwise

$$q(x, y, z) = y, \quad t(x) = (x, x, x).$$

This completes the definition of Diagram 23.

Still working elementwise, it is obvious to observe that the pair (p'_0, p'_2) is monomorphic. This can be rephrased as having a monomorphism

$$(p'_0, p'_2)\colon \big(q, t\colon K \leftrightarrow Y\big) \rightarrowtail \big(d_0, s_0\colon R \leftrightarrows Y\big) \times \big(d_1, s_0\colon R \leftrightarrows Y\big)$$

in the fibre $\mathsf{Pt}_Y(\mathcal{E})$. But since R is reflexive, it contains the two inclusions of the two factors in the product. By Example 5.7 and Corollary 4.4, (p'_0, p'_2) is an isomorphism. In other words, $K \cong R \times_Y R$ in $\mathcal{E}$, which means elementwise

$$\big\{(x, y, z) \big| xRy,\ yRz,\ xRz\big\} = \big\{(x, y, z) \big| xRy,\ yRz\big\}.$$

This is precisely the transitivity of R.

To prove the symmetry of the relation R, we consider the relation S on the object R given by the pullback of Diagram 24. In the case of sets

$$S = \big\{(a, b, c, d) \big| aRb, \quad cRd, \quad aRd\big\}.$$

Working elementwise, we notice that S is trivially reflexive. By the first part of the proof, S is thus transitive.

Let now $(x, y) \in R$. Since R is reflexive and S is transitive

$$\begin{aligned}(yRy) \text{ and } (xRx) &\Rightarrow (yRy)S(xRy) \text{ and } (xRy)S(xRx)\\ &\Rightarrow (yRy)S(xRx)\\ &\Rightarrow (yRx).\end{aligned}$$

This means precisely the symmetry of the relation R. □

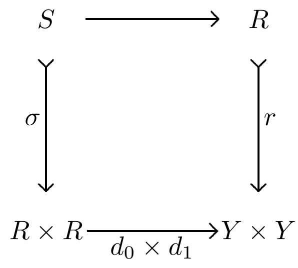


Diagram 24

A category with finite limits in which every reflexive relation is an equivalence relation is called a *Mal'cev category*, [27] and [28].

6 Intrinsic notions

Our first intrinsic notion is that of *commutative object*.

Definition 6.1 *Let $\mathcal{C}$ be a pointed unital, finitely complete category. An object $X \in \mathcal{C}$ is* commutative *when there exists a morphism φ making the Diagram 25 commutative:*

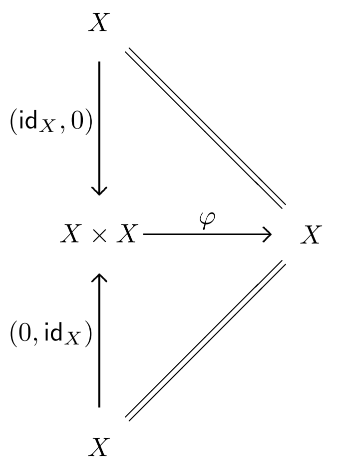


Diagram 25

Notice that since the category is *unital*, a morphism φ as in Definition 6.1 is necessarily unique, and thus *being commutative* is a property.

To justify the terminology "commutative", consider the case of the category of additive monoids. Given $x, y \in X$ we have

$$\varphi(x,y) = \varphi\big((x,0) + (0,y)\big) = \varphi(x,0) + \varphi(0,y) = x + y.$$

Analogously

$$\varphi(x,y) = \varphi\big((0,y) + (x,0)\big) = \varphi(0,y) + \varphi(x,0) = y + x.$$

This proves that $x + y = \varphi(x,y) = y + x$ and in particular, the monoid is commutative. Conversely if the addition is commutative, it is a morphism of monoids and has the required properties for being the morphism φ.

Theorem 6.2 *Let $\mathcal{C}$ be a pointed, protomodular and finitely complete category. Then every commutative object admits a unique internal abelian group structure.*

Proof The uniqueness follows at once from Corollary 4.4.

With the notation of Definition 6.1, let us prove "elementwise" that φ is the expected internal abelian group structure. For simplicity, we write $\varphi(x, y) = x + y$. The conditions on φ are thus $x + 0 = x$ and $0 + x = x$.

Using Corollary 4.2, proving $x + y = y + x$ reduces to prove it on the pairs $(x, 0)$ and $(0, x)$. This is the case since $x + 0 = x = 0 + x$.

Iterating Corollary 4.4 we get at once that the three morphisms

$$(\mathsf{id}_X, 0, 0),\ (0, \mathsf{id}_X, 0),\ (0, 0, \mathsf{id}_X)\colon X \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} X \times X \times X$$

constitute a strongly epimorphic family. So proving $x + (y + z) = (x + y) + z$ reduces again to the case where two of the elements are 0; in such cases, the result is once more trivial.

So we have already a commutative monoid structure. To prove that it is an abelian group structure, we consider Diagram 26, where we write p_0, p_1 for the two projections of $X \times X$. The split short five lemma implies

$$\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & X & \xrightarrow{(\mathsf{id}_X, 0)} & X \times X & \underset{p_1}{\overset{(0, \mathsf{id}_X)}{\rightleftarrows}} & X & \longrightarrow & \mathbf{1} \\
 & & \| & & \big\downarrow{\scriptstyle (\varphi, p_1)} & & \| & & \\
\mathbf{1} & \longrightarrow & X & \xrightarrow{(\mathsf{id}_X, 0)} & X \times X & \underset{p_1}{\overset{(\mathsf{id}_X, \mathsf{id}_X)}{\rightleftarrows}} & X & \longrightarrow & \mathbf{1}
\end{array}$$

Diagram 26

that (φ, p_2) is an isomorphism. This means

$$\forall\, (x, y) \in X \times X \quad \exists!\, (z, y) \in X \times X \quad x = z + y.$$

That is, $z = x - y$. Thus the subtraction operation is the composite

$$X \times X \xrightarrow{(\varphi, p_2)^{-1}} X \times X \xrightarrow{\ p_1\ } X. \qquad \square$$

In the non-protomodular case, not every commutative object is an internal abelian group. The example of monoids is sufficient to see this: not every commutative monoid is an abelian group.

Our second intrinsic notion is that of *normal subobject*, generalizing the notion of normal subgroup.

A normal subgroup $S \rightarrowtail G$ can be defined in two equivalent ways: as the kernel of a group homomorphism, or as the equivalence class of the unit for some (necessarily unique) congruence on G. This equivalence is by no means a general fact, even in protomodular categories, as the counter-example of topological groups will indicate.

We reserve thus the terminology *normal* to a subobject which is intuitively an "equivalence class of an internal equivalence relation".

Definition 6.3 *Let $\mathcal{E}$ be a category with finite limits. We say that a morphism $f\colon X \longrightarrow Y$ is* normal *to an equivalence relation $r\colon R \rightarrowtail Y \times Y$ when the following conditions hold:*

1. *$f \times f$ factors through the monomorphism r, yielding thus a pullback as in the left hand square of Diagram 27*

2. *the right hand square in Diagram 27 is a pullback as well.*

To understand better this notion of normality, observe first that

Lemma 6.4 *In the conditions of Definition 6.3, the morphism f is necessarily a monomorphism.*

Proof Working elementwise again, if $x, x' \in X$ with $f(x) = f(x')$, we have

$$\tilde{f}(x, x') = \big(f(x), f(x')\big) = \tilde{f}(x, x), \quad p_0(x, x') = x = p_0(x, x).$$

The second pullback in 6.3 forces $(x, x') = (x, x)$, thus $x = x'$. $\square$

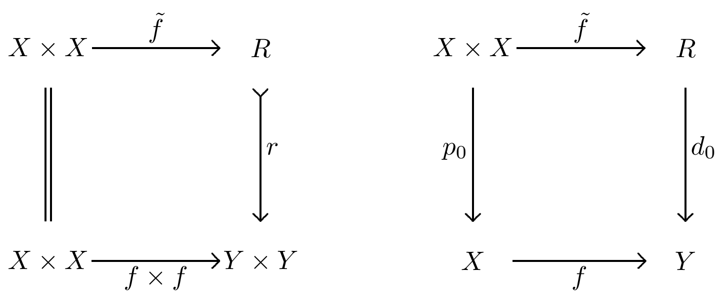


Diagram 27

In the case of sets, our notion of normality recaptures the classical notion of equivalence class:

Lemma 6.5 *In the category of sets, a subobject $X \rightarrowtail Y$ is normal to an equivalence relation R on Y when X is an equivalence class of that relation R.*

Proof Let us write f in 6.3 as a canonical inclusion. Elementwise, in the category of sets, the subobject $X \subseteq Y$ is normal to the equivalence relation R on Y when

1. $\forall x, x' \in X \quad (x, x') \in R$;
2. $\forall x \in X\ \forall y \in Y \quad (x, y) \in R \Rightarrow y \in X$.

This means exactly that X is an equivalence class for the relation R on Y. □

This suggests at once the following generic example of a normal subobject, in the pointed case.

Example 6.6 *Let $\mathcal{E}$ be a pointed category with finite limits. For an equivalence relation R on an object Y, the pullback in Diagram 28 defines a subobject $s\colon [1]_R \rightarrowtail Y$ which must of course be called the "equivalence class of the base point of Y". This subobject $s\colon [1]_R \rightarrowtail Y$ is normal to R.*

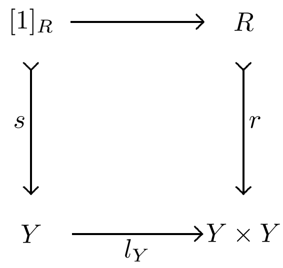


Diagram 28

Proof Immediate from 6.5. □

The following example is certainly enlightening as well.

Example 6.7 *Let $\mathcal{E}$ be a pointed category with finite limits and $g\colon A \longrightarrow B$ an arbitrary morphism. The kernel $\mathsf{ker}\, g\colon K[g] \rightarrowtail A$ of g is normal to the kernel relation $R[g]$ of g.*

Elementwise again, if $x, x' \in K[g]$, $g(x) = 1 = g(x')$ thus $(x, x') \in R[g]$. Next if $x \in K[g]$, $a \in A$ and $g(x) = g(a)$, then $g(a) = 1$ thus $a \in K[g]$. □

The following theorem underlines the power of the protomodularity condition.

Theorem 6.8 *Let $\mathcal{E}$ be a protomodular category with finite limits. When a monomorphism $X \rightarrowtail Y$ is normal to an equivalence relation R on Y, this relation R is necessarily unique (up to isomorphism).*

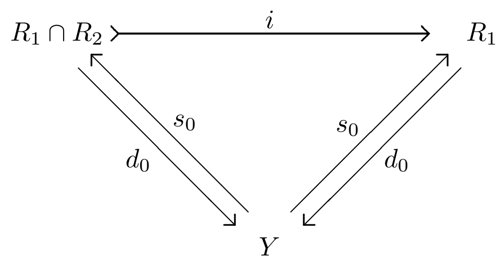


Diagram 29

Proof Let us assume that the monomorphism $f\colon X \rightarrowtail Y$ is normal to the equivalence relations R_1 and R_2 on Y. A straightforwar elementwise argument shows that X is still normal to $R_1 \cap R_2$.

By reflexivity of an equivalence relation R on the object Y, the first projection $d_0\colon R \longrightarrow Y$ is a split epimorphism, with the diagonal s_0 as section. The canonical inclusion of $R_1 \cap R_2$ in R_1 yields the commutative Diagram 29 in $\mathcal{E}$ that is, a morphism

$$i\colon \big(d_0^{R_1\cap R_2}, s_0^{R_1\cap R_2}\colon R_1 \cap R_2 \leftrightarrows Y\big) \longrightarrow \big(d_0^{R_1}, s_0^{R_1}\colon R_1 \leftrightarrows Y\big)$$

in the fibre $\mathsf{Pt}_Y(\mathcal{E})$ of the fibration of points. The consideration of condition 6.3.2 for normality indicates that $f^*(i)$ in $\mathsf{Pt}_X(\mathcal{E})$ is isomorphic to the identity on $X \times X$. Therefore i is an isomorphism by 3.2 and $R_1 \cap R_2$ is thus isomorphic to R_1 as subobject of $Y \times Y$. An analogous argument with R_2 allows to conclude the proof. $\square$

This proposition makes the notion of a normal subobject intrinsic in every protomodular category:

Definition 6.9 *Let $\mathcal{E}$ be a protomodular category with finite limits. A subobject $X \rightarrowtail Y$ is* normal *when it is normal to some equivalence relation R on Y.*

The following example is the one that you expected.

Example 6.10 *In the category of groups, a subobject $H \rightarrowtail G$ is normal in the sense of Definition 6.9 precisely when H is a normal subgroup of G.*

Suppose that the subgroup $H \rightarrowtail G$ is normal to a congruence R on G. Since the category of groups is exact, R is the kernel relation of the quotient $g\colon G \twoheadrightarrow G/K$, where K is the equivalence class of $1 \in G$ for the congruence R. But H is an equivalence class for R (see Lemma 6.5) and contains $1 \in G$, since it is a subgroup. Thus $H = K$ and H is a normal subgroup.

