LAX DISTRIBUTIVE LAWS FOR TOPOLOGY, II
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Abstract. For a small quantaloid \( \mathcal{Q} \) we consider four fundamental 2-monads \( \mathbb{T} \) on \( \mathcal{Q}-\text{Cat} \), given by the presheaf 2-monad \( \mathbb{P} \) and the copresheaf 2-monad \( \mathbb{P}^! \), as well as their two composite 2-monads, and establish that they all laxly distribute over \( \mathbb{P} \). These four 2-monads therefore admit lax extensions to the category \( \mathcal{Q}-\text{Dist} \) of \( \mathcal{Q} \)-categories and their distributors. We characterize the corresponding \( (\mathbb{T}, \mathcal{Q}) \)-categories in each of the four cases, leading us to both known and novel categorical structures.

1. Introduction

The syntax used in Monoidal Topology [6] is given by a quantale \( \mathcal{V} \), a \( \text{Set} \)-monad \( \mathbb{T} \) and, most importantly, by a lax extension of \( \mathbb{T} \) to the 2-category \( \mathcal{V}\text{-Rel} \) of sets and \( \mathcal{V} \)-valued relations, or, equivalently, by a lax distributive law of \( \mathbb{T} \) over the discrete \( \mathcal{V} \)-presheaf monad \( \mathbb{P}_\mathcal{V} \), the Kleisli category of which is exactly \( \mathcal{V}\text{-Rel} \). Once equipped with such a lax extension or lax distributive law, the monad \( \mathbb{T} \) may then be naturally extended to become a 2-monad on the 2-category \( \mathcal{V}\text{-Cat} \). This lax monad extension from \( \text{Set} \) to \( \mathcal{V}\text{-Cat} \) facilitates the study of greatly enriched structures. For example, for \( \mathcal{V} \) the two-element chain and \( \mathbb{T} \) the ultrafilter monad, while the Eilenberg-Moore category over \( \text{Set} \) is \( \text{CompHaus} \), over \( \mathcal{V}\text{-Cat} \) one obtains ordered compact Hausdorff spaces, and when \( \mathcal{V} \) is Lawvere’s [12] extended half-line \([0, \infty]\), metric compact Hausdorff spaces; see [14, 29, 6]. Moreover, the functorial interaction between the Eilenberg-Moore category \( (\mathcal{V}\text{-Cat})^\mathbb{T} \) and the category \( (\mathbb{T}, \mathcal{V})\text{-Cat} \) of \( (\mathbb{T}, \mathcal{V}) \)-categories is a pivotal step for a serious study of representability, a powerful property which, in the basic example of the two-chain and the ultrafilter monad, entails core-compactness, or exponentiability, of topological spaces; see [3] and [6, Section III.5].

While this mechanism for generating a 2-monad on \( \mathcal{V}\text{-Cat} \) from a \( \text{Set} \)-monad provides an indispensable tool in monoidal topology, the question arises whether it is possible to make a given 2-monad \( \mathbb{T} \) on \( \mathcal{V}\text{-Cat} \) the starting point of a satisfactory theory, preferably even in the more general context of a small quantaloid \( \mathcal{Q} \), (i.e., a \( \text{Sup} \)-enriched category), rather than just a quantale \( \mathcal{V} \) (i.e., a \( \text{Sup} \)-enriched monoid), a context that has been...
propagated in this paper’s predecessor [30]. Such theory should, as a first step, entail
the study of lax extensions of $T$ to the 2-category $Q$-Dist of $Q$-categories and their
distributors (or (bi)modules), rather than just to $Q$-Rel, or, equivalently, the study of
lax distributive laws of $T$ over the non-discrete presheaf monad $P_{Q}$, rather than over its
discrete counterpart. The fact that the non-discrete presheaf monad is, other than its
discrete version, lax idempotent (i.e., of Kock-Zöberlein type [32, 10]), serves as a first
indicator that this approach should in fact lead to a categorically more satisfactory theory.

This paper makes the case for an affirmative answer to the question raised, even in
the extended context of a given small quantaloid $Q$. It is centred around four naturally
arising monads $T$ on $Q$-Cat which do not come about via the mechanism described above,
but should nevertheless be of considerable general interest. They all distribute laxly over
$P = P_{Q}$ and, hence, are laxly extendable to $Q$-Dist, and we give a detailed description
of the respective lax algebras, or $\mathcal{T}$-categories, arising. These monads are

- the presheaf 2-monad $P$ itself (Section 4);
- the copresheaf 2-monad $P^{\dagger}$ (Section 5);
- the double presheaf 2-monad $PP^{\dagger}$ (Section 6);
- the double copresheaf 2-monad $P^{\dagger}P$ (Section 7).

In each of the four cases, the establishment of the needed lax distributive law over $P$ and
the characterization of the corresponding lax algebras, or, equivalently, $(\mathcal{T}, Q)$-categories,
takes considerable “technical” effort, especially in the absence of any noticeable formal
resemblance between the four cases. However, the lax algebras pertaining to both, $P$
and $PP^{\dagger}$, are identified as $Q$-closure spaces, as considered in [21, 23]. Most challenging
has been the identification of the lax algebras pertaining to $PP^{\dagger}$, which we describe as
$Q$-interior spaces, a structure considered here for the first time. Also the lax algebras
pertaining to $P^{\dagger}$ are of a novel flavour; they are monoid objects in $Q$-Dist. Given that
their discrete cousins, i.e., the monoid objects in $Q$-Rel, are $Q$-categories, they surely
deserve further study.

We believe that we have given sufficiently many details in order to make the proofs
verifiable for the reader, also since all needed basic tools are comprehensively listed in
Section 2. In contrast to Sections 4−7, the introduction of lax distributive laws of a 2-
monad over the non-discrete presheaf monad and of their lax algebras (as given in Section
3), as well as the proof of the fact that they correspond bijectively to lax extensions of $T$
to $Q$-Dist, with lax algebras corresponding to $(\mathcal{T}, Q)$-categories (as given in Section 8),
are straightforward extensions of their “discrete” treatment in [30] and should therefore
constitute a relatively easy read. We have nevertheless given complete proofs, so that
prior reading of [30] is not required for the purpose of understanding this paper.
2. Quantaloid-enriched categories and their distributors

A quantaloid [18] is a category enriched in the monoidal-closed category Sup [9] of complete lattices and sup-preserving maps. Explicitly, a quantaloid $Q$ is a 2-category with its 2-cells given by an order "≤", such that each hom-set $Q(r,s)$ is a complete lattice and the composition of morphisms from either side preserves arbitrary suprema. Hence, $Q$ has “internal homs”, denoted by $\langle$ and $\rangle$, as the right adjoints of the composition functors:

$$- \circ u \dashv \langle u : Q(r,t) \to Q(s,t) \quad \text{and} \quad v \circ - \dashv \rangle v : Q(r,t) \to Q(r,s);$$

explicitly,

$$u \leq v \rangle w \iff v \circ u \leq w \iff v \leq w \langle u$$

for all morphisms $u : r \to s, v : s \to t, w : r \to t$ in $Q$.

Throughout this paper, we let $Q$ be a small quantaloid. From $Q$ one forms a new (large) quantaloid $Q\text{-Rel}$ of $Q$-relations with the following data: its objects are those of $\text{Set}/Q_0$ (with $Q_0 := \text{ob } Q$), i.e., sets $X$ equipped with an array (or type) map $\lvert - \rvert : X \to Q_0$, and a morphism $\varphi : X \to Y$ in $Q\text{-Rel}$ is a map that assigns to every pair $x \in X, y \in Y$ a morphism $\varphi(x,y) : \lvert x \rvert \to \lvert y \rvert$ in $Q$; its composite with $\psi : Y \to Z$ is defined by

$$(\psi \circ \varphi)(x,z) = \bigvee_{y \in Y} \psi(y,z) \circ \varphi(x,y),$$

and $1^\circ_X : X \to X$ with

$$1^\circ_X(x,y) = \begin{cases} 1_{\lvert x \rvert} & \text{if } x = y, \\ \bot & \text{else} \end{cases}$$

serves as the identity morphism on $X$. As $Q$-relations are equipped with the pointwise order inherited from $Q$, internal homs in $Q\text{-Rel}$ are computed pointwise as

$$(\theta \langle \varphi)(y,z) = \bigwedge_{x \in X} \theta(x,z) \langle \varphi(x,y) \quad \text{and} \quad (\psi \rangle \theta)(x,y) = \bigvee_{z \in Z} \psi(y,z) \rangle \theta(x,z)$$

for all $\varphi : X \to Y, \psi : Y \to Z, \theta : X \to Z$.

A (small) $Q$-category is precisely an (internal) monad in the 2-category $Q\text{-Rel}$; or equivalently, a monoid in the monoidal-closed category $(Q\text{-Rel}(X,X), \circ)$, for some $X$ over $Q_0$. Explicitly, a $Q$-category consists of an object $X$ in $\text{Set}/Q_0$ and a $Q$-relation $a : X \to X$ (its “hom”), such that $1^\circ_X \leq a$ and $a \circ a \leq a$. For every $Q$-category $(X,a)$, the underlying (pre)order on $X$ is given by

$$x \leq x' \iff \lvert x \rvert = \lvert x' \rvert \text{ and } 1_{\lvert x \rvert} \leq a(x,x'),$$

and we write $x \preceq x'$ if $x \leq x'$ and $x' \leq x$.

A map $f : (X,a) \to (Y,b)$ between $Q$-categories is a $Q$-functor (resp. fully faithful $Q$-functor) if it lives in $\text{Set}/Q_0$ and satisfies $a(x,x') \leq b(fx,fx')$ (resp. $a(x,x') = b(fx,fx')$) for all $x,x' \in X$. With the pointwise order of $Q$-functors inherited from $Y$, i.e.,

$$f \leq g : (X,a) \to (Y,b) \iff \forall x \in X : fx \leq gx \iff \forall x \in X : 1_{\lvert x \rvert} \leq b(fx,gx),$$
\(Q\)-categories and \(Q\)-functors are organized into a 2-category \(Q\)-\textsc{Cat}.

The one-object quantaloids are the (unital) quantales (see [17]); equivalently, a quantale is a complete lattice \(V\) with a monoid structure whose binary operation \(\otimes\) preserves suprema in each variable. The \(\otimes\)-neutral element is generally denoted by \(k\); so \(k = 1\) if we denote by \(\ast\) the only object of the monoid \(V\), considered as a category.

2.1. Example.

(1) The initial quantale is the two-element chain \(2 = \{\bot < \top\}\), with \(\otimes = \wedge\), \(k = \top\), and \(2\)-\textsc{Cat} is the category \textnormal{Ord} of preordered sets and monotone maps.

(2) The extended real line \([0, \infty]\), ordered by the natural \(\geq\), is a quantale with \(\otimes = +\), naturally extended to \(\infty\) (see [12]). We write \(\text{Met} = [0, \infty]\)-\textsc{Cat} for the resulting category of (generalized) metric spaces and non-expansive maps.

(3) Every frame may be considered as a quantale. In fact, these are precisely the commutative quantales in which every element is idempotent. For example, \([0, \infty]_{\max}\) may be considered as a quantale \([0, \infty]_{\max}\) with \(\alpha \otimes \beta = \max\{\alpha, \beta\}\). The resulting category \([0, \infty]_{\max}\)-\textsc{Cat} is the category \textnormal{UMet} of (generalized) ultrametric spaces (see [19]).

(4) From a small site \((C, F)\) one can construct a small quantaloid \(R := \mathcal{R}(C, F)\) (see [31]), defined by the following data:

- objects: the objects of \(C\);
- morphisms: for objects \(u, v\) in \(C\), an arrow from \(u\) to \(v\) is a closed subfunctor \(\alpha \subseteq \hat{u} \times \hat{v}\) with respect to the coverage \(F\) in \([C^{\text{op}}, \text{Set}]\), where \(\hat{u}\) and \(\hat{v}\) are the representable presheaves \(C(-, u)\) and \(C(-, v)\), respectively;
- composition: \(\beta \circ \alpha = \beta \circ \alpha\), with \(\circ\) denoting the composition of relations in the topos \([C^{\text{op}}, \text{Set}]\) and \((-)\) the closure with respect to the coverage \(F\).

It is shown in [4, 5] that Cauchy complete \(R\)-categories are equivalent to internal ordered objects in the category \(\textnormal{Sh}(C, F)\) of sheaves on the site \((C, F)\).

Recall that a quantale \(V\) is called \textit{divisible} (see [7]) if for all \(u \preceq v\) in \(V\), there are \(w, w' \in V\) such that \(u = v \otimes w = w' \otimes v\), or equivalently, \(u = v \otimes (v \setminus u) = (u \vee v) \otimes v\). A divisible quantale \(V\), since \(k \preceq \top\) guarantees the existence of some \(w \in V\) with \(\top = k \otimes \top = w \otimes \top = w \otimes \top = k\), must be \textit{integral}, i.e., \(k = \top\).

