IDEAL-DETERMINED CATEGORIES

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Dedicated to Francis Borceux at the occasion of his sixtieth birthday

Abstract

We clarify the role of Hofmann's Axiom in the old-style definition of a semi-abelian category. By removing this axiom we obtain the categorical counterpart of the notion of an ideal-determined variety of universal algebras – which we therefore call an ideal-determined category. Using known counter-examples from universal algebra we conclude that there are ideal-determined categories which fail to be Mal'tsev. We also show that there are ideal-determined Mal'tsev categories which fail to be semi-abelian.

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1. Introduction

In modern terms, a pointed category C with finite limits and finite colimits is semi-abelian if it is Barr-exact and Bourn-protomodular. As shown in [JMT], these two conditions may be equivalently replaced by the following older-style axioms:

(A) Every morphism admits a pullback-stable (normal epi, mono)-factorization (where "normal epimorphism" means "cokernel of some morphism").

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(B) For every commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{q} & C \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ E & \xrightarrow[p]{} & B \end{array} \qquad (1.1)$$

with normal epimorphisms *p, q* and monomorphisms *v, w*, one has

(B1) if *w* is normal, then so is *v*;

(B2) (*"Hofmann's Axiom"*) if v is normal and ker(*p*) ≤ *w* as subobjects of *E,* then *w* is also normal.

While the equivalence proof for the new-versus-old-style definitions given in [JMT] went a long way towards Mac Lane's [M] original quest for an appropriate categorical setting that would allow for a generalization of various classical group-theoretic constructions and results (see in particular [BB]), the following rather obvious question remained unanswered:

Question 1.1. Is Hofmann's Axiom redundant in the list of old-style axioms (i.e., does (B2) follow from (A), (B1) for pointed finitely complete and finitely cocomplete categories)?

This question draws particular relevance from the fact that some authors worked in settings that do not include Hofmann's Axiom, especially those working in Kurosh-Amitsur radical theory.

By exploiting known results and counterexamples from universal algebra, in this paper we provide the expected negative answer to Question 1.1. In fact, we will show that a pointed finitely-complete and finitely-cocomplete Barr-exact category satisfying canditions (A), (B1)

– may fail to be Mal'tsev (which is a necessary condition for protomodularity in this context) and
– may fail to be protomodular even when it is Mal'tsev.

The pivotal step for this exploitation is the surprising realization that pointed varieties of universal algebras satisfying (A), (B1) were studied already in the 1970s and 80s under different names: they were called *BIT* (*"buona teoria degli ideali"*) in [U1] and *ideal-determined* in [GU], and this fact leads us not just to a single counterexample but to an interesting class of them. We use the latter term to introduce the categorical notion given in the title of this paper and use results from [JMU1 and [JMU2] to demonstrate its relevance beyond the resolution of Question 1.1. We conclude the paper with some open questions that should form the basis for future work in this context, work that should also clarify more comprehensively the status of the notion of ideal-determined category vis-a-vis Z. Janelidze's subtractive categories [J1].

2. Ideal-determined categories

Let us recall some notions from universal algebra used in this paper:

Definition 2.1. A pointed variety **C** of universal algebras is said to be *BIT* in the sense of [U1], or, equivalently, *ideal determined* in the sense of [GU], if its congruences are determined by its ideals, i.e., if the following two conditions hold:

(a) if every congruence on any algebra in **C** is generated by its 0-class (i.e., no smaller congruence has the same 0-class);

(b) every ideal in every algebra in **C** is normal, i.e., it is the 0-class of a congruence.

Varieties of universal algebras satisfying condition 2.1(a) are called 0-regular. On the other hand, as mentioned in [JMU1], in the language of categorical algebra condition 2.1(a) simply says that every regular epimorphism in **C** is normal. Since every variety of universal algebras admits a pullback-stable (regular epi, mono)-factorization, condition 2.1(a) is nothing but the algebraic version of condition (A) of the Introduction.

As shown in [JMU2], a subalgebra S of an algebra A in a pointed variety **C** is an ideal if, and only if, there exist a surjective homomorphism $f : A' \to A$ in **C** and a normal subalgebra N in A' for which $f(N) = A'$. Therefore, under condition (A), condition 2.1(b) is nothing but the algebraic version of condition (B1) of the Introduction.

