LECTURES ON

Lax-Algebraic Methods in General Topology

SUMMER SCHOOL IN CATEGORICAL METHODS IN ALGEBRA AND TOPOLOGY

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Lecture 1 : $\mathcal{V}$-categories, $\mathcal{V}$-modules, Lawvere completeness

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1.1. Guiding examples.

(1) Ord: objects are *(pre)ordered sets* (= sets with a reflexive and transitive relation, no antisymmetry condition), with monotone maps. Formally:

$$\begin{array}{lll} (X,a) \text{ with} & 1. & \top \vDash a(x,x), \\ & 2. & a(x,y) \wedge a(y,z) \vDash a(x,z), \\ f:(X,a) \longrightarrow (Y,b) \text{ with} & & a(x,y) \vDash b(f(x),f(y)). \end{array}$$

(2) Met: objects are (generalized) *metric spaces* (=sets with a function $a : X \times X \longrightarrow [0,\infty]$ that is 0 on the diagonal and satisfies the triangle inequality), with contractions (=non-expansive maps). Formally:

$$\begin{array}{lll} (X,a) \text{ with} & 1. & 0 \geq a(x,x), \\ & 2. & a(x,y) + a(y,z) \geq a(x,z), \\ f:(X,a) \longrightarrow (Y,b) \text{ with} & & a(x,y) \geq b(f(x),f(y)). \end{array}$$

(3) UMet: the full subcategory of Met containing all *ultrametric spaces*, for which (2) 2 is strengthened to

$$2'. \quad \max\{a(x,y), a(y,z)\} \geq a(x,z).$$

(4) Top : *topological spaces* and continuous maps. In order to expose the analogy with (1), we describe topological spaces in terms of *ultrafilter convergence*, e.g. as sets X with a suitable relation $a \subseteq \beta X \times X$, with maps that preserve this relation; here βX is the set of all ultrafilters on X. Formally:

$$\begin{array}{lll} (X,a) \text{ with} & 1. & \top \vDash a(\dot{x},x), \\ & 2. & a(\mathfrak{X},\mathfrak{y}) \wedge a(\mathfrak{y},z) \vDash a(\sum \mathfrak{X},z), \\ f:(X,a) \longrightarrow (Y,b) \text{ with} & & a(\mathfrak{x},y) \vDash b(f[\mathfrak{x}],f(y)). \end{array}$$

Here we have used the following notation:

$$\begin{array}{lcl} \text{for } x \in X, A \subseteq X & : & A \in \dot{x} \iff x \in A; \\ \text{for } \mathfrak{X} \in \beta\beta X, A \subseteq X & : & A \in \sum \mathfrak{X} \iff A^{\#} \in \mathfrak{X}; \\ \text{where for } \mathfrak{x} \in \beta X & : & \mathfrak{x} \in A^{\#} \iff A \in \mathfrak{x}; \\ \text{for } \mathfrak{x} \in \beta X, B \subseteq Y & : & B \in f[\mathfrak{x}] \iff f^{-1}[B] \in \mathfrak{x}. \end{array}$$

We also used an extension of the relation a to a relation $a \subseteq \beta\beta X \times \beta X$:

$$a(\mathfrak{X}, \mathfrak{y}) \iff \forall \mathcal{A} \in \mathfrak{X},\ B \in \mathfrak{y}\ \exists \mathfrak{x} \in \mathcal{A},\ y \in B : a(\mathfrak{x}, y).$$

Motivation for the axioms as well as explanations for their equivalence with the usual presentation of topological spaces (in terms of open sets or neighborhood systems) will be provided in Lecture 2.

(5) App: *approach spaces* and contractions. Approach spaces constitute the natural generalization of (2) and (4) and may be described in terms of sets with a function $a : \beta X \times X \longrightarrow [0, \infty]$. Formally:

$$\begin{array}{ll} (X, a) \text{ with} & 1.\ \ 0 \geq a(\dot{x}, x), \\ & 2.\ \ a(\mathfrak{X}, \mathfrak{y}) + a(\mathfrak{y}, z) \geq a(\sum \mathfrak{X}, z), \\ f : (X, a) \longrightarrow (Y, b) \text{ with} & \quad\ a(\mathfrak{x}, y) \geq b(f[\mathfrak{x}], f(y)). \end{array}$$

Approach spaces may be equivalently described in terms of distance functions $d : PX \times X \longrightarrow [0, \infty]$; further explanations and references to follow in Lecture 3.

1.2. Quantales Throughout the lectures, $\mathcal{V}$ is a non-trivial commutative, unital *quantale*, e.g. a complete lattice with an associative and commutative binary operation $\otimes$ that preserves suprema in each variable:

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i,$$

such that there is a $\otimes$-neutral element $k > \bot$ (= bottom element). Every non-trivial frame can be considered as such a quantale, with $\otimes = \wedge$ and $k = \top$ (= top element). But there may be different ways of putting a quantalic structure on a frame.

Examples. (1) $2 = \{\bot \vDash \top\}$, $\otimes = \wedge$, $k = \top$.

(2) $\mathbb{P}_+ = ([0, \infty], \geq)$, $\otimes = +$ (extended to $[0, \infty]$ by $x + \infty = \infty + x = \infty$), $k = 0$.

(3) $\mathbb{P}_{\max} = ([0, \infty], \geq), \otimes = \max, k = 0.$

1.3. $\mathcal{V}$-relations The category $\mathcal{V}$-Rel is defined by:

$$\begin{aligned}
\mathrm{ob}(\mathcal{V}\text{-}\mathsf{Rel}) &= \mathrm{ob}\mathsf{Set}\\
\mathcal{V}\text{-}\mathsf{Rel}(X,Y) &= \mathsf{Set}(X\times Y,\mathcal{V});\ \text{write } r: X \nrightarrow Y \text{ for } r\in \mathcal{V}\text{-}\mathsf{Rel}(X,Y));\\
(s\cdot r)(x,z) &= \bigvee_{y\in Y} r(x,y)\otimes s(y,z)\ \ (\text{with } s: Y\nrightarrow Z);\\
1_X(x,y) &= \begin{cases} k & \text{if } x=y;\\ \bot & \text{else.}\end{cases}
\end{aligned}$$

$\mathcal{V}$-Rel is enriched over Ord, by

$$r\le r' \iff \forall x\in X, y\in Y: r(x,y)\le r'(x,y),$$

and there is an order-preserving involution $(-)^\circ : \mathcal{V}\text{-}\mathsf{Rel}^{\mathrm{op}} \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}$, which maps objects identically, with $r^\circ(y,x)=r(x,y)$ for all $x\in X$, $y\in Y$. There is a faithful functor

$$\begin{aligned}
\mathsf{Set} &\longrightarrow \mathcal{V}\text{-}\mathsf{Rel}\\
(f: X\longrightarrow Y) &\mapsto (f: X\nrightarrow Y) \text{ with } f(x,y)=\begin{cases} k & \text{if } f(x)=y;\\ \bot & \text{else.}\end{cases}
\end{aligned}$$

We continue to write $f: X\longrightarrow Y$ even when f is considered as a morphism in $\mathcal{V}$-Rel. One has

$$1_X \le f^\circ\cdot f,\quad f\cdot f^\circ \le 1_Y,$$

eg. $f\vdash f^\circ$ in the "thin" 2-category $\mathcal{V}$-Rel, so that f is actually a *map* (in Lawvere's sense) in $\mathcal{V}$-Rel.

Examples. (1) 2-Rel = Rel is the category of sets and relations.
(2) $\mathbb{P}_+$-Rel is the category of numerical (or fuzzy) relations.
(3) $\mathbb{P}_{\max}$-Rel is the category of "ultra-numerical" relations.