Every quantaloid \(Q\) gives rise to the quantaloid \(DQ\) of “diagonals of \(Q\)” (see [26]), which has an easy description when the quantaloid is a divisible quantale \(V\) (see [8, 15]): the objects of the quantaloid \(DV\) are the elements of \(V\), and a morphism \(d\) from \(u\) to \(v\) is an element in \(V\) with \(d \preceq u \wedge v\), we write \(d : u \rightsquigarrow v\) in this case. The composition of \(d\) with \(e : v \rightsquigarrow w\) in \(DV\) is defined by \(e \circ d = e \otimes (v \setminus d) = (e \vee v) \otimes d\) in \(V\). The order of the hom-sets of \(DV\) is inherited from \(V\).
Given a DV-category \((X, a)\), since in the quantaloid \(DV\) one has \(1_{|x|} = |x| = a(x, x)\), the conditions on the DV-category structure \(a\), given by a map \(X \times X \to V\), may be reformulated as

\[
a(x, y) \leq a(x, x) \land a(y, y) \quad \text{and} \quad a(y, z) \otimes (a(y, y) \triangleright a(x, y)) \leq a(x, z),
\]

for all \(x, y, z \in X\).

There are lax homomorphisms, called forward and backward globalization functors (see [15, 28]),

\[
\gamma : DV \to V, (d : u \rightsquigarrow v) \mapsto d \not\rightsquigarrow u,
\]
\[
\delta : DV \to V, (d : u \rightsquigarrow v) \mapsto v \not\rightsquigarrow d,
\]

which induce two functors from \(DV-Cat\) to \(V-Cat\).

When one considers \(V\) as a \(V\)-category \((V, h)\) with \(h(u, v) = v \not\rightsquigarrow u\), there is a full reflective embedding

\[
E_\gamma : DV-Cat \to V-Cat/V.
\]

Indeed, given a DV-category \((X, a)\), the \(V\)-relation \(d\), defined with the forward globalization functor by

\[
\forall x, y \in X : d(x, y) = a(x, y) \not\rightsquigarrow a(x, x),
\]

makes \(X\) a \(V\)-category over \(V\), via the \(V\)-functor \(t : (X, d) \to (V, h)\) with \(tx = a(x, x)\) for all \(x \in X\). Conversely, for a \(V\)-category \((X, d)\) equipped with a \(V\)-functor \(t : (X, d) \to (V, h)\), define

\[
\forall x, y \in X : a(x, y) = d(x, y) \otimes tx.
\]

To see that \((X, a)\) is indeed a DV-category with array map \(t : X \to V\), let us first note that, since \(t : (X, d) \to (V, h)\) is a \(V\)-functor, \(d(x, y) \leq ty \not\rightsquigarrow tx\), so that \(d(x, y) \otimes tx \leq ty\) and then \(a(x, y) = d(x, y) \otimes tx \leq tx \land ty\) follows, for all \(x, y \in X\). Secondly, for all \(x, y, z \in X\),

\[
a(y, z) \circ a(x, y) = d(y, z) \otimes ty \otimes (ty \triangleright (d(x, y) \otimes tx))
\]
\[
= d(y, z) \otimes d(x, y) \otimes tx \leq d(x, z) \otimes tx = a(x, z).
\]

Thus, \((X, a)\) is a DV-category, as desired.

Let \(V^*\) be the \(V\)-category with underlying set \(V\) and \(V\)-category structure

\[
h^*(u, v) = v \not\rightsquigarrow u.
\]

Of course, when \(V\) is commutative, \(V^*\) is the dual of \(V\). One obtains another full reflective embedding

\[
E_\delta : DV-Cat \to V-Cat/V^*,
\]

as follows. Given a DV-category \((X, a)\), the \(V\)-relation \(d\), defined by the backward globalization functor,

\[
\forall x, y \in X, d(x, y) = a(y, y) \triangleright a(x, y),
\]

makes \(X\) a \(V\)-category over \(V^*\), via the \(V\)-functor \(t : (X, d) \to (V, h^*)\) with \(ty = a(y, y)\) for all \(y \in X\).
2.2. Example. A \( D[0,\infty] \)-category \((X, a)\) is exactly a (generalized) partial metric space (see \([13, 2, 8, 15]\)). The category structure \( a \) is a map \( a : X \times X \rightarrow [0,\infty] \) that must satisfy

1. \( \max\{a(x, x), a(y, y)\} \leq a(x, y) \) for all \( x, y \in X \),
2. \( a(x, z) \leq a(x, y) - a(y, y) + a(y, z) \) for all \( x, y, z \in X \).

A \( D[0,\infty] \)-functor \( f : (X, a) \rightarrow (Y, b) \) is a map \( f : X \rightarrow Y \) such that

3. \( b(f x, f y) \leq a(x, y) \) for all \( x, y \in X \),
4. \( b(f x, f x) = a(x, x) \) for all \( x \in X \).

We write \( \text{ParMet} \) for the category of partial metric spaces. For \( V = [0,\infty] \), both

\[
E_\gamma : \text{ParMet} \rightarrow \text{Met}/[0,\infty] \quad \text{and} \quad E_\delta : \text{ParMet} \rightarrow \text{Met}/[0,\infty]^*
\]
give isomorphisms of categories.

A \( Q \)-relation \( \varphi : X \rightarrow Y \) becomes a \( Q \)-distributor \( \varphi : (X, a) \rightleftarrows (Y, b) \) if it is compatible with the \( Q \)-categorical structures \( a \) and \( b \); that is,

\[
b \circ \varphi \circ a \leq \varphi.
\]

\( Q \)-categories and \( Q \)-distributors constitute a quantaloid \( \text{Q-Dist} \) that contains \( \text{Q-Rel} \) as a full subquantaloid, in which the composition and internal homs are calculated in the same way as those of \( Q \)-relations; the identity \( Q \)-distributor on \((X, a)\) is given by its hom \( a : (X, a) \rightleftarrows (X, a) \).

Each \( Q \)-functor \( f : (X, a) \rightarrow (Y, b) \) induces an adjunction \( f_* \dashv f^* \) in \( \text{Q-Dist} \), given by

\[
\begin{align*}
&f_* : (X, a) \rightleftarrows (Y, b), \quad f_*(x, y) = b(f x, y) \quad \text{and} \\
&f^* : (Y, b) \rightleftarrows (X, a), \quad f^*(y, x) = b(y, f x),
\end{align*}
\]

and called the graph and cograph of \( f \), respectively. Obviously, \( a = (1_X)_* = 1_X^* \) for any \( Q \)-category \((X, a)\); hence, \( a = 1_X^* \) will be our standard notation for identity morphisms in \( \text{Q-Dist} \).

2.3. Lemma. \([20, 23]\) Let \( f : X \rightarrow Y \) be a \( Q \)-functor.

1. \( f \) is fully faithful if, and only if, \( f^* \circ f_* = 1_X^* \).
2. If \( f \) is essentially surjective, in the sense that, for any \( y \in Y \), there exists \( x \in X \) with \( y \cong f x \), then \( f_* \circ f^* = 1_Y^* \).

For an object \( s \) in \( Q \), and with \( \{s\} \) denoting the singleton \( Q \)-category, the only object of which has array \( s \) and hom \( 1_s \), \( Q \)-distributors of the form \( \sigma : X \rightleftarrows \{s\} \) are called \emph{preshaves} on \( X \) and constitute a \( Q \)-category \( \text{P}^!X \), with \( 1^*_\text{P}^!X(\sigma, \sigma') = \sigma' \vee \sigma \). Dually, the \emph{copresheaf} \( Q \)-category \( \text{P}^\dagger X \) consists of \( Q \)-distributors \( \tau : \{s\} \rightleftarrows X \) with \( 1^*_{\text{P}^\dagger X}(\tau, \tau') = \tau' \cap \tau \).
2.4. Remark. For any \( Q \)-category \( X \), it follows from the definition that the underlying order on \( P^+X \) is the reverse local order in \( Q\text{-Dist} \), i.e.,

\[
\tau \leq \tau' \text{ in } P^+X \iff \tau' \leq \tau \text{ in } Q\text{-Dist}.
\]

That is why we use a different symbol “\( \leq \)” for the underlying order of \( Q \)-categories and the 2-cells in \( Q\text{-Cat} \), while the 2-cells in \( Q \) and \( Q\text{-Dist} \) are denoted by “\( \preceq \)”.

A \( Q \)-category \( X \) is complete if the Yoneda embedding

\[
y_X : X \longrightarrow PX, \quad x \mapsto 1^*_X (-, x)
\]

has a left adjoint \( \text{sup}_X : PX \longrightarrow X \) in \( Q\text{-Cat} \); that is,

\[
1^*_X (\text{sup}_X \sigma, -) = 1^*_P X (\sigma, y_X -) = 1^*_X \vee \sigma
\]

for all \( \sigma \in PX \). It is well known that \( X \) is a complete \( Q \)-category if, and only if, \( X^\text{op} := (X, (1_X^\text{op})) \) with \( (1_X^\text{op})(x, x') = 1^*_X (x', x) \) is a complete \( Q^\text{op} \)-category (see [24]), where the completeness of \( X^\text{op} \) may be translated as the co-Yoneda embedding

\[
y_X^\text{op} : X \longrightarrow P^+X, \quad x \mapsto 1^*_X (x, -)
\]

admitting a right adjoint \( \text{inf}_X : P^+X \longrightarrow X \) in \( Q\text{-Cat} \).

2.5. Lemma. [23, 24] Let \( X \) be a \( Q \)-category.

(1) (Yoneda Lemma) For all \( \sigma \in PX \), \( \tau \in P^+X \),

\[
\sigma = (y_X)_*(-, \sigma) = 1^*_P X (\sigma, y_X -) \quad \text{and} \quad \tau = (y_X^\text{op})^*(-, \tau) = 1^*_P X (\tau, y_X^\text{op} -).
\]

In particular, both \( y_X : X \longrightarrow PX \) and \( y_X^\text{op} : X \longrightarrow P^+X \) are fully faithful.

(2) \( \text{sup}_X y_X \cong 1_X \), \( \text{inf}_X y_X^\text{op} \cong 1_X \).

(3) Both \( PX \) and \( P^+X \) are separated\(^1\) and complete, with

\[
\text{sup}_{PX} \sigma = \sigma \circ (y_X)_*, \quad \text{and} \quad \text{inf}_{P^+X} \tau = (y_X^\text{op})^* \circ \tau,
\]

for all \( \sigma \in PX \), \( \tau \in P^+X \).

Each \( Q \)-distributor \( \varphi : X \dashv Y \) induces Kan adjunctions [23] in \( Q\text{-Cat} \) given by

\[
\begin{array}{ccc}
P Y & \overset{\varphi^\circ}{\leftarrow} & PX \\
\downarrow & & \downarrow \\
P^+ Y & \overset{\varphi^\circ}{\leftarrow} & P^+ X
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
P Y & \underset{\varphi^\circ}{\rightarrow} & PX \\
\downarrow & & \downarrow \\
P^+ Y & \underset{\varphi^\circ}{\rightarrow} & P^+ X
\end{array}
\]

\[\varphi^\circ \tau = \tau \circ \varphi, \quad \varphi \circ \sigma = \sigma \vee \varphi \quad \text{and} \quad \varphi^\circ \tau = \varphi \vee \tau, \quad \varphi^\circ \sigma = \varphi \circ \sigma.\]

\(^1\)A \( Q \)-category \( X \) is separated if \( x \preceq x' \) implies \( x = x' \) for all \( x, x' \in X \).
Moreover, all the assignments in (2.i) and (2.ii) are 2-functorial, and one has two pairs of adjoint 2-functors [4] described by

\[
\begin{array}{c}
X \xrightarrow{\varphi} Y \\
\downarrow \quad \downarrow \\
Y \xrightarrow{\varphi} \mathcal{P}X \\
\end{array}
\]

\[\varphi y = \varphi(-, y)\]

\[
\begin{array}{c}
\mathcal{Q}\text{-Cat} \xrightarrow{(-)^*} \mathcal{Q}\text{-Dist}^{\text{op}}, \\
\downarrow \quad \downarrow \\
\mathcal{Q}\text{-Cat} \xrightarrow{(-)^*} (\mathcal{Q}\text{-Dist})^{\text{co}}, \\
\end{array}
\]

\[\varphi^\circ : \mathcal{P}Y \rightarrow \mathcal{P}X \iff (\varphi : X \rightarrow Y)\]  \hspace{1cm} (2.iii)

\[
\begin{array}{c}
X \xrightarrow{\varphi} Y \\
\downarrow \quad \downarrow \\
X \xrightarrow{\varphi} \mathcal{P}\dag Y \\
\end{array}
\]

\[\varphi x = \varphi(x, -)\]

\[
\begin{array}{c}
\mathcal{Q}\text{-Cat} \xrightarrow{(-)^*} \mathcal{Q}\text{-Dist}^{\text{co}}, \\
\downarrow \quad \downarrow \\
(\varphi^\dag : \mathcal{P}\dag X \rightarrow \mathcal{P}\dag Y) \iff (\varphi : X \rightarrow Y)\]

where “co” refers to the dualization of 2-cells. The unit \(y\) and the counit \(\varepsilon\) of the adjunction \((-)^* \dashv \mathcal{P}\) are respectively given by the Yoneda embeddings and their graphs:

\[\varepsilon_X := (y_X)_* : X \rightarrow \mathcal{P}X.\]