Accordingly we introduce:

Definition 2.2. A category pointed finitely complete and finitely cocomplete category **C** is said to be *ideal determined* if it satisfies conditions (A) and ((B1).

We obtain immediately:

Proposition 2.3. A pointed variety of universal algebras is ideal determined as a category if and only if it is ideal determined in the sense of universal algebra.

Furthermore, the universal-algebraic motivation for “ideal determined” can be reformulated categorically as follows:

According to [JMU2], a monomorphism $v : C \to B$ in a pointed category **C** with finite limits and colimits satisfying condition (A) should be called an *ideal* if there exists a commutative diagram of the form (1.1) in **C**, in which p and q are normal epimorphisms and w is a normal monomorphism. Hence, with this terminology **C** is ideal determined if, and only if, its ideals are normal monomorphisms. On the other hand, condition (A) simply says that **C** is a regular category in which every regular epimorphism is normal.

Hence, in terms of the correspondence between normal monomorphisms and normal epimorphisms we may briefly say that i*deal-determined categories are regular categories in which regular epimorphisms are determined by ideals.*

Let us now recall when a variety of universal algebras is semi-abelian, combining past work from both category theory and universal algebra. The universal-algebraic side of the story was discovered in [JMU1] (see Theorems 1.3 and 1.4 in [JMU1]), with another crucial remark made in [JMU2]. The equivalence (a)⇔(c) in the following theorem follows also from the main result of [BJ], while (b)⇔(c) had been proved originally in [Be]:

Theorem 2.4. The following conditions on a pointed variety **C** of universal algebras are equivalent:

(a) **C** is a semi-abelian category;

(b) **C** satisfies the Split Short Five Lemma (see [JMT]);

(c) **C** is 0-coherent in the sense of E. Beutler [Be], i.e., for every A in **C**, every subalgebra A' in A, and every congruence R on A, one has:

$$\{a \in A \mid (0,a) \in R\} \subseteq A' \text{ implies } \{a \in A \mid (a',a) \in R\} \subseteq A' \text{ for all } a' \text{ in } A.$$

(d) **C** is *BIT speciale* in the sense of [U2] (=*classically ideal determined* in the sense of [U3]), i.e., there are binary terms $t_1, ..., t_n$, and an $(n+1)$-ary term t satisfying the identities $t(x,t_1(x,y),...,t_n(x,y)) = y$ and $t_i(x,x) = 0$ for each $i = 1, ..., n$.

This theorem shows that various semi-abelian categorical constructions are closely related to the universal-algebraic theory of Magari ideals as developed by Ursini and his collaborators in [U1], [U2], and by the authors of various subsequent papers.

Remark 2.5. Already from [U2] it is well known that not every ideal determined (=*BIT*) variety of universal algebras is classically ideal determined (=*BIT speciale*). Hence, not every ideal-determined category is semi-abelian, but is it always a Mal'tsev category? The negative answer is again provided by universal algebra. The first of a string of counter-examples was provided in [GU] ("implication algebras"), which eventually led to the proof of the following much stronger result by G. D. Barbour and J.G. Raftery [BR]: For every natural number $n \geq 2$ there is a pointed ideal-determined variety of universal algebras which has $(n+1)$-permutable congruences but not n-permutable congruences.

3. Not every ideal determined Mal'tsev category/variety is semi-abelian

Throughout this section **C** denotes a pointed variety of universal algebras. We shall write M(**C**) for the (pointed) variety obtained from **C** by adding a ternary operation p satisfying the Mal'tsev identities:

$$p(x,y,y) = x = p(y,y,x). \tag{3.1}$$

Given a morphism $\alpha : A \to B$ in **C** with B in M(**C**), we can always make A an object in M(**C**), by choosing any map (not necessarily a homomorphism) $\beta : \alpha(A) \to A$ with $\beta(0) = 0$ and $\alpha\beta(b) = b$ for each $b \in \alpha(A)$, and then by defining p on A by

$$p(x,y,z) = \begin{cases} x \text{ if } y = z, \\ z \text{ if } x = y, \\ \beta(p(\alpha(x),\alpha(y),\alpha(z))) \text{ if } x \neq y \neq z; \end{cases}$$

we will denote that object by $A[\alpha,\beta]$. The morphism α determines a morphism $A[\alpha,\beta] \to B$ in M(**C**), and if β is a morphism in **C,** it actually determines a morphism $B \to A[\alpha,\beta]$ in M(**C**).

