Lectures on

Lax-Algebraic Methods in General Topology

Summer School in Categorical Methods in Algebra and Topology

Haute Bodeux (Belgium), June 3 – 10, 2007

Lecture 2 : $(\mathbb{T},\mathcal{V})$-categories, $(\mathbb{T},\mathcal{V})$-modules

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2.1. Some monads on Set and their lax extensions. For sets X, Y the natural bijection

$$\mathsf{Rel}(X,Y) \xrightarrow{\sim} \mathsf{Rel}(Y,X), \qquad r \longmapsto r^\circ ,$$

translates to

$$\mathsf{Set}(X,PY) \xrightarrow{\sim} \mathsf{Set}(Y,PX) = \mathsf{Set}^{\mathrm{op}}(PX,Y) ,$$

showing the self-adjointness of the contravariant powerset functor $P \vdash P^{\mathrm{op}}$, with

$$P : \mathsf{Set} \longrightarrow \mathsf{Set}^{\mathrm{op}}, \quad (f : X \longrightarrow Y) \longmapsto (Pf : B \longmapsto f^{-1}[B]).$$

The induced monad $\mathbb{P}^2 = (P^{\mathrm{op}}P, e, m)$ is given by

$$\begin{aligned} e_X &: X \longrightarrow P^2X, \ \ x \longmapsto \dot{x} \ \ \text{(the \textit{principal filter} on } x), \\ m_X &: P^2P^2X \longrightarrow P^2X, \ \ \mathfrak{X} \longmapsto \sum \mathfrak{X} \ \ \text{(the \textit{Kowalsky sum} of } \mathfrak{X}), \end{aligned}$$

with

$$\begin{aligned} A &\in \dot{x} \iff x \in A, \\ A &\in \sum \mathfrak{X} \iff A^{\#} \in \mathfrak{X} \\ \mathfrak{x} &\in A^{\#} \iff A \in \mathfrak{x}, \end{aligned}$$

for all $x \in X$, $A \subseteq X$, $\mathfrak{x} \subseteq PX$, $\mathfrak{X} \subseteq PPPX$. Submonads

$$\mathbb{I} \longrightarrow \beta \longrightarrow \mathbb{F} \longrightarrow \mathbb{P}^2$$

are given by

$$\begin{aligned} FX &= \{\mathfrak{x} \subseteq PX \mid \mathfrak{x} \text{ filter on } X\} \\ \beta X &= \{\mathfrak{x} \subseteq PX \mid \mathfrak{x} \text{ ultrafilter on } X\} \\ 1_{\mathsf{Set}}X &= X \cong \{\dot{x} \mid x \in X\}. \end{aligned}$$

These monads can be thought of as being induced by the adjunction

$$\mathcal{C}^{\mathrm{op}} \underset{P}{\overset{\mathcal{C}(-,2)}{\rightleftarrows}} \mathsf{Set}$$

with successively $\mathcal{C} = \mathsf{SLat}$ ($\wedge$-semilattices), Lat (lattices), Frm (frames).

For any Set-monad $\mathbb{T} = (T, e, m)$ the Barr extension $\hat{T}$ of T to Rel is defined by

$$(r : X \relbar\!\!\mapsto Y) \longmapsto (\hat{T}r : TX \relbar\!\!\mapsto TY) = Tq \cdot (Tp)^{\circ};$$

here r is identified with the span

$$\begin{array}{ccc} & R & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & & Y \end{array}$$

given by the projections of $R = \{(x, y) \in X \times Y | xry\}$; explicitly:

$$\mathfrak{x}(\hat{T}r)\mathfrak{y} \iff \exists \mathfrak{z} \in TR : Tp(\mathfrak{z}) = \mathfrak{x} \text{ and } Tq(\mathfrak{z}) = \mathfrak{y}.$$

For a map $f : X \longrightarrow Y$ in Set, one has $\hat{T}f = Tf$, so that the diagram

$$\begin{array}{ccc} \mathsf{Rel} & \xrightarrow{\hat{T}} & \mathsf{Rel} \\ \uparrow & & \uparrow \\ \mathsf{Set} & \xrightarrow{T} & \mathsf{Set} \end{array}$$

commutes. Furthermore, $\hat{T}(r^{\circ}) = (\hat{T}r)^{\circ}$, and

$$\hat{T}(s \cdot r) \leq \hat{T}s \cdot \hat{T}r$$

for all $r : X \relbar\!\!\mapsto Y$, $s : Y \relbar\!\!\mapsto Z$, *with equality holding* (so that $\hat{T}$ becomes a functor) *when T transforms (weak) pullbacks into weak pullbacks.* Here is an explanation for the last statement. For a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

one has $q \cdot p^{\circ} = g^{\circ} \cdot f$ in Rel precisely when $\mathrm{can} : W \longrightarrow X \times_Z Y$ is surjective, that is: when the diagram is a weak pullback in Set. Hence, functoriality of $\hat{T}$ forces T to preserve weak pullback diagrams. Conversely, given r and s as spans with domains R and S, respectively, one considers the pullback diagram

$$\begin{array}{ccc} R \times_Y S & \longrightarrow & S \\ \downarrow & & \downarrow \\ R & \longrightarrow & Y \end{array}$$

which T transforms into a weak pullback.

For $T = \beta$ and $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ one has

$$\begin{aligned} \mathfrak{x}(\hat{\beta}r)\mathfrak{y} &\iff \exists \mathfrak{z} \in \beta R : \beta p(\mathfrak{z}) = \mathfrak{x} \text{ and } \beta q(\mathfrak{z}) = \mathfrak{y} \\ &\iff \forall A \in \mathfrak{x}\ \forall B \in \mathfrak{y}\ : p^{-1}[A] \cap q^{-1}[B] \neq \emptyset \\ &\iff \forall A \in \mathfrak{x}\ \forall B \in \mathfrak{y}\ \exists x \in A\ \exists y \in B : xry; \end{aligned}$$

the necessity of this last condition is obvious, and for its sufficiency one considers any ultrafilter $\mathfrak{z}$ on R containing the filter-base $\{p^{-1}[A] \cap q^{-1}[B] \mid A \in \mathfrak{x}, B \in \mathfrak{y}\ \}$. It is then easy to see that one has also

$$\begin{aligned} \mathfrak{x}(\hat{\beta}r)y &\iff \forall A \in \mathfrak{x}\ :\ r[A] \in \mathfrak{y} \\ &\iff \forall B \in \mathfrak{y}\ :\ r^{\circ}[B] \in \mathfrak{x}, \end{aligned} \qquad (*)$$

with $r[A] = \{y \in Y \mid \exists x \in A : xry\}$.

For $T = F$ one generally does not have the logical equivalences as just displayed in case $T = \beta$. Rather, there are several ways of defining an "extension" of $\hat{F}$ of F. For example, (*) would give:

$$\begin{aligned} \mathfrak{x}(\hat{F}r)\mathfrak{y} &\iff \forall B \in \mathfrak{y} : r^{\circ}[B] \in \mathfrak{x} \\ &\iff \forall B \in \mathfrak{y} \exists A \in \mathfrak{x} \forall x \in A \exists y \in B : xry. \end{aligned}$$

But we have to be careful now about the meaning of "extension", since for a map $f : X \longrightarrow Y$ one has:

$$\begin{aligned} \mathfrak{x}(Ff)\mathfrak{y} &\iff \mathfrak{y} = Ff(\mathfrak{x}) \\ &\iff \forall C \subseteq Y\ (B \in \mathfrak{y} \iff f^{-1}[B] \in \mathfrak{x}), \end{aligned}$$

while for $\hat{F}$f one has only

$$x(\hat{F}f)\mathfrak{y} \iff \mathfrak{y} \subseteq Ff(\mathfrak{x}).$$

Consequently, $Ff \leq \hat{F}f$ and, similarly $(Ff)^{\circ} \leq \hat{F}(f^{\circ})$, with equality holding only in rare cases. In particular, while $F1_X$ is the identity relation in FX, $\hat{F}1_X$ is the natural order on FX given by the "finer" relation:

$$\begin{aligned} \mathfrak{x}_1(\hat{F}1_X)\mathfrak{x}_2 &\iff \mathfrak{x}_2 \subseteq \mathfrak{x}_1 \\ &\iff \mathfrak{x}_1 \text{ finer than } \mathfrak{x}_2. \end{aligned}$$

2.2 Topological spaces. Let a denote the usual (ultra)filter convergence relation in a topological space X. Hence, for $\mathfrak{x} \in \beta X$ and $y \in X$, writing $\mathfrak{x} \xrightarrow{a} y$ or just $\mathfrak{x} \longrightarrow y$ when $\mathfrak{x}ay$ holds true, one has:

$$\mathfrak{x} \longrightarrow y \iff \forall B \text{ nbh. of } y : B \in \mathfrak{x}.$$

Furthermore, the following two properties are immediate consequences of the definitions:

(1) $\dot{x} \longrightarrow x$

(2) $\mathfrak{X} \longrightarrow \mathfrak{y}$ and $\mathfrak{y} \longrightarrow z \implies \sum \mathfrak{X} \longrightarrow z$,

for all $x, z \in X, \mathfrak{y} \in \beta X, \mathfrak{X} \in \beta\beta X$; of course, $\mathfrak{X} \longrightarrow \mathfrak{y}$ means $\mathfrak{X}(\hat{\beta}a)y$. Indeed, for every open neighborhood C of z in X one has $C \in \mathfrak{y}$ since $\mathfrak{y} \longrightarrow z$, and then $a^\circ[C] \in \mathfrak{X}$ since $X \longrightarrow \mathfrak{y}$. But

$$a^\circ[C] = \{\mathfrak{x} \in \beta X \mid \exists y \in C : \mathfrak{x} \longrightarrow y\} \subseteq \{x \in \beta X \mid X \in \mathfrak{x}\} = C^\#,$$

so that we have $C^\# \in \mathfrak{X}$ and therefore $C \in \sum \mathfrak{X}$, as desired.