The presheaf 2-monad \(\mathcal{P} = (\mathcal{P}, s, y)\) on \(\mathcal{Q}\text{-Cat}\) induced by \((-)^* \dashv \mathcal{P}\) sends each \(\mathcal{Q}\)-functor \(f : X \rightarrow Y\) to

\[f_1 := (f^*)^\circ : \mathcal{P}X \rightarrow \mathcal{P}Y,\]

which admits a right adjoint \(f_1^\dag := (f_1^*)^\circ = (f_1^*)^\circ : \mathcal{P}Y \rightarrow \mathcal{P}X\) in \(\mathcal{Q}\text{-Cat}\); the monad multiplication \(s\) is given by

\[s_X = \varepsilon_X^\circ = \sup_{\mathcal{P}X} = y_X^1 : \mathcal{P}\mathcal{P}X \rightarrow \mathcal{P}X,\]  \hspace{1cm} (2.iv)

where \(\sup_{\mathcal{P}X} = y_X^1\) is an immediate consequence of Lemma 2.5. Similarly, the unit \(y_1^\dag\) is given by the co-Yoneda embeddings, and \(\varepsilon_1^\dag := (y_1^1)^*\) is the counit of the adjunction \((-)_* \dashv \mathcal{P}_1\). The induced copresheaf 2-monad \(\mathcal{P}_1^\dag = (\mathcal{P}_1^\dag, s_1^\dag, y_1^\dag)\) on \(\mathcal{Q}\text{-Cat}\) sends \(f\) to

\[f_1^\dag := (f_1^*)^\circ : \mathcal{P}_1^\dag X \rightarrow \mathcal{P}_1^\dag Y,\]

which admits a left adjoint \(f_1^\dag := (f_1^\ast)^\circ = (f_1^*)^\circ : \mathcal{P}_1^\dag Y \rightarrow \mathcal{P}_1^\dag X\) in \(\mathcal{Q}\text{-Cat}\), and the monad multiplication is given by

\[s_1^\dag_X = \varepsilon_1^\dag_X = \inf_{\mathcal{P}_1^\dag X} = (y_1^1)^i : \mathcal{P}_1^\dag \mathcal{P}_1^\dag X \rightarrow \mathcal{P}_1^\dag X.\]  \hspace{1cm} (2.v)

2.6. Lemma. Let \(f : X \rightarrow Y\) be a \(\mathcal{Q}\)-functor.

1. \(f\) is fully faithful \iff \(f_1^\ast f_1^\ast = 1_{\mathcal{P}X} \iff f_1^\ast f_1^\ast = 1_{\mathcal{P}_1^\dag X} \iff f_1 : \mathcal{P}X \rightarrow \mathcal{P}Y\) is fully faithful \iff \(f_1 : \mathcal{P}_1^\dag X \rightarrow \mathcal{P}_1^\dag Y\) is fully faithful.

2. If \(f\) is essentially surjective, then \(f_1^\ast f_1^\ast = 1_{\mathcal{P}Y}\), \(f_1^\ast f_1^\ast = 1_{\mathcal{P}_1^\dag Y}\) and both \(f_1 : \mathcal{P}X \rightarrow \mathcal{P}Y\), \(f_1^\dag : \mathcal{P}_1^\dag X \rightarrow \mathcal{P}_1^\dag Y\) are surjective.

Proof. Straightforward, with Lemma 2.3 and the definitions of \(f_1, f_1^\ast, f_1^\dag, f_1^\ast\).
2.7. Lemma. [16, 24] For all $Q$-functors $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$,
\[
 f \dashv g \iff f_* = g^* \iff f^! = g_! \iff f_i = g^! \iff f_! = g_i.
\]

2.8. Lemma. For all $Q$-functors $f, g : X \longrightarrow Y$ and $Q$-distributors $\varphi, \psi : X \longrightarrow Y$,
\begin{enumerate}
\item $f \leq g \iff f_* \geq g_* \iff f^* \leq g^* \iff f_i \leq g_i \iff f^! \geq g^! \iff f_i \geq g_i$.
\item $\varphi \leq \psi \iff \varphi^ \leq \psi^ \iff \varphi^ \geq \psi^ \iff \varphi \geq \psi$.
\end{enumerate}

2.9. Lemma. [21, 24] Let $f : X \longrightarrow Y$ be a $Q$-functor between complete $Q$-categories. Then
\[
\sup_Y \cdot f_i \leq f \cdot \sup_X \quad \text{and} \quad f \cdot \inf_X \leq \inf_Y \cdot f_i.
\]
Furthermore, $f$ is a left (resp. right) adjoint in $Q\text{-Cat}$ if, and only if, $\sup_Y \cdot f_i = f \cdot \sup_X$ (resp. $f \cdot \inf_X = \inf_Y \cdot f_i$).

The above lemma shows that left (resp. right) adjoint $Q$-functors between complete $Q$-categories are exactly sup-preserving (resp. inf-preserving) $Q$-functors. Thus we denote the 2-subcategory of $Q\text{-Cat}$ consisting of separated complete $Q$-categories and sup-preserving (resp. inf-preserving) $Q$-functors by $Q\text{-Sup}$ (resp. $Q\text{-Inf}$).

2.10. Lemma. The following identities hold for all $Q$-distributors $\varphi : X \longrightarrow Y$.
\begin{enumerate}
\item $y_X = \overline{\left(\overline{y_X}\right)}^\varphi, \quad y^*_X = \overline{\left(\overline{y_X}\right)}_{\varphi}$.
\item $1_{PX} = \overline{\left(\overline{y_X}\right)}_{\varphi}, \quad \overline{1_{PX}} = \overline{\left(\overline{y_X}\right)}_{\varphi}$.
\item $\overline{\varphi} = \varphi^\circ \cdot y_Y, \quad \overline{\varphi} = \varphi^\circ \cdot y^*_X$.
\item $\varphi = \overline{\varphi^*} \circ (y_X)_*, \quad \varphi = \overline{\varphi^*} \circ (y_X^*)_*$.
\item $(y_Y)_* \circ \varphi = \varphi^\circ \circ (y_X)_*, \quad \varphi \circ (y_X^*)_* = (y_Y^*)_* \circ (\varphi^\circ)_*.$
\end{enumerate}

Proof. (1), (3) are trivial, and (2), (4) are immediate consequences of the Yoneda lemma. For (5), note that the 2-functor
\[
P : (Q\text{-Dist})^{op} \longrightarrow Q\text{-Cat}, \quad (\varphi : X \longrightarrow Y) \mapsto (\varphi^\circ : PY \longrightarrow PX)
\]
is faithful, and
\[
((y_Y)_* \circ \varphi)^\circ = \varphi^\circ \cdot y^*_Y = \varphi^\circ \cdot \sup_{PY} = \sup_{PX} \cdot (\varphi^\circ)_! = y^*_X \cdot \varphi^\circ \circ (y_X)_* \circ (\varphi^\circ)_*
\]
follows by applying Lemma 2.9 to the left adjoint $Q$-functor $\varphi^\circ : PY \longrightarrow PX$. The other identity can be verified analogously. \]
2.11. Lemma. The following identities hold for all $\mathcal{Q}$-functors $f : X \rightarrow Y$.

(1) $f_! = f^!$, $f_* = f^\flat$, $(f_!)^i = (f^!)_i$, $(f_*)^i = (f^\flat)_i$.

(2) $\overline{f^\flat} = f^! \cdot y_Y$, $\overline{f_*} = y_Y \cdot f = f_1 \cdot y_X$.

(3) $\overline{f^\flat} = f^! \cdot y_Y$, $\overline{f_*} = y_Y \cdot f = f_1 \cdot y_X$.

(4) $(y_X)_* \circ f^* = (f^!)_* \circ (y_Y)_*$, $f_1 \cdot y_X = y_Y \cdot f_!$, $(y_X)_! \cdot f^! = (f^!)_! \cdot (y_Y)_!$.

(5) $f_* \circ (y_X^t)^* = (y_Y^t)_* \circ (f_!)^*$, $f_1 \cdot (y_X^t)^i = (y_Y^t)_! \circ f_!$, $(y_Y^t)_! \cdot f^! = (f^!)_! \cdot (y_Y^t)_!$.

**Proof.** For (1), $f_! = f^!$ since $(f^!)_! \circ f^!$ and $(f^!)_! \circ f_!$, and the other identities can be checked similarly. The non-trivial identities in (2) and (3) follow respectively from the naturality of $y^!$ and $y$, while (4) and (5) are immediate consequences of Lemma 2.10(5). ■

2.12. Lemma. The following identities hold for all $\mathcal{Q}$-distributors $\varphi : X \rightarrow Y$, $\psi : Y \rightarrow Z$ and $\mathcal{Q}$-functors $f$ whenever the operations make sense:

(1) $\overline{\varphi \circ \psi} = \overline{\varphi} \cdot \overline{\psi} = y_X^\flat \cdot \overline{\varphi}_1 \cdot \overline{\psi}$, $\overline{\psi \circ f^*} = f_1 \cdot \overline{\psi}$, $\overline{f_* \circ \varphi} = \overline{\varphi} \cdot f$.

(2) $\overline{\varphi \circ \psi} = \overline{\psi} \cdot \overline{\varphi} = (y_Z^t)_! \cdot \overline{\varphi}_1 \cdot \overline{\psi}$, $\overline{\psi \circ f_*} = \overline{\psi} \cdot f = f_* \circ \overline{\varphi} = f_1 \cdot \overline{\varphi}$.

**Proof.** Straightforward calculations with the help of Lemmas 2.10 and 2.11. ■

2.13. Lemma. For $\mathcal{Q}$-functors $f, g : \mathcal{P} X \rightarrow Y$ (resp. $f, g : \mathcal{P}^\flat X \rightarrow Y$), if $f$ (resp. $g$) is a left (resp. right) adjoint in $\mathcal{Q}$-$\text{Cat}$, then

$$fy_X \leq gy_X \text{ (resp. } f_! y_X \leq g_! y_X) \iff f \leq g.$$ 

**Proof.** For the non-trivial direction, suppose that $f \dashv h : Y \rightarrow \mathcal{P} X$, then $fy_X \leq gy_X$ implies $y_X \leq hgy_X$. Consequently, the Yoneda lemma and the $\mathcal{Q}$-functoriality of $hg : \mathcal{P} X \rightarrow \mathcal{P} X$ imply

$$\sigma = (y_X)_* (\sigma) = 1^*_{\mathcal{P}X} (y_X \cdot \sigma) \leq 1^*_{\mathcal{P}X} (hgy_X \cdot \sigma) \leq 1^*_{\mathcal{P}X} (y_X \cdot hgs) = hgs$$

and thus $\sigma \leq hgs$, hence $f \sigma \leq g \sigma$ for all $\sigma \in \mathcal{P} X$.