1.4. $\mathcal{V}$-categories The category $\mathcal{V}$-Cat is defined by:

objects: $(X, a: X\nrightarrow X)$ with
1. $1_X\le a$ $(\forall x: k\le a(x,x))$
2. $a\cdot a\le a$ $(\forall x,y,z: a(x,y)\otimes a(y,z)\le a(x,z))$

morphisms: $f:(X,a)\longrightarrow (Y,b)$: $f\cdot a\le b\cdot f$ $(\forall x,y: a(x,y)\le b(f(x),f(y)))$.

We often write $X=(X,a)$, and then put $a:=a_X$, if the structure need to be named. Note that one actually has $a\cdot a=a$ since $1_X\le a$ implies $a\le a\cdot a$. $\mathcal{V}$-Cat becomes Ord-enriched (or a "thin" 2-category) by

$$\begin{aligned}
f\le g &\iff 1_X\le g^\circ\cdot b\cdot f \ (\text{in } \mathcal{V}\text{-}\mathsf{Rel})\\
&\iff \forall x\in X: k\le b(f(x),g(x)),
\end{aligned}$$

for $f,g: X\longrightarrow Y=(Y,b)$. The involutive endofunctor

$$(-)^{\mathrm{op}}: \mathcal{V}\text{-}\mathsf{Cat}\longrightarrow \mathcal{V}\text{-}\mathsf{Cat},\ X=(X,a)\mapsto X^{\mathrm{op}}=(X,a^\circ),$$

reverses the order on the hom-sets:

$$f\le g \iff g^{\mathrm{op}}\le f^{\mathrm{op}}.$$

There is a full embedding

$$\mathsf{Set} \hookrightarrow \mathcal{V}\text{-}\mathsf{Cat},\ X \mapsto (X, 1_X),$$

which regards sets as *discrete* $\mathcal{V}$-categories.

Examples.

$$(1) 2\text{-}\mathsf{Cat} = \mathsf{Ord},\quad (2)\ \mathbb{P}_+\text{-}\mathsf{Cat} = \mathsf{Met},\quad (3)\ \mathbb{P}_{\max}\text{-}\mathsf{Cat} = \mathsf{UMet}.$$

1.5. The $\mathcal{V}$-category $\mathcal{V}$ For all $u \in \mathcal{V}$, let $u \multimap (-)$ be right adjoint to $u \otimes (-) : \mathcal{V} \longrightarrow \mathcal{V}$, so that for all $u, v, z \in \mathcal{V}$:

$$z \le u \multimap v \iff u \otimes z \le v.$$

Sometimes we also write $v \circ\!\!- u$ instead of $u \multimap v$, and we have

$$u \multimap v = \bigvee\{z \mid u \otimes z \le v\}.$$

From $k \otimes v = v$ one has $k \le v \multimap v$, and from

$$u \otimes (u \multimap v) \otimes (v \multimap w) \le v \otimes (v \multimap w) \le w$$

one obtains

$$(u \multimap v) \otimes (v \multimap w) \le u \multimap w.$$

Hence, $(\mathcal{V}, \multimap) = \mathcal{V}$ is a $\mathcal{V}$-category.

Examples. (1) $2 \in \mathrm{ob}(2\text{-}\mathsf{Cat}) = \mathrm{ob}\mathsf{Ord}$ carries its original order.

(2) For $\mathbb{P}_+ \in \mathrm{ob}(\mathbb{P}_+\text{-}\mathsf{Cat}) = \mathrm{ob}\mathsf{Met}$, $v \circ\!\!- u = \begin{cases} v - u & \text{if } u \le v \text{ (in } [0,\infty]); \\ 0 & \text{else.} \end{cases}$

(Here $\infty - u = \infty$ for $u < \infty$, and $v - \infty = 0$ for all $v \in [0, \infty]$.)

(3) For $\mathbb{P}_{\max} \in \mathrm{ob}(\mathbb{P}_{\max}\text{-}\mathsf{Cat}) = \mathrm{ob}\mathsf{UMet}$, $u \multimap v = \begin{cases} v & \text{if } u < v; \\ 0 & \text{else.} \end{cases}$

1.6. $\mathcal{V}$-Cat as a symmetric monoidal closed category For $X = (X, a),\ Y = (Y, b)$ in $\mathcal{V}$-Cat put

$$\begin{aligned} X \otimes Y &= (X \times Y, a \otimes b), (a \otimes b)((x,y),(x',y')) = a(x,x') \otimes b(y,y'), \\ E &= (E, 1_E), withasingletonsetE \\ Y^X &= (\mathcal{V}\text{-}\mathsf{Cat}(X,Y), d),\ d(f,g) = \bigwedge_{x \in X} b(f(x), g(x)). \end{aligned}$$

Then $(\mathcal{V}\text{-}\mathsf{Cat}, \otimes, E)$ is a symmetric monoidal category, and there is a natural bijective correspondence of $\mathcal{V}$-functors,

$$\frac{Z \xrightarrow{\ t\ } Y^X}{Z \otimes X \xrightarrow{\ h\ } Y}$$

given by $t(z)(x) = h(z, x)$ for all $x \in X, z \in Z$. For that, with $Z = (Z, c)$, one must verify the equivalence of

(i) $c(z, z') \le b\big(t(z)(x), t(z')(x)\big)$ for all $x \in X$,

(ii) $c(z, z') \otimes a(x, x') \le b\big(h(z, x), h(z', x')\big)$ for all $x, x' \in X$, whenever $z, z' \in Z$. In fact, using $\mathcal{V}$-functoriality of $t(z')$, one obtains from (i)

$$\begin{aligned} c(z,z') \otimes a(x,x') &\leq b\big(h(z,x), h(z',x)\big) \otimes b\big(h(z',x), h(z',x')\big) \\ &\leq b\big(h(z,x), h(z',x')\big). \end{aligned}$$

Conversely, (ii) implies (i) when one specializes $x' = x$.

The endofunctor $(-)^{\mathrm{op}} : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ preserves the monoidal structure strictly: $(X \otimes Y)^{\mathrm{op}} = X^{\mathrm{op}} \otimes Y^{\mathrm{op}}, E^{\mathrm{op}} = E$, while

$$(Y^X)^{\mathrm{op}} \cong (Y^{\mathrm{op}})^{(X^{\mathrm{op}})}.$$

1.7. $\mathcal{V}$-modules For $\mathcal{V}$ categories $X = (X,a), Y = (Y,b)$, a $\mathcal{V}$-relation $\varphi : X \nrightarrow Y$ is a $\mathcal{V}$-*module* (also: bimodule, profunctor or distributor) if $\varphi \cdot a \leq \varphi$ and $b \cdot \varphi \leq \varphi$; that is, if

$$a(x,x') \otimes \varphi(x',y) \leq \varphi(x,y) \text{ and } \varphi(x,y') \otimes b(y',y) \leq \varphi(x,y)$$

for all $x, x' \in X,\ y, y' \in Y$. Since always $\varphi = \varphi \cdot 1_X \leq \varphi \cdot a$ and $\varphi = 1_Y \cdot \varphi \leq b \cdot \varphi$, one actually has

$$\varphi \cdot a = \varphi \text{ and } b \cdot \varphi = \varphi \tag{*}$$

for a $\mathcal{V}$-module φ; we write $\varphi : X \nrightarrow\!\!\circ\, Y$ in this case. In fact, $\mathcal{V}$-modules are closed under $\mathcal{V}$-relational composition, and (*) shows that the structure a serves as the identity morphism on $X = (X,a)$. This defines the category

$$\mathcal{V}\text{-}\mathsf{Mod},$$

which is (like $\mathcal{V}$-Rel and $\mathcal{V}$-Cat) Ord-enriched, or a "thin" 2-category.