Conversely, given any relation $a : \beta X \to\!\!\!\!\!\!\mapsto X$ which (when we write $\mathfrak{x} \longrightarrow y$ for $\mathfrak{x}ay$) satisfies conditions 1, 2, we show that one can define a topology on X by

$$C \ a\text{-open in } X \iff a^\circ[C] \subseteq C^\#.$$

Indeed, one sees immediately $X^\# = \beta X$, and

$$a^\circ[C \cap D] \subseteq a^\circ[C] \cap a^\circ[D] \subseteq C^\# \cap D^\# = (C \cap D)^\#,$$
$$a^\circ[\bigcup_i C_i] = \bigcup_i a^\circ[C_i] \subseteq \bigcup_i C_i^\# \subseteq (\bigcup_i C_i)^\#,$$

when C, D, C_i are all a-open.

It is not difficult to show that, when a is the convergence relation of a topology, a-openness just means openness in the given topology. To show that any relation a satisfying 1, 2 is the convergence relation of the topology given by the a-open sets is trickier, see [Barr], [Wyl], [HT].

In all of the preceding considerations, the ultrafilter monad may be replaced by the filter monad, with the extension $\hat{F}$ of F to be defined as at the end of 2.1. That the two axioms 1, 2 still suffice to describe topological spaces is a relatively recent observation by [Seal], despite the fact that it had been known that topological spaces are describable via filter convergence axioms practically since Cartan introduced filter convergence in the 1930s.

2.3. The $(\mathbb{T}, \mathcal{V})$-setting. As in Lecture 1, $\mathcal{V}$ is a non-trivial commutative unital quantale, with $\otimes, k$. In addition, we now consider the monad $\mathbb{T} = (T, e, m)$ on Set which comes with a fixed *lax extension*

$$\hat{T} : \mathcal{V}\text{-}\mathsf{Rel} \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}$$

That is, for all $f : X \longrightarrow Y$, $r, r' : X \to\!\!\!\!\!\!\mapsto Y$, $s : Y \to\!\!\!\!\!\!\mapsto Z$, $g : Z \longrightarrow Y$ we require the following conditions:

(0) $\hat{T}X = TX$;
(1) $\hat{T}$ is a lax functor of the 2-category $\mathcal{V}$-Rel, that is:
 (a) $1_{TX} \leq \hat{T}1_X$, (b) $\hat{T}s \cdot \hat{T}r \leq \hat{T}(s \cdot r)$, (c) $r \leq r' \implies \hat{T}r \leq \hat{T}r'$;
(2) (a) $Tf \leq \hat{T}f$, (b) $(Tf)^\circ \leq \hat{T}(f^\circ)$;
(3) e and m are op-lax in $\mathcal{V}$-Rel, that is:
 (a) $e_Y \cdot r \leq \hat{T}r \cdot e_X$,
 (b) $m_Y \cdot \hat{T}\hat{T}r \leq \hat{T}r \cdot m_X$,
 which may be equivalently expressed by:
 (a°) $r \cdot e_X^\circ \leq e_Y^\circ \cdot \hat{T}r$,
 (b°) $\hat{T}\hat{T}r \cdot m_X^\circ \leq m_Y^\circ \cdot \hat{T}r$.

Of course, (2)(a) entails (1)(a). The following convenient rules follow from (0)-(2) and are due to [Seal]:

(4) (a) $\hat{T}(s\cdot f)=\hat{T}s\cdot\hat{T}f=\hat{T}s\cdot Tf$, (b)$\hat{T}(g^\circ\cdot r)=\hat{T}(g^\circ)\cdot\hat{T}r=(Tg)^\circ\cdot\hat{T}r$.

Hence, despite being just a lax functor, $\hat{T}$ *preserves "whiskering" a $\mathcal{V}$-relation from the right with a map and from the left with the converse of a map*, and evaluation of $\hat{T}$ on the map can be done using T in this case.

Proof of (4)(a):

$$\begin{aligned}
\hat{T}(s\cdot f)\le\hat{T}(s\cdot f)\cdot(Tf)^\circ\cdot Tf &\le \hat{T}(s\cdot f)\cdot\hat{T}(f^\circ)\cdot Tf && ((2)(\mathrm{b}))\\
&\le \hat{T}(s\cdot f\cdot f^\circ)\cdot Tf && ((1)(\mathrm{b}))\\
&\le \hat{T}s\cdot Tf && ((1)(\mathrm{c}))\\
&\le \hat{T}s\cdot\hat{T}f && ((2)(\mathrm{a}))\\
&\le \hat{T}(s\cdot f). && ((1)(\mathrm{b}))
\end{aligned}$$

2.4. $(\mathbb{T},\mathcal{V})$-relations, Kleisli composition. For sets X, Y, a $(\mathbb{T},\mathcal{V})$-*relation* $r:X\rightharpoondown Y$ is simply a $\mathcal{V}$-relation $r:TX\nrightarrow Y$. With $s:Y\rightharpoondown Z$, one has the *Kleisli composition* (introduced in this context by [Hof]):

$$s*r=s\cdot\hat{T}r\cdot m_X^\circ:X\rightharpoondown Z,$$

for which one can establish the following rules:

(0) $r\le r', s\le s' \implies s*r\le s'*r'$;
(1) (a) $r\le r*e_X^\circ$, (b) $r\le e_Y^\circ*r$;
(2) (a) $r*e_X^\circ=r\cdot\hat{T}1_X=r*(e_X^\circ\cdot\hat{T}1_X)$, (b) $e_Y^\circ*r=(e_Y^\circ\cdot\hat{T}1_Y)*r$;
(3) (a) $t*(s*r)\le(t*s)*r$ if $\hat{T}$ preserves the composition in $\mathcal{V}$-Rel, and if $\hat{T}(m_X^\circ)=(Tm_X)^\circ$;
(b) $t*(s*r)\ge(t*s)*r$, if $m:\hat{T}\hat{T}\longrightarrow\hat{T}$ is a (strict) natural transformation;
(4) $\hat{T}1_X=\hat{T}(e_X^\circ)\cdot m_X^\circ$.

Proofs:

(1)

$$\begin{aligned}
r &= r\cdot(Te_X)^\circ\cdot m_X^\circ\le r\cdot\hat{T}(e_X^\circ)\cdot m_X^\circ=r*e_X^\circ, && (2.3(2)(\mathrm{b}))\\
r &= r\cdot e_{TX}^\circ\cdot m_X^\circ\le e_Y^\circ\cdot\hat{T}r\cdot m_X^\circ=e_Y^\circ*r. && (2.3(3)(\mathrm{a}^\circ))
\end{aligned}$$

(2)(a)

$$\begin{aligned} r * (e_X^\circ \cdot \hat{T}1_X) &= r \cdot \hat{T}(e_X^\circ \cdot \hat{T}1_X) \cdot m_X^\circ \\ &= r \cdot (Te_X)^\circ \cdot \hat{T}\hat{T}1_X \cdot m_X^\circ && (2.3(4)(b)) \\ &\le r \cdot (Te_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X && (2.3(3)(b^\circ)) \\ &= r \cdot \hat{T}1_X \\ &= r \cdot \hat{T}1_X \cdot (Te_X)^\circ \cdot m_X^\circ \\ &\le r \cdot \hat{T}1_X \cdot \hat{T}(e_X)^\circ \cdot m_X^\circ && (2.3(2)(b)) \\ &\le r \cdot \hat{T}(e_X^\circ) \cdot m_X^\circ && (2.3(1)(b)) \\ &= r * e_X^\circ \\ &\le r * (e_X^\circ \cdot \hat{T}1_X). \end{aligned}$$