As one already has the isomorphisms of ordered hom-sets

$$\mathcal{Q}$-$\text{Dist}(X, Y) \cong \mathcal{Q}$-$\text{Cat}(Y, \mathcal{P} X) \cong (\mathcal{Q}$-$\text{Cat})^{co}(X, \mathcal{P}^\flat Y)$$

then from the adjunctions (2.iii), more isomorphisms can be formed in $\mathcal{Q}$-$\text{Sup}$ and $\mathcal{Q}$-$\text{Inf}$:
2.14. Lemma. [23] For all \( Q \)-categories \( X, Y \), one has the natural isomorphisms of ordered hom-sets

\[
\QDist(X, Y) \cong (\QSup)^\circ(PX, P^!Y) \cong \QInf(P^!Y, PX)
\]

\[
\cong \QSup(PY, PX) \cong (\QInf)^\circ(PX, PY).
\]

Proof. Each \( Q \)-distributor \( \varphi : X \rightarrow Y \) induces the Isbell adjunction \( \varphi^\uparrow \dashv \varphi^\downarrow : P^!Y \rightarrow PX \) [23] with

\[
\varphi^\uparrow \sigma = \sigma \searrow \varphi \quad \text{and} \quad \varphi^\downarrow \tau = \tau \nearrow \varphi
\]

for all \( \sigma \in PX, \tau \in P^!Y \). It is straightforward to check that

\[
\QDist(X, Y) \cong (\QSup)^\circ(PX, P^!Y) \cong \QInf(P^!Y, PX)
\]

\[
\varphi \overset{\sim}{\rightarrow} \varphi^\uparrow \overset{\sim}{\rightarrow} \varphi^\downarrow \overset{\sim}{\rightarrow} \varphi^\circ
\]

\[
\cong \QSup(PY, PX) \cong (\QInf)^\circ(PX, PY)
\]

\[
\overset{\sim}{\rightarrow} \varphi^\circ \overset{\sim}{\rightarrow} \varphi^\circ
\]

gives the required isomorphisms. The readers may refer to [23, Theorems 4.4 & 5.7] for details.

\[
\square
\]

3. The non-discrete version of lax distributive laws and their lax algebras

In this section we establish the non-discrete version of the lax distributive laws considered in [30]. For a 2-monad \( T = (T, m, e) \) on \( \QCat \), a lax distributive law \( \lambda : TP \rightarrow PT \) is given by a family

\[
\{\lambda_X : TPX \rightarrow PTX\}_{X \in \ob(\QCat)}
\]

of \( \Q \)-functors satisfying the following inequalities for all \( \Q \)-functors \( f : X \rightarrow Y \):

\( (a) \)

\[
\begin{array}{ccc}
TPX & \xrightarrow{T(f)} & TPY \\
\lambda_X & \leq & \lambda_Y \\
Ptx & \xrightarrow{(Tf)^!} & PTY \\
\end{array}
\]

\( (Tf)^! \cdot \lambda_X \leq \lambda_Y \cdot T(f) \) \quad (lax naturality of \( \lambda \));

\( (b) \)

\[
\begin{array}{ccc}
TPX & \xrightarrow{T_X} & PTX \\
\gamma_{TX} & \geq & \gamma_{TX} \\
\end{array}
\]

\[ y_{TX} \leq \lambda_X \cdot T_y X \] \quad (lax \( \mathbb{P} \)-unit law);

\( (c) \)

\[
\begin{array}{ccc}
TPX & \xrightarrow{T_X} & PTX \\
\lambda_X & \geq & \lambda_X \\
\end{array}
\]

\[ s_{TX} \cdot (\lambda_X)^! \cdot \lambda_p X \leq \lambda_X \cdot Ts_X \] \quad (lax \( \mathbb{P} \)-mult. law);
3.2. Definition. For a distributive law over the 2-monad $Q$ on the 2-category $\mathcal{C}$, i.e., $\lambda : T \mathcal{C} \rightarrow T \mathcal{C}$ is a distributive law for $\mathcal{C}$, then $\lambda$ is said to be strictly everywhere. For simplicity, in what follows, we refer to a lax distributive law $\lambda : T \mathcal{C} \rightarrow T \mathcal{C}$, which indirectly emphasizes the fact that the ambient 2-cell structure is given by order; we also say that $\mathcal{T}$ distributes over $\mathcal{P}$ by $\lambda$ in this case, adding strictly when $\lambda$ is strict.

3.1. Remark. Recall that in the discrete case (see [30]), a distributive law $\lambda$ of a monad $\mathcal{T} = (T, m, e)$ on $\mathcal{P} = \text{Set}/Q_0$ over the discrete presheaf monad $\mathcal{P}$ on $\mathcal{P} = \text{Set}/Q_0$ is usually required to be monotone, i.e.,

$$f \leq g \Rightarrow \lambda_X \cdot T f \leq \lambda_X \cdot T g$$

for all $Q$-functors $f, g : X \rightarrow PX$. As for the non-discrete case, $\mathcal{T} = (T, m, e)$ becomes a 2-monad on the 2-category $\mathcal{Q}$-Cat and, hence, the monotonicity of a distributive law of $\mathcal{T}$ over the 2-monad $\mathcal{P}$ on $\mathcal{Q}$-Cat is automatically satisfied through the 2-functoriality of $\mathcal{T}$.

3.2. Definition. For a distributive law $\lambda : T \mathcal{P} \rightarrow T \mathcal{P}$, a lax $\lambda$-algebra $(X, p)$ over $\mathcal{Q}$ is a $\mathcal{Q}$-category $X$ with a $\mathcal{Q}$-functor $p : TX \rightarrow PX$ satisfying

$$y_X \leq p \cdot e_X$$

(lax unit law);
The resulting 2-category is denoted by \((\lambda, Q)\)-Alg, with the local order inherited from \(Q\)-Cat.

**3.3. Proposition.** \((\lambda, Q)\)-Alg is topological over \(Q\)-Cat and, hence, totally complete and totally cocomplete.

**Proof.** For any family of \(\lambda\)-algebras \((Y_j, q_j)\) and \(Q\)-functors \(f_j : X \rightarrow Y_j \ (j \in J)\),

\[
p := \bigwedge_{j \in J} (f_j)^! \cdot q_j \cdot T f_j
\]

gives the initial structure on \(X\) with respect to the forgetful functor \((\lambda, Q)\)-Alg \(\rightarrow Q\)-Cat, and thus establishes the topologicity of \((\lambda, Q)\)-Alg over \(Q\)-Cat (see [1]). The total completeness and total cocompleteness of \((\lambda, Q)\)-Alg then follow from that of \(Q\)-Cat (see [22, Theorem 2.7]).

4. The distributive law of the presheaf 2-monad

The presheaf 2-monad \(P\) on \(Q\)-Cat is lax idempotent, or of Kock-Zöberlein type [25], in the sense that

\[
(y_X)_! \leq y_{PX}
\]

for all \(Q\)-categories \(X\). This fact makes it possible to establish the distributivity of \(P\) over itself:

**4.1. Theorem.** The presheaf 2-monad \(P\) distributes over itself by \(\lambda\) with

\[
\lambda_X = y_{PX} \cdot y_X^! = y_{PX} \cdot \sup_{PX} : PPX \rightarrow PPX.
\]

**Proof.** We show that \(\lambda\) satisfies the laws (a), (b), (c) and (e) strictly and (d) laxly.

(a) \(f_!! \cdot \lambda_X = \lambda_Y \cdot f_!!\) for any \(Q\)-functor \(f : X \rightarrow Y\). The commutativity of the upper square and the lower square of the diagram

respectively follow from Lemma 2.9 and the naturality of \(y\).
(b) $y_{PX} = \lambda_X \cdot (y_X)!$. Since $y_X$ is fully faithful, one has $y'_X \cdot (y_X)! = 1_{PX}$ and thus the diagram

is commutative.

(c) $s_{PX} \cdot (\lambda_X)! \cdot \lambda_{PX} = \lambda_X \cdot (s_X)!$. In the following diagram, the commutativity of the left and the middle trapezoids both follow from the naturality of $y$, and the right triangle commutes since $y_{PX}$ is fully faithful.

(d) $(y_X)! \leq \lambda_X \cdot y_{PX}$. In the following diagram, $\text{sup}_{PX} \cdot y_{PX} = 1_{PX}$ and $\mathbb{P}$ being lax idempotent guarantees $(y_X)! \leq y_{PX}$.

(e) $(s_X)! \cdot \lambda_{PX} \cdot (\lambda_X)! = \lambda_X \cdot s_{PX}$. The naturality of $y$ ensure that the left and the right
trapezoids of the diagram

\[
\begin{array}{ccc}
PPX & \xrightarrow{\lambda_X} & PPPX \\
& (\lambda_X)^! & \\
\downarrow_{PPX} & \downarrow & \downarrow_{PPPX} \\
PPX & \xrightarrow{y_{PPX}} & PPPX \\
\begin{array}{c}
p_{PPX} = y_{PPX}
\end{array} & \begin{array}{c}
(\lambda_X)
\end{array} & \begin{array}{c}
\lambda_{PPX}
\end{array} \\
\end{array}
\]

are commutative, and the commutativity of the middle triangle follows from the full faithfulness of \(y_{PPX}\).

\[\begin{array}{l}
\text{A } \mathcal{Q}\text{-closure space} \ [21, 23] \text{ is a pair } (X, c) \text{ that consists of a } \mathcal{Q}\text{-category } X \text{ and a } \mathcal{Q}\text{-closure operation } c \text{ on } PX; \text{ that is, a } \mathcal{Q}\text{-functor } c : PX \longrightarrow PX \text{ satisfying } 1_{PX} \leq c \text{ and } c \cdot c = c. \text{ A continuous } \mathcal{Q}\text{-functor } f : (X, c) \longrightarrow (Y, d) \text{ between } \mathcal{Q}\text{-closure spaces is a } \mathcal{Q}\text{-functor } f : X \longrightarrow Y \text{ such that }
\end{array}\]

\[f \cdot c \leq d \cdot f : PX \longrightarrow PY.\]

\[
\mathcal{Q}\text{-closure spaces and continuous } \mathcal{Q}\text{-functors constitute a 2-category } \mathcal{Q}\text{-Cls with the local order inherited from } \mathcal{Q}\text{-Cat.}
\]

**4.2. Theorem.** \((\lambda, \mathcal{Q})\text{-Alg} \cong \mathcal{Q}\text{-Cls.}\)

**Proof.** For any \(\mathcal{Q}\text{-category } X, \) we show that a \(\mathcal{Q}\text{-functor } c : PX \longrightarrow PX \) gives a lax \(\lambda\)-algebra structure on \(X\) if, and only if, \((X, c)\) is a \(\mathcal{Q}\text{-closure space.}\)

\(c\) satisfies (f) \(\iff\) \(1_{PX} \leq c\): This is an immediate consequence of Lemma 2.13.
\(c\) satisfies (g) \(\iff\) \(c \cdot c \leq c\): Note that

\[
c \cdot c = \sup_{PX} \cdot y_{PX} \cdot c \cdot \sup_{PX} \cdot y_{PX} \cdot c
= \sup_{PX} \cdot c ! \cdot y_{PX} \cdot \sup_{PX} \cdot c ! \cdot y_{PX}
= \sup_{PX} \cdot c ! \cdot \lambda_X \cdot c ! \cdot y_{PX},
\]

and thus

\[
c \cdot c \leq c \iff \sup_{PX} \cdot c ! \cdot \lambda_X \cdot c ! \cdot y_{PX} \leq c
\iff \sup_{PX} \cdot c ! \cdot \lambda_X \cdot c ! \leq c \cdot \sup_{PX},
\]

\((\sup_{PX} \dashv y_{PX})\)

which is precisely the condition (g).

Therefore, the isomorphism between \((\lambda, \mathcal{Q})\text{-Alg} \text{ and } \mathcal{Q}\text{-Cls}\) follows since a continuous \(\mathcal{Q}\text{-functor } f : (X, c) \longrightarrow (Y, d) \) is exactly a \(\mathcal{Q}\text{-functor } f : X \longrightarrow Y \) satisfying the condition (h).
5. The strict distributive law of the copresheaf 2-monad

5.1. Theorem. The copresheaf 2-monad $\mathbb{P}^\dagger$ distributes strictly over $\mathbb{P}$ by $\lambda^\dagger$ with

$$\lambda^\dagger_X = ((y_X)_i)^\dagger \cdot y_{\mathbb{P}^\dagger \mathbb{P} \mathbb{P} \mathbb{P} X} : \mathbb{P}^\dagger \mathbb{P} X \longrightarrow \mathbb{P} \mathbb{P} X.$$ 

Proof. We show that $\lambda^\dagger$ satisfies the laws (a)-(e) strictly.

(a) $f_i^! \cdot \lambda^\dagger_X = \lambda^\dagger_Y \cdot f_i$ for any $\mathcal{Q}$-functor $f : X \longrightarrow Y$. Indeed, both the upper square and the lower square of the diagram

$$\begin{array}{ccc}
\mathbb{P}^\dagger \mathbb{P} X & \xrightarrow{f_i} & \mathbb{P}^\dagger \mathbb{P} Y \\
\downarrow \gamma_{\mathbb{P}^\dagger \mathbb{P} X} & & \downarrow \gamma_{\mathbb{P}^\dagger \mathbb{P} Y} \\
\mathbb{P} \mathbb{P}^\dagger \mathbb{P} X & \xrightarrow{((y_X)_i)^\dagger} & \mathbb{P} \mathbb{P}^\dagger \mathbb{P} Y \\
\downarrow \lambda^\dagger_X & & \downarrow \lambda^\dagger_Y \\
\mathbb{P} \mathbb{P}^\dagger X & \xrightarrow{f_i = f_i^!} & \mathbb{P} \mathbb{P}^\dagger Y
\end{array}$$

are commutative, respectively by the naturality of $y$ and Lemma 2.11(4).

