INJECTIVITY VERSUS EXPONENTIABILITY

WALTER THOLEN*

To my friend Jirka

Abstract. At the morphism level, exponentiability implies the existence of certain injective hulls. We prove a converse statement, thus showing an intimate link between the concepts given by the title.

1. Introduction

Injectivity and projectivity of objects are important concepts which category theory inherited from homological and commutative algebra. Their treatment in many standard books on general categories (like [ML]) is nevertheless quite brief ([Bo] being a notable exception), probably because the must-know facts like closure under (co)products and retracts yield little more than a series of easy exercises. However, the theme plays a central role in categorical model theory and receives excellent attention in Jiří Adámek's work, notably in his monograph [AR] with Rosický, for example in their characterization theorem for accessible categories with products, as the small-injectivity classes of locally presentable categories.

That injectivity and projectivity of morphisms (considered as objects of the sliced categories over their codomains and domains, respectively) are hidden features of the left- and right-lifting properties as used to define Quillen model structures does not seem to have been spelled out clearly until fairly recently (see [H], [AHRT1]), and there has not been a lot of work which exploits this aspect intensively. Exceptions are two existence theorems for weak factorization systems, one based on a Quillen-type "Generalized Small Object Argument" as given in [AHRT2], and the other one based on a general existence criterion for injective hulls given by Banaschewski [Ba] for general algebras and presented in a general categorical context in [T1].

In [T2] we began our investigation of injective hulls of morphisms, with injectivity to be understood relative to a class $\mathcal{H}$ of exponentiable monomorphisms, and proved an existence criterion, a refined version of which was given in [T3], Theorem 3.5. There remained, however, an unsatisfactory aspect, as follows. When factoring a morphism $f : A \to B$ as $f = q \cdot k$ with $k \in \mathcal{H}$ and $q \in \mathcal{P}$, where $(\mathcal{H}, \mathcal{P})$ is a weak factorization system, one may regard q as an $\mathcal{H}$-injective object over B or k as a $\mathcal{P}$-projective object under A. In [T3] we considered two distinct refinements of this situation, requiring q

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to be (the object part of) an $\mathcal{H}$-injective hull of f, or k to be the $\mathcal{P}$-projective cover of f. This led us to the notions of left-essential and right-essential weak factorization system, respectively, but for which we could not give any differentiating examples. Also from a theoretical point of view it was disappointing to see the self-dual notion of weak factorization system split up into two no longer self-dual subnotions.

In this paper we begin by revisiting the general notion of injective hull (or projective cover) and observe that the undesirable effects just described are easily circumvented when we follow the lead of [R] (see Remark 3.4 of [R]) and, avoiding the standard notion of $\mathcal{H}$-essential morphism, resort to a more traditional notion of $\mathcal{H}$-injective hull. As a consequence we are able to give a further refinement of the existence theorem first presented in [T2, T3]. The spirit of this theorem (see 5.1 below) remains: exponentiability implies injectivity. The main point of this paper, however, is to prove that the converse slogan is also valid (see 4.1 below). This observation came as a surprise to us, since exponentiability seemed to be a rather strong and/or foreign property in the context of injectivity. Nevertheless, here we prove that weak exponentiability is a necessary condition for the existence of certain essential weak factorizations. Moreover, this injectivity-implies-exponentiability result actually appears to be "cleaner" than its counterpart, which is why we present it first.

At the end of this paper we discuss the two results in more specialized contexts, considering first extensive categories and then the Abelian category of k-vector spaces.