(b)

$$\begin{aligned} (e_Y^\circ \cdot \hat{T}1_Y) * r &= e_Y^\circ \cdot \hat{T}1_Y \cdot \hat{T}r \cdot m_X^\circ && (2.3(1)(a)) \\ &\le e_Y^\circ \cdot \hat{T}r \cdot m_X^\circ && (2.3(1)(b)) \\ &= e_Y^\circ * r \\ &\le (e_Y^\circ \cdot \hat{T}1_Y) * r && (2.3(1)(a)) \end{aligned}$$

(3)

$$\begin{aligned} t * (s * r) &= t \cdot \hat{T}(s \cdot \hat{T}r \cdot m_X^\circ) \cdot m_X^\circ \\ &\ge t \cdot \hat{T}s \cdot \hat{T}\hat{T}r \cdot \hat{T}(m_X^\circ) \cdot m_X^\circ && (2.3(1)(b)) \\ &\ge t \cdot \hat{T}s \cdot \hat{T}\hat{T}r \cdot (Tm_X)^\circ \cdot m_X^\circ && (2.3(2)(b)) \\ &= t \cdot \hat{T}s \cdot \hat{T}\hat{T}r \cdot m_{TX}^\circ \cdot m_X^\circ \\ &\le t \cdot \hat{T}s \cdot m_Y^\circ \cdot \hat{T}r \cdot m_X^\circ && (2.3(3)(b^\circ)) \\ &= (t * s) * r, \end{aligned}$$

and the given provisions make the respective inequality signs into equality signs.

(4)

$$\begin{aligned} \hat{T}1_X &= \hat{T}1_X \cdot \hat{T}1_X = \hat{T}1_X * e_X^\circ && (2.3(4)(a),\ (2)(a)) \\ &= \hat{T}1_X \cdot \hat{T}(e_X^\circ) \cdot m_X^\circ = \hat{T}(e_X^\circ) \cdot m_X^\circ. && (2.3(2)(b)) \end{aligned}$$

One calls a $(\mathbb{T}, \mathcal{V})$-relation $r : X \to\!\!\!\!\!\mapsto Y$ *unitary* if

$$r * e_X^\circ \le r \text{ and } e_Y^\circ * r \le r.$$

In general, e_X° is not unitary, but

$$1_X^\# := e_X^\circ \cdot \hat{T}1_X$$

is: from (2)(a) one has $e_X^\circ * 1_X^\# = 1_X^\#$ and from (2)(a)(b) one obtains $1_X^\# * e_X^\circ = e_X^\circ * e_X^\circ = e_X^\circ \cdot \hat{T}1_X = 1_X^\#$. Hence, under the provisions guaranteeing associativity, sets and unitary $(\mathbb{T},\mathcal{V})$-relations form a category

$$(\mathbb{T},\mathcal{V})\text{-}\mathsf{Rel},$$

with Kleisli composition. But even in the absence of associativity, it turns out to be useful to refer to $(\mathbb{T},\mathcal{V})$-Rel as a *generalized ordered category*, irrespective of the annoying fact that the Kleisli composite of two unitary $(\mathbb{T},\mathcal{V})$-relations may no longer be unitary. As we will see, similarly to 2.3(4), "whiskering" with maps is still possible and well behaved.

Note that for $\mathcal{V}=2$ and $\mathbb{T}=\beta$, with its Barr extension $\hat{\beta}$, $\hat{\beta}$ is a functor and $m:\hat{\beta}\hat{\beta}\longrightarrow\hat{\beta}$ a natural transformation (but $e:1\longrightarrow\hat{\beta}$ remains just op-lax!), so that $(\mathbb{T},\mathcal{V})$-Rel is a genuine category in this case.

2.5. $(\mathbb{T},\mathcal{V})$**-categories.** A $(\mathbb{T},\mathcal{V})$-category $X=(X,a)$ is a set X with a $(\mathbb{T},\mathcal{V})$-relation $a:X\rightharpoondown X$ which is

(1) *reflexive*: $e_X^\circ\leq a$;
(2) *transitive*: $a*a\leq a$.

These conditions are equivalently expressed by

1′. $1_X\leq a\cdot e_X$,

2′. $a\cdot\hat{T}a\leq a\cdot m_X$

$$\begin{array}{ccccc} TTX & \xrightarrow{\hat{T}a} & TX & \xleftarrow{e_X} & X \\ {\scriptstyle m_X}\downarrow & \geq & \downarrow{\scriptstyle a} & \geq \swarrow {\scriptstyle 1_X} & \\ TX & \xrightarrow[a]{} & X & & \end{array}$$

which exhibit X as a *lax Eilenberg-Moore algebra*. In pointwise notation 1', 2' amount to:

1". $k\leq a(e_X(x),x)$,
2". $\hat{T}a(\mathfrak{X},\mathfrak{y})\otimes a(\mathfrak{y},z)\leq a(m_X(\mathfrak{X}),z)$

for all $x,z\in X$, $\mathfrak{X}\in TTX$, $\mathfrak{y}\in TX$.

A $(\mathbb{T},\mathcal{V})$-*functor* $f:X\longrightarrow Y=(Y,b)$ is a mapping $f:X\longrightarrow Y$ with

$$f\cdot a\leq b\cdot Tf:\quad \begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow & \leq & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Equivalently, $a\leq f^\circ\cdot b\cdot Tf$, or $a(\mathfrak{x},y)\leq b((Tf)\mathfrak{x},f(y))$ for all $\mathfrak{x}\in TX, y\in X$.

With composition as in Set one obtains the category $(\mathbb{T},\mathcal{V})$-Cat.

We note that the structure a of X is necessarily unitary. Indeed, from $e_X^\circ\leq a$ and $a*a\leq a$ one obtains with the monotonicity of $*$:

$$e_X^\circ*a\leq a*a\leq a \text{ and } a*e_X^\circ\leq a*a\leq a.$$

Also, the condition $e_X^\circ \leq a$ implies the formally stronger condition $1_X^\# \leq a$, with the help of 2.4(4):

$$e_X^\circ \leq 1_X^\# = e_X^\circ \cdot \hat{T}1_X \leq a \cdot \hat{T}(e_X^\circ) \cdot m_X^\circ = a * e_X^\circ \leq a.$$

We also note that, since $a \leq a * e_X^\circ \leq a * a$ by 2.4(1), one in fact has $a * a = a$. Finally, it seems debatable whether the defining condition for a $(\mathbb{T}, \mathcal{V})$-functor should rather be $f \cdot a \leq b \cdot \hat{T}f$. But it turns out that this is a mute point since with 2.4(4) one obtains: *for every map* $f : X \longrightarrow Y$ *and every unitary* $(\mathbb{T}, \mathcal{V})$*-relation* $s : Y \rightharpoondown Z$,

$$s \cdot \hat{T}f = s \cdot Tf.$$

Indeed:

$$s \cdot \hat{T}f = s \cdot \hat{T}(1_Y \cdot f) = s \cdot \hat{T}1_Y \cdot Tf = s \cdot \hat{T}(e_Y^\circ) \cdot m_Y^\circ \cdot Tf = (s * e_Y^\circ) \cdot Tf = s \cdot Tf.$$

Examples:

$\mathbb{T}\backslash\mathcal{V}$	2	$\mathbb{P}_+$	$\mathbb{P}_{\max}$
$\mathbb{I}$	Ord	Met	UMet
β	Top	App	UApp

Here App $= (\beta, \mathbb{P}_+)$-Cat is the category of *approach spaces* (see [Low]), [CH1], [HLV]), which may also be described in terms of distance functions $PX \times X \longrightarrow [0, \infty]$; the unltrafilter functor β has been extended to $\mathbb{P}_+$-Rel by:

$$\hat{\beta}r(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{A \in \mathfrak{x}, B \in \mathfrak{y}} \bigvee_{x \in A, y \in B} r(x, y)$$

for $r : X \rightharpoondown Y$, $\mathfrak{x} \in \beta X, \mathfrak{y} \in \beta Y$ (see [CT]). UApp is the "ultrametric analogue" of App.

In the chart one may replace β by the filter monad $\mathbb{F}$, without any changes to the category entries in the chart: see [Seal], [SS]. Further examples may be found in the papers quoted.