There is a full embedding

$$\mathcal{V}\text{-}\mathsf{Rel} \hookrightarrow \mathcal{V}\text{-}\mathsf{Mod},\ X \mapsto (X, 1_X)$$

and, more importantly, there are functors

$$(-)_* : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Mod}, \quad (-)^* : \big(\mathcal{V}\text{-}\mathsf{Cat}\big)^{\mathrm{op}} \longrightarrow \mathcal{V}\text{-}\mathsf{Mod}$$

which map objects identically and send $f : X \longrightarrow Y$ respectively to

$$\begin{aligned} &f_* : X \nrightarrow\!\!\circ\, Y \quad \text{with} \quad f_* = b \cdot f, \text{ hence: } f_*(x,y) = b(f(x), y), \\ &f^* : Y \nrightarrow\!\!\circ\, X \quad \text{with} \quad f^* = f^\circ \cdot b, \text{ hence: } f^*(y,x) = b(y, f(x)), \end{aligned}$$

for all $x \in X,\ y \in Y$. We can leave all verifications to the reader, but let us point out that for $X = (X,a)$ it makes sense to continue to denote the identity morphism on X in $\mathcal{V}$-Rel by 1_X while $1_X^* = (1_X)_* = a$ is the identity morphism on X in $\mathcal{V}$-Mod. We also show $f_* \dashv f^*$ for $f : X \longrightarrow Y$ in $\mathcal{V}$-Cat:

$$\begin{aligned} f_* \cdot f^* &= b \cdot f \cdot f^\circ \cdot b \leq b \cdot 1_Y \cdot b \leq b = 1_Y^*, \\ 1_X^* &= a \leq 1_X \cdot a \cdot a \leq f^\circ \cdot f \cdot a \cdot a \leq f^\circ \cdot b \cdot f \cdot a \leq f^\circ \cdot b \cdot b \cdot f = f^* \cdot f_*. \end{aligned}$$

There are commutative diagrams

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow{(-)_*} & \mathcal{V}\text{-}\mathsf{Mod} \\ \uparrow & & \uparrow \\ \mathsf{Set} & \longrightarrow & \mathcal{V}\text{-}\mathsf{Rel} \end{array} \qquad \begin{array}{ccc} (\mathcal{V}\text{-}\mathsf{Cat})^{\mathrm{op}} & \xrightarrow{(-)^*} & \mathcal{V}\text{-}\mathsf{Mod} \\ \uparrow & & \uparrow \\ \mathsf{Set}^{\mathrm{op}} & \xrightarrow{(-)^\circ} & \mathcal{V}\text{-}\mathsf{Rel}, \end{array}$$

with $\mathcal{V}$-Rel $\hookrightarrow$ $\mathcal{V}$-Mod preserving adjointness.

The (covariant) endofunctor $(-)^{\mathrm{op}}$ of $\mathcal{V}$-Cat may be extended to

$$(-)^{\mathrm{op}} : (\mathcal{V}\text{-}\mathsf{Mod})^{\mathrm{op}} \longrightarrow \mathcal{V}\text{-}\mathsf{Mod}, \ \ (\varphi : X \multimap\!\!\!\!\!\rightarrow Y) \mapsto (\varphi^\circ : Y^{\mathrm{op}} \multimap\!\!\!\!\!\rightarrow X^{\mathrm{op}}),$$

in the sense that it makes the following diagram commute:

$$\begin{array}{ccc} (\mathcal{V}\text{-}\mathsf{Mod})^{\mathrm{op}} & \xrightarrow{(-)^{\mathrm{op}}} & \mathcal{V}\text{-}\mathsf{Mod} \\ {\scriptstyle (-)^*}\uparrow & & \uparrow{\scriptstyle (-)_*} \\ \mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow{(-)^{\mathrm{op}}} & \mathcal{V}\text{-}\mathsf{Cat} \end{array}$$

1.8. Proposition *For $\mathcal{V}$-categories X, Y and a $\mathcal{V}$-relation $\varphi : X \to\!\!\!\!\!\!+\ Y$, one has*

$$\varphi : X \multimap\!\!\!\!\!\rightarrow Y \ \textit{in}\ \mathcal{V}\text{-}\mathsf{Mod} \iff \varphi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathcal{V} \ \textit{in}\ \mathcal{V}\text{-}\mathsf{Cat}.$$

Proof. "⇒" For $X = (X, a)$, $Y = (Y, b)$ one obtains from $\varphi \cdot a \leq \varphi$ and $b \cdot \varphi \leq \varphi$:

$$\varphi(x, y) \otimes a^\circ(x, x') \otimes b(y, y') = a(x', x) \otimes \varphi(x, y) \otimes b(y, y') \leq \varphi(x', y'),$$

hence,

$$(a^\circ \otimes b)((x, y), (x', y')) \leq \varphi(x, y) \multimap \varphi(x', y') \qquad (\dagger)$$

for all $x, x' \in X,\ y, y' \in Y$, as desired.

"⇐" From $k \leq b(y, y)$ and (†) one concludes

$$a(x, x') \otimes \varphi(x', y) \leq \varphi(x', y) \otimes a^\circ(x', x) \otimes b(y, y) \leq \varphi(x, y)$$

for all $x, x' \in X,\ y \in Y$, hence $\varphi \cdot a \leq \varphi$. Likewise, $b \cdot \varphi \leq \varphi$. □

A $\mathcal{V}$-functor $f : X \longrightarrow Y$ is *fully faithful* if it satisfies the following equivalent conditions:

(i) $f^* \cdot f_* = 1_X^*$,
(ii) $f^* \cdot f_* \leq 1_X^*$,
(iii) $f^\circ \cdot b \cdot f \leq a$,
(iv) $b(f(x), f(x')) \leq a(x, x')$ for all $x, x' \in X$.

f is a *full embedding* when it is fully faithful and, as a function, injective. Every subset $M \subseteq X$ becomes a $\mathcal{V}$-category such that $M \hookrightarrow X$ is a full embedding, by simply restricting the structure $a : X \times X \longrightarrow \mathcal{V}$ to $M \times M$.

1.9. The Yoneda $\mathcal{V}$-functor, the Yoneda Lemma By 1.8, for a $\mathcal{V}$-category $X = (X, a)$, the $\mathcal{V}$-module $1_X^* : X \nrightarrow X$ gives rise to a $\mathcal{V}$-functor $a : X^{\mathrm{op}} \otimes X \longrightarrow \mathcal{V}$ which, by 1.6, corresponds to the *Yoneda $\mathcal{V}$-functor*

$$\begin{aligned} y : X &\longrightarrow \hat{X} := \mathcal{V}^{X^{\mathrm{op}}} \\ x &\mapsto \big(y(x) = a(-, x) : X^{\mathrm{op}} \longrightarrow \mathcal{V}\big). \end{aligned}$$

Denoting the structure of $\hat{X}$ by $\hat{a}$, for all $f \in \hat{X}$ and $x, y \in X$ one has:

$$\begin{aligned} a(x, y) &\leq f(y) \multimap f(x) \\ \Longrightarrow f(y) &\leq a(x, y) \multimap f(x) \\ \Longrightarrow f(y) &\leq \hat{a}(a(-, y), f) \leq a(y, y) \multimap f(y) \leq k \multimap f(y) = f(y). \end{aligned}$$

Hence, we have proved the *Yoneda Lemma*:

$$\hat{a}\big(y(y), f\big) = f(y)$$

which, when exploited in case $f = y(x)$, gives: *y is fully faithful.*

1.10. Examples

(1) $\mathcal{V} = 2$: $\hat{X}$ may be identified with

$$P_{\downarrow} X = \{A \subseteq X | \downarrow A = A\},$$

the set of down-closed subsets on X (ordered by inclusion), and y is then the down-map $x \mapsto \downarrow x = \{y \in X \mid y \leq x\}$.

(2) $\mathcal{V} = \mathbb{P}_+$: $\hat{X}$ carries the (non-symmetric) sup-metric

$$\hat{a}(f, g) = \sup\{g(x) - f(x) \mid x \in X, f(x) \leq g(x)\}.$$

(Here "sup" refers to the natural order of $[0, \infty]$.)