2. $\mathcal{H}$-injective hulls

Let $\mathcal{H}$ be a class of morphisms in a category $\mathcal{C}$ containing all isomorphisms and being closed under composition with them. Recall that an object A is $\mathcal{H}$-injective if the hom-map $\mathcal{C}(k, A) : \mathcal{C}(Y, A) \to \mathcal{C}(X, A)$ is surjective for all $k : X \to Y$ in $\mathcal{H}$. An *$\mathcal{H}$-injective hull* of an object X in $\mathcal{C}$ is a morphism $h : X \to A$ in $\mathcal{H}$ with A $\mathcal{H}$-injective, such that $th = h$ for an endomorphism t of A is possible only if t is an automorphism. Instead of the last condition, one often requires h to be *$\mathcal{H}$-essential*, so that $fh \in \mathcal{H}$ always implies $f \in \mathcal{H}$. This makes no difference, provided that $\mathcal{H}$ contains all split monomorphisms of $\mathcal{C}$:

Proposition 2.1. *Consider the following statements for $h : X \to A$ in $\mathcal{H}$ with A $\mathcal{H}$-injective:*

(i) *h is an $\mathcal{H}$-injective hull of X;*
(ii) *whenever $fh \in \mathcal{H}$, then f is a split monomorphism;*
(iii) *h is $\mathcal{H}$-essential.*

Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii), and all three statements are equivalent when $\mathsf{SplitMono} \subseteq \mathcal{H}$.

Proof. (i) $\Rightarrow$ (ii): When $fh \in \mathcal{H}$, since A is $\mathcal{H}$-injective, h factors through fh, so $gfh = h$ for some g. By (i), the endomorphism gf is an isomorphism, so that f must be a split monomorphism.

(ii) $\Rightarrow$ (i): If $th = h$, there is s with $st = 1$ by (ii). But then $sh = h$, so that s is also a split monomorphism. Hence, both s and t must be isomorphisms.

(iii) $\Rightarrow$ (i): If $th = h$ is $\mathcal{H}$-essential, then $t \in \mathcal{H}$, and one obtains s with $st = 1$ since A is $\mathcal{H}$-injective. Now $sh = h$ gives $s \in \mathcal{H}$, so that $\mathcal{H}$-injectivity of A makes also s a split monomorphism. Again, both s and t must be isomorphisms. ☐

Remark 2.2. (1) $\mathcal{H}$-injective hulls are uniquely determined, up to isomorphism: if both h and k are $\mathcal{H}$-injective hulls of X, then $fh = k$ for an isomorphism f.

(2) If $\mathcal{H}$-injective hulls exist and are given by extremal monomorphisms, they cannot be chosen naturally, unless every object is $\mathcal{H}$-injective; more precisely (see Theorem 3.2 of [AHRT3]): *if a natural transformation $\eta : 1 \to E$ with an endofunctor E of $\mathcal{C}$ is pointwise an $\mathcal{H}$-injective hull and a monomorphism, then it is also an epimorphism and, hence, an isomorphism when every morphism in $\mathcal{H}$ is an extremal monomorphism.*

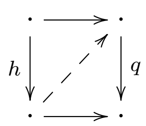
(3) If $\mathcal{H}$ is part of an orthogonal $(\mathcal{E}, \mathcal{H})$-factorization system for morphisms in $\mathcal{C}$, then it suffices to have

$$\mathcal{E} \cap \mathsf{SplitMono} \subseteq \mathcal{H}$$

in order for the statements (i) – (iii) of the Proposition to be equivalent. Indeed, if $h \in \mathcal{H}$ satisfies (ii) and $fh \in \mathcal{H}$, then also $eh \in \mathcal{H}$ for an $(\mathcal{E}, \mathcal{H})$-factorization $f = k \cdot e$, and $e \in \mathcal{E} \cap \mathsf{SplitMono}$ by hypothesis. Hence (iii), follows.