2.6. Theorem. *The forgetful functor* $U : (\mathbb{T}, \mathcal{V})$-Cat $\longrightarrow$ Set *is topological. In particular,* U *is a bifibration and has both, a fully faithful left adjoint and a fully faithful right adjoint. Consequently,* $(\mathbb{T}, \mathcal{V})$-Cat *is complete and cocomplete and has a projective generator and an injective cogenerator.*

Proof. Let us first show that U is a fibration. Given a $(\mathbb{T}, \mathcal{V})$-category (Y, b) and a Set-map $f : X \longrightarrow Y$,

$$a := f^\circ \cdot b \cdot Tf$$

is obviously the largest structure on X that can make f into a $(\mathbb{T},\mathcal{V})$-algebra, if it is a structure at all. But reflexivity and transitivity of a are easily shown:

$$\begin{aligned} 1_X \le f^\circ \cdot f &\le f^\circ \cdot b \cdot e_Y \cdot f \le f^\circ \cdot b \cdot Tf \cdot e_X = a \cdot e_X; \\ a \cdot \hat{T}a &= f^\circ \cdot b \cdot Tf \cdot \hat{T}(f^\circ \cdot b \cdot Tf) \\ &= f^\circ \cdot b \cdot Tf \cdot (Tf)^\circ \cdot \hat{T}b \cdot TTf && (2.3(4)) \\ &\le f^\circ \cdot b \cdot \hat{T}b \cdot TTf \\ &\le f^\circ \cdot b \cdot m_Y \cdot TTf \\ &= f^\circ \cdot b \cdot Tf \cdot m_X = a \cdot m_X. \end{aligned}$$

For a family $f_i : X \longrightarrow Y_i$ of Set-maps with $Y = (Y, b_i)$ in $(\mathbb{T},\mathcal{V})$-Cat,

$$a = \bigwedge_{i\in I} a_i \text{ with } a_i = f_i^\circ \cdot b_i \cdot Tf_i$$

is the *U-initial structure* on X w.r.t $(f_i)_{i\in I}$:

$$\begin{aligned} 1_X &\le \bigwedge_{i\in I}(a_I \cdot e_X) = (\bigwedge_{i\in I} a_i) \cdot e_X = a \cdot e_X; \\ a \cdot \hat{T}a &= (\bigwedge_{i\in I} a_i) \cdot \hat{T}(\bigwedge_{i\in I} a_i) \le \bigwedge_{i\in I}(a_i \cdot \hat{T}a_i) \le \bigwedge_{i\in I}(a_i \cdot m_X) = (\bigwedge_{i\in I} a_i) \cdot m_X = a \cdot m_X. \end{aligned}$$

For $I = \emptyset$, $a = \top$ is constant, and

$$X \longmapsto (X, \top)$$

describes the fully faithful right adjoint of U (the *indiscrete structure on* X). The fully faithful left adjoint to U is given by

$$X \longmapsto (X, 1_X^\#)$$

(the *discrete structure on* X). In fact, $1_X^\# = e_X^\circ \cdot \hat{T}1_X \ge e_X^\circ$ is trivially reflexive, and

$$\begin{aligned} 1_X^\# \cdot \hat{T}1_X^\# \cdot m_X^\circ &= 1_X^\# \cdot \hat{T}(e_X^\circ \cdot \hat{T}1_X) \cdot m_X^\circ \\ &= 1_X^\# \cdot (Te_X)^\circ \cdot \hat{T}\hat{T}1_X \cdot m_X^\circ && (2.3(4)(\mathrm{b})) \\ &\le e_X^\circ \cdot \hat{T}1_X \cdot (Te_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X && (2.3(3)(\mathrm{b}^\circ)) \\ &= e_X^\circ \cdot \hat{T}1_X \cdot \hat{T}1_X = 1_X^\# \end{aligned}$$

shows reflexivity. Furthermore, for any (TV)-structure a on X one has $a = a \cdot \hat{T}1_X \ge e_X^\circ \cdot \hat{T}1_X = 1_X^\#$, since a is unitary (see 2.5).

The remaining statements follow from the general theory of topological functors: existence of all U-initial structures implies existence of all *U-final structures*, although it is much harder to describe these efficiently. Limits in $(\mathbb{T},\mathcal{V})$-Cat are formed by providing the limit of the underlying Set-diagram with the U-initial structure w.r.t. the limit projections in Set; dually for colimits. A generator in Set becomes a projective generator in $(\mathbb{T},\mathcal{V})$-Cat when provided with the discrete structure; dually for cogenerators. □

Note that there are (at least) three interesting structures that one may want to put on a one-element set $1 = E = \{*\}$:

$1 = (1, \top)$ is a terminal object in $(\mathbb{T}, \mathcal{V})$-Cat;
$P = (1, 1_1^{\#})$ is a projective generator in $(\mathbb{T}, \mathcal{V})$-Cat;
$E = (1, k)$ is the $\otimes$-neutral object in $(\mathbb{T}, \mathcal{V})$-Cat,

with $\otimes$ yet to be defined. Here

$$1_1^{\#}(\mathfrak{x}, *) = \hat{T}1_1(\mathfrak{x}, e_1(*))$$

for all $\mathfrak{x} \in T1$. With respect to a description of the U-final structure on Y w.r.t. a family $f_i : X_i \longrightarrow Y$, $X_i = (X_i, a_i), i \in I$, let us note only that certainly

$$b_0 := e_Y^\circ \vee \bigvee_{i \in I} f_i \cdot a \cdot Tf_i^\circ$$

is the least reflexive structure on X making every f_i a $(\mathbb{T}, \mathcal{V})$-functor, but it needs to be "made" transitive. Transfinite processes describing the transitive hull of a V-relation are described in [Hof].

2.7 $(\mathbb{T}, \mathcal{V})$**-modules.** $(\mathbb{T}, \mathcal{V})$-*modules* are structure-preserving $(\mathbb{T}, \mathcal{V})$-relations between $(\mathbb{T}, \mathcal{V})$-categories. Hence, for $(\mathbb{T}, \mathcal{V})$-categories $X = (X, a)$, $Y = (Y, b)$, a $(\mathbb{T}, \mathcal{V})$-module $\varphi : X \multimap Y$ is a $(\mathbb{T}, \mathcal{V})$-relation $\varphi : X \rightharpoonup Y$ with

$$\varphi * a \leq \varphi \text{ and } b * \varphi \leq \varphi,$$

that is:

$$\varphi \cdot \hat{T}a \cdot m_X^\circ \leq \varphi \text{ and } b \cdot \hat{T}\varphi \cdot m_X^\circ \leq \varphi$$

with $\varphi : TX \rightharpoonup Y$ in $\mathcal{V}$-Rel. Equivalently, this means

$$\hat{T}a(\mathfrak{X}, \mathfrak{x}) \otimes \varphi(\mathfrak{x}, y) \leq \varphi(m_X(\mathfrak{X}), y) \text{ and } \hat{T}\varphi(\mathfrak{X}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \leq \varphi(m_X(\mathfrak{X}), y)$$

for all $x \in TX$, $\mathfrak{X} \in TTX$, $\mathfrak{y} \in TY$, $y \in Y$. Since

$$\varphi \leq \varphi * e_X^\circ \leq \varphi * a \leq \varphi \text{ and } \varphi \leq e_Y^\circ * \varphi \leq b * \varphi \leq \varphi,$$

a $(\mathbb{T}, \mathcal{V})$-module is unitary and satisfies $\varphi * a = \varphi = b * \varphi$. But only when $*$ is associative will the $(\mathbb{T}, \mathcal{V})$-modules be the morphisms of an (ordered) category

$$(\mathbb{T}, \mathcal{V})\text{-Mod}$$

where objects are $(\mathbb{T}, \mathcal{V})$-categories. Nevertheless, also in the absence of associativity will we consider $(\mathbb{T}, \mathcal{V})$-Mod, as a generalized category just like $(\mathbb{T}, \mathcal{V})$-Rel, consider the forgetful functor $(\mathbb{T}, \mathcal{V})$-Mod $\longrightarrow$ $(\mathbb{T}, \mathcal{V})$-Rel and, more importantly, define the functors

$$(-)_* : (\mathbb{T}, \mathcal{V})\text{-Cat} \longrightarrow (\mathbb{T}, \mathcal{V})\text{-Mod}, \ \ (-)^* : ((\mathbb{T}, \mathcal{V})\text{-Cat})^{\mathrm{op}} \longrightarrow (\mathbb{T}, \mathcal{V})\text{-Mod},$$

which are identical on objects and send $f : X \longrightarrow Y$ in $(\mathbb{T}, \mathcal{V})$-Cat to

$$f_* = b \cdot Tf : X \multimap Y \text{ and } f^* = f^\circ \cdot b : Y \multimap X,$$

respectively. Recall that in 2.5 we proved $f_* = b \cdot \hat{T}f$. Let us prove first that f_*, f^* *are indeed* $(\mathbb{T}, \mathcal{V}) -$ *modules, with*

$$f_* \dashv f^*.$$

In particular, we note that $1_X^* = a = (1_X)_*$ is the identity morphism on X in $(\mathbb{T},\mathcal{V})$-Mod.