The converse follows at once from Example 6.7, choosing for g the quotient $G \twoheadrightarrow G/H$. $\square$

Example 6.11 *In the category of rings without unit, a subobject $I \rightarrowtail A$ is normal in the sense of Definition 6.9 precisely when I is a two-sided ideal of the ring A.*

Suppose that the subring $I \rightarrowtail A$ is normal to a congruence R on A. Since the category of rings is exact, R is the kernel relation of the quotient $g\colon A \twoheadrightarrow G/K$, where K is the equivalence class of $0 \in A$ for the congruence R. But I is an equivalence class for R (see 6.5) and contains $0 \in A$, since it is a subring. Thus $I = K$ and I is a two-sided ideal.

The converse follows at once from Example 6.7, choosing for g the quotient $A \twoheadrightarrow A/I$. $\square$

Let us finally exhibit an interesting link between the two intrinsic notions studied in this section:

Proposition 6.12 *In a protomodular category with finite limits, the following conditions are equivalent:*

1. *the object X is abelian;*
2. *the diagonal $s_0\colon X \rightarrowtail X \times X$ is a normal subobject.*

Proof Again, we freely work "elementwise".

Let us suppose that the object X is abelian, that is, is provided with an abelian group operation, which we write additively.

On the object $X \times X$, we consider the relation:

$$(x, y) \approx (x', y') \text{ iff } x - x' = y - y'.$$

This is trivially an equivalence relation and the class of $(0, 0)$ is exactly the diagonal. One concludes by Example 6.6.

Conversely, suppose that the diagonal of $s_0 \colon X \rightarrowtail X \times X$ is normal to the equivalence relation R on $X \times X$. We consider Diagram 30 where

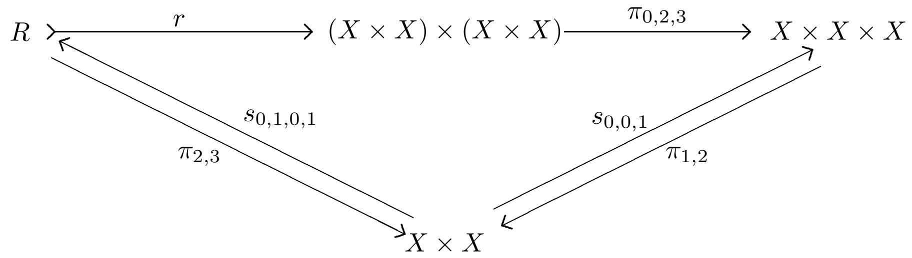


Diagram 30

- the factors of a n-fold product are as usual denoted by indices $0, \ldots, n-1$;
- a notation like $\pi_{0,2,3}$ indicates the projection which, elementwise, is described by $\pi_{0,2,3}(a, b, c, d) = (a, c, d)$;
- a notation like $s_{0,0,1}$ indicates the injection which, elementwise, is described by $s_{0,0,1}(a, b) = (a, a, b)$;
- the morphism $\pi_{2,3}$ is thus the second projection of R and the morphism $s_{0,1,0,1}$ is the diagonal of R.

Diagram 30 is obviously commutative and both composites

$$\pi_{2,3} \circ s_{0,1,0,1}, \quad \pi_{1,2} \circ s_{0,0,1}$$

are the identity on $X \times X$. This means that the horizontal composite is a morphism in the fibre over $X \times X$ in the fibration of points. Let us write γ for this composite.

Writing $s_0 \colon X \rightarrowtail X \times X$ for the diagonal of X, $s_0^{-1}(\pi_{2,3})$ is $X \times X$, because s_0 is normal to R (second condition in 6.3, via the symmetry of R). But obviously $s_0^{-1}(\pi_{1,2})$ is given by

$$s_0^{-1}(\pi_{1,2}) = \big\{ \big(a, (b, c, d)\big) \big| a, b, c, d \in X, \quad (c, d) = (a, a) \big\} \cong X \times X.$$

This shows that $s_0^{-1}(\gamma)$ is isomorphic to the identity on $X \times X$, thus γ is an isomorphism by protomodularity (see 3.2).

We are now able to define a ternary operation on X as the composite

$$p \colon X \times X \times X \xrightarrow{\gamma^{-1}} R \xrightarrow{\pi_1} X.$$

We claim that

$$a + b =_{\text{def}} p(a, 0, b),$$

defines an abelian group structure on X. Of course this is an abbreviation for the composite

$$X \times X \xrightarrow{\mathsf{id}_X \times 0 \times \mathsf{id}_X} X \times X \times X \xrightarrow{\quad p \quad} X.$$

To conclude the proof, by Theorem 6.2, it suffices to prove that $x + 0 = x$ and $0 + x = x$.

Let us recall that the two conditions for the diagonal being normal to R mean, elementwise,

1. $\big((a, a), (b, b)\big) \in R$ for all $a, b \in X$;
2. $\big((a, a), (b, c)\big) \in R$ implies $b = c$.

On the other hand,

$$\gamma\big((a,b),(c,d)\big) = (a,c,d)$$

and by definition of p,

$$\gamma^{-1}(a,c,d) = \Big(\big(a, p(a,c,d)\big), \big(c,d\big)\Big) \in R.$$

Moreover, since γ is an isomorphism, thus a monomorphism,

$$\big((a,b),(c,d)\big) = \big((a,b'),(c,d)\big) \in R \Rightarrow b = b'$$

that is

$$\big((a,b),(c,d)\big) \in R \Rightarrow b = p(a,c,d).$$

The last implication forces in particular $p(a,d,d) = a$ since by normality of the diagonal, $\big((a,a),(d,d)\big) \in R$. Choosing $d = 0$, this yields already $a + 0 = a$. On the other hand, by reflexivity of R and definition of p, we have both

$$\big((a,d),(a,d)\big) \in R, \quad \Big(\big(a, p(a,a,d)\big), \big(a,d\big)\Big) \in R$$

from which $p(a,a,d) = d$, since γ is a monomorphism. Choosing $a = 0$, this yields this time $0 + d = d$. □

In the proof of Proposition 6.12, we have in particular constructed a ternary operation $p(x,y,z)$ satisfying the axioms

$$p(a,a,b) = b, \quad p(a,b,b) = a.$$

Such an operation is called a *Mal'cev operation*. Of course, given a group operation $+$,

$$p(x,y,z) = x - y + z$$

defines at once a Mal'cev operation.

One can prove that given an algebraic theory $\mathbb{T}$, the category $\mathsf{Set}^{\mathbb{T}}$ of models is a Mal'cev category (that is, every equivalence relation in $\mathsf{Set}^{\mathbb{T}}$ is an equivalence relation; see Section 5) precisely when the theory $\mathbb{T}$ contains a Mal'cev operation.

One should not be mislead by the fact that putting

$$a + b = p(a,0,b), \quad a - b = p(a,b,0)$$

in the proof of Proposition 6.12, we have constructed an abelian group operation from a Mal'cev operation. This is by no means possible in general. There is here a very strong – somehow hidden – assumption: p is a morphism in the protomodular category! For example every group $(G,+)$ is of course provided with the group operation $+ \ldots$ but

$$+\colon G \times G \longrightarrow G$$

is by no means a morphism in the category of groups, except when G is abelian.

In the exact case, we have further

Proposition 6.13 *Let $\mathcal{E}$ be a pointed, exact and protomodular category. The following conditions are equivalent for a monomorphism $s\colon S \rightarrowtail X$:*

1. *s is normal;*
2. *s is the kernel of some morphism;*

Proof The implication $(2 \Rightarrow 1)$ is the content of 6.7.

Conversely, assume that the monomorphism $s\colon S \rightarrowtail X$ is normal to the equivalence relation R on X. Since the category $\mathcal{E}$ is exact, the quotient X/R exists and R is the kernel pair of the quotient map $q\colon X \twoheadrightarrow X/R$. We must prove that $s = \mathsf{ker}\, q$. We work elementwise, using Baqrr's metatheorem. Writing 1 for the base point of an object in $\mathcal{E}$, the first condition in 6.3 implies $(x,1) \in R$ for every $x \in S$, thus $q(x) = q(1) = 1$. Conversely if $y \in X$ is such that $q(y) = 1 = q(1)$, then $(1,y) \in R$ since R is the kernel pair of q; since $1 \in S$, we conclude that $y \in S$, by the second condition in 6.3. □

7 Homological categories

The central notion of this section, which allows giving its full power to the notion of an exact sequence, is :

Definition 7.1 *A category $\mathcal{E}$ is* homological *when*

1. *$\mathcal{E}$ is pointed;*
2. *$\mathcal{E}$ is regular;*
3. *$\mathcal{E}$ is protomodular.*

First let us observe that in this context, the image of a morphism can be computed as in the case of abelian categories, that is, as the cokernel of its kernel. This will imply in particular the existence of some cokernels.

Proposition 7.2 *Let $\mathcal{E}$ be an homological category. The image factorization $f = i \circ p$ of a morphism as in Diagram 31 is such that $p = \mathsf{coker}\,\mathsf{ker}\, f$.*

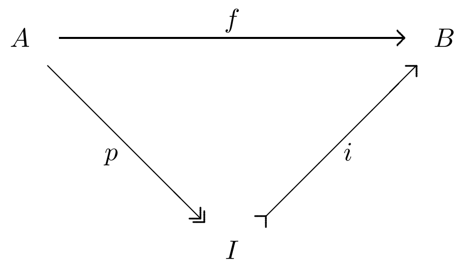


Diagram 31

Proof The image factorization exists because the category is regular and $p = \mathsf{coker}\,\mathsf{ker}\, p$ by Proposition 4.5. But since i is a monomorphism, $\mathsf{ker}\, f = \mathsf{ker}\, p$. □

Corollary 7.3 *In an homological category $\mathcal{E}$, every kernel map has a cokernel and is the kernel of that cokernel.*

Proof If $k = \mathsf{ker}\, f$ and $f = i \circ p$ is the image factorization of f, then $p = \mathsf{coker}\,\mathsf{ker}\, f = \mathsf{coker}\, k$. Of course $k = \mathsf{ker}\, p$, since i is a monomorphism. □

The next observation is crucial. In the context of regular categories, our Proposition 5.8 admits an elegant generalization, which is essential for proving the various diagram lemmas of homological algebra.

Proposition 7.4 *Let $\mathcal{E}$ be an homological category. In the commutative Diagram 32, where g is a regular epimorphism: if the outer rectangle and the square* (1) *are both pullbacks, the square* (2) *is a pullback as well.*

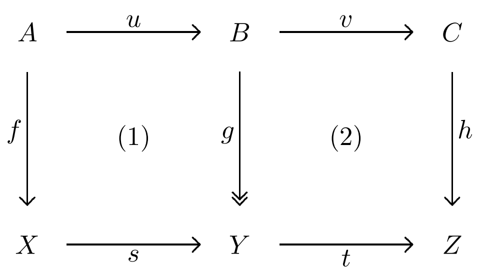


Diagram 32

Proof Consider the diagram of the statment and extend it with the kernel pairs $(\alpha_|, \beta_|)$ of the vertical morphisms, the corresponding diagonals $\delta_|$ and the obvious factorizations u', v' through these, yielding Diagram 33. Notice that f is a regular epimorphism, since so is g and the square (1) is a pullback.

An easy chase on the diagram shows that both downward directed squares (3) are pullbacks and both downward directed rectangles (3)+(4) are pullbacks. But all morphisms $\alpha_|$ and $\beta_|$ are split epimorphisms, with section the diagonal $\delta_|$. We can thus apply Proposition 5.8 and conclude that both downward directed squares (4) are pullbacks as well. Another easy chase on the diagram shows then that the square (2) is a pullback as well. □

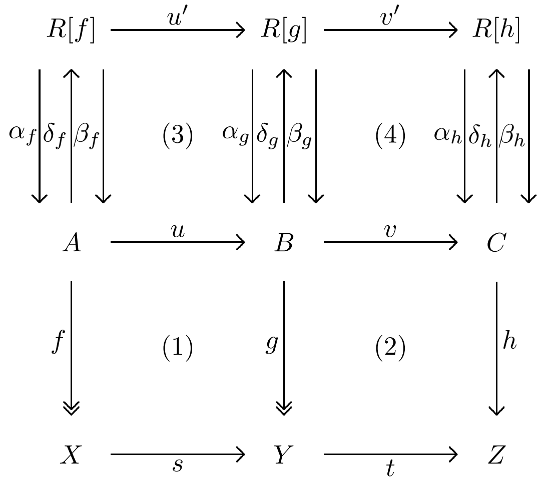


Diagram 33

With Proposition 4.5 in mind, one defines

Definition 7.5 *In a pointed category, a sequence of morphisms*

$$\mathbf{1} \longrightarrow K \rightarrowtail^{k} A \overset{q}{\twoheadrightarrow} Q \longrightarrow \mathbf{1}$$

is a short exact sequence *when* $k = \mathsf{ker}\, q$ *and* $q = \mathsf{coker}\, k$.

Observe at once that

Lemma 7.6 *In a pointed protomodular category $\mathcal{E}$ with finite limits, and in particular in an homological category, a sequence*

$$\mathbf{1} \longrightarrow K \rightarrowtail^{k} A \overset{q}{\twoheadrightarrow} Q \longrightarrow \mathbf{1}$$

is a short exact sequence precisely when $k = \mathsf{ker}\, q$ *and q is a regular epimorphism.*

Proof In a protomodular pointed category, every regular epimorphism is the cokernel of its kernel (see 4.5). □

Definition 7.7 *In a pointed regular category with finite limits, a sequence of morphisms*

$$A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C$$

is an exact sequence *when, considering the image factorizations of f and g as in Diagram 34, the following*

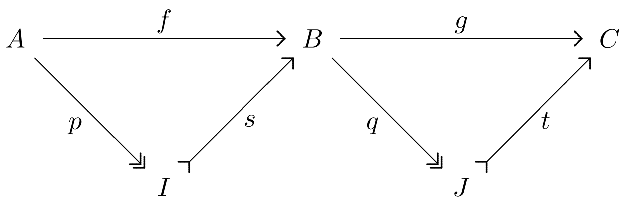


Diagram 34

sequence is a short exact sequence

$$\mathbf{1} \longrightarrow I \rightarrowtail^{s} B \overset{q}{\twoheadrightarrow} J \longrightarrow \mathbf{1}.$$

A long sequence of composable morphisms is exact when each pair of consecutive morphisms is exact.