The class $\mathcal{H}$ defines, for every object B of $\mathcal{C}$, the class $\mathcal{H}_B = \Sigma_B^{-1}\mathcal{H}$ of morphisms in the sliced category $\mathcal{C}/B$ (with $\Sigma_B : \mathcal{C}/B \to \mathcal{C}$ the domain functor). A morphism $q : Y \to B$ of $\mathcal{C}$, considered as an object of $\mathcal{C}/B$, is $\mathcal{H}_B$-injective precisely when it has the right lifting property w.r.t. all morphisms h of $\mathcal{H}$:

(1)



any outer commutative rectangle with $h \in \mathcal{H}$ admits a diagonal making both triangles commutative; one writes $q \in \mathcal{H}^{\square}$ in this case (see [AHRT1]). A factorization $f = q \cdot k$ of a morphism f is called *essential* if $qt = q$ and $tk = k$ always implies that t is an isomorphism. Hence, if $k \in \mathcal{H}$ and $q \in \mathcal{H}^{\square}$, this means that *$q$ is an $\mathcal{H}_B$-injective hull of f in $\mathcal{C}/B$.* In what follows we will give necessary and sufficient conditions for certain morphisms f with codomain B to admit an $\mathcal{H}_B$-injective hull.

3. Preliminaries on weakly exponentiable morphisms

Recall that, for a functor $G : \mathcal{A} \to \mathcal{X}$ a morphism $w : X \to GA$ in $\mathcal{X}$ with an object A in $\mathcal{A}$ is called a *weakly G-universal arrow* for X if every morphism $f : X \to GB$ with $B \in \mathrm{ob}\,\mathcal{A}$ factors as $f = Gg \cdot w$, for some $g : A \to B$ in $\mathcal{A}$. We call the weakly G-universal arrow *essential* if every $t : A \to A$ in $\mathcal{A}$ with $Gt \cdot w = w$ is an isomorphism. (*Stable* is used in [R] instead of essential, but stable seems to be too reminiscent of pullback stable.) The dual notion is (essential) weakly G-couniversal arrow. Of course, every (strict) G-(co)universal arrow (so that the factorization $f = Gg \cdot w$ is unique) is essential.

For a morphism $h : C \to B$ in a category $\mathcal{C}$ with pullbacks, we consider the functor "pulling back along h"

$$h^* : \mathcal{C}/B \to \mathcal{C}/C$$

and its left adjoint $h_!$, "composing with h". Recall that h is an exponentiable morphism in $\mathcal{C}$ (see [N]) when h^* has a right adjoint, e.g. when for every $p : A \to C$ in $\mathcal{C}$ there is $\bar{p} : D \to B$ and $w : C \times_B D \to A$ in $\mathcal{C}$ such that $w : h^*(\bar{p}) \to p$ is an h^*-couniversal arrow for p in $\mathcal{C}/C$.

It is well known that for h exponentiable, every such w is an isomorphism precisely when h is a monomorphism in $\mathcal{C}$, because in the double adjunction

$$h_! \dashv h^* \dashv h_*$$

the right adjoint h_* is full and faithful if, and only if, the left adjoint $h_!$ is full and faithful, and the latter properly holds precisely when h is a monomorphism (see [DT]).

Here is what we can prove when w is just weakly h^*-couniversal:

Proposition 3.1. *For a monomorphism $h : C \to B$ of $\mathcal{C}$ every weakly h^*-couniversal arrow is a split epimorphism in $\mathcal{C}/C$.*

Proof. When h is a monomorphism,

$$\begin{array}{ccc} A & \xrightarrow{p} & C \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{hp} & B \end{array}$$

is a pullback diagram for all p. Hence, for a weakly h^*-couniversal arrow $w : h^*(\bar{p}) \to p$ in $\mathcal{C}/C$ with $\bar{p} : D \to B$ in $\mathcal{C}$ one obtains $\bar{h} : A \to D$ in $\mathcal{C}$ with $\bar{p}\bar{h} = hp$ that makes

$$\begin{array}{ccc} h^*(\bar{p}) & \xrightarrow{w} & p \\ {\scriptstyle v=h^*(\bar{h})}\uparrow & \nearrow{\scriptstyle 1} & \\ h^*(hp) & & \end{array}$$

commute (Here v is determined by $h'v = \bar{h}$ and $p'v = p$ with h', p' forming the pullback of h, p.) Consequently, w is a split epimorphism in $\mathcal{C}/C$. □