(b) $y_{\mathbb{P}^\dagger X} = \lambda^\dagger_X \cdot (y_X)_i$. For this, note that both, the left square and the right triangle of the diagram

$$\begin{array}{ccc}
\mathbb{P}^\dagger X & \xrightarrow{y_{\mathbb{P}^\dagger X}} & \mathbb{P} \mathbb{P}^\dagger X \\
\downarrow (y_X)_i & & \downarrow (y_X)_i \\
\mathbb{P} \mathbb{P}^\dagger \mathbb{P} X & \xrightarrow{y_{\mathbb{P}^\dagger \mathbb{P} X}} & \mathbb{P} \mathbb{P} \mathbb{P}^\dagger X \\
\downarrow \lambda^\dagger_X & & \downarrow \lambda^\dagger_X \\
\mathbb{P} \mathbb{P} \mathbb{P} \mathbb{P} \mathbb{P} X
\end{array}$$

are commutative, by the naturality of $y$ and the full faithfulness of $(y_X)_i$, respectively.

(c) $s_{\mathbb{P}^\dagger X} \cdot (\lambda^\dagger_X)_i \cdot \lambda^\dagger_{\mathbb{P} X} = \lambda^\dagger_X \cdot (s_X)_i$. In the following diagram, the naturality of $y$ guarantees the commutativity of the left square and the right triangle, and together with the full
faithfulness of $y_{P \uparrow P \downarrow X}$ it forces the commutativity of the middle square.

(d) $(y^!_{P}) = \lambda^!_{X} \cdot y_{P \uparrow X}$. From $y^!_{X} = \sup_{P \downarrow X}$ one sees that the upper triangle of the diagram

\[ \begin{array}{ccc}
PX & \xrightarrow{y_{P \uparrow P \downarrow X}} & PP \uparrow P \downarrow X \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
P \uparrow P \downarrow X & \xrightarrow{y_{P \uparrow P \downarrow X}} & PP \uparrow P \downarrow X \\
\end{array} \]

is commutative, and the commutativity of the left and the right trapezoids follow respectively from the naturality of $y$ and Lemma 2.11(5).

(e) $(s^!_{X})_{i} \cdot \lambda^!_{P \uparrow X} \cdot (\lambda^!_{X})_{i} = \lambda^!_{X} \cdot s^!_{P \downarrow X}$. This follows from the commutativity of the following
5.2. REMARK. Stubbe described a strict distributive law of $\mathbb{P}$ over $\mathbb{P}^\dagger$ given by

$$\mathbb{P} \mathbb{P}^\dagger X \xrightarrow{(y_{\mathbb{P} \mathbb{P}^\dagger X})_{\circ}} \mathbb{P} \mathbb{P}^\dagger X \xrightarrow{\sup_{\mathbb{P} \mathbb{P}^\dagger X}} \mathbb{P} \mathbb{P}^\dagger X$$  \hspace{1cm} (5.1)$$

in [27]. In fact, the strict distributive law $\lambda^X_\mathbb{P} : \mathbb{P} \mathbb{P}^\dagger X \rightarrow \mathbb{P} \mathbb{P}^\dagger X$ defined in Theorem 5.1 is precisely the right adjoint of (5.1) in $\mathcal{Q}$-Cat.

Recall that a $\mathcal{Q}$-category is a monad in $\mathcal{Q}$-Rel. Similarly, a monad in $\mathcal{Q}$-Dist gives “a $\mathcal{Q}$-category over a base $\mathcal{Q}$-category”; that is, a $\mathcal{Q}$-category $X$ equipped with a $\mathcal{Q}$-distributor $\alpha : X \rightarrow X$, such that $1^*_X \preceq \alpha$ and $\alpha \circ \alpha \preceq \alpha$. The latter two inequalities actually force the $\mathcal{Q}$-relation $\alpha$ on $X$ to be a $\mathcal{Q}$-distributor, since with $a = 1^*_X$ one has

$$a \circ (\alpha \circ a) \preceq a \circ (\alpha \circ a) \preceq a \circ a \preceq \alpha \circ a \preceq \alpha \circ \alpha \preceq \alpha.$$

Thus, a monad in $\mathcal{Q}$-Dist is given by a set $X$ over $\mathcal{Q}_0$ that comes equipped with two $\mathcal{Q}$-category structures, comparable by “$\preceq$”. With morphisms to laxly preserve both structures we obtain the 2-category $\textbf{Mon}(\mathcal{Q}$-Dist$)$; hence, its morphisms $f : (X, \alpha) \rightarrow (Y, \beta)$ are precisely $\mathcal{Q}$-functors $f : X \rightarrow Y$ with

$$f \cdot \overleftarrow{\alpha} \preceq \overleftarrow{\beta} \cdot f.$$
or, equivalently, \( \alpha(x, x') \preceq \beta(f x, f x') \) for all \( x, x' \in X \).

We also point out that the copresheaf 2-monad \( \mathbb{P}^\dagger \) on \( \textbf{Q-Cat} \) is \textit{ooplax idempotent}, or \textit{of dual Kock-Zöberlein type}, in the sense that

\[
y_{\mathbb{P}X}^\dagger \preceq (y_X^\dagger)_i
\]

for all \( \mathcal{Q} \)-categories \( X \). We shall use this fact to characterize \((\lambda^\dagger, \mathcal{Q})\)-algebras as monads in \( \mathcal{Q} \)-\textbf{Dist}:

5.3. Theorem. \((\lambda^\dagger, \mathcal{Q})\)-\textbf{Alg} \( \cong \textbf{Mon}(\mathcal{Q} \text{-Dist}) \).

Proof. Step 1. We show that if \( (X, p) \) is a \((\lambda^\dagger, \mathcal{Q})\)-algebra, then

\[
p = \inf_{\mathbb{P}X} \cdot p_1 \cdot (y_X^\dagger)_i. \tag{5.ii}
\]

Indeed, the conditions (f) and (g) for the \((\lambda^\dagger, \mathcal{Q})\)-algebra \( (X, p) \) read as

(f) \( y_X \preceq p \cdot y_X^\dagger \) and

(g) \( y_X^\dagger \cdot p_1 \cdot \lambda_X^\dagger \cdot p_1 \preceq p \cdot (y_X^\dagger)_i \),

and consequently

\[
p = \inf_{\mathbb{P}X} \cdot y_{\mathbb{P}X}^\dagger \cdot p
\]

\[
= \inf_{\mathbb{P}X} \cdot p_1 \cdot (y_X^\dagger)_i \tag{y^\dagger is natural}
\]

\[
\leq \inf_{\mathbb{P}X} \cdot p_1 \cdot (y_X^\dagger)_i \tag{\mathbb{P}^\dagger is oplax idempotent}
\]

\[
= (y_X^\dagger)_i \cdot (y_X^\dagger)_i \cdot \inf_{\mathbb{P}X} \cdot p_1 \cdot (y_X^\dagger)_i
\]

\[
= (y_X^\dagger)_i \cdot \lambda_X^\dagger \cdot y_{\mathbb{P}X}^\dagger \cdot \inf_{\mathbb{P}X} \cdot p_1 \cdot (y_X^\dagger)_i
\]

\[
\leq (y_X^\dagger)_i \cdot \lambda_X^\dagger \cdot p_1 \cdot (y_X^\dagger)_i
\]

\[
\leq (y_X^\dagger)_i \cdot p_1 \cdot \lambda_X^\dagger \cdot p_1 \cdot (y_X^\dagger)_i
\]

\[
\leq p \cdot (y_X^\dagger)_i \cdot (y_X^\dagger)_i
\]

\[
= p. \tag{y_X^\dagger is fully faithful}
\]

Step 2. As an immediate consequence of (5.ii), \( p \) is a right adjoint in \( \textbf{Q-Cat} \). For any \( \mathcal{Q} \)-category \( X \), as one already has

\[
\mathcal{Q} \text{-Dist}(X, X) \cong \mathcal{Q} \text{-Inf}(\mathbb{P}^\dagger X, \mathbb{P}X)
\]

from Lemma 2.14, with the isomorphism given by

\[
(\alpha : X \to X) \mapsto (\alpha^\dagger : \mathbb{P}^\dagger X \to \mathbb{P}X, \quad \alpha^\dagger \tau = \tau \backslash \alpha),
\]

in order for us to establish a bijection between monads on \( X \) (in \( \mathcal{Q} \)-\textbf{Dist}) and \((\lambda^\dagger, \mathcal{Q})\)-algebra structures on \( X \), it suffices to prove
isomorphisms

\[ 1_X^* \preceq \alpha \iff \alpha^\downarrow \text{ satisfies (f), and} \]

\[ \alpha \circ \alpha \preceq \alpha \iff \alpha^\downarrow \text{ satisfies (g)} \]

for all \( Q \)-distributors \( \alpha : X \to X \).

First, \( 1_X^* \preceq \alpha \iff \alpha^\downarrow \text{ satisfies (f). Since } \overline{1_X^*} = y_X \text{ and, as one easily sees, } \overline{\alpha} = \alpha^\downarrow \cdot y_X^\dagger, \)
the equivalence \( 1_X^* \preceq \alpha \iff y_X \preceq \alpha^\downarrow \cdot y_X^\dagger \text{ follows immediately.} \)

Second, \( \alpha \circ \alpha \preceq \alpha \iff \alpha^\downarrow \text{ satisfies (g), i.e.,} \)

\[ y_X^\dagger \cdot (\alpha^\downarrow)_! \cdot \lambda_X^\dagger \cdot (\alpha^\downarrow)_i \preceq \alpha^\downarrow \cdot (y_X^\dagger)_i = \alpha^\downarrow \cdot \inf_{P^\dagger X}. \]

Note that

\[ \overline{\alpha \circ \alpha} = y_X^\dagger \cdot \overline{\alpha^\downarrow} \cdot \overline{\alpha^\downarrow} \text{ (Lemma 2.12(1))} \]

\[ = y_X^\dagger \cdot (\alpha^\downarrow)_! \cdot (y_X^\dagger)_! \cdot \alpha^\downarrow \cdot y_X^\dagger \quad (\alpha^\downarrow \cdot y_X^\dagger = \overline{\alpha^\downarrow}) \]

\[ = y_X^\dagger \cdot (\alpha^\downarrow)_! \cdot \lambda_X^\dagger \cdot y_{P^\dagger X}^\dagger \cdot y_X^\dagger \quad (\lambda^\dagger \text{ satisfies (d))} \]

\[ = y_X^\dagger \cdot (\alpha^\downarrow)_! \cdot \lambda_X^\dagger \cdot (\alpha^\downarrow)_i \cdot y_{P^\dagger X}^\dagger \cdot y_X^\dagger, \quad (y^\dagger \text{ is natural}) \]

and hence

\[ \alpha \circ \alpha \preceq \alpha \iff y_X^\dagger \cdot (\alpha^\downarrow)_! \cdot \lambda_X^\dagger \cdot (\alpha^\downarrow)_i \cdot y_{P^\dagger X}^\dagger \cdot y_X^\dagger \leq \alpha^\downarrow \cdot y_X^\dagger \quad \text{(Lemma 2.13)} \]

\[ \iff y_X^\dagger \cdot (\alpha^\downarrow)_! \cdot \lambda_X^\dagger \cdot (\alpha^\downarrow)_i \leq \alpha^\downarrow \cdot \inf_{P^\dagger X}, \quad \text{(Lemma 2.13)} \]

as desired.

**Step 3.** \( f : (X, \alpha) \to (Y, \beta) \) is a morphism in \( \textbf{Mon}(Q\text{-}\text{Dist}) \) if, and only if, \( f : (X, \alpha^\downarrow) \to (Y, \psi^\downarrow) \) satisfies (h). Indeed,

\[ f_! \cdot \alpha^\downarrow \preceq \psi^\downarrow \cdot f_! \iff f_! \cdot \alpha^\downarrow \cdot y_X^\dagger \preceq \psi^\downarrow \cdot f_! \cdot y_X^\dagger \quad \text{(Lemma 2.13)} \]

\[ \iff f_! \cdot \alpha^\downarrow \cdot y_X^\dagger \preceq \psi^\downarrow \cdot y_{P^\dagger}^\dagger \cdot f \quad \text{(y^\dagger \text{ is natural})} \]

\[ \iff f_! \cdot \overline{\alpha^\downarrow} \preceq \overline{\beta^\downarrow} \cdot f, \]

which completes the proof.