$$\begin{aligned} f_* * a &= b\cdot \hat{T}f\cdot \hat{T}a\cdot m_X^\circ \\ &\le b\cdot \hat{T}(f\cdot a)\cdot m_X^\circ \\ &\le b\cdot \hat{T}(b\cdot Tf)\cdot m_X^\circ \\ &= b * f_* \\ &= b\cdot \hat{T}(b\cdot Tf)\cdot m_X^\circ \\ &\le b\cdot \hat{T}b\cdot m_Y^\circ\cdot Tf \\ &= (b*b)\cdot Tf \\ &= b\cdot Tf = f_* \end{aligned} \qquad \begin{aligned} a * f^* &= a\cdot \hat{T}(f^\circ\cdot b)\cdot m_Y^\circ \\ &= a\cdot (Tf)^\circ\cdot \hat{T}b\cdot m_Y^\circ \\ &\le f^\circ\cdot b\cdot \hat{T}b\cdot m_Y^\circ \\ &= f^* * b \\ &= f^\circ\cdot b\cdot \hat{T}b\cdot m_Y^\circ \\ &= f^\circ\cdot (b*b) \\ &= f^\circ\cdot b \\ &= f^* \end{aligned}$$

$$\begin{aligned} f_* * f^* &= b\cdot \hat{T}f\cdot \hat{T}(f^\circ\cdot b)\cdot m_Y^\circ \\ &= b\cdot \hat{T}f\cdot \hat{T}(f^\circ)\cdot \hat{T}b\cdot m_Y^\circ \\ &\le b\cdot \hat{T}(f\cdot f^\circ)\cdot \hat{T}b\cdot m_Y^\circ \\ &\le b\cdot \hat{T}(1_Y)\cdot \hat{T}b\cdot m_Y^\circ \\ &= b\cdot \hat{T}b\cdot m_Y^\circ \\ &= b*b = b = 1_Y^* \end{aligned} \qquad \begin{aligned} f^* * f_* &= f^\circ\cdot b\cdot \hat{T}(b\cdot Tf)\cdot m_X^\circ \\ &\ge a\cdot (Tf)^\circ\cdot \hat{T}(b\cdot Tf)\cdot m_X^\circ \\ &= a\cdot \hat{T}(f^\circ\cdot b\cdot Tf)\cdot m_X^\circ \\ &\ge a\cdot \hat{T}(a\cdot (Tf)^\circ\cdot Tf)\cdot m_X^\circ \\ &\ge a\cdot \hat{T}a\cdot m_X^\circ \\ &= a*a = a = 1_X^*. \end{aligned}$$

Functoriality of $(-)_*$ and $(-)^*$ is best shown after:

2.8. Proposition (Whiskering). *For* $f : X \longrightarrow Y$ *in* $(\mathbb{T},\mathcal{V})$-Cat *and* $\varphi : Y \,{\circ\!\!\rightharpoonup}\, Z$, $\psi : Z \,{\circ\!\!\rightharpoonup}\, Y$ *in* $(\mathbb{T},\mathcal{V})$-Mod,

$$f^* * \psi = f^\circ\cdot\psi : Z \,{\circ\!\!\rightharpoonup}\, X \text{ and } \varphi * f_* = \varphi\cdot Tf = \varphi\cdot \hat{T}f : X \,{\circ\!\!\rightharpoonup}\, Z$$

are also in $(\mathbb{T},\mathcal{V})$-Mod. *Furthermore, if* $\varphi \dashv \psi$ *and* $m_Y^\circ\cdot Tf \le TTf\cdot m_X^\circ$, *then* $\varphi * f_* \dashv f^* * \psi$.

Proof. With $X = (X,a)$, $Y = (Y,b)$, $Z = (Z,c)$ we have for $f^* * \psi$:

$$\begin{aligned} &f^* * \psi = f^\circ\cdot b\cdot \hat{T}\psi\cdot m_Z^\circ = f^\circ\cdot (b*\psi) = f^\circ\cdot\psi, \\ &(f^* * \psi) * c = f^\circ\cdot\psi\cdot \hat{T}c\cdot m_Z^\circ = f^\circ\cdot(\psi * c) = f^*\cdot\psi = f^* * \psi, \\ &a * (f^* * \psi) = a\cdot \hat{T}(f^\circ\cdot\psi)\cdot m_Z^\circ = a\cdot (Tf)^\circ\cdot \hat{T}\psi\cdot m_Z^\circ \le f^\circ\cdot b\cdot \hat{T}\psi\cdot m_Z^\circ = f^* * \psi. \end{aligned}$$

For $\varphi * f_*$ we compute:

$$\begin{aligned}\varphi * f_* &= \varphi \cdot \hat{T}(b \cdot Tf) \cdot m_X^\circ = \varphi \cdot \hat{T}b \cdot TTf \cdot m_X^\circ \leq \varphi \cdot \hat{T}b \cdot m_Y^\circ \cdot Tf \\ &= (\varphi * b) \cdot Tf = \varphi \cdot Tf \leq \varphi \cdot \hat{T}f = \varphi \cdot \hat{T}f \cdot \hat{T}1_X \\ &\leq \varphi \cdot \hat{T}f \cdot \hat{T}(a \cdot e_X) = \varphi \cdot \hat{T}f \cdot \hat{T}a \cdot Te_X \leq \varphi \cdot \hat{T}(f \cdot a) \cdot Te_X \\ &\leq \varphi \cdot \hat{T}(b \cdot Tf) \cdot m_X^\circ = \varphi * f_*,\end{aligned}$$

where we used $Te_X \leq m_X^\circ$ for the last inequality. Now:

$$\begin{aligned}(\varphi * f_*) * a &= (\varphi \cdot Tf) \cdot \hat{T}a \cdot m_X^\circ \leq \varphi \cdot \hat{T}(f \cdot a) \cdot m_X^\circ \leq \varphi \cdot \hat{T}(b \cdot Tf) \cdot m_X^\circ \\ &= \varphi \cdot \hat{T}b \cdot TTf \cdot m_X^\circ \leq \varphi \cdot \hat{T}b \cdot m_Y^\circ \cdot Tf = (\varphi * b) \cdot Tf \\ &= \varphi \cdot Tf = \varphi * f_*, \\ c * (\varphi * f_*) &= c \cdot \hat{T}(\varphi \cdot Tf) \cdot m_X^\circ = c \cdot \hat{T}\varphi \cdot TTf \cdot m_X^\circ \\ &\leq c \cdot \hat{T}\varphi \cdot m_Y^\circ \cdot Tf = (c * \varphi) \cdot Tf = \varphi \cdot Tf = \varphi * f_*.\end{aligned}$$

Finally, assume that the m-naturality diagram for f is a weak pullback, so that $m_Y^\circ \cdot Tf = TTf \cdot m_Y^\circ$, and let $\varphi \dashv \psi$, so that

$$b = 1_Y^* \leq \psi * \varphi \text{ and } \varphi * \psi \leq 1_Z^* = c.$$

Then:

$$\begin{aligned}(\varphi * f_*) * (f^* * \psi) &= (\varphi \cdot Tf) * (f^\circ \cdot \psi) = \varphi \cdot Tf \cdot \hat{T}(f^\circ \cdot \psi) \cdot m_Z^\circ \\ &= \varphi \cdot Tf \cdot (Tf)^\circ \cdot \hat{T}\psi \cdot m_Z^\circ \\ &\leq \varphi \cdot \hat{T}\psi \cdot m_Z^\circ = \varphi * \psi \leq 1_Z^*, \\ (f^* * \psi) * (\varphi * f_*) &= (f^\circ \cdot \psi) * (\varphi \cdot Tf) = f^\circ \cdot \psi \cdot \hat{T}(\varphi \cdot Tf) \cdot m_X^\circ \\ &= f^\circ \cdot \psi \cdot \hat{T}\varphi \cdot TTf \cdot m_X^\circ \\ &= f^\circ \cdot \psi \cdot \hat{T}\varphi \cdot m_Y^\circ \cdot Tf \\ &= f^\circ \cdot (\psi * \varphi) \cdot Tf \geq f^\circ \cdot b \cdot Tf \geq a = 1_X^*.\end{aligned}$$

□

Functoriality of $(-)_*$ and $(-)^*$ follows immediately from the Whiskering formulae:

$$\begin{aligned}(g \cdot f)_* &= c \cdot T(g \cdot f) = c \cdot Tg \cdot Tf = g_* \cdot Tf = g_* * f_* \\ (g \cdot f)^* &= (g \cdot f)^\circ \cdot c = f^\circ \cdot g^\circ \cdot c = f^\circ \cdot g^* = f^* * g^*.\end{aligned}$$