Without assuming the exponentiability of h in full, we can now still prove:

Proposition 3.2. *For a monomorphism h of $\mathcal{C}$, every h^*-couniversal arrow is an isomorphism.*

Proof. In the notation of the Proof of 3.1, we must show $vw = 1$. For the morphisms

$$hp' \xrightarrow{vw} hp' \xrightarrow{h'} \bar{p}$$

in $\mathcal{C}/B$ one has

$$p' = h^*(hp') \xrightarrow{h^*(vw)=vw} p' = h^*(hp') \xrightarrow{h^*(h')=1} p' = h^*(\bar{p})$$

in $\mathcal{C}/C$. Hence,

$$w \cdot h^*(h'vw) = w \cdot h^*(h') \cdot h^*(vw) = wvw = w = w \cdot h^*(h'),$$

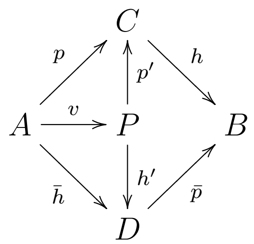
and therefore $h'vw = h'$, by the couniversality of w. Since also $p'vw = p'$, $vw = 1$, follows. □

4. Injectivity implies weak exponentiability

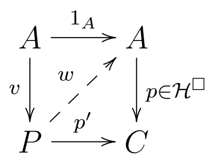
Throughout this section *we assume $\mathcal{C}$ to have pullbacks and $\mathcal{H}$ to be stable under pullback and weakly left cancellable* (so that $hk \in \mathcal{H}$, $h \in \mathcal{H}$ implies $k \in \mathcal{H}$). Note that the latter property is implied by the former when $\mathcal{H}$ is a class of monomorphisms; and both properties hold for any $\mathcal{H}$ that is part of an orthogonal $(\mathcal{E}, \mathcal{H})$-factorization system. Without any additional hypothesis on $\mathcal{H}$ one can prove:

Theorem 4.1. *Let $f = hp$ be a morphism with $h : C \to B$ in $\mathcal{H}$ and $p \in \mathcal{H}^\square$. If f allows a factorization $f = \bar{p}\bar{h}$ with $\bar{h} \in \mathcal{H}$, $\bar{p} \in \mathcal{H}^\square$, then there exists a weakly h^*-couniversal arrow $w : h^*(\bar{p}) \to p$, which is a split epimorphism in $\mathcal{C}/C$, and even an isomorphism when the factorization $f = \bar{p}\bar{h}$ is essential. In that case the weakly h^*-couniversal arrow is also essential.*

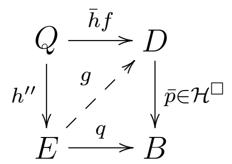
Proof. One forms the pullback h', p' of h, $\bar{p}$. The comparison morphism v making the diagram

(2)


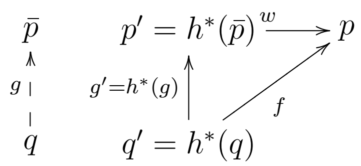
commutative must lie in $\mathcal{H}$, by the weak left cancellability, since $\bar{h} = h'v \in \mathcal{H}$ and $h' \in \mathcal{H}$, by pullback stability. Now one obtains w rendering



commutative, and we claim that $w : p' = h^*(\bar{p}) \to p$ is weakly h^*-couniversal. Hence, for $q : E \to B$ in $\mathcal{C}$ and $f : q' = h^*(q) \to p$ in $\mathcal{C}/C$ one considers

(3)


where h'' is the pullback of h along q. Since $\bar{p}\bar{h}f = hpf = hq' = qh''$, one obtains g with $gh'' = \bar{h}f$, $qg = \bar{p}$. We must show that

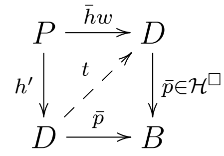
(4)


commutes, where g' is determined by $h'g' = gh''$, $p'g' = q'$. But

$$h'vf = \bar{h}f = gh'' = h'g' \text{ and } p'vf = pf = q' = p'g'.$$

Hence, $vf = g'$, which implies $f = wg'$, as desired.