6. The distributive law of the double presheaf 2-monad

Recall that the adjunctions \((-)^* \dashv P\) and \((-)_* \dashv P^\dagger\) displayed in (2.iii) give rise to the isomorphisms

\[ Q\text{-}\text{Cat}(Y, P X) \cong Q\text{-}\text{Dist}(X, Y) \cong Q\text{-}\text{Cat}(X, P^\dagger Y), \quad (6.i) \]
for all $Q$-categories $X$, $Y$. In fact, (6.i) induces another pair of adjoint 2-functors [27]

$$P_c^\dagger \vdash P_c : Q-Cat \longrightarrow (Q-Cat)^{coop},$$

which map objects as $P^\dagger$ and $P$ do, but with $P_c^\dagger f = f^i$ and $P_c f = f^!$ for all $Q$-functor $f$. The units and counits of this adjunction are respectively given by

$$y_{P^\dagger X} \cdot y_X^\dagger = (y_X^\dagger)_! \cdot y_X : X \longrightarrow PP^\dagger X \quad \text{and} \quad y_{P^\dagger X} \cdot y_X = (y_X)_! \cdot y_X^\dagger : X \longrightarrow P^\dagger P X$$

for all $Q$-categories $X$. This adjunction induces the double presheaf 2-monad $(P_cP_c^\dagger, \eta, s)$ on $Q-Cat$ with the multiplication given by

$$s_X = ((y_{P^\dagger X})_! \cdot y_{P^\dagger X}^\dagger)_! = (y_{PP^\dagger X} \cdot y_{P^\dagger X})^\dagger = s_{P^\dagger X} \cdot (y_{PP^\dagger X})^\dagger : PP^\dagger PP^\dagger X \longrightarrow PP^\dagger X. \quad (6.iii)$$

As Lemma 2.11(1) implies $P_cP_c^\dagger = PP^\dagger$, the double presheaf 2-monad on $Q-Cat$ may be alternatively written as

$$PP^\dagger = (PP^\dagger, \eta, s).$$

6.1. Theorem. The double presheaf 2-monad $PP^\dagger$ distributes over $P$ by $\Lambda$ with

$$\Lambda_X = y_{PP^\dagger X} \cdot ((y_X)_!^\dagger) : PP^\dagger P X \longrightarrow PPP^\dagger X.$$

Proof. First note that

$$\Lambda_X = \lambda_{P^\dagger X} \cdot (\lambda_X)_!^\dagger. \quad (6.iv)$$

Indeed, from the naturality of $y$ one soon obtains the commutativity of the diagram

for any $Q$-category $X$. Now we check the laws (a)-(e) for $\Lambda$:
(a) $\Lambda_Y \cdot f_\Pi = f_\Pi \cdot \Lambda_X$ for every $Q$-functor $f : X \to Y$. This is a direct consequence of the naturality of $\lambda$ and $\lambda^\uparrow$.

(b) $y_{PP^\uparrow X} = \Lambda_X \cdot (y_X)_\Pi$. This is easy since $\lambda$ and $\lambda^\uparrow$ both satisfy the $P$-unit law (b) strictly.

(c) $s_{PP^\uparrow X} \cdot (\Lambda_X)_\Pi \cdot \Lambda_P X = \Lambda_X \cdot (s_X)_\Pi$. This follows from the naturality of $\lambda$ and the fact that $\lambda$ and $\lambda^\uparrow$ both satisfy the $P$-multiplication law (c) strictly.

(d) $(\eta_X)_\Pi \leq \Lambda_X \cdot \eta_{P X}$. Since $\lambda$ satisfies the lax $P$-unit law (d) and $\lambda^\uparrow$ strictly satisfies the $P^\uparrow$-unit law (d), one obtains the upper and the lower right-hand triangles of the following diagram. Moreover, the naturality of $y$ guarantees the commutativity of the
lower-left square.

\[
\begin{array}{ccc}
P X & \xrightarrow{y^*_X} & (\gamma^*_X)_! \\
\eta_X & \downarrow & \downarrow (\eta_X)_! \\
\Pi \Pi P X & \xrightarrow{\lambda_X^\dagger} & \Pi \Pi \Pi \Pi P X \\
\gamma_{\Pi \Pi P X} & \downarrow & \lambda_{\Pi \Pi X} \\
P \Pi \Pi P X & \xrightarrow{(\lambda_X)_!} & \Pi \Pi \Pi \Pi P X \end{array}
\]

\[
\begin{array}{ccc}
P \Pi \Pi P X & \xrightarrow{(\lambda_X)_!} & \Pi \Pi \Pi \Pi P X \\
\lambda_{\Pi \Pi X} & \downarrow & \lambda_{\Pi \Pi X} \\
\Pi \Pi \Pi \Pi P X & \xrightarrow{(\lambda_X)_!} & \Pi \Pi \Pi \Pi P X \\
\gamma_{\Pi \Pi \Pi \Pi P X} & \downarrow & \lambda_{\Pi \Pi X} \\
P \Pi \Pi P X & \xrightarrow{(\lambda_X)_!} & \Pi \Pi \Pi \Pi P X \end{array}
\]

\[
\begin{array}{ccc}
P \Pi \Pi P X & \xrightarrow{(\lambda_X)_!} & \Pi \Pi \Pi \Pi P X \\
\lambda_{\Pi \Pi X} & \downarrow & \lambda_{\Pi \Pi X} \\
\Pi \Pi \Pi \Pi P X & \xrightarrow{(\lambda_X)_!} & \Pi \Pi \Pi \Pi P X \\
\gamma_{\Pi \Pi \Pi \Pi P X} & \downarrow & \lambda_{\Pi \Pi X} \\
P \Pi \Pi P X & \xrightarrow{(\lambda_X)_!} & \Pi \Pi \Pi \Pi P X \end{array}
\]

\[
\begin{array}{ccc}
P \Pi \Pi P X & \xrightarrow{(\lambda_X)_!} & \Pi \Pi \Pi \Pi P X \\
\lambda_{\Pi \Pi X} & \downarrow & \lambda_{\Pi \Pi X} \\
\Pi \Pi \Pi \Pi P X & \xrightarrow{(\lambda_X)_!} & \Pi \Pi \Pi \Pi P X \\
\gamma_{\Pi \Pi \Pi \Pi P X} & \downarrow & \lambda_{\Pi \Pi X} \\
P \Pi \Pi P X & \xrightarrow{(\lambda_X)_!} & \Pi \Pi \Pi \Pi P X \end{array}
\]

We explain the commutativity of the following diagram:

1. The definition of \( \lambda \).
2. \( \lambda^\dagger \) satisfies the \( \Pi \) unit law (b) strictly.
3. Note that \( \lambda^\dagger_X = ((y_X)_!)^\dagger \cdot y_{\Pi \Pi P X} \) is a right adjoint in \( Q \text{-Cat} \) (with \( \sup_{\Pi \Pi P X} \cdot (y_X)_! \) as its left adjoint), thus so is \( (\lambda_X^\dagger)_! \) by Lemma 2.7. Hence

\[
(\lambda_X^\dagger)_! \cdot (y_{\Pi \Pi P X}^\dagger)_! = (\lambda_X^\dagger)_! \cdot (\inf_{\Pi \Pi P X})_! \\
= (\inf_{\Pi \Pi P X})_! \cdot (\lambda_X^\dagger)_!_! \\
= (y_{\Pi \Pi P X}^\dagger)_! \cdot (\lambda_X^\dagger)_!_!.
\]

(Comment 2.7)
(4): As \( y^t_{\mathcal{P}X} \) is a right adjoint in \( \mathcal{Q}\text{-Cat} \), similar to (3) one deduces
\[
(y^t_{\mathcal{P}X})! \cdot (y^t_{\mathcal{P}P\mathcal{P}X})^! = (y^t_{\mathcal{P}X})! \cdot (\inf_{\mathcal{P}\mathcal{P}P\mathcal{P}X})!
\]
(Lemma 2.7)
\[
= (\inf_{\mathcal{P}P\mathcal{P}X})! \cdot (y^t_{\mathcal{P}X})!.
\]
(Lemma 2.9)
\[
= (y^t_{\mathcal{P}P\mathcal{P}X})^! \cdot (y^t_{\mathcal{P}X})!.
\]
(Lemma 2.7)

(5): Since \( y^t_{\mathcal{P}P\mathcal{P}X} \), \( y^t_{\mathcal{P}P\mathcal{P}X} \) is a left adjoint in \( \mathcal{Q}\text{-Cat} \). It follows that
\[
(y_{\mathcal{P}P\mathcal{P}X})! \cdot (y^t_{\mathcal{P}P\mathcal{P}X})^! = \sup_{\mathcal{P}P\mathcal{P}X} (y^t_{\mathcal{P}P\mathcal{P}X})^!
\]
(Lemma 2.7)
\[
= ((y^t_{\mathcal{P}P\mathcal{P}X})!') \cdot \sup_{\mathcal{P}P\mathcal{P}X}
\]
(Lemma 2.9)
\[
= ((y^t_{\mathcal{P}P\mathcal{P}X})!') \cdot (y^t_{\mathcal{P}P\mathcal{P}X})!.
\]
(Lemma 2.11(1))

(6): From \( y^t_{\mathcal{P}P\mathcal{P}X} \) one has
\[
((y^t_{\mathcal{P}P\mathcal{P}X})!') \cdot \lambda_{\mathcal{P}P\mathcal{P}X} = (\inf_{\mathcal{P}P\mathcal{P}X})! \cdot \lambda_{\mathcal{P}P\mathcal{P}X}
\]
(Lemma 2.7)
\[
= \lambda_{\mathcal{P}P\mathcal{P}X} \cdot (\inf_{\mathcal{P}P\mathcal{P}X})!.
\]
(\( \lambda \) satisfies (a))
\[
= \lambda_{\mathcal{P}P\mathcal{P}X} \cdot (y^t_{\mathcal{P}P\mathcal{P}X})!.
\]
(Lemma 2.7)

(7): Since \( (\lambda^t_{\mathcal{P}X})! \cdot (\lambda^t_{\mathcal{P}X})! \), Lemma 2.9 implies
\[
(\lambda^t_{\mathcal{P}X})! \cdot \mathcal{S}_{\mathcal{P}X} = (\lambda^t_{\mathcal{P}X})! \cdot \sup_{\mathcal{P}X} = \sup_{\mathcal{P}P\mathcal{P}X} \cdot (\lambda^t_{\mathcal{P}X})^! = \mathcal{S}_{\mathcal{P}X} \cdot (\lambda^t_{\mathcal{P}X})^!.
\]

(6): \( \lambda \) satisfies the \( \mathcal{P} \)-multiplication law (e) strictly.

A \( \mathcal{Q} \)-interior space is a pair \((X, c)\) consisting of a \( \mathcal{Q} \)-category \( X \) and a \( \mathcal{Q} \)-closure operation \( c \) on \( \mathcal{P}X \). A continuous \( \mathcal{Q} \)-functor \( f : (X, c) \to (Y, d) \) between \( \mathcal{Q} \)-interior spaces is a \( \mathcal{Q} \)-functor \( f : X \to Y \) such that
\[
c \cdot f^i : f^i \cdot d : \mathcal{P}Y \to \mathcal{P}X.
\]

\( \mathcal{Q} \)-interior spaces and continuous \( \mathcal{Q} \)-functors constitute a 2-category \( \mathcal{Q}\text{-Int} \), with the local order inherited from \( \mathcal{Q}\text{-Cat} \).

6.2. REMARK. When \( \mathcal{Q} \) is a commutative quantale, \( \mathcal{V} \), one has \( u \not\leq v = v \wedge u \) for all \( u, v \in \mathcal{V} \). Considering a set \( X \) as a discrete \( \mathcal{V} \)-category one can display \( \mathcal{P}X \) and \( \mathcal{P}^tX \) as having the same underlying set \( \mathcal{V}^X \), and for all \( \varphi, \psi \in \mathcal{V}^X \) one has
\[
\mathcal{P}X(\varphi, \psi) = \mathcal{P}^tX(\psi, \varphi),
\]
i.e., \( \mathcal{P}^tX \) is the dual of \( \mathcal{P}X \). Thus, for a closure operation \( c : \mathcal{P}^tX \to \mathcal{P}^tX \) one has
\[
1_{\mathcal{P}^tX} \leq c \iff c \leq 1_{\mathcal{P}X},
\]
that is, \( c \) is an interior operation on \( \mathcal{P}X \) (see [11]). Particularly, when \( \mathcal{V} = 2, \mathcal{P}X \) is just the powerset of \( X \), and a closure operation \( c \) on \( \mathcal{P}^tX \) is exactly an interior operation on the powerset of \( X \). So, an interior space \((X, c)\) as defined here coincides with the usual notion.
6.3. **Theorem.** \((\Lambda, \mathcal{Q})\text{-Alg} \cong \mathcal{Q}\text{-Int}\).  