2.9. We can now establish "$(\mathbb{T},\mathcal{V})$-versions" of the commutative diagrams given in 1.7 for $\mathbb{T}=\mathbb{I}$, namely:

$$\begin{array}{ccc} (\mathbb{T},\mathcal{V})\text{-}\mathsf{Cat} & \xrightarrow{(-)_*} & (\mathbb{T},\mathcal{V})\text{-}\mathsf{Mod} \\ \uparrow & & \uparrow \\ \mathsf{Set} & \xrightarrow{(-)_\#} & (\mathbb{T},\mathcal{V})\text{-}\mathsf{Rel} \end{array} \qquad \begin{array}{ccc} \big((\mathbb{T},\mathcal{V})\text{-}\mathsf{Cat}\big)^{\mathrm{op}} & \xrightarrow{(-)^*} & (\mathbb{T},\mathcal{V})\text{-}\mathsf{Mod} \\ \uparrow & & \uparrow \\ \mathsf{Set}^{\mathrm{op}} & \xrightarrow{(-)^\#} & (\mathbb{T},\mathcal{V})\text{-}\mathsf{Rel} \end{array}$$

Here the vertical full embeddings are given by $X \longmapsto (X, 1_X^\#)$ (see 2.6). Since for a unitary $(\mathbb{T},\mathcal{V})$-relation $r : X \longrightarrow\!\!\!\!\!\!\!+\;\; Y$ one has with 2.4(2)

$$r * e_X^\circ = r * 1_X^\# \le r \text{ and } e^\circ * r = 1_Y^\# * r \le r,$$

$r : (X, 1_X^\#) \multimap (Y, 1_Y^\#)$ is indeed a $(\mathbb{T},\mathcal{V})$-module. Since the diagrams are supposed to commute, one is forced to take for $f_\#$ and $f^\#$ the underlying $(\mathbb{T},\mathcal{V})$-relations of

$$\big(f : (X, 1_X^\#) \longrightarrow (Y, 1_Y^\#)\big)_* \text{ and } \big(f : (X, 1_X^\#) \longrightarrow (Y, 1_Y^\#)\big)^*,$$

respectively. Hence,

$$f_\# = 1_Y^\# \cdot Tf = e_Y^\circ \cdot \hat{T}1_Y \cdot Tf = e_Y^\circ \cdot \hat{T}f : X \longrightarrow\!\!\!\!\!\!\!+\;\; Y,$$
$$f^\# = f^\circ \cdot 1_Y^\# = f^\circ \cdot e_Y^\circ \cdot \hat{T}1_Y = e_X^\circ \cdot Tf^\circ \cdot \hat{T}1_Y = e_X^\circ \cdot \hat{T}(f^\circ) : Y \longrightarrow\!\!\!\!\!\!\!+\;\; X,$$

and $(-)_\#$, $(-)^\#$ are well-defined functors since $(-)_*$, $(-)^*$ are. Furthermore, $f_\# \dashv f^\#$. "Translation" of the Whiskering formulae 2.8 gives

$$t * f^\# = t \cdot \hat{T}(f^\circ) \text{ and } s * f_\# = s \cdot \hat{T}f$$

for a map $f : X \longrightarrow Y$, and for $t : X \longrightarrow\!\!\!\!\!\!\!+\;\; Z$ and $s : Y \longrightarrow\!\!\!\!\!\!\!+\;\; Z$ in $(\mathbb{T},\mathcal{V})$-Rel. A priori, these formulae hold for unitary $(\mathbb{T},\mathcal{V})$-relations, but with calculations similar to the proof of 2.6 one can show that they actually hold true *for all* $(\mathbb{T},\mathcal{V})$*-relations.*

2.10 Intermezzo: what $(\mathbb{T},\mathcal{V})$-Mod really is. There is a perfectly satisfactory description of $\mathcal{M} = (\mathbb{T},\mathcal{V})$-$\mathsf{Mod}$, as a special type of *distributor*, or of *equipment with scalar category* $\mathcal{K} = (\mathbb{T},\mathcal{V})$-$\mathsf{Cat}$, as discussed at various levels of generality by [Wood 1,2], [CKW] and [CKVW], up to a change in variance for the left action. Very briefly: we simply have a functor

$$\mathcal{M} : \mathcal{K}^{\mathrm{op}} \longrightarrow [\mathcal{K}^{\mathrm{op}}, \mathsf{Ord}]$$

with $\mathcal{M}(Y)(X) := \mathcal{M}(X,Y) := (\mathbb{T},\mathcal{V})\text{-}\mathsf{Mod}(X,Y)$,

$$\begin{aligned} \mathcal{M}(Y)(f) := \mathcal{M}(f,Y) : \mathcal{M}(X,Y) &\longrightarrow \mathcal{M}(X',Y), \\ \varphi &\longmapsto \varphi * f_* \\ \mathcal{M}(g)_X := \mathcal{M}(X,g) : \mathcal{M}(X,Y) &\longrightarrow \mathcal{M}(X,Y') \\ \varphi &\longmapsto g^* * \varphi \end{aligned}$$

for all $f : X' \longrightarrow X$, $g : Y' \longrightarrow Y$ in $\mathcal{K} = (\mathbb{T},\mathcal{V})$-$\mathsf{Cat}$. Well-definedness of $\mathcal{M}$ is reflected by the following rules which, with 2.8, are easily established.

$$\begin{array}{ll}\mathcal{M}(Y) \text{ is a functor:} & \varphi * (1_X)_* = \varphi,\ \varphi * (f \cdot f')_* = (\varphi * f_*) * f'_*;\\ \mathcal{M}(g) \text{ is a nat. tr.:} & g^* * (\varphi * f_*) = (g^* * \varphi) * f_*;\\ \mathcal{M} \text{ is a functor:} & 1_Y^* * \varphi = \varphi,\ (g \cdot g')^* * \varphi = (g')^* * (g^* * \varphi).\end{array}$$

Moreover, the equipment is *starred* whenever $*$ is associative (see 2.4(3)), since then

$$\mathcal{M}(f, Y) : \mathcal{M}(X, Y) \longrightarrow \mathcal{M}(X', Y) \text{ has a right adjoint: } \psi \longmapsto \psi * f^*,$$
$$\mathcal{M}(X, g) : \mathcal{M}(X, Y) \longrightarrow \mathcal{M}(X, Y') \text{ has a left adjoint: } \psi \longmapsto g_* * \psi,$$

for all X, Y, f, g as above. Even without the associativity assumption one has always the adjointness conditions:

$$\begin{array}{ll}(\varphi * f_*) * f^* \leq \varphi, & (\psi * f^*) * f_* \geq \psi,\\ g_* * (g^* * \varphi) \leq \varphi, & g^* * (g_* * \psi) \geq \psi.\end{array}$$

Of course, what we have said here about $\mathcal{K} = (\mathbb{T}, \mathcal{V})$-$\mathsf{Cat}$ and $\mathcal{M} = (\mathbb{T}, \mathcal{V})$-$\mathsf{Mod}$ is valid also for $\mathcal{K} = \mathsf{Set}$ and $\mathcal{M} = (\mathbb{T}, \mathcal{V})$-$\mathsf{Rel}$ where $f_\#$, $f^\#$ will take the role of f_*, f^*.

2.11 Algebraic functors. Let us, for a moment, consider a second Set-monad $\mathbb{S} = (S, d, n)$ with lax extension $\hat{S}$, and a morphism $j : \mathbb{S} \longrightarrow \mathbb{T}$ of monads (so that $j : S \longrightarrow T$ satisfies $j \cdot d = e$ and $j \cdot n = m \cdot jj$ in Set), such that $j : \hat{S} \longrightarrow \hat{T}$ is op-lax, that is: $j_Y \cdot \hat{S}r \leq \hat{T}r \cdot j_X$, or $\hat{S}r \cdot j_X^\circ \leq j_Y^\circ \cdot \hat{T}r$ for all $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ in $\mathcal{V}$-Rel. Then j induces a functor

$$\begin{array}{rl}J : (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat} & \longrightarrow (\mathbb{S}, \mathcal{V})\text{-}\mathsf{Cat}\\ (X, a) & \longmapsto (X, a \cdot j_X)\end{array}$$

JX is indeed a $(\mathbb{S}, \mathcal{V})$-algebra:

$$a \cdot j_X \cdot d_X = a \cdot e_X \geq 1_X,$$
$$a \cdot j_X \cdot \hat{S}(a \cdot j_X) \leq a \cdot j_X \cdot \hat{S}a \cdot Sj_X \leq a \cdot \hat{T}a \cdot j_{TX} \cdot Sj_X \leq a \cdot m_X \cdot (jj)_X = a \cdot j_X \cdot n_X;$$

and for $f : (X, a) \longrightarrow (Y, b)$ in $(\mathbb{T}, \mathcal{V})$-$\mathsf{Cat}$ one has

$$f \cdot a \cdot j_X \leq b \cdot Sf \cdot j_X = b \cdot j_Y \cdot Sf,$$

so that $f : JX \longrightarrow JY$ is in $(\mathbb{S}, \mathcal{V})$-$\mathsf{Cat}$. In the particular case $\mathbb{S} = \mathbb{I}$ and $j = e$ one obtains

$$\begin{array}{rl}J : (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat} & \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}\\ (X, a) & \longmapsto (X, a \cdot e_X),\end{array}$$

which has a left adjoint given by

$$\begin{array}{rl}I : \mathcal{V}\text{-}\mathsf{Cat} & \longrightarrow (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat}\\ (X, a) & \longmapsto (X, e_X^\circ \cdot \hat{T}a)\end{array}$$