Let us now assume that the factorization $f = \bar{p}\bar{h}$ is essential. From $\bar{p}\bar{h}w = hpw = hp' = \bar{p}h'$ one obtains t with $\bar{p}t = \bar{p}$, $th' = \bar{h}w$:



Then $t\bar{h} = th'v = \bar{h}wv = \bar{h}$, so that essentiality of the factorization makes t an isomorphism. Now

$$th'vw = \bar{h}wvw = \bar{h}w = th'$$

gives $h'vw = h'$ which, in conjunction with $p'vw = p'$, proves $vw = 1$.

Finally, in order to show that the weakly h^*-couniversal arrow w is essential, consider any $s : \bar{p} \to \bar{p}$ in $\mathcal{C}/B$ with $ws' = w$, where $s' = h^*(s)$ is determined by $h's' = sh'$ and $p's' = p'$. But since w is an isomorphism, $s' = 1_P$, hence $h' = sh'$. Consequently, $s\bar{h} = sh'v = h'v = \bar{h}$, so that essentiality of the factorization $f = \bar{p}\bar{h}$ renders s an isomorphism. □

Remark 4.2. If in Theorem 4.1 one has not just $\bar{p} \in \mathcal{H}^{\Box}$ but $\bar{p} \in \mathcal{H}^{\perp}$ (= the class of morphisms q such that for every outer commutative square (1) with $h \in \mathcal{H}$, there is a *unique* diagonal making both triangles commute), so that in particular any morphism g making diagram (3) commutative is uniquely determined, then the weakly h^*-commutative arrow $w : h^*(\bar{p}) \to p$ is actually h^*-couniversal. This holds because any morphism g rendering (4) commutative will also make (3) commute: with $g' = h^*(g)$ one has

$$\bar{h}f = \bar{h}wg' = h'vwg' = h'g' = gh'',$$

since $\bar{p} \in \mathcal{H}^{\perp}$ makes the factorization $f = \bar{p}\bar{h}$ essential, so that $vw = 1$ follows.

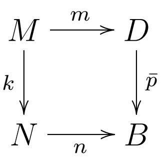
Corollary 4.3. *Let $(\mathcal{E}, \mathcal{H})$ be an orthogonal factorization system in a category with pullbacks, such that $\mathcal{E} \subseteq \mathcal{H}^{\Box}$ and every morphism $f : A \to B$ has an $\mathcal{H}_B$-injective hull in $\mathcal{C}/B$. Then for all $p : A \to C$ in $\mathcal{E}$ and $h : C \to B$ in $\mathcal{H}$ there exists an essential weakly h^*-couniversal arrow for p which is an isomorphism.*

5. Exponentiability implies injectivity

$\mathcal{H}$ continues to be a pullback-stable class of morphisms in a category $\mathcal{C}$ with pullbacks.

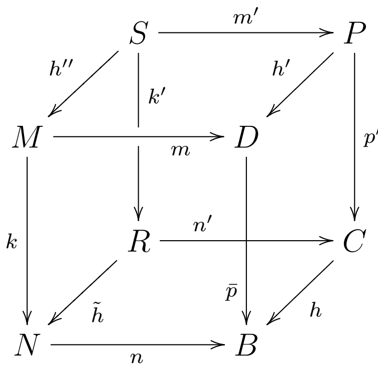
Theorem 5.1. *For a morphism $f = hp$ with $h : C \to B$ in $\mathcal{H}$ and $p \in \mathcal{H}^{\Box}$, let $w : h^*(\bar{p}) \to p$ be an h^*-couniversal arrow for p in $\mathcal{C}/C$ with $\bar{p}$ in $\mathcal{C}/B$. Furthermore, assume that h is a monomorphism,* **or** *that w is a split epimorphism in $\mathcal{C}/C$ and $\mathcal{H} \cdot \mathsf{SplitMono} \subseteq \mathcal{H}$. Then f has a factorization $f = \bar{p}\bar{h}$ with $\bar{h} \in \mathcal{H}$ and $\bar{p} \in \mathcal{H}^{\Box}$, and this factorization is essential when w is an isomorphism, in particular when h is a monomorphism.*