**Proof.** **Step 1.** We show that if \((X, p)\) is a \((\Lambda, \mathcal{Q})\)-algebra, then

\[
p = (\inf_{PPX} \cdot p^! \cdot y_{PPX} \cdot \mathcal{X})^! \cdot y_{PPP} = (y_{\mathcal{X}})^{!} \cdot (y_{PPX})^! \cdot p^! \cdot \inf_{PPP} \cdot y_{PPX}. \tag{6.v}
\]

Indeed, from the definition of the 2-monad \([PP]\) one may translate the conditions \((f)\) and \((g)\) for \((X, p)\) respectively as

\[
y_X \leq p \cdot (y_X^!) \cdot y_X \quad \text{and} \quad y_X^! \cdot p \cdot \Lambda_X \cdot p^! \leq p \cdot \sup_{PP} \cdot (y_{PPX})^!.
\]

Since from Lemma 2.13 one has

\[
y_X \leq p \cdot (y_X^!) \cdot y_X \iff 1_{PPX} \leq p \cdot (y_X^!);
\]

and since \(\Lambda_X = y_{PPX} \cdot ((y_X^!))^!\) implies

\[
y_X^! \cdot p \cdot \Lambda_X \cdot p^! = y_X^! \cdot p \cdot y_{PPX} \cdot ((y_X^!))^! \cdot p^!
\]

\[
= \sup_{PPX} \cdot y_{PPX} \cdot p \cdot ((y_X^!))^! \cdot p^! \quad \text{(y is natural)}
\]

\[
= p \cdot ((y_X^!))^! \cdot p^!,
\]

the conditions \((f)\) and \((g)\) may be simplified to read as

\[
(f) \quad 1_{PPX} \leq p \cdot (y_X^!); \quad \text{and}
\]

\[
(g) \quad p \cdot ((y_X^!))^! \cdot p^! \leq p \cdot \sup_{PP} \cdot (y_{PPX})^!.
\]

Therefore,

\[
p = \sup_{PPX} \cdot y_{PPX} \cdot p
\]

\[
= (y_X)^! \cdot p! \cdot y_{PPX} \quad \text{(y is natural)}
\]

\[
= (y_X)^! \cdot p! \cdot (\inf_{PPX})^! \cdot \inf_{PP} \cdot y_{PPX} \quad \text{(\(\inf_{PPX}\) is surjective)}
\]

\[
\leq (y_X)^! \cdot (\inf_{PPX})^! \cdot p! \cdot \inf_{PP} \cdot y_{PPX} \quad \text{(Lemma 2.9)}
\]

\[
= (y_X^! \cdot (y_{PPX})^! \cdot p! \cdot \inf_{PP} \cdot y_{PPX} \quad \text{(\(y^! \inf_{PPX}\))}
\]

\[
= (y_X^! \cdot (y_{PPX})^! \cdot p! \cdot \inf_{PP} \cdot y_{PPX} \quad \text{(\(y^! \inf_{PPX}\))}
\]

\[
\leq p \cdot (y_X^!) \cdot (y_X^! \cdot (y_{PPX})^! \cdot p! \cdot \inf_{PP} \cdot y_{PPX} \quad \text{(\(p\) satisfies \(f)\))}
\]

\[
\leq p \cdot (y_X^!) \cdot (y_X^! \cdot \inf_{PP} \cdot y_{PPX} \quad \text{(\(y^! \inf_{PPX}\))}
\]

\[
\leq p \cdot \sup_{PP} \cdot (y_{PPX})^! \cdot \inf_{PP} \cdot y_{PPX} \quad \text{(\(p\) satisfies \(g\))}
\]

\[
= p.
\]

**Step 2.** As an immediate consequence of \((6.v)\), \(p\) is a right adjoint in \(\mathcal{Q}\text{-Cat}\). For every \(\mathcal{Q}\)-category \(X\), as one already has

\[
\mathcal{Q}\text{-Dist}(P^! X, X) \cong (\mathcal{Q}\text{-Cat})^{\mathcal{Q}}(P^! X, P^! X) \cong (\mathcal{Q}\text{-Inf})^{\mathcal{Q}}(PP^! X, PX)
\]
from Lemma 2.14, with the isomorphism given by

\[(\varphi : P^\dagger X \rightarrow X) \mapsto (\varphi : P^\dagger X \rightarrow P^\dagger X) \mapsto (\varphi : PP^\dagger X \rightarrow PX),\]

in order to establish a bijection between \(Q\)-closure operations on \(P^\dagger X\) and \((\Lambda, Q)\)-algebra structures on \(X\), it suffices to prove

- \(1_{P^\dagger X} \leq \varphi \iff \varphi \circ \) satisfies (f), and
- \(\varphi \cdot \varphi \leq \varphi \iff \varphi \circ \) satisfies (g)

for all \(Q\)-distributors \(\varphi : P^\dagger X \rightarrow X\).

First, \(1_{P^\dagger X} \leq \varphi \iff \varphi \circ \) satisfies (f). Indeed,

\[\overrightarrow{(y_X)} = 1_{P^\dagger X} \leq \varphi \iff \varphi \circ \leq \overrightarrow{(y_X)} = (\varphi \circ y_X) \quad \text{(Lemma 2.8(2))}\]

\[\iff 1_{PX} \leq \varphi \circ (y_X). \quad (\varphi \circ \downarrow \varphi)\]

Second, \(\varphi \cdot \varphi \leq \varphi \iff \varphi \circ \) satisfies (g), i.e.,

\[\varphi \circ ((y_X)_i) \cdot (\varphi \circ)_i \leq \varphi \circ \cdot \sup_{PP^\dagger X} \cdot (y_{PP^\dagger X})^i.\]

Since

\[\varphi \circ ((y_X)_i) \cdot (\varphi \circ)_i = \varphi \circ ((y_X)_i) \cdot (\varphi \circ)^i \quad \text{(Lemma 2.11(1))}\]
\[= \varphi \circ ((y_X)_i) \cdot (\varphi \circ)_i^i \quad (\varphi \circ \downarrow \varphi)\]
\[= \varphi \circ (\varphi \circ)^i, \quad \text{(Lemma 2.10(3))}\]

and since from (6.iii) one already knows

\[s_X = \sup_{PP^\dagger X} \cdot (y_{PP^\dagger X})^i = (y_{P^\dagger X})^i \cdot (y_{P^\dagger X})_i^i,\]

the condition (g) for \(\varphi \circ \) may be alternatively expressed as

\[\varphi \circ (\varphi_i) \leq \varphi \circ (y_{P^\dagger X}) \cdot (y_{P^\dagger X})_i^i.\]

Moreover, from Lemma 2.10(4) one has

\[\varphi^\circ = (\varphi^* \circ (y_X)^\circ) = (y_{P^\dagger X})_i^i, \quad (6.vi)\]
and, consequently,
\[ \varphi \cdot \varphi \leq \varphi \]
\[ \iff \varphi^\otimes \cdot y_{P|^X}^! \cdot \varphi^\otimes \cdot y_{P|^X}^! \leq \varphi^\otimes \cdot y_{P|^X}^! \quad \text{(Lemma 2.10(3))} \]
\[ \iff \varphi^\otimes \cdot y_{P|^X}^! \cdot \varphi^\otimes \leq \varphi^\otimes \quad \text{(Lemma 2.13)} \]
\[ \iff \varphi^\otimes \cdot (\varphi^\otimes)_i \cdot y_{P|^X}^! \leq \varphi^\otimes = \varphi^\otimes \cdot \inf_{P|^X} y_{P|^X}^! \quad \text{(y\textsuperscript{1} is natural)} \]
\[ \iff \varphi^\otimes \cdot (\varphi^\otimes)_i \leq \varphi^\otimes \cdot \inf_{P|^X} y_{P|^X}^! = \varphi^\otimes \cdot (y_{P|^X}^!)_i \quad \text{(Lemma 2.13)} \]
\[ \iff (\varphi \circ (\varphi^\otimes)_i)^\otimes \leq (\varphi \circ (\varphi^\otimes)_i)^\otimes \quad \text{(Lemma 2.8(2))} \]
\[ \iff (y_{P|^X}^!)_i \cdot \varphi^\otimes \leq ((\varphi^\otimes)^!)_i \cdot \varphi^\otimes = ((y_{P|^X}^!)_i)^! \cdot (\varphi^\otimes)^! \cdot \varphi^\otimes \quad \text{(Equation (6.vi))} \]
\[ \iff \varphi^\circ \cdot (\varphi^\otimes)_i \leq \varphi^\circ \cdot (y_{P|^X}^!)_i \cdot ((y_{P|^X}^!)_i)^! \]
\[ \iff \varphi \text{ satisfies (g);} \]

here the penultimate equivalence is an immediate consequence of
\[ (y_{P|^X}^!)_i \cdot \varphi^\otimes \dashv \varphi^\circ \cdot (y_{P|^X}^!)_i \quad \text{and} \quad \varphi^\circ \cdot \varphi^\otimes \dashv \varphi^\circ \cdot (\varphi^\otimes)_i. \]

**Step 3.** \( f : (X, \varphi) \dashv (Y, \psi) \) is a continuous \( Q \)-functor if, and only if, \( f : (X, \varphi^\circ) \dashv (Y, \psi^\circ) \) satisfies (h), i.e.,
\[ f_! \cdot \varphi^\circ \leq \psi^\circ \cdot f_! \]
Indeed,
\[ \varphi \cdot f_! \leq f_! \cdot \psi \]
\[ \iff \varphi^\otimes \cdot y_{P|^X}^! \cdot f_! \leq f_! \cdot \psi^\otimes \cdot y_{P|^X}^! \quad \text{(Lemma 2.10(3))} \]
\[ \iff \varphi^\otimes \cdot (f_!)_i \cdot y_{P|^X}^! \leq f_! \cdot \psi^\otimes \cdot y_{P|^X}^! \quad \text{(y\textsuperscript{1} is natural)} \]
\[ \iff \varphi^\otimes \cdot (f_!)_i \leq f_! \cdot \psi^\otimes \quad \text{(Lemma 2.13)} \]
\[ \iff (\varphi \circ (f_!)_i)^\otimes \leq (f^\circ \circ \psi)^\otimes \quad \text{(Lemma 2.8(2))} \]
\[ \iff (f^\circ \circ \psi)^\otimes \leq (\varphi \circ (f_!)_i)^\otimes \]
\[ \iff \psi^\circ \cdot f_! \leq f_! \cdot \varphi^\circ \]
\[ \iff f_! \cdot \varphi^\circ \leq \psi^\circ \cdot f_! \quad \text{implies } \varphi^\circ \dashv \varphi^\circ \text{ and } \psi^\circ \dashv \psi^\circ \]

here Lemma 2.13 is applicable to the third equivalence because \( f_! = (f^\circ)^\otimes \), as well as \( \psi^\otimes \),

is a right adjoint in \( Q\text{-Cat} \). This completes the proof. 

7. The distributive law of the double copresheaf 2-monad

As the adjunction (6.ii) has its dual
\[ P^\text{coop} \dashv (P^\text{coop})^\text{coop} : Q\text{-Cat} \dashv (Q\text{-Cat})^\text{coop}, \quad (7.i) \]
one naturally constructs the double copresheaf 2-monad
\[ \mathcal{P}^{\dagger}\mathcal{P} = (\mathcal{P}^{\dagger}\mathcal{P}, \eta^{\dagger}, \sigma^{\dagger}) \]
on \textbf{Q-Cat}, with the units given by
\[ \eta^{\dagger}_X = y^{\dagger}_{P\mathcal{X}} \cdot y_X = (y_X)_! \cdot y^{\dagger}_X : X \to \mathcal{P}^{\dagger}\mathcal{P}X \]
and the multiplication by
\[ s^{\dagger}_X = (y^{\dagger}_{P\mathcal{X}} \cdot y^{\dagger}_{P\mathcal{X}})_! = ((y^{\dagger}_{P\mathcal{X}})_! \cdot y^{\dagger}_{P\mathcal{X}})_! = s^{\dagger}_{P\mathcal{X}} \cdot y^{\dagger}_{P\mathcal{P}\mathcal{P}\mathcal{X}} : \mathcal{P}^{\dagger}\mathcal{P}\mathcal{P}\mathcal{P}\mathcal{X} \to \mathcal{P}^{\dagger}\mathcal{P}X. \]

\textbf{7.1. Theorem.} The double copresheaf 2-monad \( \mathcal{P}^{\dagger}\mathcal{P} \) distributes over \( \mathcal{P} \) by \( \Lambda^{\dagger} \) with
\[ \Lambda^{\dagger}_X = y^{\dagger}_{P\mathcal{P}\mathcal{P}\mathcal{X}} \cdot y^{\dagger}_{P\mathcal{P}\mathcal{X}} : \mathcal{P}^{\dagger}\mathcal{P}\mathcal{P}\mathcal{X} \to \mathcal{P}^{\dagger}\mathcal{P}X. \]