$I(X,a)$ is indeed a $(\mathbb{T},\mathcal{V})$-algebra:

$$\begin{aligned} e_X^\circ \cdot \hat{T}a \cdot e_X &\geq e_X^\circ \cdot e_X \cdot a \geq a \geq 1_X, \\ e_X^\circ \cdot \hat{T}a \cdot \hat{T}(e_X^\circ \cdot \hat{T}a) &\leq e_X^\circ \cdot \hat{T}(a \cdot e_X^\circ \cdot \hat{T}a) \leq e_X^\circ \cdot \hat{T}(e_X^\circ \cdot \hat{T}a \cdot \hat{T}a) \\ &\leq e_X^\circ \cdot \hat{T}(e_X^\circ \cdot \hat{T}(a \cdot a)) \leq e_X^\circ \cdot \hat{T}(e_X^\circ \cdot \hat{T}a) \\ &\leq e_X^\circ \cdot (Te_X)^\circ \cdot \hat{T}\hat{T}a \leq e_X^\circ \cdot m_X \hat{T}\hat{T}a \leq e_X^\circ \cdot \hat{T}a \cdot m_X. \end{aligned}$$

I is easily seen to be functorial, and 1_X serves as both unit and counit for the adjunction: for $(X,a) \in \mathrm{ob}(\mathcal{V}\text{-}\mathsf{Cat})$ one has $JI(X,a) = (X, e_X^\circ \cdot \hat{T}a \cdot e_X)$ and $e_X^\circ \cdot \hat{T}a \cdot e_X \geq e_X^\circ \cdot e_X \cdot a \geq a$, and for $(X,a) \in \mathrm{ob}((\mathbb{T},\mathcal{V})\text{-}\mathsf{Cat})$ one has $IJ(X,a) = (X, e_X^\circ \cdot \hat{T}a \cdot Te_X)$ and $e_X^\circ \cdot \hat{T}a \cdot Te_X \leq e_X^\circ \cdot \hat{T}a \cdot m_X^\circ = e_X^\circ * a \leq a$.

Example: For $\mathbb{T} = \beta$ and $\mathcal{V} = 2$ one obtains the functor $J : \mathsf{Top} \longrightarrow \mathsf{Ord}$ which provides a topological space X with the dual of its specialization order: $x \leq y \iff \dot{x} \longrightarrow y \iff y \in \overline{\{x\}}$. Its left adjoint I provides an ordered set with the topology whose open sets are generated by the down sets $\downarrow x = \{z \in X \mid x \leq x\}$, $x \in X$.

2.12 Monad extension from Set to $\mathcal{V}$-Cat. The monad $\mathbb{T} = (T, e, m)$ of Set with its lax extension to $\mathcal{V}$-Rel can be (laxly) extended to a (proper) monad of $\mathcal{V}$-Cat (which we will denote by $\mathbb{T}$ again) and will then allow for a lax extension to $\mathcal{V}$-Mod (which we will denote by $\hat{T}$ again), as follows. For a $\mathcal{V}$-category $X = (X,a)$, put $TX = (TX, \hat{T}a)$; since

$$1_{TX} \leq \hat{T}1_X \leq \hat{T}a, \quad \hat{T}a \cdot \hat{T}a \leq \hat{T}(a \cdot a) = \hat{T}a,$$

this is a $\mathcal{V}$-category. Also, when $f : (X,a) \longrightarrow (Y,b)$ is a $\mathcal{V}$-functor, so is $Tf : (TX, \hat{T}a) \longrightarrow (TY, \hat{T}b)$, since

$$Tf \cdot \hat{T}a \leq \hat{T}(f \cdot a) \leq \hat{T}(b \cdot f) = \hat{T}b \cdot Tf.$$

Since $e : 1 \longrightarrow \hat{T}$ is op-lax, each $e_X : X \longrightarrow TX$ is a $\mathcal{V}$-functor, and since $m : \hat{T}\hat{T} \longrightarrow \hat{T}$ is op-lax, each $m_X : TTX \longrightarrow TX$ is a $\mathcal{V}$-functor. Hence, $\mathbb{T}$ is a monad of $\mathcal{V}$-Cat, but note that the diagram

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow{T} & \mathcal{V}\text{-}\mathsf{Cat} \\ \uparrow & & \uparrow \\ \mathsf{Set} & \xrightarrow{T} & \mathsf{Set} \end{array}$$

with the vertical embedding $X \longmapsto (X, 1_X)$ commutes only laxly since the inequality $1_{TX} \leq \hat{T}1_X$ may be strict.

Now, $\mathbb{T}$ of $\mathcal{V}$-Cat allows for a (proper) extension to a lax monad of $\mathcal{V}$-Mod when we put

$$\big(\varphi : (X,a) \multimap (Y,b)\big) \overset{\hat{T}}{\longmapsto} \big(\hat{T}\varphi : (TX, \hat{T}a) \multimap (TY, \hat{T}b)\big).$$

$\hat{T} : \mathcal{V}\text{-}\mathsf{Mod} \longrightarrow \mathcal{V}\text{-}\mathsf{Mod}$ is a lax functor, which extends $T : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ properly, in the sense that the following diagrams commute strictly:

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Mod} & \xrightarrow{\hat{T}} & \mathcal{V}\text{-}\mathsf{Mod} \\ {\scriptstyle (-)_*}\uparrow & & \uparrow{\scriptstyle (-)_*} \\ \mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow{T} & \mathcal{V}\text{-}\mathsf{Cat} \end{array} \qquad \begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Mod} & \xrightarrow{\hat{T}} & \mathcal{V}\text{-}\mathsf{Mod} \\ {\scriptstyle (-)^*}\uparrow & & \uparrow{\scriptstyle (-)^*} \\ \big(\mathcal{V}\text{-}\mathsf{Cat}\big)^{\mathrm{op}} & \xrightarrow{T^{\mathrm{op}}} & \big(\mathcal{V}\text{-}\mathsf{Cat}\big)^{\mathrm{op}} \end{array}$$

2.13 $(\mathbb{T}, \mathcal{V})$**-**$\mathsf{Cat}$ **versus** $(\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$**.** With the monad $\mathbb{T}$ of $\mathcal{V}\text{-}\mathsf{Cat}$, we can consider its Eilenberg-Moore category and define

$$K : (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}} \longrightarrow (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat}$$

as follows: for $X = (X, a)$ in $\mathcal{V}\text{-}\mathsf{Cat}$ with $\mathbb{T}$-algebraic structure $\xi : TX \longrightarrow X$, let $K(X, \xi) = (X, a \cdot \xi)$. K is well defined on objects since

$$\begin{aligned} 1_X \le a \cdot 1_X &= a \cdot (\xi \cdot e_X) = (a \cdot \xi) \cdot e_X, \\ a \cdot \xi \cdot \hat{T}(a \cdot \xi) &= a \cdot \xi \cdot \hat{T}a \cdot T\xi \\ &\le a \cdot a \cdot \xi \cdot T\xi \qquad \text{since } \xi \text{ is a } \mathcal{V}\text{-functor} \\ &= a \cdot \xi \cdot m_X. \end{aligned}$$

For a $\mathbb{T}$-homomorphism $f : (X, \xi) \longrightarrow (Y, \zeta)$ with $Y = (Y, \zeta)$ one has

$$f \cdot (a \cdot \xi) \le b \cdot f \cdot \xi = b \cdot \zeta \cdot Tf,$$

so that we can simply put $Kf = f$. With the forgetful functor $U^{\mathbb{T}}$ and J of 2.10, the diagram

$$\begin{array}{ccc} (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}} & \xrightarrow{\quad K \quad} & (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat} \\ & {\scriptstyle U^{\mathbb{T}}}\searrow \quad \swarrow{\scriptstyle J} & \\ & \mathcal{V}\text{-}\mathsf{Cat} & \end{array}$$

commutes. Note that $\mathsf{Set}^{\mathbb{T}}$ is coreflectively embedded in $(\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$, via

$$(X, \xi) \longmapsto ((X, 1_X), \xi);$$

here one uses the fact that ξ is a unitary $(\mathbb{T}, \mathcal{V})$-relation in order to see that ξ becomes indeed a $\mathcal{V}$-functor.