Proof. Initially, we will just work with a weakly h^*-couniversal arrow w for p in $\mathcal{C}/C$ which is a split epimorphism in $\mathcal{C}/C$, so that there is v making diagram (2) of 4.1 commutative, with $\bar{h} := vh'$. Of course, when w is (strictly) h^*-couniversal and h a monomorphism, w and v are isomorphisms by Prop. 3.2, and $\bar{h} \in \mathcal{H}$ follows since $\mathcal{H}$ is stable. But one obtains $\bar{h} \in \mathcal{H}$ also under the hypothesis $\mathcal{H} \cdot \mathsf{SplitMono} \subseteq \mathcal{H}$. In order to prove $\bar{p} \in \mathcal{H}^{\Box}$ one considers the commutative diagram

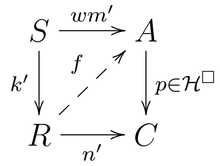


with $k \in \mathcal{H}$. Using just weak h^*-couniversality we will show the existence of $g : N \to D$ with $\bar{p}g = n$, but we will need (strict) h^*-couniversality of w to obtain also $gk = m$.

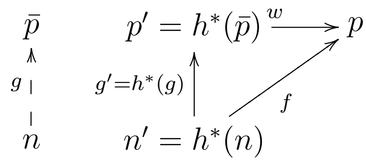
The commutative diagram



is obtained by pulling back along h, hence all but possibly the front and back faces are pullback diagrams. Since $pwm' = p'm' = n'k'$, one obtains f making

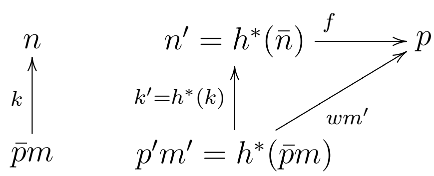


commute, and then $g : n \to \bar{p}$ in $\mathcal{C}/B$ rendering



commutative; here $g' : R \to P$ is determined by $p'g' = n'$, $h'g' = g\tilde{h}$.

As a morphism in $\mathcal{C}/B$ g satisfies $\bar{p}g = n$, and we are left with having to show $gk = m$. But since



commutes, from

$$wh^*(gk) = wg'k' = fk' = wm' = wh^*(m)$$

with $m : \bar{p}m \to \bar{p}$ in $\mathcal{C}/B$ one obtains $gk = m$ when w is h^*-couniversal.

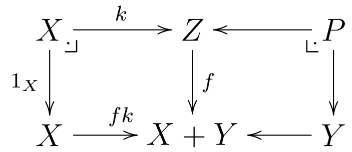
Finally, for the essentiality of the factorization $f = \bar{p}\bar{h}$, one considers $s : D \to D$ with $\bar{p}s = \bar{p}$ and $s\bar{h} = \bar{h}$, hence $sh'v = h'v$, and $sh' = h'$ when w is an isomorphism. For the morphism $s' = h^*(s) : h^*(p) \to h^*(\bar{p})$ with $p's' = p' = h^*(\bar{p})$ and $h's' = sh'$ one then has $s' = 1$, and $wh^*(s) = w$ gives $s = 1$. □

Corollary 5.2. *Let $(\mathcal{E}, \mathcal{H})$ be an orthogonal factorization system in a category $\mathcal{C}$ with pullbacks and $\mathcal{E} \subseteq \mathcal{H}^{\Box}$. If every morphism in $\mathcal{H}$ is an exponentiable monomorphism in $\mathcal{C}$, then every morphism $A \to B$ in $\mathcal{C}$ has an $\mathcal{H}_B$-injective hull in $\mathcal{C}/B$.*