With the notation of 7.7, one should be aware that requiring that a sequence

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C$$

is exact forces f to have an image s which is a kernel. Thus not every morphism f can appear as first arrow in an exact sequence. Therefore we define

Definition 7.8 *In a pointed regular category $\mathcal{E}$, a morphism $f\colon A \longrightarrow B$ is* proper *when its image is a kernel.*

As expected, one has:

Proposition 7.9 *Let $\mathcal{E}$ be an homological category.*

1. *a morphism $f\colon A \longrightarrow B$ is a monomorphism precisely when the sequence*

$$\mathbf{1} \xrightarrow{\quad \alpha_A \quad} A \xrightarrow{\quad f \quad} B$$

is exact;

2. *$k = \mathsf{ker}\, f$ precisely when the sequence*

$$\mathbf{1} \xrightarrow{\quad \alpha_A \quad} K \xrightarrow{\quad k \quad} A \xrightarrow{\quad f \quad} B$$

is exact;

3. *a morphism $f\colon A \longrightarrow B$ is a regular epimorphism precisely when the sequence*

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad \tau_B \quad} \mathbf{1}$$

is exact;

4. *$q = \mathsf{coker}\, f$ with f a proper morphism precisely when the sequence*

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad q \quad} Q \xrightarrow{\quad \tau_Q \quad} \mathbf{1}$$

is exact.

Proof Every monomorphism is equal to its image. The morphism $\alpha_A\colon \mathbf{1} \longrightarrow A$ is a monomorphism. Therefore the sequence (α_A, f) is exact when $\mathsf{ker}\, f = \alpha_A$, that is, when f is a monomorphism (see Theorem 4.2). In the second statement, the exactness of the sequence (α_K, k, f) means thus first that k is a monomorphism, and second that $k = \mathsf{ker}\, f$.

A morphism $f\colon A \longrightarrow B$ is a regular epimorphism precisely when it admits the identity on B as image. The morphism $\tau_B\colon B \longrightarrow \mathbf{1}$ admits the identity on B as kernel. Therefore the sequence (f, τ_B) is exact when the image of f is the identity on B, that is, when f is a regular epimorphism.

To prove the last statement, let us write $f = i \circ p$ for the image factorization of f. Since p is an epimorphism, $\mathsf{coker}\, f = \mathsf{coker}\, i \circ p = \mathsf{coker}\, i$. The exactness of the sequence (f, q, τ_Q) means thus first that q is a regular epimorphism (see 7.9.3), and second that $i = \mathsf{Im}\, f = \mathsf{ker}\, q$. By 4.5, this implies $q = \mathsf{coker}\, i = \mathsf{coker}\, f$.

Conversely, let $q = \mathsf{coker}\, f$ and $i = \mathsf{ker}\, g$, for some morphism g. If $g = j \circ q'$ is the image factorization of g, one has still $i = \mathsf{ker}\, q'$ because j is a monomorphism. By Theorem 4.2 this implies

$$q' = \mathsf{coker}\, i \cong \mathsf{coker}\, f = q.$$

Thus $i = \mathsf{ker}\, q' = \mathsf{ker}\, q$ and the sequence (f, q) is exact. ☐

Let us now infer the general form of the short five lemma.

Theorem 7.10 (Short five lemma) *The short five lemma holds in every homological category $\mathcal{E}$. That is, given the commutative Diagram 35, where the horizontal lines are short exact sequences, if a and c are isomorphisms, b is an isomorphism as well.*

Proof Consider the commutative Diagram 36. The square (2) is a pullback because $u = \mathsf{ker}\, p$. The outer part of the diagram is a pullback because $v = \mathsf{ker}\, q$. Since the left hand horizontal morphisms are isomorphisms, the rectangle (2)+(3) is a pullback. By Proposition 7.4, the square (3) is a pullback, thus b is an isomorphism, since so is c. ☐

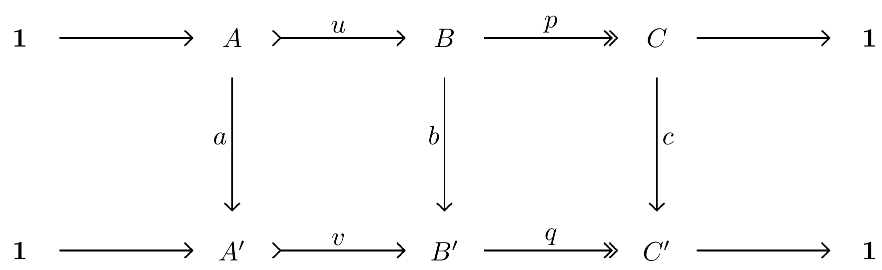


Diagram 35

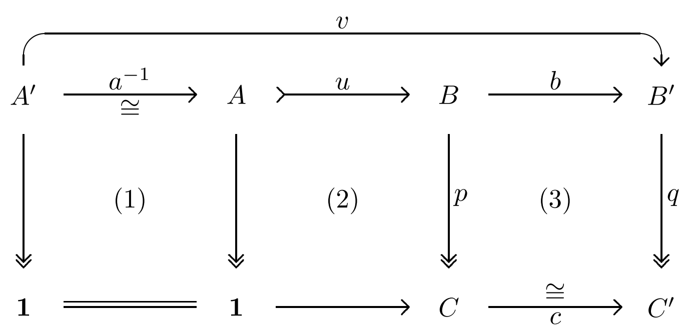


Diagram 36

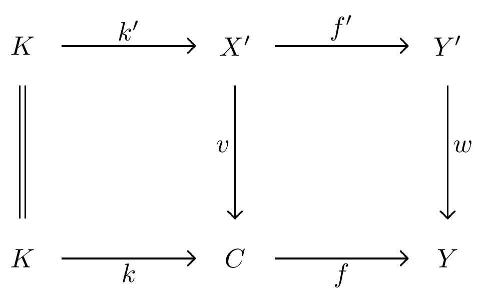


Diagram 37

Let us now switch to the "nine lemma". For the sake of clarity, we split the proof in various partial results. First of all, we consider some properties which are valid in every pointed category.

Lemma 7.11 *Let $\mathcal{E}$ be a pointed category. Consider the commutative Diagram 37 where*

- $k = \mathsf{ker}\, f$*;*
- *the right hand square is a pullback;*
- $f' \circ k' = \omega_{K,Y'}$*.*

In those conditions, $k' = \mathsf{ker}\, f'$*.*

Proof We work again elementwise. In the case of pointed sets, an element of X' has the form (x, y) with $x \in X$, $y \in Y'$ and $f(x) = w(y)$. Therefore

$$\begin{aligned} K[f'] &= \big\{(x,y) \big| x \in X,\ y \in Y',\ f(x) = w(y),\ y = 1\big\} \\ &= \big\{(x,1) \big| x \in K[f]\big\} \\ &\cong K[f]. \end{aligned}$$

which concludes the proof. □

Lemma 7.12 *Let $\mathcal{E}$ be a pointed category with pullbacks. Consider Diagram 38, where the horizontal composites are zero morphisms. Such a diagram admits a decomposition as in the commutative Diagram 39 where*

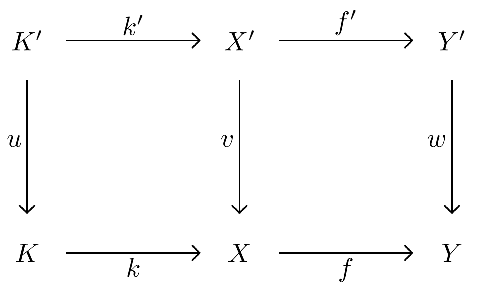


Diagram 38

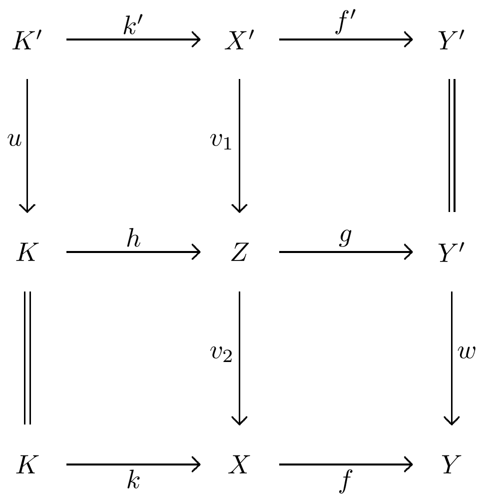


Diagram 39

- *the lower right hand square is a pullback;*
- $g \circ h$ *is the zero morphism;*

- $v_2 \circ v_1 = v$.

Proof The decomposition is constructed in the following way:

- the square $f \circ v_2 = w \circ g$ is a pullback by definition;
- v_1 is the factorization of the square $f \circ v = w \circ f'$ through this pullback, thus $v_2 \circ v_1 = v$ and $g \circ v_1 = f'$;
- h is the factorization of the square $w \circ \omega_{K,Y'} = f \circ k$ through the same pullback, thus $g \circ h = \omega_{K,Y'}$ and $v_2 \circ h = k$;

To prove the commutativity of the upper left hand square, it suffices to check it after composition with the projections g and v_2 of the pullback. And indeed, $g \circ h \circ u = f' \circ k'$ because both composites $h \circ g$ and $f' \circ k'$ are zero. On the other hand $v_2 \circ v_1 \circ k' = v \circ k' = k \circ u$. □

Corollary 7.13 *In the conditions of Lemma 7.12:*

1. *if* $k = \mathsf{ker}\, f$*, then* $h = \mathsf{ker}\, g$*;*
2. *if moreover* $k' = \mathsf{ker}\, f'$*, the upper left hand square is a pullback;*
3. *if the category* $\mathcal{E}$ *is regular and moreover* (k', f') *is a short exact sequence,* (h, g) *is a short exact sequence as well.*

Proof The first assertion follows from 7.11.

Next if $h \circ \gamma = v_1 \circ \delta$, then

$$f' \circ \delta = g \circ v_1 \circ \delta = g \circ h \circ \gamma = \omega_{K,Y'} \circ \gamma$$

is the zero morphism, thus δ factors through $k' = \mathsf{ker}\, f'$ as $\delta = k' \circ \varepsilon$. Since h is a monomorphism, this implies at once $u \circ \varepsilon = \gamma$.

Finally if $f' = g \circ v_1$ is a regular epimorphism, so is g. □

Lemma 7.14 *Let* $\mathcal{E}$ *be a pointed category. Consider the commutative Diagram 40, where* $k = \mathsf{ker}\, f$*.*

$$\begin{array}{ccccc} K' & \xrightarrow{k'} & X' & \xrightarrow{f'} & Y' \\ {\scriptstyle u}\big\downarrow & (1) & {\scriptstyle v}\big\downarrow & (2) & \big\downarrow{\scriptstyle w} \\ K & \xrightarrow[k]{} & X & \xrightarrow[f]{} & Y \end{array}$$

Diagram 40

1. *When* w *is a monomorphism, one has* $k' = \mathsf{ker}\, f'$ *if and only if the square* (1) *is a pullback.*
2. *When the square* (2) *is a pullback and* $f' \circ k'$ *is the zero morphism, one has* $k' = \mathsf{ker}\, f'$ *if and only if* u *is an isomorphism.*

Proof Again working elementwise, it suffices to check the result for pointed sets.

Assume first that w is a monomorphism. If (1) is a pullback, k' is injective since so is k. For simplicity, let us write k and k' as canonical inclusions. Since w is injective,

$$w \circ f' \circ k' = f \circ k \circ u = \omega_{K,Y} \circ u = \omega_{K',Y} = w \circ \omega_{K',Y'}$$

implies $f' \circ k' = \omega_{K',Y'}$ that is, $K' \subseteq K[f']$. If $x \in K[f']$, $(f \circ v)(x) = (w \circ f')(x) = 1$, thus $v(x) \in K$. Thus $x \in v^{-1}(K) = K'$ and finally, $K' = K[f']$.

Now assume that $k' = \mathsf{ker}\, f'$. We must prove that $K' = v^{-1}(K)$ that is,

$$\forall x \in X' \quad v(x) \in K \Rightarrow x \in K'.$$

But $v(x) \in K$ implies $(w \circ f')(x) = (f \circ v)(x) = 1$, thus $f'(x) = 1$ since w is injective. This means precisely $x \in \ker f' = K'$.