(3) $\mathcal{V} = \mathbb{P}_{\max}$: $\hat{X}$ carries the metric

$$\hat{a}(f, g) = \sup\{g(x) \mid x \in X, f(x) < g(x)\}.$$

1.11. L-separation and L-completeness The functor $(-)_*$ of 1.7 induces, for all $X, Z \in \mathrm{ob}\mathcal{V}\text{-}\mathsf{Cat}$, a mapping

$$\alpha_{Z,X} : \mathcal{V}\text{-}\mathsf{Cat}(Z, X) \longrightarrow (\mathcal{V}\text{-Map})(Z, X), \quad f \mapsto f_*,$$

where the codomain is the set of all left-adjoint modules $Z \nrightarrow X$. We call X *L-separated* if $\alpha_{Z,X}$ is injective for all Z, and *L-complete* (complete in the sense of Lawvere) if $\alpha_{Z,X}$ is surjective for all Z. Hence, L-completeness of X means that maps (in the 2-category $\mathcal{V}$-Mod) into X are represented by $\mathcal{V}$-functors, and L-separation means that such a representation (whenever it exists) must necessarily be unique.

First we prove that, for L-separation, it suffices to consider the case $Z = E$ (see 1.6). Note that every $x \in X$ corresponds to a $\mathcal{V}$-functor $x : E \longrightarrow X$.

1.12. Proposition *The following are equivalent for a $\mathcal{V}$-category X:*

(i) *X is L-separated;*

(ii) *for all $x, y, \in X$, $x_* = y_*$ implies $x = y$;*

(iii) *for all* $x, y \in X$, $x^* = y^*$ *implies* $x = y$;
(iv) *for all* $x, y \in X$, $a(x, y) \geq k$ *and* $a(y, x) \geq k$ *imply* $x = y$;
(v) *the function* $y : X \longrightarrow \hat{X}$ *is injective;*
(vi) $y : X \longrightarrow \hat{X}$ *is a full embedding.*

Proof. (i)⇒(ii) is trivial. (ii)⇒(i): From $f_* = g_*$ with $f, g : Z \longrightarrow X$ in $\mathcal{V}$-Cat one obtains

$$(f \cdot x)_* = f_* \cdot x_* = g_* \cdot x_* = (g \cdot x)_*$$

and then $f(x) = g(x)$ for all $x \in X$, by hypothesis. (ii)⇒(iii): Adjoints determine each other uniquely. (iii)⇒(v): Under the correspondence

$$\frac{X^{\mathrm{op}} \cong X^{\mathrm{op}} \otimes E \longrightarrow \mathcal{V}}{X \;{-\!\!\circ\!\!\!\rightarrow}\; E}$$

of 1.8, $y(x) = a(-, x)$ corresponds to $x^* = x^\circ \cdot a$ $\big($with $x \in X = (X, a)\big)$. (iv)⇔(v): From $a(z, x) = a(z, y)$ for all $z \in X$ one trivially obtains $k \leq a(x, y)$ and $k \leq a(y, x)$. Vice versa, from $k \leq a(x, y)$ one derives

$$a(z, x) \leq a(z, x) \otimes a(x, y) \leq a(z, y)$$

for all z, and symmetrically $a(z, y) \leq a(z, x)$. (v)⇔(vi) follows from 1.9. □

Examples. (1) $\mathcal{V} = 2$: X is L-separated iff its order it antisymmetric.
(2) $\mathcal{V} = \mathbb{P}_+$ or $\mathbb{P}_{\max}$: $X = (X, a)$ is L-separated iff $a(x, y) = 0 = a(y, x)$ implies $x = y$, for all $x, y \in X$.

1.13. Corollary *$\mathcal{V}$ is L-separated. For all $\mathcal{V}$-categories X, Y, the $\mathcal{V}$-category Y^X is L-separated; in particular $\hat{X}$ is L-separated.*

Proof. $k \leq u \multimap v$ and $k \leq v \multimap u$ means $u \leq v$ and $v \leq u$ in $\mathcal{V}$, hence $u = v$. For $Y = (Y, b)$, $k \leq c(f, g)$ in Y^X means $k \leq b(f(x), g(x))$ for all $x \in X$, which makes the second assertion trivial, and the third follows from the previous two. □

Also for L-completeness one can restrict oneself to the case $Z = E$ in 1.11, but the Axiom of Choice is needed to prove equivalence, as we show next. We could avoid AC if we would restrict ourselves in 1.11 to $Z = E$; everything that follows would remain valid.

1.14. Proposition *The following are equivalent for a $\mathcal{V}$-category X:*
(i) *X is L-complete;*
(ii) *every left-adjoint $\mathcal{V}$-module $\varphi : E \;{-\!\!\circ\!\!\!\rightarrow}\; X$ is of the form $\varphi = x_*$, for some $x \in X$.*
(iii) *every right-adjoint $\mathcal{V}$-module $\psi : X \;{-\!\!\circ\!\!\!\rightarrow}\; E$ is of the form $\psi = x^*$, for some $x \in X$.*

Proof. It suffices to show (ii)⇒(i) and (iii)⇒(i). Hence, we consider

$$\varphi \dashv \psi : X \;{-\!\!\circ\!\!\!\rightarrow}\; Z$$

in $\mathcal{V}$-Mod with $Z = (Z, c)$ and obtain, for all $z \in Z$, the composite adjunction

$$\varphi \cdot z_* \dashv z^* \cdot \psi : X \to\!\!\!\!\circ\; E$$

which, by hypothesis, is represented by some $f(z) \in X$: $(f \cdot z)_* = \varphi \cdot z_*$. This defines a mapping $f : Z \longrightarrow X$ with $a \cdot f \cdot z = \varphi \cdot c \cdot z$ for all $z \in Z$ which implies

$$\varphi = \varphi \cdot c = a \cdot f \text{ and } \psi = c \cdot \psi = f^\circ \cdot a.$$

From $c \leq \psi \cdot \varphi$ one now obtains

$$f \cdot c \leq f \cdot \psi \cdot \varphi = f \cdot f^\circ \cdot a \cdot \varphi \leq a \cdot \varphi = a \cdot a \cdot f \leq a \cdot f,$$

so that $f : Z \longrightarrow X$ is indeed a $\mathcal{V}$-functor. This proves (ii)⇒(i), and (iii)⇒(i) follows analogously. □

1.15. Tight $\mathcal{V}$-forms Adjoint $\mathcal{V}$-modules $\varphi \dashv \psi : X \to\!\!\!\!\circ\; E$ correspond to $\mathcal{V}$-functors

$$\begin{array}{rcl} j & = & \varphi(*, -) : X \cong E^{\mathrm{op}} \otimes X \longrightarrow \mathcal{V} \\ h & = & \psi(-, *) : X^{\mathrm{op}} \cong X^{\mathrm{op}} \otimes E \longrightarrow \mathcal{V} \end{array}$$

(see proof of 1.12). The adjointness condition $\varphi \cdot \psi \leq 1_X^*$ and $1_E^* \leq \psi \cdot \varphi$ may be equivalently expressed in terms of j and h as

$$h(x) \otimes j(y) \leq a(x, y) \text{ and } k \leq \bigvee_{y \in X} h(y) \otimes j(y)$$

$(x, y \in X = (X, a))$. The first inequality implies

$$j(y) \leq \bigwedge_{x \in X} \big(h(x) \multimap a(x, y)\big)$$

so that with the second inequality we obtain

$$k \leq \bigvee_{y \in X} \big(h(y) \otimes \bigwedge_{x \in X} (h(x) \multimap a(x, y))\big) \tag{*}$$

A $\mathcal{V}$-functor $h : X^{\mathrm{op}} \longrightarrow \mathcal{V}$ is a *tight $\mathcal{V}$-form of X* if it satisfies (*).