Corollary 5.3. *In a category with pullbacks and (regular epi, mono)-factorizations, assume that every regular epimorphism $A \to B$ in $\mathcal{C}$ is injective in $\mathcal{C}/B$. Then if monomorphisms in $\mathcal{C}$ are exponentiable, every morphism $A \to B$ in $\mathcal{C}$ has an injective hull in $\mathcal{C}/B$.*

6. Examples

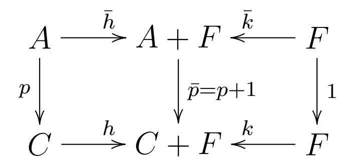
(1) In every category with binary coproducts one has a weak factorization system $(\mathcal{H}, \mathsf{SplitEpi})$, with every coproduct injection lying in $\mathcal{H}$. Moreover, if the category is extensive (see [CLW]), then $\mathcal{H}$ consists precisely of the class of coproduct injections (see [T3], Prop. 2.6 and Thm. 2.7). In this case $\mathcal{H}$ is stable under pullback, and also left cancellable (not just weakly). In fact, if fk is a coproduct injection (with any morphism f), k is also one, as the diagram



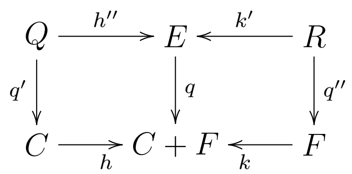
shows. Consequently, by Theorem 4.1, for every coproduct injection $h : C \to C + F$ there is a weakly h^*-couniversal arrow for every split epimorphism $p : A \to C$. But in fact, in this context, a much stronger statement can be proved which, in turn, could be used to reproduce Thm. 2.7 of [T3], as an application of Theorem 5.1:

Proposition 6.1. *In an extensive category every coproduct injection is exponentiable.*

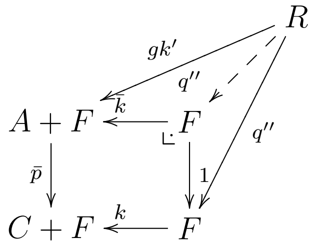
Proof. For the coproduct injection $h : C \to C+F$ and any morphism $p : A \to C$ consider the diagram



in which both rows are coproducts. Both rectangles are therefore pullback diagrams, and we can show that $1 : h^*(\bar{p}) \to p$ is h^*-couniversal for p, as follows. Given $f : q' = h^*(q) \to p$ as in the proof of 3.1, one forms the pullback diagram

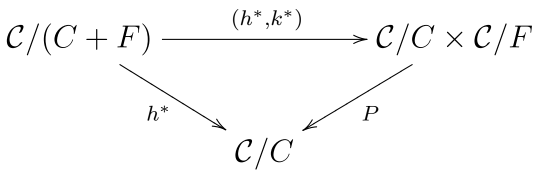


and finds $g : q \to \bar{p}$ in $\mathcal{C}/C + F$ with $h^*(g) = f$, and $g := f + g'' : E \cong Q + R \to A + F$. In order to see that $g : q \to \bar{p}$ is uniquely determined by $h^*(g) = f$ one needs to show that such g satisfies the defining conditions of $f + q''$, namely $gh'' = \bar{h}f$ and $gk' = kq''$, only the second of which is not immediate. But this equation follows with the pullback property:



□

There is a much shorter proof of Proposition 5.1, based on Schanuel's defining property of extensive category, as follows: The top arrow of

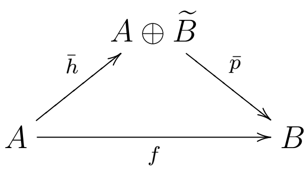


is an equivalence of categories, and the projection P has a trivial right adjoint R, given by $R(p) = (p, 1_F)$. Hence, h^* has also a right adjoint. We nevertheless included the longer proof since it shows the construction explicitly.