Next, we suppose that (2) is a pullback and $f' \circ k'$ is the zero morphism. When $k' = \ker f'$, the morphism u is simply $u(x, 1) = x$ and is an isomorphism, as follows at once from 7.11. Conversely given the situation of the statement with u an isomorphism, k' is a monomorphism because so are u and k. The subobject K' of X' is contained in $K[f']$ because $k' \circ f' = \omega_{K',Y'}$. To prove the equality, choose $(x, 1) \in K[f']$, that is $x \in K[f]$; we must prove that $(x, 1) \in K'$. Since u is an isomorphism, we find $(x', y') \in K'$ such that $u(x', y') = x$. Since v is the first projection of the pullback (2), $v \circ k' = k \circ u$ forces $x' = x$. Since $(x', y') \in K[f']$, we have $y = 1$. Thus $(x, 1) = (x', y') \in K'$. □

In the homological case, we have a partial converse of Lemma 7.14:

Lemma 7.15 *Let $\mathcal{E}$ be an homological category. Consider Diagram 41 where both rows are exact sequences.*

$$\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & K' & \xrightarrow{k'} & X' & \xrightarrow{f'} & Y' & \longrightarrow & \mathbf{1} \\
 & & {\scriptstyle u}\downarrow & (1) & {\scriptstyle v}\downarrow & (2) & \downarrow{\scriptstyle w} & & \\
\mathbf{1} & \longrightarrow & K & \xrightarrow{k} & X & \xrightarrow{f} & Y & &
\end{array}$$

Diagram 41

1. *if the square* (1) *is a pullback, w is a monomorphism;*
2. *if u is an isomorphism, the square* (2) *is a pullback;*
3. *if w is an isomorphism, then v is a regular epimorphism if and only if u is a regular epimorphism.*
4. *if w and u are regular epimorphisms, v is a regular epimorphism;*
5. *if w and u are monomorphisms, v is a monomorphism;*
6. *if u is a regular epimorphism, the restriction*

$$f'' \colon K[v] \twoheadrightarrow K[w]$$

of f' through the kernels of v and w is still a regular epimorphism.

Proof 1. Suppose that (1) is a pullback, thus $K' = v^{-1}(K)$. Working elementwise, we observe that

$$K' = \{x \in X' | v(x) \in K\} = \{x \in X' | (f \circ v)(x) = 1\} = K[f \circ v].$$

Thus $k' = \ker(w \circ f')$ and $f' = \operatorname{coker} k'$. By 7.2, $w \circ f'$ is the image factorization of $w \circ f'$ and thus w is a monomorphism.

2. Suppose that u is an isomorphism and consider the commutative Diagram 42. The middle square is a pullback because $k' = \ker f'$, the outer part is a pullback because $k = \ker f$ and the left hand square is a pullback because u is an isomorphism. It follows at once from 7.4 that the right hand square is a pullback as well.

3. If w is an isomorphism, the square (1) is a pullback by 7.14.1. Thus u is a regular epimorphism, since so is v. Conversely suppose that u is a regular epimorphism. Since w is an isomorphism and f' is a regular epimorphism, f is a regular epimorphism. Consequently the lower sequence is exact. Let $v = \alpha \circ \beta$ be the epi-mono factorization of v and consider Diagram 43. Since the map u is a regular epimorphism and the map α is a monomorphism, there is a factorization t which makes the left hand lower square a pullback, in Diagram 43. Consequently, the map t is the kernel of $f \circ \alpha$. On the other hand, the map $f \circ \alpha \circ \beta = w \circ f'$ is a regular epimorphism and thus the map $f \circ \alpha$ is a regular epimorphism as well. Accordingly, the middle row is an exact sequence. The Short Five Lemma (see 7.10) implies that α is an isomorphism. Thus $v = \alpha \circ \beta$ is a regular epimorphism.

4. If w and u are regular epimorphisms, consider the decomposition given by Lemma 7.12:

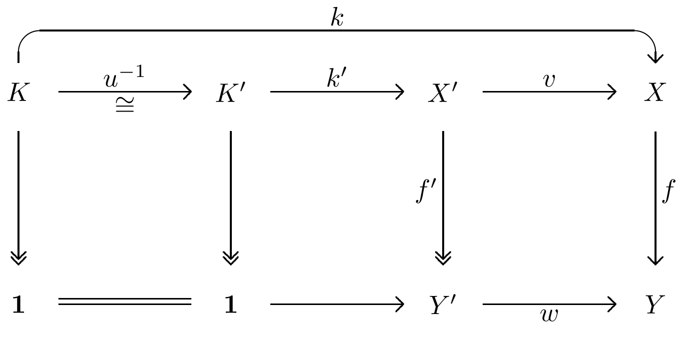


Diagram 42

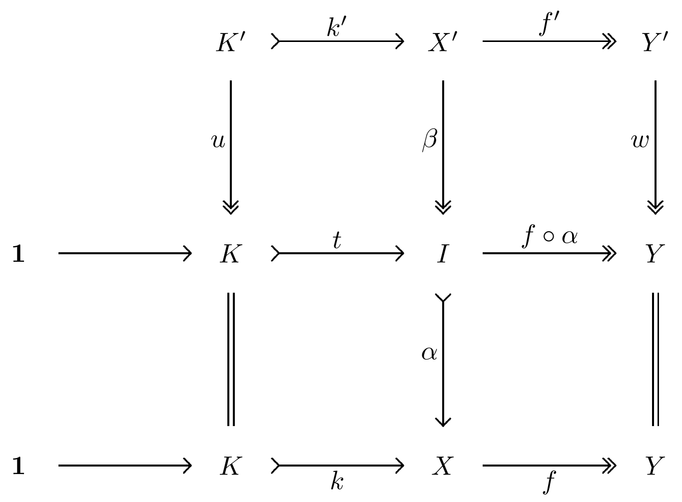


Diagram 43

- the upper left square is a pullback by 7.14.1;
- g is a regular epimorphism, since so is f';
- v_2 is a regular epimorphism because the lower right hand square is a pullback.

To prove that v is a regular epimorphism, it remains to prove that so is v_1. But the two sequences (k', f') and (h, g) are exact. Thus by (3), the map v_1 is a regular epimorphism since u is a regular epimorphism.

5. If w an u are monomorphisms, v_2 is a monomorphism because the lower right hand square is a pullback. Since the upper left hand square is a pullback and u is a monomorphism, v_1 is a monomorphism as well, by 4.1. Thus $v = v_2 \circ v_1$ is a monomorphism.

6. Finally if u is a regular epimorphism, statement 4 of the present lemma, together with 7.13.3, implies that v_1 is a regular epimorphism. Since the lower right hand square in 7.12 is a pullback, Lemma 7.11 implies that, writing k_{v_2} and k_w for the kernels of v_2 and w, both kernel objects are equal and $g \circ k_{v_2} = k_w$. This yields Diagram 44, where k_v is the kernel of v and g' is the obvious factorization through the kernels. By 7.14.1, the

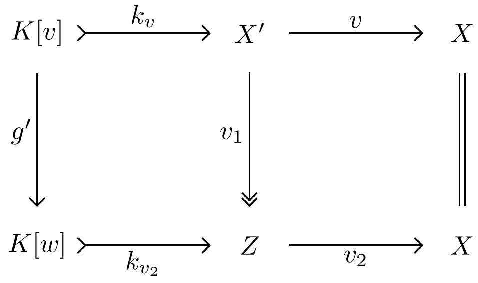


Diagram 44

left hand square is a pullback and thus g' is a regular epimorphism, since so is v_1. Composing the left hand square with g yields

$$k_w \circ g' = g \circ k_{v_2} \circ g' = g \circ v_1 \circ k_v = f' \circ k_v.$$

Thus the regular epimorphism g' is indeed the factorization mentioned in the statement. □

Now a technical lemma which emphasizes a condition forcing a map to be a kernel map:

Lemma 7.16 *Let $\mathcal{E}$ be an homological category. Consider the commutative Diagram 45 where $k' = \operatorname{ker} f'$ and*

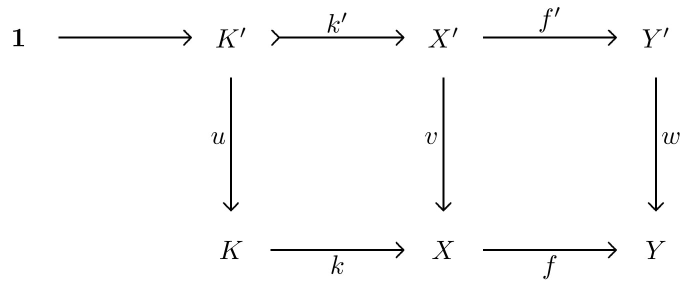


Diagram 45

$f \circ k$ is the zero morphism. In the decomposition of Lemma 7.12, suppose that v_1 is a regular epimorphism and h is a monomorphism. Then $k = \operatorname{ker} f$.

Proof We consider Diagram 46 where $a = \operatorname{ker} f$. Then and $a' = \operatorname{ker} g$, since the lower right hand square is a pullback (see 7.11). Since $g \circ h = \omega_{K,Y'}$, the morphism h factors through a' via a morphism φ. Since h is a monomorphism by assumption, φ is a monomorphism as well. By 7.14.1, the upper left hand square is a pullback and therefore $\varphi \circ u$ is a regular epimorphism, since so is v_1 by assumption. Thus φ is a regular epimorphism and, since it is also a monomorphism, it is an isomorphism. But

$$a \circ \varphi = v_2 \circ a' \circ \varphi = v_2 \circ h = k.$$

Thus k is isomorphic via φ to $a = \operatorname{ker} f$; this proves that $k = \operatorname{ker} f$. □

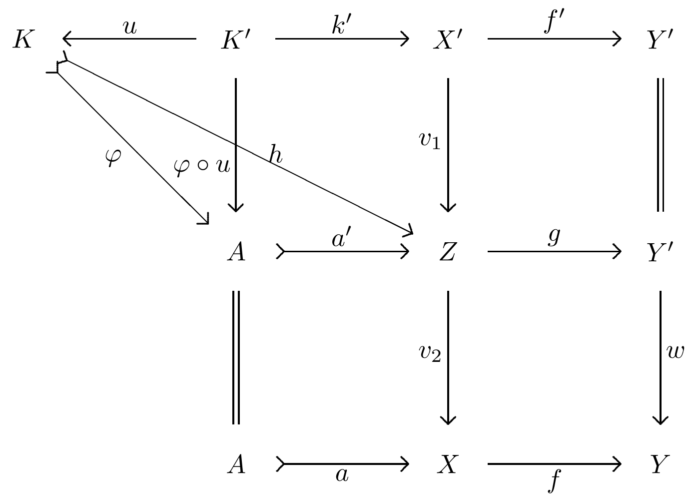


Diagram 46

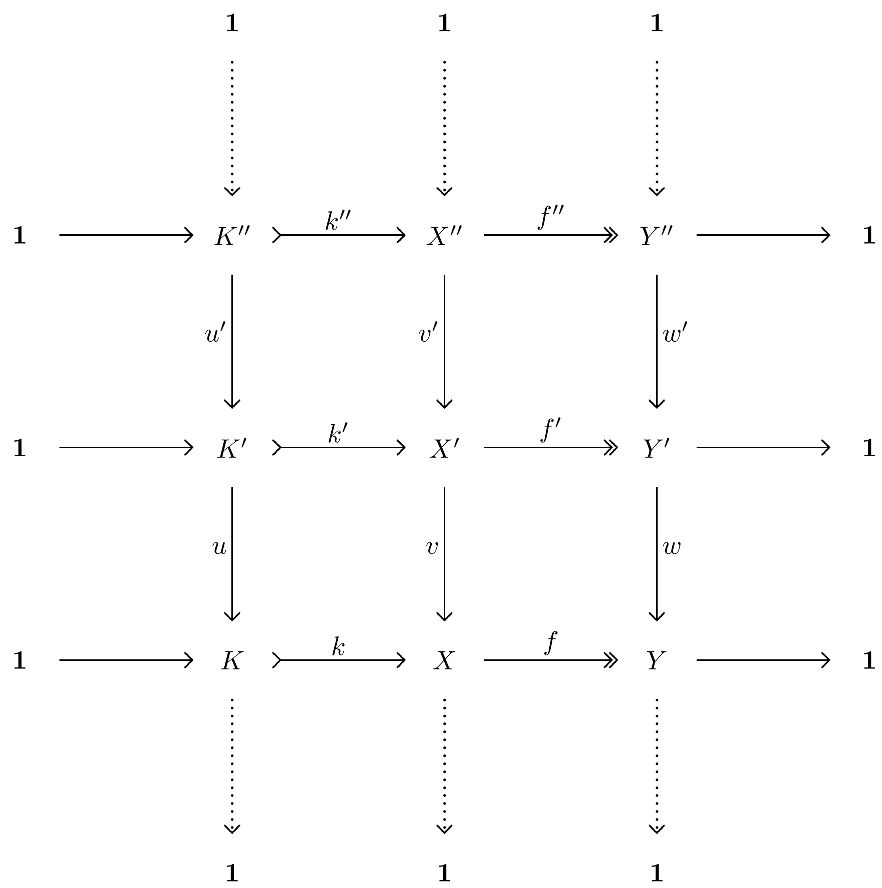


Diagram 47

Theorem 7.17 (Nine lemma) *Let $\mathcal{E}$ be an homological category. Consider the commutative Diagram 47 where the horizontal lines are short exact sequences and $v \circ v'$ is the zero morphism. If two columns are short exact sequences, the third column is a short exact sequence as well.*

Proof 1. Let us first assume that the first and the second column are short exact sequences. Since

$$w \circ w' \circ f'' = f \circ v \circ v' = f \circ \omega_{X'',X} = \omega_{X'',Y} = \omega_{Y'',Y} \circ f''$$

and f'' is an epimorphism (see 7.9), $w \circ w'$ is the zero morphism. Since f and v are regular epimorphisms (see 7.9), $f \circ v = w \circ f'$ is a regular epimorphism and thus w is a regular epimorphism. It remains to prove that $w' = \mathsf{ker}\, w$.