Examples. (1) $\mathcal{V} = 2$: $h : X^{\mathrm{op}} \longrightarrow 2$ describes a down-closed set A in X, and (*) reads as

$$(\exists y \in A)(\forall x \in A)\ x \leq y,$$

so that $A = \downarrow y$.

(2) $\mathcal{V} = \mathbb{P}_+$: a tight $\mathbb{P}_+$-form on (X, a) is a function $h : X \longrightarrow [0, \infty]$ with

$$h(y) \leq h(x) \implies h(x) - h(y) \leq a(x, y) \ \ (x, y \in X),$$
$$\inf_{y \in X} \Big(h(y) + \sup_{x \in X : h(x) \leq a(x,y)} \big(a(x, y) - h(x)\big)\Big) = 0.$$

(3) $\mathcal{V} = \mathbb{P}_{\max}$: A tight $\mathbb{P}_{\max}$-form on (X, a) must satisfy

$$h(y) < h(x) \implies h(x) \leq a(x, y) \ \ (x, y \in X),$$
$$\inf_{y \in X} \Big(\max\big(h(y), \sup_{x \in X : h(x) < a(x,y)} a(x, y)\big)\Big) = 0.$$

We put

$$\tilde{X} := \{h \in \hat{X} \mid h \text{ tight}\},$$

considered as a full $\mathcal{V}$-subcategory of $\hat{X} = \mathcal{V}^{X^{\mathrm{op}}}$. The Yoneda $\mathcal{V}$-functor takes values in $\tilde{X}$, because $\mathsf{y}(x) = a(-,x)$ corresponds to the right-adjoint $\mathcal{V}$-module x^*, and (as we just saw) adjointness implies tightness.

1.16. Theorem *A $\mathcal{V}$-category $X = (X,a)$ is L-complete if, and only if the underlying map of $\mathsf{y} : X \longrightarrow \tilde{X}$ is surjective, that is: if every tight $\mathcal{V}$-form on X has the form $a(-,x)$ for some $x \in X$.*

Proof. According to 1.14 it suffices to show that tight $\mathcal{V}$-forms h on X correspond bijectively to right-adjoint $\mathcal{V}$-modules $\psi : X \,-\!\!\!\!\circ\!\!\!\to E$. In fact, after 1.15, it suffices to show that the $\mathcal{V}$-module ψ with $\psi(x,*) = h(x)$ $(x \in X)$ corresponding to $h : X^{\mathrm{op}} \longrightarrow \mathcal{V}$ has a left-adjoint φ. Hence, we define a $\mathcal{V}$-relation φ by

$$\varphi(*,y) = j(y) := \bigwedge_{x \in X} \big(h(x) \multimap a(x,y)\big)$$

and obtain $k \le \psi \cdot \varphi$ from the hypothesis of tightness of h. The definition gives also $h(x) \otimes j(y) \le a(x,y)$ $(x,y \in X)$, eg. $\varphi \cdot \psi \le a$. It remains to be shown that j is a $\mathcal{V}$-functor. But from

$$h(z) \otimes \big(h(z) \multimap a(z,x)\big) \otimes a(x,y) \le a(z,x) \otimes a(x,y) \le a(z,y)$$

one obtains

$$j(x) \otimes a(x,y) \le \big(h(z) \multimap a(z,x)\big) \otimes a(x,y) \le \big(h(z) \multimap a(z,y)\big)$$

for all $x,y,z \in X$, hence

$$j(x) \otimes a(x,y) \le j(y) \text{ and } a(x,y) \le \big(j(x) \multimap j(y)\big)$$

for all $x,y \in X$, as desired. □

1.17. Proposition *For a $\mathcal{V}$-category (X,a) and a $\mathcal{V}$-functor $H : \hat{X}^{\mathrm{op}} \longrightarrow \mathcal{V}$, let*

$$h := \big(\, X^{\mathrm{op}} \xrightarrow{\mathsf{y}^{\mathrm{op}}} \hat{X}^{\mathrm{op}} \xrightarrow{H} \mathcal{V} \,\big).$$

Then $H = \hat{a}(-,h)$ if, and only if, $k \le H(h)$.

Proof. $H = \hat{a}(-,h)$ trivially implies $k \le \hat{a}(h,h) = H(h)$. Conversely, as a $\mathcal{V}$-functor H satisfies

$$\begin{aligned} \hat{a}(f,h) &\le H(h) \multimap H(f) \\ &\le k \multimap H(f) \quad (\text{ since } k \le H(h)) \\ &\le H(f) \end{aligned}$$

for all $f \in \hat{X}$. For "$\ge$", from the Yoneda Lemma(1.9) one obtains

$$f(y) = \hat{a}(\mathsf{y}(y), f) \le \big(H(f) \multimap H(\mathsf{y}(y))\big);$$

equivalently,

$$H(f) \leq f(y) \multimap h(y)$$

for all $y \in X$, and therefore $H(f) \leq \hat{a}(f, h)$. □

1.18. Corollary *$\mathcal{V}$ is L-complete*

Proof. From 1.17, applied to $X = E$, one obtains that a $\mathcal{V}$-functor $H : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{V}$ satisfies $H = \big((-) \multimap H(k)\big)$ whenever $k \leq H(H(k))$. According to 1.16, it suffices to verify the latter inequality when H is tight. In fact, one trivially has for all $v \in \mathcal{V}$

$$\bigwedge_{u\in\mathcal{V}} \big(H(u) \multimap (u \multimap v)\big) \leq H(k) \multimap (k \multimap v) = H(k) \multimap v.$$

Hence, with the tightness of H one has

$$\begin{aligned} k &\leq \bigvee_{v\in\mathcal{V}} \Big(H(v) \otimes \bigwedge_{u\in\mathcal{V}} \big(H(u) \multimap (u \multimap v)\big)\Big) \\ &\leq \bigvee_{v\in\mathcal{V}} \big(H(v) \otimes (H(k) \multimap v)\big) \\ &\leq H(H(k)), \end{aligned}$$

where the last inequality is obtained as follows: $w \multimap v \leq H(v) \multimap H(w)$ implies

$$\bigvee_{v\in\mathcal{V}} H(v) \otimes (w \multimap v) \leq H(w),$$

for all $w \in \mathcal{V}$, which yields the desired result with $w := H(k)$. □

We can now mimic the previous proof in a more general fashion:

1.19. Theorem *$\hat{X}$ is L-complete, for every $\mathcal{V}$-category X.*

Proof. According to 1.16, 1.17, it suffices to show that a tight $\mathcal{V}$-form on $\hat{X}$ satisfies $k \leq H(h)$, with $h = H\mathsf{y}^{\mathrm{op}}$. For all $g \in \hat{X}$ and $y \in X$ one has with 1.9

$$\begin{aligned} \bigwedge_{f\in\hat{X}} \big(H(f) \multimap \hat{a}(f, g)\big) &\leq H(\mathsf{y}(y)) \multimap \hat{a}(\mathsf{y}(y), g) \\ &\leq h(y) \multimap g(y), \end{aligned}$$

hence

$$\bigwedge_{f\in\hat{X}} \big(H(f) \multimap \hat{a}(f, g)\big) \leq \hat{a}(h, g).$$

With the tightness of H one concludes (as in the proof of 1.18):

$$\begin{aligned} k &\leq \bigvee_{g\in\hat{X}} \Big(H(g) \otimes \bigwedge_{f\in\hat{X}} \big(H(f) \multimap \hat{a}(f, g)\big)\Big) \\ &\leq \bigvee_{g\in\hat{X}} H(g) \otimes \hat{a}(h, g) \leq H(h). \end{aligned}$$

□

Remark. More generally, using the definition of L-completeness directly, one can show that Y^X is L-complete whenever Y is L-complete.

1.20. Examples

(1) We saw already in 1.15 that every tight 2-form on an ordered set X is of the form $y(y)$ for some $y \in X$. Hence, every ordered set is L-complete (in the sense of Lawvere). This initial disappointment is alleviated by the next standard example.