We remark in passing that, without the extensitivity assumption for $\mathcal{C}$, while the functor (h^*, k^*) has always a left adjoint, given by coproduct, when the coproduct injections j, k are exponentiable monomorphisms (h^*, k^*) has also a right adjoint. Indeed, an (h^*, k^*)-couniversal arrow for $(p, q) \in \mathrm{ob}(\mathcal{C}/C \times \mathcal{C}/F)$ is obtained by forming the pullback of $\bar{p}$ and $\bar{q}$ in $\mathcal{C}$, where $h^*(\bar{p}) = p$, $k^*(\bar{q}) = q$ represent couniversal arrows for p, q, respectively.

(2) Just like in Set, $(\mathsf{Mono}, \mathsf{Epi})$ is also a weak factorization system in the category Vec_k of k-vector spaces (for a field k), with $\mathsf{Mono} = \{$coproduct injections$\}$ and $\mathsf{Epi} = \mathsf{SplitEpi}$ (granting the Axiom of Choice). But unlike Set, the category Vec_k is not extensive. In fact, the (coproduct) injection of the x-axis into $\mathbb{R}^2$ is not exponentiable, as one easily sees by considering the y-axis and the line $x = y$. However, by Theorem 4.1, for every monomorphism $h : C \to B$ and every epimorphism $p : A \to C$, from an essential $(\mathsf{Mono}, \mathsf{Epi})$-factorization $f = \bar{p}\bar{h}$ of $f = hp$ we will still obtain a weakly h^*-couniversal arrow for p.

The question remains how to obtain $\bar{p}$, $\bar{h}$, e.g., how to obtain the injective hull of f? For that, take a subspace $\widetilde{B}$ of B with $B = \mathrm{im}\, f \oplus \widetilde{B}$ and consider the factorization



with $\bar{h} = \text{incl.}$, $\bar{p}|_A = f$, $\bar{p}|_{\widetilde{B}} = \text{incl.}$ We show that every endomorphism t of $A \oplus \widetilde{B}$ with $t\bar{h} = \bar{h}$, $\bar{p}t = \bar{p}$ must be an isomorphism. By hypothesis then, $t(a) = a$ for all $a \in A$. For $b \in \widetilde{B}$ we have $\bar{p}(t(b)) = \bar{p}(b) = b$, and writing $t(b) = \widetilde{a} + \widetilde{b}$ with $\widetilde{a} \in A$, $\widetilde{b} \in \widetilde{B}$, we obtain

$$b = \bar{p}(t(b)) = \bar{p}(\widetilde{a}) + \bar{p}(\widetilde{b}) = f(\widetilde{a}) + \widetilde{b},$$

so that $f(\widetilde{a}) = b - \widetilde{b} \in \operatorname{im} f \cap \widetilde{B} = 0$. Consequently, $b = \widetilde{b}$, hence $t(b) = \widetilde{a} + b = t(\widetilde{a}) + b$, so that b is in the range of t. It follows that t is surjective, and t has also a trivial kernel. For, if $t(a + b) = 0$ with any $a \in A$, then $0 = \bar{p}(t(a + b)) = f(a) + b$, hence $b = 0$ and then $a = t(a) = 0$.

What is the size and the structure of the group $G = \{t \in \text{End}_k(A \oplus \widetilde{B}) \mid t\bar{h} = \bar{h}, \bar{p}t = \bar{p}\}$? Interestingly, the map

$$G \to \hom_k(\widetilde{B}, \ker f), \quad t \mapsto (b \mapsto t(b) - b).$$

is a group isomorphism. Hence G is isomorphic to the additive group of a k-vector space; in particular, G is Abelian.

Of course, the above example may more generally be pursued in the context of R-modules provided that the needed direct summands exist.

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Department of Mathematics and Statistics, York University, Toronto ON, M3J 1P3, Canada