The restriction of K to $\mathsf{Set}^{\mathbb{T}}$ has been used in [CH] to show that $\mathcal{V}$ itself can be considered as a $(\mathbb{T}, \mathcal{V})$-category whenever it comes equipped with a $\mathbb{T}$-algebra structure over Set.

2.14. The categories $(\mathbb{T}, \mathcal{V})$-$\mathsf{ModCat}$ **and** $(\mathbb{T}, \mathcal{V})$**-**$\mathsf{CatCat}$**.** We are now going to enlarge the domain of definition of K considerably. The objects of the category

$$(\mathbb{T}, \mathcal{V})\text{-}\mathsf{ModCat}$$

are triples (X, a, c) with a $\mathcal{V}$-category (X, a) and a $(\mathbb{T}, \mathcal{V})$-category (X, c) such that $c \cdot \hat{T}a \le c$, $a \cdot c \le c$; briefly, $c : (TX, \hat{T}a) \longrightarrow\!\!\!\!\!\!\circ\;\; (X, a)$ is a $\mathcal{V}$-*module.*

Morphisms are mappings that live simultaneously in $\mathcal{V}$-Cat and in $(\mathbb{T},\mathcal{V})$-Cat. The forgetful functor

$$U : (\mathbb{T},\mathcal{V})\text{-}\mathsf{ModCat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$$

turns out to be *topological*, with U-initial structures formed as for $(\mathbb{T},\mathcal{V})$-$\mathsf{Cat} \longrightarrow \mathsf{Set}$. Its left adjoint $\bar{I}$ is induced by I of 2.11:

$$\bar{I}(X,a) = (X, a, e_X^\circ \cdot \hat{T}a).$$

Every object $((X,a),\xi)$ in $(\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$ gives the $(\mathbb{T},\mathcal{V})$-ModCat object (X,a,ξ_*); hence, there is a natural functor

$$C : (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}} \longrightarrow (\mathbb{T},\mathcal{V})\text{-}\mathsf{ModCat}.$$

Now, let us "extend" the functor K of 2.13 to

$$\bar{K} : (\mathbb{T},\mathcal{V})\text{-}\mathsf{ModCat} \longrightarrow (\mathbb{T},\mathcal{V})\text{-}\mathsf{Cat},$$

putting $\bar{K}(X,a,c) = (X,c)$. Then, for $((X,a),\xi) \in \mathrm{ob}(\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$,

$$\bar{K}C((X,a),\xi) = (X,\xi_*) = (X, a\cdot\xi) = K((X,a),\xi),$$

so that $\bar{K}$ does indeed extend K along C. We can extend the domain of definition of $\bar{K}$ by considering the category

$$(\mathbb{T},\mathcal{V})\text{-}\mathsf{CatCat}$$

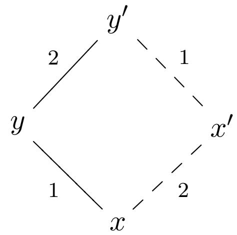
as follows: objects are triples (X,a,c) with $(X,a) \in \mathrm{ob}(\mathcal{V}\text{-}\mathsf{Cat})$, $(X,c) \in \mathrm{ob}((\mathbb{T},\mathcal{V})\text{-}\mathsf{Cat})$, and

$$c \cdot \hat{T}a \leq a \cdot c \text{ and } \hat{T}(a\cdot c) = \hat{T}a \cdot \hat{T}c;$$

as for $(\mathbb{T},\mathcal{V})$-ModCat,

morphisms $f : (X,a,c) \longrightarrow (Y,b,d)$ are mappings with $f : (X,c) \longrightarrow (Y,b)$ in $\mathcal{V}$-Cat and $f : (X,c) \longrightarrow (Y,d)$ in $(\mathbb{T},\mathcal{V})$-Cat.

Examples: 1. Objects in $(\mathbb{I},2)$-CatCat are sets X equipped with two order relations $\leq_1$, $\leq_2$ such that, whenever $x \leq_1 y$ and $y \leq_2 y'$, then there is x' with $x' \leq_1 y'$ and $x \leq_2 x'$:



K endows X with the composite order.

2. Objects in $(\beta,2)$-CatCat are ordered topological spaces, eg. topological spaces X which carry an order relation $\leq$ such that, whenever $\mathfrak{x} \leq \mathfrak{y}$ and $\mathfrak{y} \longrightarrow y$, then $\mathfrak{x} \longrightarrow x$ for some $x \leq y$; here the order of X is extended to βX as in 2.1:

$$\mathfrak{x} \leq \mathfrak{y} \iff \forall B \in \mathfrak{y} : \downarrow B \in \mathfrak{x},$$

with $\downarrow B = \{x \in X \mid \exists y \in B : x \leq y\}$. K equips the ordered topological space with a new topology in which $\mathfrak{x}$ converges to y precisely when, in the original topology, $\mathfrak{x}$ converges to some $x \leq y$.

$(\mathbb{T}, \mathcal{V})$-$\mathsf{ModCat}$ is a full reflective subcategory of $(\mathbb{T}, \mathcal{V})$-$\mathsf{CatCat}$: the left adjoint W to the inclusion functor Z is given by

$$W(X, a, c) = (X, a, a \cdot c)$$

for all $(X, a, c) \in \mathrm{ob}(\mathbb{T}, \mathcal{V})$-$\mathsf{CatCat}$. When we compose W with $\bar{K}$ we obtain

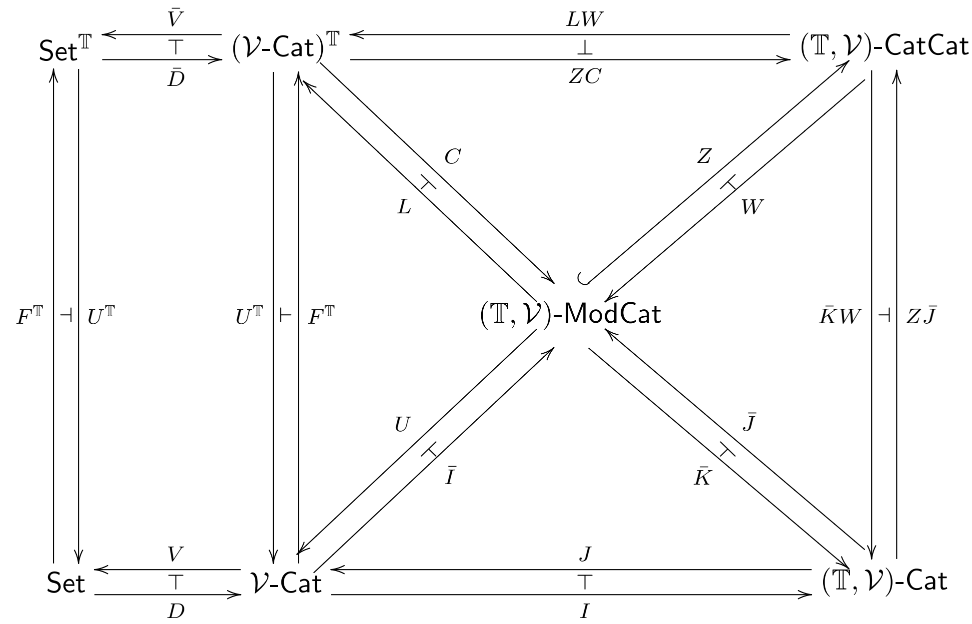
$$\bar{K}W(X, a, c) = (X, a \cdot c).$$

We finally note that $\bar{K}$ has a right adjoint $\bar{J} : (\mathbb{T}, \mathcal{V})\text{-}\mathsf{Cat} \longrightarrow (\mathbb{T}, \mathcal{V})\text{-}\mathsf{ModCat}$, induced by J of 2.11:

$$\bar{J}(X, c) = (X, c \cdot e_X, c).$$

We must leave all verifications to the reader.

2.15 A commutative diagram. The following commutative diagram of adjunctions (compose right adjoints with right adjoints and left adjoints with left adjoints) summarizes the interrelations established so far.



Here V is the forgetful functor, with left adjoint $D : X \longmapsto (X, 1_X)$. This adjunction "lifts" to $\bar{D} \dashv \bar{V}$ where $\bar{D}$ is the coreflective embedding encountered in 2.13. We have no explicit description of the left adjoint $L \dashv C$ but know that it exists, from "Wyler's Taut Lift Theorem", since C preserves initial structures (see [Tho1]). In addition to $VD = 1$, $\bar{V}\bar{D} = 1$ one has some other adjunction units or co-units being identity morphisms: since $\bar{K}\bar{J} = 1$, and $U\bar{I} = 1$, $\bar{J}$ is a reflective embedding and $\bar{I}$ is a coreflective embedding; finally, since $WZ = 1$, $(\mathbb{T}, \mathcal{V})$-$\mathsf{ModCat}$ is reflectively embedded into $(\mathbb{T}, \mathcal{V})$-$\mathsf{CatCat}$.

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