(2) Let (x_n) be a Cauchy sequence in $X = (X, a) \in \mathsf{Met}$. Then one can define $h : X^{\mathrm{op}} \longrightarrow \mathcal{V}$ and $j : X \longrightarrow \mathcal{V}$ by

$$h(y) = \lim_n a(y, x_n) \text{ and } j(y) = \lim_n a(x_n, y) \qquad (*)$$

and obtain $h(x) + j(y) \geq a(x, y)$ $(x, y \in X)$ and $\inf_y \big(h(y) + j(y)\big) = 0$. Hence, j and h represent an adjoint pair $\varphi \dashv \psi : X \,\circ\!\!\!\!\longrightarrow E$, as in 1.15. Conversely, given such j and h, one produces a Cauchy sequence (x_n), as follows: for every n, choose $x_n \in X$ with

$$h(x_n) + j(x_n) \leq \frac{1}{n}.$$

Since $a(x_n, x_m) \leq h(x_n) + j(x_m) \leq \frac{1}{n} + \frac{1}{m}$, (x_n) is indeed a Cauchy sequence such that the formulae (*) hold true. Consequently, a tight $\mathbb{P}_+$-form h corresponds to an equivalence class of Cauchy sequences, and any $x \in X$ representing such h as $h = a(-, x)$ corresponds to a limit of any Cauchy sequence (x_n) in that equivalence class:

$$0 = a(x, x) = h(x) = \lim_n a(x, x_n).$$

Hence, L-completeness of X means Cauchy completeness of X.

(3) Also in UMet L-completeness means Cauchy completeness.

1.21. L-dense $\mathcal{V}$-functors, L-closure Recall that a $\mathcal{V}$-functor $f : X \longrightarrow Y$ is fully faithful if $f^* \cdot f_* = 1_X^*$. One calls f *L-dense* if $f_* \cdot f^* = 1_Y^*$. L-dense $\mathcal{V}$-functors have good composition-cancellation properties. Let $g : Y \longrightarrow Z$ be a $\mathcal{V}$-functor; then:

(1) f, g L-dense $\Longrightarrow$ $g \cdot f$ L-dense;
(2) $g \cdot f$ L-dense $\Longrightarrow$ g L-dense;
(3) $g \cdot f$ L-dense, g fully faithful $\Longrightarrow$ f L-dense;
(4) $g \cdot f$ fully faithful, f L-dense $\Longrightarrow$ g fully faithful.

Moreover: every surjective $\mathcal{V}$-functor is L-dense, since $1_Y \leq f \cdot f^\circ$ implies $b \leq b \cdot b \leq b \cdot f \cdot f^\circ \cdot b$, hence $1_Y^* \leq f_* \cdot f^*$.

For $M \subseteq X$ we define the *L-closure $\overline{M}$ of M in X* by

$$\overline{M} = \bigcup \{N \subseteq X \mid M \hookrightarrow N \quad L\text{-dense}\};$$

of course, here we consider every subset $N \subseteq X$ as a $\mathcal{V}$-category such that $N \hookrightarrow X$ is fully faithful, see 1.8.

Considering the factorization $f = \big(X \longrightarrow f(X) \hookrightarrow Y\big)$ one sees immediately that, when f is L-dense if $\overline{f(X)} = Y$, and the converse statement is also true. For that one uses the following Proposition, in order to first show that for all $M \subseteq X$, $M \hookrightarrow \overline{M}$ is L-dense, and if $M \subseteq Y \subseteq X$, then $\overline{M}^Y = \overline{M}^X \cap Y$.

1.22. Proposition *For a $\mathcal{V}$-category $X = (X, a)$, $M \subseteq X$ and $x \in X$, the following conditions are equivalent:*

(*i*) $x \in \overline{M}$;
(*ii*) $a(x,x) \leq \bigvee_{z\in M} a(x,z) \otimes a(z,x)$;
(*iii*) $x^* \cdot x_* = x^* \cdot m_* \cdot m^* \cdot x_*$ (*where m denotes the full embedding $m : M \hookrightarrow X$*);
(*iv*) $k \leq \bigvee_{z\in M} a(x,z) \otimes a(z,x)$;
(*v*) $1_E^* \leq x^* \cdot m_* \cdot m^* \cdot x_*$;
(*vi*) $m^* \cdot x_* \dashv x^* \cdot m_*$;
(*vii*) *$x_* : E \,-\!\!\circ\!\!\to X$ factors through $m_* : M \,-\!\!\circ\!\!\to X$ by a map $\varphi : E \,-\!\!\circ\!\!\to M$ in* $\mathcal{V}$-Mod.

Proof. (*i*) $\Rightarrow$ (*ii*): For any $N \subseteq X$ with $M \subseteq N$, $x \in N$, L-density of $j : M \hookrightarrow N$ means $1_N^* \leq j_* \cdot j^*$, that is:

$$a(y,t) \leq \bigvee_{z\in M} a(y,z) \otimes a(z,t) \tag{\dagger}$$

for all $y, t \in N$. Putting $y = t = x$ shows (*i*) $\Longrightarrow$ (*ii*), while (*ii*) $\Longrightarrow$ (*i*) follows when one considers $N = M \cup \{x\}$, because (†) holds trivially when $y \in M$ or $t \in M$. (*ii*) $\Longleftrightarrow$ (*iii*): Note that in (*ii*), "$\geq$" always holds, so that one can put equivalently "$=$". Now (*iii*) is a mere translation of that equality:

$$x^* \cdot x_* = x^\circ \cdot a \cdot a \cdot x = x^\circ \cdot a \cdot x = (x^\circ \cdot a \cdot m) \cdot (m^\circ \cdot a \cdot x) = x^* \cdot m_* \cdot m^* \cdot x_*.$$

The equivalence (*iv*) $\Longleftrightarrow$ (*v*) follows similarly, and (*ii*) $\Longrightarrow$ (*iv*) is trivial. For (*iv*) $\Longrightarrow$ (*ii*) we compute

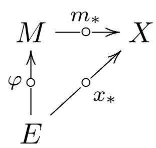
$$\begin{aligned} a(x,x) &= a(x,x) \otimes k \\ &\leq \bigvee_{z\in M} a(x,x) \otimes a(x,z) \otimes a(z,x) \qquad (\text{since } \otimes \text{ preserves } \textstyle\bigvee) \\ &\leq \bigvee_{z\in M} a(x,z) \otimes a(z,x). \end{aligned}$$

Since $(m^* \cdot x_*) \cdot (x^* \cdot m_*) \leq m^* \cdot m_* = 1_M^*$ comes for free, (*v*) $\Longleftrightarrow$ (*vi*) is trivial. (*vi*) $\Longrightarrow$ (*vii*): $1_E^* \leq (x^* \cdot m_*) \cdot (m^* \cdot x_*)$ implies

$$x_* \leq x_* \cdot x^* \cdot m_* \cdot m^* \cdot x_* \leq m_* \cdot m^* \cdot x_* \leq x_*,$$

hence $m_* \cdot (m^* \cdot x_*) = x_*$.