Let us consider for this Diagram 48, which yields the decomposition of the diagram constituted of the last two lines in Diagram 47, as in Lemma 7.12. The lower row being exact, the middle row is exact too. Since the

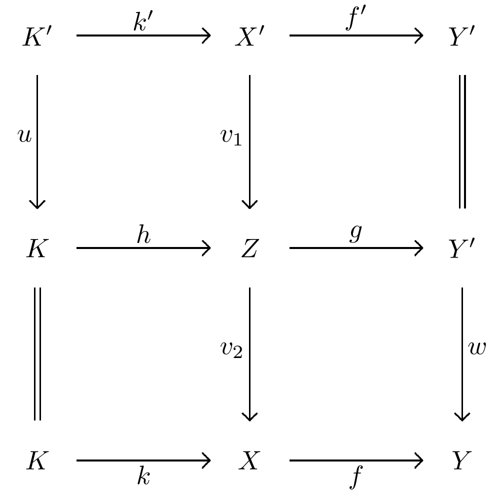


Diagram 48

first row is exact, and u is a regular epimorphism, the map v_1 is a regular epimorphism by 7.15.3.

Let us also consider Diagram 49, yielding the decomposition of the diagram constituted of the last two columns of Diagram 47, and which is built up from the same pullback. We can apply Lemma 7.16 to the

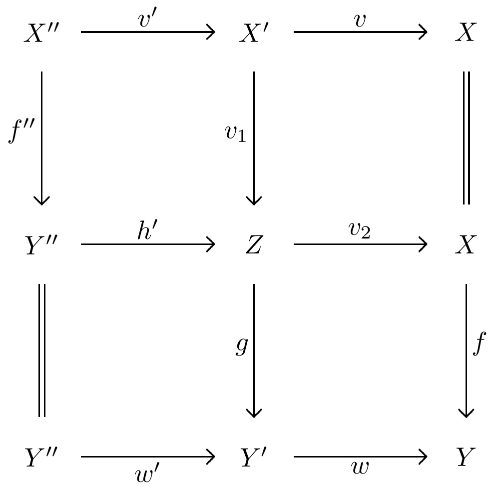


Diagram 49

diagram constituted of the last two columns of Diagram 49 in order to prove that $w' = \mathsf{ker}\, w$. We just observed that v_1 is a regular epimorphism. It remains to prove that h' is a monomorphism. In Diagram 50 the left hand square is a pullback because $k'' = \mathsf{ker}\, f''$ and, moreover, the morphism f'' is a regular epimorphism by assumption. But the outer rectangle can be equivalently obtained as the outer part of Diagram 51. The left hand square is now a pullback because $u' = \mathsf{ker}\, u$ and the right hand square is a pullback according to the

$$
\begin{array}{ccccc}
K'' & \xrightarrow{k''} & X'' & \xrightarrow{v'} & X' \\
\big\downarrow & & \big\downarrow{\scriptstyle f''} & (*) & \big\downarrow{\scriptstyle v_1} \\
\mathbf{1} & \longrightarrow & Y'' & \xrightarrow[h']{} & Z
\end{array}
$$

Diagram 50

$$
\begin{array}{ccccc}
K'' & \xrightarrow{u'} & K' & \xrightarrow{k'} & X' \\
\big\downarrow & & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle v_1} \\
\mathbf{1} & \longrightarrow & K & \xrightarrow[h]{} & Z
\end{array}
$$

Diagram 51

previous observation. Thus the outer rectangle is also pullback and one concludes by 7.4 that the square $(*)$ in Diagram 50 is a pullback. Since v' is a monomorphism by assumption, h' is a monomorphism too, by 4.1.

2. Next we suppose that the first and the last column are exact. By assumption u and w are thus regular epimorphisms, from which v is a regular epimorphism as well, by 7.15. It remains to prove that $v' = \mathsf{ker}\, v$. Still we have $k = \mathsf{ker}\, f$, $k' = \mathsf{ker}\, f'$, $k'' = \mathsf{ker}\, f''$: moreover f'' and v_1 are regular epimorphisms. Therefore all the arguments used above to prove that the square (*) is a pullback remain valid in the present case. By Corollary 7.13, we know also that $h' = \mathsf{ker}\, v_2$ since $w' = \mathsf{ker}\, w$. The consideration of the commutative

$$
\begin{array}{ccccc}
X'' & \xrightarrow{v'} & X' & \xrightarrow{v} & X \\
{\scriptstyle f''}\big\downarrow & & {\scriptstyle v_1}\big\downarrow & & \big\| \\
Y'' & \xrightarrow[h']{} & Z & \xrightarrow[v_2]{} & X
\end{array}
$$

Diagram 52

Diagram 52 shows at once that $v' = \mathsf{ker}\, v$. Indeed we can apply 3.5.4.1 with $h' = \mathsf{ker}\, v_2$.

3. It remains to consider the case where the last two columns are exact. Then w' is a monomorphism and by 7.14.1, the upper left hand square in Diagram 47 is a pullback.

Having this pullback implies that $u' = \mathsf{ker}\,(v \circ k')$, by 7.14.1 applied to the first two columns. Finally $u' = \mathsf{ker}\,(v \circ k') = \mathsf{ker}\,(k \circ u)$ implies $u' = \mathsf{ker}\, u$ because k is a monomorphism.

It remains to prove that u is a regular epimorphism. But this is just 7.15.6 applied to the last two columns. Indeed f'' is a regular epi, thus the induced factorization between $K' = K[f']$ and $K = K[f]$, i.e. the map u, is a regular epimorphism. □

Of course, the Nine Lemma could be stated equivalently by interchanging the roles of lines and columns: this is just a "typographical" modification. But let us emphasize the fact that since no duality principle exists as in abelian categories, the roles of the first and the last column cannot be interchanged: two independent proofs are needed.

8 Semi-abelian categories

Let us give at once the central definition of this section.

Definition 8.1 *A category $\mathcal{E}$ is* semi-abelian *[30] when*

1. *$\mathcal{E}$ is pointed;*
2. *$\mathcal{E}$ is finitely complete;*
3. *$\mathcal{E}$ is finitely cocomplete.*
4. *$\mathcal{E}$ is exact;*
5. *$\mathcal{E}$ is protomodular;*

Of course, every semi-abelian category is in particular homological. It should be mentioned that our Definition 8.1 is redundant: for example, requiring only the existence of binary coproducts in condition 3 would have been sufficient: in presence of the other axioms, this forces the existence of coequalizers.

Example 8.2 *A variety is semi-abelian if and only if it is pointed, protomodular (see Theorem 3.5 for a characterization). In particular the varieties of groups, Ω-groups, rings without unit, modules on a ring, are semi-abelian.*

A variety is always complete, cocomplete and exact. □

And of course:

Example 8.3 *Every abelian category is semi-abelian.*

By 3.10. □

It can even be proved that a category $\mathcal{E}$ is abelian if and only if the category $\mathcal{E}$ and its dual $\mathcal{E}^{\mathsf{op}}$ are both semi-abelian. This justifies somehow the terminology "semi-abelian".

Example 8.4 *When $\mathcal{E}$ is semi-abelian, every category of points $\mathsf{Pt}_I(\mathcal{E})$ is semi-abelian as well.*

The zero point is the identity on I. The product of two points over I is obtained via the pullback of the two epi-parts, and their coproduct via the pushout of the two section-parts. Equalizers, pullbacks, kernels, coequalizers, pushouts and cokernels are computed as in $\mathcal{E}$. In particular, a morphism of points is a regular epimorphism if and only if it is a regular epimorphism in $\mathcal{E}$. The result follows thus from straightforward chase on diagrams. □

The following example is certainly worth being mentioned, even if the proof cannot be given here:

Example 8.5 *Let $\mathcal{E}$ be an elementary topos. The category $\mathcal{E}_*^{\mathsf{op}}$, dual of the category of pointed objects of $\mathcal{E}$, is semi-abelian.*

See [4] or [19] for a proof. □

The property of semi-abelian categories that we want to emphasize here is the existence of "semi-direct products". Here is the key result for defining them.

Theorem 8.6 *Consider a morphism $v\colon W \longrightarrow Y$ of a semi-abelian category $\mathcal{E}$. The inverse image functor*

$$v^*\colon \mathsf{Pt}_Y(\mathcal{E}) \longrightarrow \mathsf{Pt}_W(\mathcal{E})$$

of the fibration of points is monadic.

Proof The idea is to use Beck's criterion for monadicity.

First, the existence of finite colimits implies the existence of a left adjoint $v_!$ to v^*, namely, the pushout along v: it is easy chase on diagrams to check the details.

On the other hand v^* reflects isomorphisms, by protomodularity.

It remains to prove that given a reflexive pair

$$u, v, w\colon \big(p, s\colon A \leftrightarrows V\big) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \big(q, t\colon B \leftrightarrows V\big)$$

in $\mathsf{Pt}_V(\mathcal{E})$, whose image along v^* admits a split coequalizer in $\mathsf{Pt}_W(\mathcal{E})$, then (u, w) has a coequalizer in $\mathsf{Pt}_V(\mathcal{E})$ which is preserved by v^*. But the categories of points trivially have coequalizers computed as in $\mathcal{E}$, so that it suffices to prove that v^* preserves the coequalizer of (u, v).

By Example 8.4, the categories of points in $\mathcal{E}$ are still semi-abelian. But computing the cokernel of two morphisms $u, v\colon A \rightrightarrows B$ in a regular category is the same as computing the cokernel of the two projections of the relation R obtained by factoring the pair (u, v) through its image, as in Diagram 53. When the pair (u, v) is

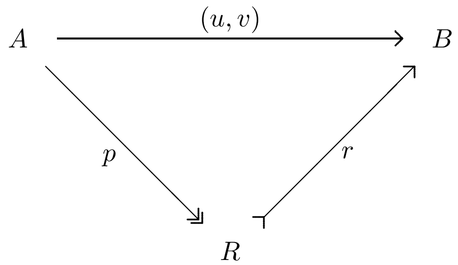


Diagram 53

reflexive, the relation R is reflexive. And in the situation of this proof, the relation R is an equivalence relation by Theorem 5.9. And by exactness of the categories of points, R is even the kernel pair of his cokernel, that is, of the cokernel of (u, v).

The functor v^* preserves finite limits (it has an adjoint) and by regularity it preserves regular epimorphisms, since it acts by pullbacks. Thus v^* preserves kernel pairs of regular epimorphisms. But since every regular epimorphism is the coequalizer of its kernel pair, v^* preserves the coequalizers of kernel pairs. By the argument above, v^* preserves thus also the coequalizers of reflexive pairs of morphisms. $\square$

Let us instead recall the basic elements concerning semi-direct products of groups.

Definition 8.7 *Let $(G, \cdot)$ be a group. A G-*group *is a pair consisting of a group $(X, +)$ and an action*

$$m\colon G \times X \longrightarrow X, \quad (g, x) \mapsto gx$$

which satisfies the axioms

$$1x = x, \quad g'(gx) = (g \cdot g')x, \quad g(x + x') = gx + gx'$$

for all elements $g, g' \in G$ and $x, x' \in X$. (Let us make clear that G and X are arbitrary groups, not necessarily abelian.)

Of course one infers at once easy consequences of this definition, like

$$g0 = 0, \quad g(-x) = -(gx)$$

for all elements $g \in G$ and $x \in X$.

Proposition 8.8 *Let $(G, \cdot)$ be a group and $(X, +, m)$ a G-group. The set $X \times G$, provided with the multiplication*

$$(x, g) \star (x', g') = (x + gx', g \cdot g')$$

is a group, called the semi-direct product *of $(X, +, m)$ and G and written $X \rtimes G$.*

Proof The unit for the multiplication $\star$ is the pair $(0, 1)$ while the inverse of (x, g) is the pair $(-g^{-1}x, g^{-1})$. The rest is routine computation. $\square$

Corollary 8.9 *Let $(G, \cdot)$ be a group and $(X, +, m)$ a G-group. We obtain a split exact sequence of groups*

$$\mathbf{1} \longrightarrow X \xrightarrow{\ k\ } X \rtimes G \underset{p}{\overset{s}{\rightleftarrows}} G \longrightarrow \mathbf{1}.$$

Proof In $X \rtimes G$, the pairs $k(x) = (x, 1)$ constitute a normal subgroup isomorphic to X while the pairs $s(g) = (0, g)$ constitute a subgroup isomorphic to G. Trivially, k is the kernel of the projection $p(x, g) = g$. It is routine to observe that k, s, p are indeed group homomorphisms. □

Here is now the result which provides the "key" for defining semi-direct products in a semi-abelian category:

Proposition 8.10 *Let* Gp *be the category of groups and group homomorphisms and let G be a fixed group. The category of points* $\mathsf{Pt}_G(\mathsf{Gp})$ *is equivalent to the category of G-groups and their morphisms.*

Proof A morphism $f\colon (X, +, m) \longrightarrow (Y, +, n)$ of G-groups is, of course, a group homomorphism $f\colon X \longrightarrow Y$ which commutes with the action of G, that is, the equality $f(gx) = gf(x)$ holds for all elements $x \in X$ and $g \in G$.