Now, consider the diagram

$$\begin{array}{ccccc} K & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B \\ \| & & \downarrow \iota & & \| \\ K & \underset{\kappa}{\longrightarrow} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array} \tag{3.2}$$

in **C** constructed as follows:

- α and β are arbitrary morphisms in **C** with $\alpha\beta = 1_B$;
- $K = \alpha^{-1}(0)$ is the kernel of α, and A' is a subalgebra in A containing K and $\beta(B)$;
- ι, κ, and κ' are the inclusion maps, and α' and β' the induced maps determined by $\alpha' = \alpha\iota$ and $\iota\beta' = \beta$, respectively.

There are many ways of making B an object in M(**C**); let us put

$$p(x,y,z) = \begin{cases} x \text{ if } y = z, \\ z \text{ if } x = y, \\ 0 \text{ if } x \neq y \neq z \end{cases}$$

in B and denote this object by B_0. After that we can form $A[\alpha,\beta]$, and, since A' contains $\beta(B)$, it determines a subalgebra in $A[\alpha,\beta]$; moreover, that subalgebra is nothing but $A'[\alpha',\beta']$, and the diagram (3.2) determines a similar diagram in M(**C**), namely,

$$\begin{array}{ccccc} K_0 & \xrightarrow{\kappa'} & A'[\alpha',\beta'] & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B_0 \\ \| & & \Big\downarrow \iota & & \| \\ K_0 & \xrightarrow[\kappa]{} & A[\alpha,\beta] & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B_0, \end{array} \qquad (3.3)$$

where K_0 is constructed similarly to B_0. This proves:

Theorem 3.1. M(**C**) is semi-abelian if and only if so is **C**.

In particular, since we know that not every (pointed) ideal-determined variety is semi-abelian, we immediately conclude that not every ideal-determined Mal'tsev variety is semi-abelian. Therefore there exist Barr-exact, Mal'tsev and ideal-determined categories that are not semi-abelian.

Remark 3.2. (a) The arguments used for obtaining (3.3) from (3.2) and then deducing Theorem 3.1, apply obviously not just for (3.1) but also for similar conditions involving equalities of "one-step" terms.

(b) One could use similar arguments with β in (3.2) being merely a map with $\beta(0) = 0$ and $\alpha\beta(b) = b$ for every $b \in \alpha(A)$, not a homomorphism, as we originally required in the construction of $A[\alpha,\beta]$.

4. Four open questions

Question 4.1. Is every ideal-determined category Barr exact?

An obvious candidate for a counter-example would be a quasi-variety of universal algebras that generates a familiar ideal-determined variety. However, this does not work: in fact, it is easy to show that if a quasi-variety generates an ideal-determined variety then it is a variety. This indicates that one should begin by studying exact completions of ideal-determined categories.

Question 4.2. Is a pointed finitely complete and finitely cocomplete Barr-exact category ideal determined if, and only if, it satisfies condition (A) and is subtractive in the sense of Z. Janelidze [J1]?

Since the subtractive categories that are varieties are the same as subtractive varieties, in the "varietal" case the affirmative answer is well known [GU], and this is in fact the reason why we are interested in this question. Barr exactness is essential here, and the reason for that is clear not only by the comment to Question 4.3 given below, but also by the fact that, say, the category of torsion-free abelian groups is subtractive and satisfies condition (A) but fails to be ideal determined.

Question 4.3. Is a category **C** abelian whenever both **C** and $\mathbf{C}^{op}$ are ideal determined?

This question is closely related to the previous ones since, as shown by Z. Janelidze [J2], under mild additional conditions (which are much weaker than the conjunction of (A) and our standard assumption of beinf pointed and finitely complete and finitely cocomplete), **C** is additive whenever both **C** and $\mathbf{C}^{op}$ are subtractive. And together with Barr exactness additivity would imply abelianness.

Our fourth question is rather vague:

Question 4.4. What is the role of finite cocompleteness in this work?

Finite cocompleteness is not very often used, but it holds in all algebraic examples. It is not clear to us to what extent it would be interesting to study the classes of categories considered above without this assumption.

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