$(vii) \implies (vi)$: Assuming that



commutes, with $\varphi \dashv \psi : M \nrightarrow E$, one has $m_* \cdot \varphi = x_*$ and, necessarily, $\psi \cdot m^* = x^*$. Since $m^* \cdot m_* = 1_M^*$, $\varphi = m^* \cdot x_*$ and $\psi = x^* \cdot m_*$ follows. □

1.23. Proposition *For a $\mathcal{V}$-functor $f : X \longrightarrow Y$ and $M, M' \subseteq X$, $N \subseteq Y$, one has:*

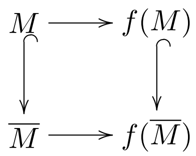
1. *$M \subseteq \overline{M}$, and $M \subseteq M'$ implies $\overline{M} \subseteq \overline{M'}$;*
2. *$\overline{\emptyset} = \emptyset$ and $\overline{\overline{M}} = \overline{M}$;*
3. *$f(\overline{M}) \subseteq \overline{f(M)}$ and $\overline{f^{-1}(N)} \subseteq f^{-1}(\overline{N})$;*
4. *if k is $\vee$-irreducible (so that $k \leq u \vee v$ implies $k \leq u$ or $k \leq v$), then $\overline{M \cup M'} = \overline{M} \cup \overline{M'}$.*

Proof. (1) is trivial. (2) The existence of $x \in \overline{\emptyset}$ would mean

$$\bot < k \leq a(x,x) \leq \bigvee_{z \in \emptyset} a(x,z) \otimes a(z,x) = \bot$$

which is impossible. Hence, $\overline{\emptyset} = \emptyset$. Furthermore, as a composite of L-dense $\mathcal{V}$-functors, $M \hookrightarrow \overline{M} \hookrightarrow \overline{\overline{M}}$ is L-dense, hence $\overline{\overline{M}} \subseteq \overline{M}$.

(3) Applying composition-cancellation rules to



one sees that $f(M) \hookrightarrow f(\overline{M})$ is L-dense, hence $f(\overline{M}) \subseteq \overline{f(M)}$. With $M = f^{-1}(N)$, this implies $\overline{f^{-1}(N)} \subseteq f^{-1}(\overline{N})$. (4) Trivially, $\overline{M} \cup \overline{M'} \subseteq \overline{M \cup M'}$. Conversely, for $x \in \overline{M \cup M'}$ one has

$$k \leq \bigvee_{z \in M \cup M'} a(x,z) \otimes a(z,x) = \big(\bigvee_{z \in M} a(x,z) \otimes a(z,x)\big) \vee \big(\bigvee_{z \in M'} a(x,z) \otimes a(z,x)\big),$$

so that with the given condition on $\mathcal{V}$ one can conclude $x \in \overline{M}$ or $x \in \overline{M'}$. □

1.24. Corollary *If k is $\vee$-irreducible in $\mathcal{V}$, then the L-closure operation defines a topology on X such that every $\mathcal{V}$-functor becomes continuous. Hence, L-closure defines a functor $L : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Top}$.*

1.25. Examples

1. For $X = (X, \leq)$ in $2\text{-}\mathsf{Cat} = \mathsf{Ord}$ and $M \subseteq X$, one has $x \in \overline{M}$ precisely when $x \leq z \leq x$ for some $z \in M$. Also, $M \subseteq X$ is open in LX if every $x \in M$ satisfies

$$\forall z \in X \ \ (x \leq z \leq x \implies z \in M).$$

(2) In Met, $\overline{M} = \{x \in X = (X,a) \mid \inf_{z\in M}\big(a(x,z)+a(z,x)\big) = 0\}$, and in UMet

$$\overline{M} = \{x \in X \mid \inf_{z\in M}\big(\max(a(x,z),a(z,x))\big) = 0\}$$

which, for a symmetric (ultra)metric space, describes the ordinary topological closure.

1.26. Proposition *For a $\mathcal{V}$-category X the following assertions are equivalent:*

- (i) *X is L-separated;*
- (ii) *for all $j : Z \longrightarrow Y$, $f,g : Y \longrightarrow X$ in $\mathcal{V}$-Cat with j L-dense and $f\cdot j = g\cdot j$, one has $f = g$;*
- (iii) *$\Delta = \{(x,x)|x \in X\}$ is L-closed in the direct product $X \times X$.*

Proof. $(i) \Rightarrow (ii)$: $f\cdot j = g\cdot j$ implies $f_* = f_*\cdot j_*\cdot j^* = g_*\cdot j_*\cdot j^* = g_*$ when j is L-dense, and then $f = g$ when X is L-separated.
$(ii) \Rightarrow (iii)$: Consider $j : \Delta \hookrightarrow \overline{\Delta}$ and the projections $p_1,p_2 : \overline{\Delta} \longrightarrow X$ which trivially satisfy $p_1\cdot j = p_2\cdot j$, hence $p_1 = p_2$ by hypothesis, which is possible only if $\overline{\Delta} = \Delta$.
$(iii) \Rightarrow (i)$: First note that the structure $\tilde{a}$ of the direct product $X \times X$ is given by

$$\tilde{a}((x,y),(x',y')) = a(x,x') \wedge a(y,y').$$

Assume $x_* = y_*$ for $x,y \in X$, hence also $x^* = y^*$. Consequently, for $z \in X$, $a(x,z) = a(y,z)$ and $a(z,x) = a(z,y)$, and we obtain with $a = a\cdot a$:

$$\begin{aligned}\tilde{a}((x,y),(x,y)) &= a(x,x)\wedge a(y,y) = a(x,y) = \bigvee_{z\in X} a(x,z)\otimes a(z,y)\\ &= \bigvee_{z\in X}(a(x,z)\wedge a(y,z))\otimes(a(z,x)\wedge a(z,y))\\ &= \bigvee_{z\in X}\tilde{a}((x,y),(z,z))\otimes\tilde{a}((z,z),(x,y)).\end{aligned}$$

With 1.22 $(ii) \Rightarrow (i)$ we conclude $(x,y) \in \overline{\Delta} = \Delta$, hence $x = y$, so that X is L-separated by 1.12. □

1.27. Proposition *The full subcategory $\mathcal{V}$-Cat$_{\mathrm{sep}}$ of L-separated $\mathcal{V}$-categories is epireflective in $\mathcal{V}$-Cat. Hence, limits of L-separated $\mathcal{V}$-categories are formed in $\mathcal{V}$-Cat, while colimits are obtained by reflecting the colimit formed in $\mathcal{V}$-Cat. The epimorphisms in $\mathcal{V}$-Cat$_{\mathrm{sep}}$ are precisely the L-dense $\mathcal{V}$-functors.*

Proof. Consider the equivalence relation

$$x \sim y :\iff x_* = y_* \iff x^* = y^*$$

on $X \in \mathcal{V}$-Cat. With $p : X \longrightarrow X/{\sim}$ denoting the projection, $X/{\sim}$ becomes a $\mathcal{V}$-category via

$$c(p(x),p(y)) := a(x,y).$$

The structure $c = p \cdot a \cdot p^\circ$ on $X/\!\sim$ is in fact well-defined since, if $p(x) = p(u)$, $p(y) = p(v)$, then

$$a(x,y) = y^\circ \cdot a \cdot x = (y^\circ \cdot a) \cdot (a \cdot x) = y^* \cdot x_* = v^* \cdot u_* = a(u,v).$$

$X/\!\sim$ is L-separated since $(p \cdot x)_* = (p \cdot y)_*$ implies

$$x_* = a \cdot x = p^\circ \cdot p \cdot a \cdot p^\circ \cdot p \cdot x = p^\circ \cdot c \cdot p \cdot x = p^\circ \cdot (p \cdot x)_* = p^\circ \cdot (p \cdot y)_* = y_*,$$

which means $p \cdot x = p \cdot y$. Finally, any $\mathcal{V}$-functor $f : X \longrightarrow Y$ with Y L-separated factors uniquely through p via $g : X/\sim \longrightarrow Y$. Indeed, $x \sim y$ implies

$$(f \cdot x)_* = f_* \cdot x_* = f_* \cdot y_* = (f \cdot y)_*$$

and then$f \cdot x = f \cdot y$ since Y is L-separated.

L-dense $\mathcal{V}$-functors are certainly epic in $\mathcal{V}$-Cat, by 1.26. In order to show the converse statement, one factors f through the L-closure of its image. The non-trivial step now is to show that the cokernel pair of a full L-closed embedding into an L-separated $\mathcal{V}$-category, when formed in $\mathcal{V}$-Cat, is again L-separated. □

An alternative way of constructing the reflection morphism is to simply consider the image of X under $\mathcal{y} : X \longrightarrow \hat{X}$.