Every G-group $(X, +, m)$ yields a point in $\mathsf{Pt}_G(\mathsf{Gp})$

$$p_G, i_G\colon X \rtimes G \rightleftarrows G, \quad p_G(x, g) = g, \quad i_G(g) = (0, g)$$

as attested by Corollary 8.9. This construction extends at once in a functor

$$\Pi\colon G\text{-}\mathsf{Gp} \longrightarrow \mathsf{Pt}_G(\mathsf{Gp}).$$

Conversely, given a group $(H, \star)$ and a point

$$p, s\colon H \rightleftarrows G, \quad p \circ s = \mathsf{id}_G$$

in $\mathsf{Pt}_G(\mathsf{Gp})$, we define the group X to be the kernel of p. We provide X with a G-action by defining

$$\mu\colon G \times X \longrightarrow X, \quad (g, x) \mapsto gx = s(g) \star x \star s(g)^{-1}.$$

Observe that $\mu(g, x)$ is indeed an element of X, simply because X, as a kernel, is a normal subgroup of H. Checking the axioms for a G-action is routine. Again this construction extends in a functor

$$\Gamma\colon \mathsf{Pt}_G(\mathsf{Gp}) \longrightarrow G\text{-}\mathsf{Gp}.$$

It remains to observe that both constructions are mutually inverse, which is once more routine. □

Writing $\mathbf{1}$ for the zero group, the category $\mathsf{Pt}_{\mathbf{1}}(\mathsf{Gp})$ is simply the category Gp of groups. Given a group G, the unique morphism $\alpha_G\colon \mathbf{1} \longrightarrow G$ from the zero group induces by 8.6 a monadic functor

$$\alpha_G^*\colon \mathsf{Pt}_G(\mathsf{Gp}) \longrightarrow \mathsf{Gp}$$

and thus a corresponding monad $\mathbb{T}_G$ on Gp. By Proposition 8.10, the category of algebras for this monad is thus equivalent to the category of G-groups.

This suggests the following definition:

Definition 8.11 *Let $\mathcal{E}$ be a semi-abelian category and $G \in \mathcal{E}$ an object of $\mathcal{E}$.*

1. *A G-*algebra *is an algebra for the monad $\mathbb{T}_G$ on $\mathcal{E}$ corresponding to the monadic functor*

$$\alpha_G^*\colon \mathsf{Pt}_G(\mathcal{E}) \longrightarrow \mathcal{E},$$

with $\alpha_G\colon \mathbf{1} \longrightarrow G$.

2. *The $(X, \xi) \rtimes G$ of a G-algebra (X, ξ) and the object G is the domain part H of the point*

$$p, s\colon H \rightleftarrows G$$

corresponding to (X, ξ) by the equivalence $\mathsf{Pt}_G(\mathcal{E}) \cong \mathcal{E}^{\mathbb{T}_G}$.

9 Topological groups

We observed in Example 3.12 that, when a category $\mathcal{E}$ is finitely complete, the category $\mathsf{Gp}(\mathcal{E})$ of internal groups in $\mathcal{E}$ is protomodular. In this section we shall focus on the topological context, namely on the categories $\mathsf{Gp}(\mathsf{Top})$ and $\mathsf{Gp}(\mathsf{Haus})$ of topological and Hausdorff groups. Let us begin with some very general facts.

9.1 The category $\mathsf{Gp}(\mathsf{Top})$

We shall gather here the main categorical facts. First, the forgetful functor $U\colon \mathsf{Gp}(\mathsf{Top}) \longrightarrow \mathsf{Gp}$ is topological (see [31], [1], [5] and our Theorem 10.1) and consequently cotopological. It therefore has all the good lifting properties one can hope for. This implies in particular that

- $\mathsf{Gp}(\mathsf{Top})$ is finitely complete and cocomplete
- the functor U has a left adjoint, right inverse, $D\colon \mathsf{Gp} \longrightarrow \mathsf{Gp}(\mathsf{Top})$ which associates with a group G the topological group (G, T_G^0), where T_G^0 indicates the discrete topology;
- the functor U has also a right adjoint, right inverse, $F\colon \mathsf{Gp} \longrightarrow \mathsf{Gp}(\mathsf{Top})$ which associates with a group G the topological group (G, T_G^1), where T_G^1 indicates the indiscrete topology.

Therefore the functor U is a fibration, or more precisely the pair (U, F) is a fibered reflection. A morphism $f\colon (G, T_G) \longrightarrow (G, T'_{G'})$ is *cartesian* when the following square is a pullback:

$$\begin{array}{ccc} (G, T_G) & \xrightarrow{\ f\ } & (G', T_{G'}) \\ \downarrow & & \downarrow \\ (G, T_G^1) & \xrightarrow[\ f\ ]{} & (G', T_{G'}^1) \end{array}$$

This means that the only open sets of T_G are the inverse images by f of the open sets of $T_{G'}$. When this is the case, we shall denote by T_G^f this topology on G. It is clear that when $f = i$ is the inclusion of a subgroup, T_G^i is precisely the induced topology. The class of cartesian maps is stable by composition, by pullbacks, and contains the homeomorphic isomorphisms. If the map $g = f \circ h$ and the map f are cartesian, then the map h is cartesian as well. Every map $f\colon (G, T_G) \longrightarrow (G', T_{G'})$ in $\mathsf{Gp}(\mathsf{Top})$ has a canonical decomposition $f = f_c \circ f_i$ with f_c cartesian and f_i U-invertible (i.e. such that $U(f_i)$ is an isomorphism). Moreover, every commutative square whose one pair of parallel arrows is cartesian and whose image by U is a pullback is itself a pullback. Let us first emphasize the following more specific result:

Proposition 9.1 *Given a topological group (G, T_G), the closure $\overline{\{1_G\}}$ of the unit element 1_G is such that its induced topology is indiscrete. Accordingly the functor $F\colon \mathsf{Gp} \longrightarrow \mathsf{Gp}(\mathsf{Top})$ admits as a right adjoint the functor $C\colon \mathsf{Gp}(\mathsf{Top}) \longrightarrow \mathsf{Gp}$, where $C(G, T_G) = \overline{\{1_G\}}$.*

Proof It is well known that the closure $\overline{\{1_G\}}$ is a normal subgroup of G. On the other hand, every non empty closed set W of $\overline{\{1_G\}}$ is closed in T_G. So if $1_G \in W$, then $W = \overline{\{1_G\}}$. If not, there is an $x \in W \subseteq \overline{\{1_G\}}$ such that $x^{-1}W$ is closed in $\overline{\{1_G\}}$ and contains 1_G, so that $x^{-1}W = \overline{\{1_G\}}$, and $W = x \cdot \overline{\{1_G\}} = \overline{\{1_G\}}$. Accordingly, $\overline{\{1_G\}}$ has no other non-empty closed subset but itself, and the induced topology is consequently indiscrete. So that we have $\big(\overline{\{1_G\}}, T^i\big) = F\big(\overline{\{1_G\}}\big)$. Now take a group L and a continuous homomorphism $l\colon (L, T_L^1) \longrightarrow (G, T_G)$. Then $l^{-1}\big(\overline{\{1_G\}}\big)$ is a non empty closed set of T_L, so that $l^{-1}\big(\overline{\{1_G\}}\big) = L$, and h has a factorization $\bar{h}\colon L \longrightarrow \overline{\{1_G\}}$. □

So, we have here a remarkable sequence of three adjunctions: $D \dashv U \dashv F \dashv C$.

One of the first consequences of the protomodularity of $\mathsf{Gp}(\mathsf{Top})$ is the following:

Proposition 9.2 *Given a continuous homomorphism $f\colon (X, T_X) \longrightarrow (Y, T_Y)$, split by a continuous homomorphism s in $\mathsf{Gp}(\mathsf{Top})$, a group homomorphism $h\colon X \longrightarrow H$ is continuous from (X, T_X) to (H, T_H) if and only if the group homomorphisms*

$$h \circ \mathsf{ker}\, f\colon \big(K[f], T^i\big) \longrightarrow (H, T_H), \quad h \circ s\colon (Y, T_Y) \longrightarrow (H, T_H)$$

are continuous:

$$\begin{array}{ccccc} \big(K[f],T^i\big) & \xrightarrow{\ker f} & (X,T_X) & \overset{h}{\cdots\cdots\to} & (H,T_H) \\ \downarrow & & s\uparrow\downarrow f & & \\ 1 & \xrightarrow[\tau_Y]{} & (Y,T_Y) & & \end{array}$$

Proof Let us consider the finest topology T on X which makes (X,T) a topological group and the maps $\ker f\colon \big(K[f],T^i\big)\longrightarrow(X,T)$ and $s\colon (Y,T_Y)\longrightarrow(X,T)$ continuous. This is the topology which makes universally continuous any group homomorphism $h\colon X\longrightarrow H$ such that the group homomorphisms

$$h\circ\ker f\colon \big(K[f],T^i\big)\longrightarrow(H,T_H),\quad h\circ s\colon (Y,T_Y)\longrightarrow(H,T_H)$$

are continuous. In particular the map $\mathrm{id}_X\colon (X,T)\rightarrowtail(X,T_X)$ is continuous. It is a monomorphism in $\mathsf{Gp}(\mathsf{Top})$ and its pullbacks along $\ker f$ and s are both isomorphisms, as the commutativity of the following diagram shows immediately:

$$\begin{array}{ccccc} \big(K[f],T^i\big) & \overset{\ker f}{\rightarrowtail} & (X,T) & \overset{s}{\leftarrowtail} & (Y,T_Y) \\ \Big\downarrow \mathrm{id}_{\left(K[f],T^i\right)} & & \Big\downarrow \mathrm{id}_X & & \Big\downarrow \mathrm{id}_{(Y,T_Y)} \\ \big(K[f],T^i\big) & \underset{\ker f}{\rightarrowtail} & (X,T_X) & \underset{s}{\leftarrowtail} & (Y,T_Y) \end{array}$$

Consequently id_X is an homeomorphism and $T=T_X$. □

9.2 Normal subobjects

The category $\mathsf{Gp}(\mathsf{Top})$ being protomodular, there is an intrinsic notion of normal subobject. Our aim here will be to characterize the normal topological subgroups.

We shall begin with some very general observations on internal groups. We recalled that, given any finitely complete category $\mathcal{E}$, the category $\mathsf{Gp}(\mathcal{E})$ of internal groups in $\mathcal{E}$ is protomodular. We shall present here an internal group as an object X of $\mathcal{E}$ endowed with a division, i.e. a binary operation $d\colon X\times X\longrightarrow X$ with a left unit $e\colon 1\longrightarrow X$ satisfying internally

$$d(e,x)=x,\quad d(x,x)=e,\quad d(d(x,y),d(x,z))=d(y,z).$$

We shall need specifically the internal map d_*, corresponding elementwise to

$$d_*\colon X\times X\longrightarrow X,\quad d_*(y,x)=d\big(d(y,x),x\big)=x^{-1}\cdot y\cdot x.$$

Let (I,d_I) be any subgroup of (X,d) in $\mathsf{Gp}(\mathcal{E})$, and let us consider now the following pullback:

$$\begin{array}{ccc} \Gamma_X^I & \xrightarrow{\delta} & I \\ j\Big\downarrow & & \Big\downarrow i \\ X\times X & \xrightarrow[d]{} & X \end{array}$$

This is clearly internally corresponding to $\Gamma_X^I = \{(u,v) \in X \times X \,|\, u^{-1} \cdot v \in I\}$.

Lemma 9.3 *The map $j\colon \Gamma_X^I \rightarrowtail X \times X$ defines, on the object X in $\mathcal{E}$, an internal equivalence relation to which the subobject $i\colon I \rightarrowtail X$ is normal in $\mathcal{E}$.*

Proof Straightforward, thanks to the Yoneda embedding. □

Of course, there is no reason for Γ_X^I to be a subgroup of the group $X \times X$. Categorically speaking, this is the case if and only if, in the category $\mathcal{E}$, the left hand side vertical map in the following pullback is an isomorphism, where the map $d_{X\times X}$ denotes the division of the product group $X \times X$:

$$\begin{array}{ccc} P & \longrightarrow & \Gamma_X^I \\ \downarrow & & \downarrow{\scriptstyle j} \\ \Gamma_X^I \times \Gamma_X^I \xrightarrow[j \times j]{} (X\times X)^2 & \xrightarrow[d_{X\times X}]{} & X \times X \end{array}$$

Proposition 9.4 *The subobject $i\colon I \rightarrowtail X$ is normal in $\mathsf{Gp}(\mathcal{E})$ if and only if Γ_X^I is a subgroup of the group $X \times X$.*

Proof When Γ_X^I is a subgroup of the group $X \times X$, then $i\colon I \rightarrowtail X$ becomes normal to Γ_X^I in $\mathsf{Gp}(\mathcal{E})$. Conversely suppose that $i\colon I \rightarrowtail X$ is normal to some equivalence relation R in $\mathsf{Gp}(\mathcal{E})$. Then, thanks to the Yoneda embedding, we can check that $R \cong R \cap \Gamma_X^I \cong \Gamma_X^I$. □

There is also the following characterization, which is the internal translation, thanks to the Yoneda embedding, of a well known result in the set theoretical context:

Proposition 9.5 *The map $i\colon I \rightarrowtail X$ is normal in $\mathsf{Gp}(\mathcal{E})$ if and only if, in the category $\mathcal{E}$, the left hand side vertical map in the following pullback is an isomorphism:*

$$\begin{array}{ccc} J & \longrightarrow & I \\ \downarrow & & \downarrow{\scriptstyle i} \\ I \times X \xrightarrow[i \times \mathsf{id}_X]{} X \times X & \xrightarrow[d_*]{} & X \end{array}$$