Recall that, for all $X \in \mathcal{V}$-Cat, $\mathcal{y} : X \longrightarrow \tilde{X}$ is an isomorphism in $\mathcal{V}$-Cat if, and only if, X is L-separated and L-complete (by 1.12 and 1.16).

1.28. Proposition *For a $\mathcal{V}$-category X, as a set $\tilde{X}$ (see 1.15) coincides with the L-closure of $\mathcal{y}(X)$ in $\hat{X} = \mathcal{V}^{X^{\mathrm{op}}}$. Hence, $\mathcal{y} : X \longrightarrow \tilde{X}$ is fully faithful and L-dense.*

Proof. By 1.22, a $\mathcal{V}$-functor $h : X^{\mathrm{op}} \longrightarrow \mathcal{V}$ lies in the L-closure of $\mathcal{y}(X)$ in $\hat{X}$ if, and only if,

$$k \leq \bigvee_{y \in X} \hat{a}(h, \mathcal{y}(y)) \otimes \hat{a}(\mathcal{y}(y), h)$$

Since $\hat{a}(\mathcal{y}(y), h) = h(y)$ by 1.9, this means precisely that h must be tight. □

1.29. Theorem *The following assertions are equivalent for a $\mathcal{V}$-category X:*

- (*i*) *X is L-separated and L-complete;*
- (*ii*) *X is an injective object in $\mathcal{V}$-Cat with respect to the fully faithful L-dense $\mathcal{V}$-functors;*
- (*iii*) *$\mathcal{y} : X \longrightarrow \tilde{X}$ has a retraction $F : \tilde{X} \longrightarrow X$ in $\mathcal{V}$-Cat*

Proof. (*i*) ⇒ (*ii*): Let $m : Y \longrightarrow Z$ in $\mathcal{V}$-Cat satisfy $m^* \cdot m_* = 1_Y^*$ and $m_* \cdot m^* = 1_Z^*$. Then, for $f : Y \longrightarrow X$ in $\mathcal{V}$-Cat one has the adjunction $f_* \cdot m^* \dashv m_* \cdot f^*$ in $\mathcal{V}$-Mod, for which there is $g : Z \longrightarrow X$ in $\mathcal{V}$-Cat with $f_* \cdot m^* = g_*$ when X is L-complete. Consequently, $f_* = f_* \cdot m^* \cdot m_* = g_* \cdot m_* = (g \cdot m)_*$, which implies $f = g \cdot m$ when X is L-separated. (*ii*) ⇒ (*iii*) follows trivially since $\mathcal{y} : X \longrightarrow \tilde{X}$ is fully faithful and L-dense.

(iii) ⇒ *(i)*: Let $F : \tilde{X} \longrightarrow X$ be $\mathcal{V}$-functor with $F \cdot y = 1_X$ Then X is L-separated by 1.12. We need to show $y \cdot F = 1_{\tilde{X}}$, that is: every tight $\mathcal{V}$-form h of X is represented as $h = y(F(h))$. For "≤" one uses 1.9 to obtain

$$h(y) = \hat{a}(y(y), h) \leq a\big(F(y(y)), F(h)\big) = a(y, F(h)) = y\big(F(h)\big)(y)$$

for all $y \in X$. For "≥" we first note that , similarly, one has

$$\hat{a}(h, y(y)) \leq a\big(F(h), F(y(y))\big) = a(F(h), y) \leq h(y) \multimap h(F(h)),$$

hence

$$h(y) \otimes \hat{a}(h, y(y)) \leq h(F(h))$$

for all $y \in X$. Since h is tight, one obtains

$$k \leq \bigvee_{y} h(y) \otimes \hat{a}(h, y(y)) \leq h(F(h))$$

and therefore

$$h(F(h)) \multimap h(y) \leq k \multimap h(y) \leq h(y)$$

for all $y \in X$. Finally, we conclude that

$$y(F(h))(y) = a(y, F(h)) \leq h(F(h)) \multimap h(y) \leq h(y),$$

for all $y \in X$, as desired. □

1.30. Corollary *Every $\mathcal{V}$-functor $f : X \longrightarrow Y$ with Y L-separated and L-complete admits a unique extension $g : \tilde{X} \longrightarrow Y$ with $g \cdot y = f$.*

Proof. Existence of g follows from 1.29 since, by 1.28, the fully faithful $\mathcal{V}$-functor $y : X \longrightarrow \tilde{X}$ is L-dense. Uniqueness follows with 1.26. □

1.31. Proposition *For an L-separated $\mathcal{V}$-category X, let $Y \subseteq X$ carry the full subobject structure of X. Then:*

1. *If Y is L-complete, then Y is L-closed in X.*
2. *If Y is L-closed in X and X L-complete, then Y is also L-complete.*

Proof. (1) By 1.29, the full L-dense embedding $i : Y \hookrightarrow \overline{Y}$ has a retraction $f : \overline{Y} \longrightarrow Y$. From $i \cdot f \cdot i = i$ one obtains $i \cdot f = 1_{\overline{Y}}$ with 1.26, since with X also $\overline{Y}$ is L-separated.
(2) The inclusion $Y \hookrightarrow X$ can be extended along $y : Y \longrightarrow \tilde{Y}$ to a $\mathcal{V}$-functor $g : \tilde{Y} \longrightarrow X$. Then, with 1.23, we obtain

$$g(\tilde{Y}) = g(\overline{y(Y)}) \subseteq \overline{g(y(Y))} = \overline{Y} = Y,$$

so that the assertion follows with 1.29. □

1.32. Theorem *The full subcategory $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{cpl}}$ of L-complete and L-separated $\mathcal{V}$-categories is reflective in $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sep}}$, hence in $\mathcal{V}$-Cat. Every fully faithful and L-dense $\mathcal{V}$-functor $j : X \longrightarrow Z$ with $Z \in \mathcal{V}$-$\mathsf{Cat}_{\mathrm{cpl}}$ serves as a reflection morphism for $X \in \mathcal{V}$-Cat. The $\mathcal{V}$-category $\mathcal{V}$ cogenerates both $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sep}}$ and $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{cpl}}$; in fact, a $\mathcal{V}$-category X is L-separated (and L-complete) if, and only if, it is fully embeddable (as an L-closed subobject) into some product of copies of $\mathcal{V}$ in $\mathcal{V}$-Cat.*

Proof. L-completeness of $\tilde{X}$ for every $X \in \mathcal{V}\text{-}\mathsf{Cat}$ follows from 1.19, 1.31 and 1.28, and $\tilde{X}$ is trivially L-separated. Hence, by 1.30, $y : X \longrightarrow \tilde{X}$ serves as a reflection into $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$, for every $X \in \mathcal{V}\text{-}\mathsf{Cat}$. In fact, by 1.26 and 1.29 y may be replaced by any fully-faithful L-dense $\mathcal{V}$-functor $j : X \longrightarrow Z$ with $Z \in \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$. Since $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$ is reflective in $\mathcal{V}\text{-}\mathsf{Cat}$, any product $\prod_I \mathcal{V}$ lies in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$, and so does any of its L-closed full subobjects, by 1.31. Conversely, for $X \in \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$ one has $X \cong \tilde{X}$ (see 1.28), and with Z denoting the L-closure of $\tilde{X}$ in $\prod_X \mathcal{V}$ one obtains an L-dense fully faithful $\mathcal{V}$-functor $j : \tilde{X} \longrightarrow Z$ with $Z \in \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$, which serves as a $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$-reflection for $\tilde{X}$ and must therefore be an isomorphism. Hence, X is L-closed in $\prod_X \mathcal{V}$. In the absence of L-completeness, X still embeds fully into $\prod_X \mathcal{V}$. □

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