Proof This is the categorical translation of the fact that, for all $(y,x) \in I \times X$, the element $x^{-1} \cdot y \cdot x$ must be in I. □

We know that, in a protomodular category, an object X is abelian if and only if the diagonal $s_0\colon X \rightarrowtail X \times X$ is normal. We have also:

Corollary 9.6 *An internal group X in $\mathcal{E}$ is abelian if and only if $d_* = p_0\colon X \times X \longrightarrow X$.*

Let us come back to the category $\mathsf{Gp}(\mathsf{Top})$. Let (X, T_X) be a topological group, and (I, T_I) a subobject in $\mathsf{Gp}(\mathsf{Top})$. Then T_I is such that the inclusion $i\colon I \rightarrowtail X$ is continuous, but of course T_I is not necessarily the topology T_I^i induced by T_X on I. According to the previous characterization, we obtain:

Proposition 9.7 *A subobject $i\colon (I, T_I) \rightarrowtail (X, T_X)$ is normal in the category $\mathsf{Gp}(\mathsf{Top})$ if and only if:*

1. *the subgroup I is a normal subgroup of X;*
2. *the map $d_*\colon I \times X \longrightarrow I; (y,x) \mapsto x^{-1} \cdot y \cdot x$ is continuous as a map from the topological product $(I, T_I) \times (X, T_X)$ to the topological group (I, T_I).*

The second condition shows that there is no reason for the morphism $\mathsf{id}\colon (X, T_X^0) \longrightarrow (X, T_X)$ to be a normal subobject in general. Of course, every normal subgroup I of X produces a normal subobject $i\colon (I, T_I^i) \rightarrowtail (X, T_X)$ by means of the induced topology. But a normal subobject is not necessarily of this form. For instance, as soon as I is a central subgroup, we have $d_* = p_I\colon I \times X \longrightarrow I$ since $x^{-1} \cdot y \cdot x = y$; accordingly, in this case, every subobject of the form $i\colon (I, T_I) \rightarrowtail (X, T_X)$ is normal. So that, *when A is an abelian group, and I is any subgroup of A, every continuous inclusion $i\colon (I, T_I) \rightarrowtail (A, T_A)$ produces a normal subobject.* This is the case in particular for $\mathsf{id}_A\colon (A, T_A^0) \longrightarrow (A, T_A)$. This example is particularly interesting since it emphasizes, provided T_A is not discrete, that *there are, in* $\mathsf{Gp}(\mathsf{Top})$*, normal subobjects which are not kernels.* We have the following obvious, but useful characterization:

Proposition 9.8 *A monomorphism $i\colon (I, T_I) \rightarrowtail (X, T_X)$ in $\mathsf{Gp}(\mathsf{Top})$ is a kernel if and only if it is normal and cartesian.*

9.3 $\mathsf{Gp}(\mathsf{Top})$ is homologigal

The fact that $\mathsf{Gp}(Top)$ is homological will be attested — in a much more general setting — by our Corollary 10.12. But let us already sketch here a more direct argument.

The regular epimorphisms in the category $\mathsf{Gp}(\mathsf{Top})$ are just the surjective open homomorphisms. The category $\mathsf{Gp}(\mathsf{Top})$ is regular since the regular epimorphisms are stable by pullback, and every effective equivalence relation in $\mathsf{Gp}(\mathsf{Top})$ (i.e. every kernel pair) has a coequalizer [2]. Given $h\colon (G, T_G) \longrightarrow (H, T_H)$, a regular epimorphism, and (I, T_I) a subobject of (G, T_G), the *direct image* of (I, T_I) under h is the pair $\big(h(I), T^q_{h(I)}\big)$ where

- $h(I)$ is the ordinary image subgroup of $h(I)$ in H;
- $T^q_{h(I)}$ is the relative quotient topology, namely, the topology such that V is an open set in $T^q_{h(I)}$ if and only if $h^{-1}(V)$ is an open set in T_I.

Finally, it is clear that a cartesian surjective homomorphism is necessarily a regular epimorphism. Now the category $\mathsf{Gp}(\mathsf{Top})$, being pointed, protomodular and regular, it is homological.

An exact sequence is then a sequence:

$$\mathbf{1} \longrightarrow (K, T_K) \overset{k}{\rightarrowtail} (G, T_G) \overset{h}{\twoheadrightarrow} (H, T_H) \longrightarrow \mathbf{1}$$

where h is a surjective open homomorphism and k is the (necessarily cartesian) kernel map of h. In other words $T_K = T_K^k$ and $T_H = T_H^q$.

As a first homological aspect of the protomodularity of $\mathsf{Gp}(\mathsf{Top})$, let us mention:

Proposition 9.9 *Consider the following morphism of exact sequence. Then the map f is cartesian if and only if the induced map ϕ is cartesian:*

$$\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & (K, T_K^k) & \overset{k}{\rightarrowtail} & (G, T_G) & \overset{h}{\twoheadrightarrow} & (H, T_H) & \longrightarrow & \mathbf{1} \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathsf{id}_H} & & \\
\mathbf{1} & \longrightarrow & (K', T_{K'}^{k'}) & \underset{k'}{\rightarrowtail} & (G', T_{G'}) & \underset{h'}{\twoheadrightarrow} & (H, T_H) & \longrightarrow & \mathbf{1}
\end{array}$$

Proof The left-hand square is always a pullback, so that if f is cartesian, the map ϕ is also cartesian. Conversely, let us consider the following diagram induced by the canonical decomposition $f = f_c \circ f_i$:

$$
\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & (K, T_K^k) & \overset{k}{\rightarrowtail} & (G, T_G) & \overset{h}{\twoheadrightarrow} & (H, T_H) & \longrightarrow & \mathbf{1} \\
 & & \downarrow{\scriptstyle \mathsf{id}_K} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \mathsf{id}_H} & & \\
\mathbf{1} & \dashrightarrow & (K, T_K^k) & \underset{\bar{k}}{\rightarrowtail} & (G, T_G^f) & \underset{\bar{h}}{\twoheadrightarrow} & (H, T_H) & \dashrightarrow & \mathbf{1} \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f_c} & & \downarrow{\scriptstyle \mathsf{id}_H} & & \\
\mathbf{1} & \longrightarrow & (K', T_{K'}^{k'}) & \underset{k'}{\rightarrowtail} & (G', T_{G'}) & \underset{h'}{\twoheadrightarrow} & (H, T_H) & \longrightarrow & \mathbf{1}
\end{array}
$$

The map $\bar{h}$ is a regular epimorphism, since $\bar{h} \circ f_i = h$ is a regular epimorphism. On the other hand, since the map f_i is U-invertible, the image by U of the lower left-hand square is a pullback in Gp. Since the pair (ϕ, f_c) is a pair of cartesian maps, this same square is itself a pullback in $\mathsf{Gp}(\mathsf{Top})$. Accordingly the middle row is exact. By the short five lemma, the map f_i is an isomorphism in $\mathsf{Gp}(\mathsf{Top})$, and consequently an homeomorphism. Thus $T_G = T_G^f$, and f is cartesian. □

From this, we can derive a characterization of the cartesian regular epimorphisms:

Corollary 9.10 *A regular epimorphism $h\colon (G, T_G) \twoheadrightarrow (G', T_{G'})$ is cartesian if and only if the induced topology on $\mathsf{ker}\, h$ is indiscrete.*

Proof The terminal map $\tau_H\colon (H, T_H) \longrightarrow \mathbf{1}$ is cartesian if and only if the topology T_H is indiscrete. Now consider the following diagram

$$
\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & (\mathsf{ker}\, h, T^i) & \overset{i}{\rightarrowtail} & (G, T_G) & \overset{h}{\twoheadrightarrow} & (G', T_{G'}) & \longrightarrow & \mathbf{1} \\
 & & \downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathsf{id}} & & \\
\mathbf{1} & \longrightarrow & \mathbf{1} & \rightarrowtail & (G', T_{G'}) & \underset{\mathsf{id}}{\twoheadrightarrow} & (G', T_{G'}) & \longrightarrow & \mathbf{1}
\end{array}
$$

and apply the Proposition 9.9. □

Remark 9.11 *According to Proposition 9.1, in the following exact sequence, the map h is cartesian:*

$$\mathbf{1} \longrightarrow \left(\overline{\{1_G\}}, T^i\right) \overset{i}{\rightarrowtail} (G, T_G) \overset{h}{\twoheadrightarrow} (G/\overline{\{1_G\}}, T^q) \longrightarrow \mathbf{1}$$

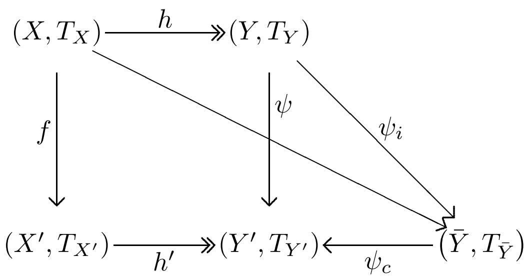
Remark 9.12 *This property of the map h tranforms the trite observation on the (Hausdorff, see [6] and [29]) topological group $\left(G/\overline{\{1_G\}}, T^q\right)$ that any non empty open or closed set is the union of its elements, into the non trivial fact that any non empty open or closed set in (G, T_G) is necessarily an* arbitrary union of cosets of $\overline{\{1_G\}}$.

In the same order of ideas we have:

Proposition 9.13 *Suppose that we are given the following pullback with h' a regular epimorphism; then f is cartesian if and only if ψ is cartesian:*

$$
\begin{array}{ccc}
(X, T_X) & \overset{h}{\longrightarrow} & (Y, T_Y) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \psi} \\
(X', T_{X'}) & \underset{h'}{\twoheadrightarrow} & (Y', T_{Y'})
\end{array}
$$

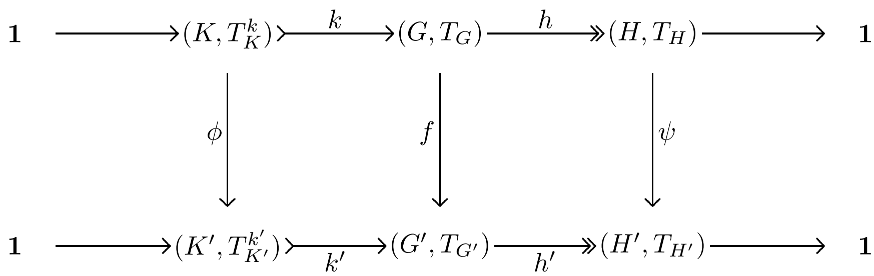
Proof Only the "only if" part needs a proof. Consider the following diagram with $\psi = \psi_c \circ \psi_i$ the canonical decomposition of ψ:



Then, ψ_i being U-invertible, the image by U of the quadrangle with sides f and ψ_c is a pullback in Gp. Accordingly, since the "parallel" arrows ψ_c and f are cartesian, the quadrangle in question is itself a pullback. The map $\psi_i \circ h$ is then a regular epimorphism, and thus a strong epimorphism, with ψ_i (as a U-invertible map) a monomorphism. Thus ψ_i is an isomorphism in $\mathsf{Gp(Top)}$. □

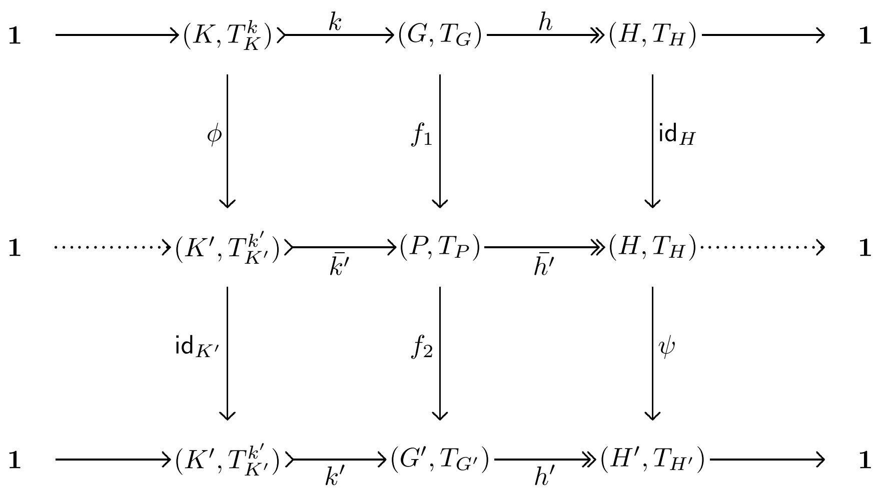
Finally we have:

Theorem 9.14 *Consider, in* $\mathsf{Gp(Top)}$*, the following morphism of exact sequences:*



1. *when ϕ and ψ are cartesian, then f is cartesian too;*
2. *when f and ψ are cartesian, then ϕ is cartesian too;*
3. *when f and ϕ are cartesian, with moreover ϕ a regular epimorphism, then ψ is cartesian too.*

Proof Consider the following decomposition, where the lower right-hand square is a pullback:



On one hand, the map $\bar{h}'$ is a regular epimorphism, since $\bar{h}' \circ f_1 = h$ is itself a regular epimorphism. On the other hand there is a unique map $\bar{k}'$ which completes the middle row as an exact sequence and makes the upper left-hand square a pullback.

1. If ϕ is cartesian, then according to Proposition 9.9, the map f_1 is cartesian. Since ψ is cartesian, the map f_2 is cartesian as well, the cartesian maps being stable by pullback. Consequently $f = f_2 \circ f_1$ is cartesian.

2. If ψ is cartesian, then the map f_2 is cartesian as well. But since $f = f_2 \circ f_1$ is cartesian, the map f_1 is cartesian. Thus ϕ is cartesian by Proposition 9.9.

3. If ϕ is a cartesian regular epimorphism, then f_1 is cartesian, and is a regular epimorphism by Proposition 8 in [13]. Since $f = f_2 \circ f_1$ is cartesian, the map f_2 is cartesian. Then, according to Proposition 9.13, the map ψ is cartesian. $\square$

10 Topological semi-abelian algebras

Let us first recall a useful well-known result.

Theorem 10.1 (Wyler [31]) *Let $\mathbb{T}$ be an algebraic theory. The functor*

$$U\colon \mathsf{Top}^{\mathbb{T}} \longrightarrow \mathsf{Set}^{\mathbb{T}}, \quad A \mapsto A$$

forgetting the "topological structure" is topological, while the functor

$$V\colon \mathsf{Top}^{\mathbb{T}} \longrightarrow \mathsf{Top}, \quad A \mapsto A$$

forgetting the $\mathbb{T}$-algebra structure is monadic. The category $\mathsf{Top}^{\mathbb{T}}$ is thus complete and cocomplete, with limits and colimits computed as in $\mathsf{Set}^{\mathbb{T}}$; limits are also computed as in Top. $\square$

Convention *Throughout this section, given a semi-abelian theory $\mathbb{T}$, the notation α_i ($i = 1, \ldots, n$) or θ will always indicate operations as in Theorem 3.5.*

The theory of topological groups uses in an intensive way the fact that given an element $g \in G$ of a topological group G (written additively), the mapping

$$-+g\colon G \longrightarrow G, \quad x \mapsto x + g$$

is an homeomorphism mapping 0 on g and admitting the subtraction by g as inverse. This homogeneity property of the topology can be partly recaptured in the case of a semi-abelian theory:

Proposition 10.2 *Let $\mathbb{T}$ be a semi-abelian theory. For every element a of a topological $\mathbb{T}$-algebra A,*

$$A \rightarrowtail A^n, \quad \mapsto \big(\alpha_1(x,a), \ldots, \alpha_n(x,a)\big)$$

presents A as a topological retract of A^n, with thus the induced topology, and maps the element $a \in A$ on $(0, \ldots, 0) \in A^n$.

Proof It suffices to observe that

$$A^n \longrightarrow A, \quad (a_1, \ldots, a_n) \mapsto \theta(a_1, \ldots, a_n, a)$$

is a retraction of the given map in the category of topological spaces. $\square$

Notice that the inclusion given in Proposition 10.2 is by no means a $\mathbb{T}$-homomorphism: it does not even preserve the constant 0.

Corollary 10.3 *Let $\mathbb{T}$ be a semi-abelian theory. Given an element $a \in A$ of a topological $\mathbb{T}$-algebra A, the subsets*

$$\bigcap_{i=1}^{n} \alpha_i(-,a)^{-1}(U), \quad U \text{ open neighborhood of } 0$$

constitute a fundamental system of open neighborhoods of a.

Proof Every open neighborhood of $(0, \ldots, 0) \in A^n$ contains a neighborhood of the form U^n, with $U \subseteq A$ open neighborhood of 0. One concludes by Proposition 10.2. $\square$

Metatheorem 10.4 *Let $\mathbb{T}$ be a semi-abelian theory and $\mathcal{P}$ a property stable under finite limits. If the property $\mathcal{P}$ is valid at the neighborhood of 0 in a given semi-abelian algebra A, that property $\mathcal{P}$ is valid at the neighborhood of every point of A.*

Proof By Proposition 10.2, since every retract of A^n is the equalizer of the identity and an idempotent morphism on A^n. □

Another useful property of topological groups is that every neighborhood V of 0 contains a symmetric neighborhood W such that $W + W \subseteq V$. The generalization to the semi-abelian case is easy:

Lemma 10.5 *Let $\mathbb{T}$ be a semi-abelian theory and V an open neighborhood of 0 in a topological $\mathbb{T}$-algebra A. For every k-ary operation τ of the theory there exists an open neighborhood U of 0 in A such that*

$$a_1, \ldots, a_k \in U \Rightarrow \tau(a_1, \ldots, a_k) \in V.$$

Proof The function

$$\tau_A \colon A^k \longrightarrow A, \quad (X_1, \ldots, X_k) \mapsto \tau(X_1, \ldots, X_k)$$

is continuous and maps $(0, \ldots, 0)$ on 0. Therefore $\tau_A^{-1}(V)$ is an open neighborhood of $(0, \ldots, 0)$ in A^k and this neighborhood contains one of the form U^k, with U a neighborhood of 0 in A. □

Obviously, every subalgebra B of the topological algebra A, provided with the induced topology, is a topological algebra on its own. As usual when we mention that the subalgebra B is open, or closed, or compact, or whatever, this is always for the topology induced by that of A. Among the subalgebras, we have the kernels, which are simply the algebraic kernel provided with the induced topology.

First, let us generalize a celebrated result on topological groups.

Proposition 10.6 *Let $\mathbb{T}$ be a semi-abelian theory. Every open subalgebra $B \subseteq A$ of a topological algebra A is closed.*

Proof Given $a \in A \setminus B$, we must prove the existence of an open subset $U \subseteq A \setminus B$ containing a. It suffices to put

$$U = \bigcap_{i=1}^{n} \alpha_i(a, -)^{-1}(B).$$

This subset is open, as a finite intersection of open subsets. It contains a because $\alpha_i(a, a) = 0 \in B$ for each index i. Moreover $U \cap B = \emptyset$, because $b \in U \cap B$ would imply

$$a = \theta\big(\alpha_1(a, b), \ldots, \alpha_n(a, b), b\big) \in B$$

since then each $\alpha_i(a, b)$ and b itself would be in the subalgebra B. □

Corollary 10.7 *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a subalgebra. The following conditions are equivalent:*

1. *B is a neighborhood of 0;*
2. *B is an open neighborhood of 0;*
3. *B is a closed neighborhood of 0.*

Proof ($2 \Rightarrow 3$) follows from Proposition 10.6 and ($3 \Rightarrow 1$) is trivial. If B is a neighborhood of 0 and $b \in B$,

$$U = \bigcap_{i=1}^{n} \alpha_i(-, b)^{-1}(B)$$

is a neighborhood of b; it is contained in B because

$$x \in U \Rightarrow x = \theta\big(\alpha_1(x, b), \ldots, \alpha_n(x, b), b\big) \in B$$

since B is a subalgebra. Thus B is open. □

Let us now investigate the behaviour of subalgebras with respect to topological closure.

Proposition 10.8 *Let $\mathbb{T}$ be an algebraic theory. The closure $\overline{B} \subseteq A$ of every subalgebra $B \subseteq A$ of a topological $\mathbb{T}$-algebra A is still a subalgebra.*

Proof Let $\tau(X_1, \ldots, X_m)$ be an m-ary operation of the theory $\mathbb{T}$. Since τ is continuous on A and $B \subseteq A$ is a subalgebra:

$$\tau\left(\overline{B}^n\right) = \tau\left(\overline{B^n}\right) \subseteq \overline{\tau(B^n)} \subseteq \overline{B}. \qquad \square$$

Next we consider quotient topological algebras.

Proposition 10.9 *Let $\mathbb{T}$ be a semi-abelian theory and $B \subseteq A$ a kernel subalgebra. Given an arbitrary subset $X \subseteq A$, the saturation $\widetilde{X}$ of X for the corresponding quotient $q\colon A \twoheadrightarrow A/B$ is given by*

$$\begin{aligned}
\widetilde{X} &= q^{-1}\big(q(X)\big) \\
&= \big\{a \in A \big| \exists x \in X\ \forall i\ \alpha_i(a,x) \in B\big\} \\
&= \big\{a \in A \big| \exists x \in X\ \forall i\ \alpha_i(x,a) \in B\big\} \\
&= \big\{a \in A \big| \exists b_1, \ldots, b_n \in B\ \theta(b_1, \ldots, b_n, a) \in X\big\} \\
&= \big\{\theta(b_1, \ldots, b_n, x) \big| b_1, \ldots, b_n \in B,\ x \in X\big\}.
\end{aligned}$$

In particular, for every $x \in A$,

$$[x] = \theta(B^n, x) = \big\{\theta(b_1, \ldots, b_n, x) \big| b_1, \ldots, b_n \in B\big\}.$$

Proof If $\theta(b_1, \ldots, b_n, a) \in X$, we have in A/B

$$\begin{aligned}
[a] &= \big[\theta(0, \ldots, 0, a)\big] \\
&= \theta\big([0], \ldots, [0], [a]\big) \\
&= \theta\big([b_1], \ldots, [b_n], [a]\big) \\
&= \big[\theta(b_1, \ldots, b_n, a)\big] \\
&\in q(X)
\end{aligned}$$

thus $a \in q^{-1}\big(q(X)\big)$. Conversely if $a \in q^{-1}\big(q(X)\big)$, there exists $x \in X$ such that $[x] = [a]$, that is, $\alpha_i(x,a) \in B$ for each index i, as follows at once from Corollary 3.6. This implies

$$\theta\big(\alpha_1(x,a), \ldots, \alpha_n(x,a), a\big) = x \in X$$

and it suffices to choose $b_i = \alpha_i(x,a)$.

Finally when $a \in \widetilde{X}$, we have already observed that

$$a = \theta\big(\alpha_1(a,x), \ldots, \alpha_n(a,x), x\big)$$

with $x \in X$ and $\alpha_i(a,x) \in B$ for each index i. Conversely if $x \in X$ and $b_i \in B$ for each index i, we obtain

$$\big[\theta(b_1, \ldots, b_n, x)\big] = \theta\big([b_1], \ldots, [b_n], [x]\big) = \theta\big([0], \ldots, [0], [x]\big) = [x]$$

thus $\theta(b_1, \ldots, b_n, x) \in \widetilde{X}$. $\square$

Proposition 10.10 *Let $\mathbb{T}$ be a semi-abelian theory. The regular epimorphisms $q\colon B \twoheadrightarrow Q$ in $\mathsf{Top}^{\mathbb{T}}$ are the surjective morphisms where Q is provided with the quotient topology. Every regular epimorphism is also an open map.*

Proof If q is a coequalizer in $\mathsf{Top}^{\mathbb{T}}$, it is also a coequalizer in $\mathsf{Set}^{\mathbb{T}}$ (see Theorem 10.1). Thus in $\mathsf{Set}^{\mathbb{T}}$, q is the cokernel of its kernel $k\colon K \rightarrowtail B$, by Theorem 4.5. If U is open in B, by Proposition 10.9,

$$q^{-1}\big(q(U)\big) = \bigcup_{k_1, \ldots, k_n \in K} \theta(k_1, \ldots, k_n, -)^{-1}(U)$$

is open, as a union of open subsets. This proves that providing Q with the quotient topology makes q an open map.

The quotient topology provides Q with the structure of a topological $\mathbb{T}$-algebra. Indeed given a k-ary operation τ, q^k is still a continuous open surjection, thus a quotient map of topological spaces. Therefore the continuity of τ on Q is inherited from its continuity on B. But then trivially, $q = \mathsf{coker}\, k$ in $\mathsf{Top}^{\mathbb{T}}$. $\square$

The category Top of topological spaces is not Barr regular; nevertheless:

Theorem 10.11 *The category $\mathsf{Top}^{\mathbb{T}}$ of topological models of a semi-abelian theory $\mathbb{T}$ is Barr regular.*

Proof In the category of topological spaces, every open surjection yields necessarily the quotient topology and open surjections are stable under pullbacks. One concludes by Proposition 10.10. □

Corollary 10.12 *Let $\mathbb{T}$ be a semi-abelian theory. The category $\mathsf{Top}^{\mathbb{T}}$ of topological $\mathbb{T}$-algebras is homological.*

Proof By Theorem 10.1 we have finite limits; by Theorem 10.11 we have the regularity; by Example 3.11 $\mathsf{Top}^{\mathbb{T}}$ is pointed protomodular. □

Finally, we prove the existence of topological semi-direct products.

Theorem 10.13 *When $\mathbb{T}$ is a semi-abelian theory, the inverse image functors of the fibration of points are monadic and therefore, topological semi-direct products and be defined as in Definition 8.11.*

Proof Given $v\colon Y \longrightarrow X$ in $\mathsf{Top}^{\mathbb{T}}$, the functor v^* has a left adjoint, namely, the pushout along v, and reflects isomorphisms, by protomodularity.

By the Beck criterion, we still have to check a condition on some coequalizers. But coequalizers in the categories $\mathsf{Pt}_X(\mathsf{Top}^{\mathbb{T}})$ and $\mathsf{Pt}_Y(\mathsf{Top}^{\mathbb{T}})$ are computed as in $\mathsf{Pt}_X(\mathsf{Set}^{\mathbb{T}})$ and $\mathsf{Pt}_Y(\mathsf{Set}^{\mathbb{T}})$, that is as in $\mathsf{Set}^{\mathbb{T}}$, and are provided with the quotient topology (see Proposition 10.10). The functor v^* in $\mathsf{Set}^{\mathbb{T}}$ preserves the coequalizers involved in the Beck criterion, because it is monadic (see Theorem 8.6). Moreover the functor v^* in $\mathsf{Top}^{\mathbb{T}}$ preserves open surjections, as every topological pullback. We conclude by Proposition 10.10. □

11 Torsion theories

(This section is presently being written and will be available very soon)

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