\textbf{Proof.} First note that
\[ \Lambda^{\dagger}_X = \lambda^{\dagger}_{P\mathcal{X}} \cdot (\lambda_X)_!. \] Indeed, with the naturality of \( y \) and the full faithfulness of \( (y_{P\mathcal{X}})_! \) one easily sees that the diagram
\[ \begin{tikzcd}
\mathcal{P}^{\dagger}\mathcal{P}\mathcal{P}\mathcal{X} \ar[r]^{(\lambda_X)_!} \ar[d]_{y_{P\mathcal{P}\mathcal{P}\mathcal{X}}} & \mathcal{P}^{\dagger}\mathcal{P}X \ar[r]^{(y_{P\mathcal{X}})_!} \ar[d]_{y_{P\mathcal{P}\mathcal{X}}} & \mathcal{P}^{\dagger}\mathcal{P}\mathcal{P}\mathcal{X} \ar[d]_{y_{P\mathcal{P}\mathcal{P}\mathcal{X}}} \\
\mathcal{P}\mathcal{P}\mathcal{P}\mathcal{X} \ar[r]_{(y^{\dagger}_X)_!} & \mathcal{P}^{\dagger}\mathcal{P}\mathcal{X} \ar[r]_{y^{\dagger}_{P\mathcal{P}\mathcal{X}}} & \mathcal{P}\mathcal{P}\mathcal{P}\mathcal{X} \\
\mathcal{P}\mathcal{P}\mathcal{P}\mathcal{X} \ar[r]_{(y^{\dagger}_{P\mathcal{X}})_!} & \mathcal{P}\mathcal{P}\mathcal{P}X & \mathcal{P}\mathcal{P}\mathcal{P}\mathcal{X} \ar[r]_{\lambda^{\dagger}_{P\mathcal{X}}} & \mathcal{P}^{\dagger}\mathcal{P}\mathcal{P}\mathcal{X} \ar[u]_{(y_{P\mathcal{X}})_!} \ar[u]_{(y^{\dagger}_{P\mathcal{X}})_!} \\
\end{tikzcd} \]
commutes for every \( \mathcal{Q} \)-category \( X \). Now we check the laws (a)-(e) for \( \Lambda^{\dagger} \):

(a) \( \Lambda^{\dagger}_f \cdot f_{\Pi i} = f_{\Pi i} \cdot \Lambda^{\dagger}_X \) for every \( \mathcal{Q} \)-functor \( f : X \to Y \). This is a direct consequence of the naturality of \( \lambda \) and \( \lambda^{\dagger} \).
(b) \( y_{P \map X} = \Lambda_X^X \cdot (y_X)_i \). This is easy since \( \lambda \) and \( \lambda^\dagger \) both satisfy the \( \mathbb{P} \)-unit law (b) strictly.

\[
\begin{array}{ccc}
P^\dagger P X & \xrightarrow{(y_X)_i} & (y_{P X}), \\
\downarrow & & \downarrow \gamma_{P \map P X} \\
(P \map P P X) & \xrightarrow{(\lambda_X)_i} & (P \map X) \lambda^\dagger_{P X} \\
\downarrow & & \downarrow \lambda^\dagger_{P P X} \\
P^\dagger PP X & \rightarrow & PP^\dagger P X \\
n & \xrightarrow{\lambda^\dagger_{P X}} & \lambda^\dagger_{P P X} \lambda^\dagger_{PP P X} \\
(\lambda^\dagger_{P X})_i & \xrightarrow{(s_{P X})_i} & (\lambda^\dagger_{P P X})_i \\
\downarrow & & \downarrow \lambda^\dagger_{PP P P X} \\
P^\dagger PP P X & \rightarrow & PP^\dagger PP P X \\
\end{array}
\]

(c) \( s_{P \map P X} \cdot (\Lambda_X^X)_i \cdot \Lambda_X^X = \Lambda_X^X \cdot (s_X)_i \). This follows from the naturality of \( \lambda^\dagger \) and the fact that \( \lambda \) and \( \lambda^\dagger \) both satisfy the \( \mathbb{P} \)-multiplication law (c) strictly.

\[
\begin{array}{ccc}
P^\dagger PP P X & \xrightarrow{\Lambda^\dagger_{P X}} & PP^\dagger PP P X \\
\downarrow & & \downarrow (\Lambda^\dagger_{P P X})_i \\
P^\dagger PPP P X & \xrightarrow{(\lambda^\dagger_{P P P X})_i} & PP^\dagger PPP P X \\
\downarrow & & \downarrow (\lambda^\dagger_{PP P P P X})_i \\
P^\dagger P P P P P X & \rightarrow & PP^\dagger P P P P P X \\
\end{array}
\]

(d) \( (\eta^\dagger_X)_i \leq \Lambda_X^X \cdot \eta^\dagger_{P X} \). Since \( \lambda \) satisfies the lax \( \mathbb{P} \)-unit law (d) and \( \lambda^\dagger \) strictly satisfies the \( \mathbb{P}^\dagger \)-unit law (d), one obtains the commutativity of the two lower triangles of the following diagram. Moreover, the naturality of \( y^\dagger \) guarantees the commutativity of the middle rhombus.
(e) $(s_X^t)_! \cdot \Lambda_{P^!P_X}^t \cdot (\Lambda_X^t)_i = \Lambda_X^t \cdot s_{P^!P_X}^t$. We explain the commutativity of the following diagram:

\[ (\Lambda_X^t)_i \xrightarrow{1} (\lambda_{P^!P_X}^t)_i \xrightarrow{2} (\lambda_{P^!P_X}^t)_i \xrightarrow{3} (\lambda_{P^!P_X}^t)_i \xrightarrow{4} (\lambda_{P^!P_X}^t)_i \xrightarrow{5} (y_{P^!P_X}^t)_i \xrightarrow{6} (y_{P^!P_X}^t)_i \xrightarrow{7} (y_{P^!P_X}^t)_i \xrightarrow{8} (y_{P^!P_X}^t)_i \xrightarrow{9} (y_{P^!P_X}^t)_i \xrightarrow{10} (\Lambda_{P^!P_X}^t)_i \]

1: Equation (6.4v).
2: The definition of \( \lambda \).
3: \( \Lambda \) satisfies the \( \mathbb{P} \)-unit law (b) strictly.
4: Since \( y_X^t = \sup_{P_X} \dashv y_{P^!} \), \( (y_X^t)_i \) is a left adjoint in \( \mathcal{Q} \)-\textsc{Cat}, by Lemma 2.7. Thus

\[
(y_X^t)_i \cdot y_{P^!P_X}^t = (y_X^t)_i \cdot (\sup_{P^!P_X})_i = (\sup_{P^!P_X})_i \cdot (y_X^t)_i \quad \text{(Lemma 2.7)}
\]

5: Follows from an application of Lemma 2.11(4) to \( y_{P^!P_X} : P^!P_X \to PP^!P_X \).
6: From \( \sup_{P^!P_X} \dashv y_{P^!P_X} \) one has

\[
(y_{P^!P_X}^t)_i \cdot \lambda_{P^!P_X}^t = (\sup_{P^!P_X})_i \cdot \lambda_{P^!P_X}^t \quad \text{(Lemma 2.7)}
\]

7: \( \lambda^t \) satisfies the \( \mathbb{P} \)-unit law (b) strictly.
Since $\lambda_X \vdash (\lambda_X)_i$, Lemma 2.9 implies
$$((\lambda_X)_i \cdot \inf_{PP_X} = (\lambda_X)_i \cdot \inf\cdot (\lambda_X))_{ii} = \inf\cdot (\lambda_X)_{ii}.$$ 

$\lambda^\dagger$ satisfies the $\mathbb{P}^\dagger$-multiplication law (e) strictly.

7.2. Theorem. $(\Lambda^\dagger, Q)$-$\text{Alg} \cong Q$-$\text{Cls}$.

Proof. Step 1. We show that if $(X, p)$ is a $(\Lambda^\dagger, Q)$-algebra, then
$$p = \inf_{PP_X} \cdot p_i \cdot (y_{PP_X}^\dagger)_i. \quad (7.\text{vi})$$

Indeed, with (7.ii) and (7.iii) one may translate the conditions (f) and (g) respectively as
$$y_X \leq p \cdot y_{PP_X}^\dagger \cdot y_X \quad \text{and} \quad \sup_{PP_X} \cdot p_i \cdot \Lambda_X^\dagger \cdot p_i \leq p \cdot (y_{PP_X}^\dagger)_i \cdot y_{PP_X}^\dagger.$$ 

To simplify the above conditions, first note that Lemma 2.13 implies
$$y_X \leq p \cdot y_{PP_X}^\dagger \cdot y_X \iff 1_{PP_X} \leq p \cdot y_{PP_X}^\dagger.$$ 

Second, from $\Lambda_X^\dagger = y_{PP_X}^\dagger \cdot (y_{PP_X}^\dagger)_i = y_{PP_X}^\dagger \cdot (\sup_{PP_X})_i$ (see the commutative diagram (7.v)) one has
$$\sup_{PP_X} \cdot p_i \cdot \Lambda_X^\dagger \cdot p_i = \sup_{PP_X} \cdot p_i \cdot (y_{PP_X}^\dagger) \cdot (\sup_{PP_X})_i \cdot p_i = \sup_{PP_X} \cdot (\sup_{PP_X})_i \cdot p_i \quad (y \text{ is natural})$$
and, moreover,
$$p \cdot (\sup_{PP_X})_i \cdot p_i \leq p \cdot (y_{PP_X}^\dagger)_i \cdot (y_{PP_X}^\dagger)_i \quad (y_{PP_X}^\dagger \vdash (y_{PP_X}^\dagger)_i)$$
$$p \cdot (\sup_{PP_X})_i \cdot (y_{PP_X})_i \cdot p_i \leq p \cdot (y_{PP_X}^\dagger)_i \quad (y \text{ is natural})$$
$$p \cdot p_i \leq p \cdot (y_{PP_X}^\dagger)_i.$$

Therefore, $(X, p)$ is a $(\Lambda^\dagger, Q)$-algebra if, and only if,

(f) $1_{PP_X} \leq p \cdot y_{PP_X}^\dagger$ and

(g) $p \cdot p_i \leq p \cdot (y_{PP_X}^\dagger)_i$. 

\[\square\]
It follows that
\[
p = \inf_{P} X \cdot y_{P_{X}}^{\dagger} \cdot p
\]
\[
= \inf_{P} X \cdot p_{i} \cdot y_{P_{X}^{i}}^{\dagger}
\]
\[
\leq \inf_{P} X \cdot p_{i} \cdot (y_{P_{X}}^{\dagger})_{i}
\]
\[
\leq p \cdot y_{P_{X}}^{\dagger} \cdot \inf_{P} X \cdot p_{i} \cdot (y_{P_{X}}^{\dagger})_{i}
\]
\[
\leq p \cdot (y_{P_{X}}^{\dagger})_{i} \cdot (y_{P_{X}}^{\dagger})_{i}
\]
\[
= p.
\]

**Step 2.** As an immediate consequence of (7.vi), \( p \) is a right adjoint in \( Q\text{-Cat} \). For every \( Q \)-category \( X \), as one already has

\[
\text{Q-Dist}(X, P_{X}) \cong Q\text{-Cat}(P_{X}, P_{X}) \cong Q\text{-Inf}(P_{X} P_{X}, P_{X})
\]

from Lemma 2.14 with the isomorphism given by

\[
(\varphi : X \rightleftharpoons P_{X}) \mapsto (\overline{\varphi} : P_{X} \rightleftharpoons P_{X}) \mapsto (\varphi^{\dagger} : P_{X} P_{X} \rightarrow P_{X})
\]

in order to establish a bijection between \( Q \)-closure operations on \( P_{X} \) and \( (\Lambda^{\dagger}, Q) \)-algebra structures on \( X \), it suffices to prove

1. \( 1_{P_{X}} \leq \overline{\varphi} \iff \varphi^{\dagger} \) satisfies (f), and
2. \( \overline{\varphi} \cdot \overline{\varphi} \leq \overline{\varphi} \iff \varphi^{\dagger} \) satisfies (g)

for all \( Q \)-distributors \( \varphi : X \rightleftharpoons P_{X} \).

First, the equivalence \( (1_{P_{X}} \leq \overline{\varphi} \iff \varphi^{\dagger} \) satisfies (f)) is trivial since \( \overline{\varphi} = \varphi^{\dagger} \cdot y_{P_{X}}^{\dagger} \).

Second, \( \overline{\varphi} \cdot \overline{\varphi} \leq \overline{\varphi} \iff \varphi^{\dagger} \) satisfies (g).

Indeed,

\[
\overline{\varphi} \cdot \overline{\varphi} \leq \overline{\varphi} \iff \overline{\varphi} \cdot y_{P_{X}}^{\dagger} \cdot \varphi^{\dagger} \cdot y_{P_{X}}^{\dagger} \leq \overline{\varphi} \cdot y_{P_{X}}^{\dagger}
\]
\[
\iff \overline{\varphi} \cdot y_{P_{X}}^{\dagger} \cdot \varphi^{\dagger} \leq \varphi^{\dagger}
\]
\[
\iff \varphi^{\dagger} \cdot (\varphi^{\dagger})_{i} \cdot y_{P_{X}^{i}}^{\dagger} \leq \varphi^{\dagger} \iff \varphi^{\dagger} \cdot \inf_{P_{X}^{i}} \cdot y_{P_{X}^{i}}^{\dagger}
\]
\[
\iff \varphi^{\dagger} \cdot (\varphi^{\dagger})_{i} \leq \varphi^{\dagger} \cdot \inf_{P_{X}^{i}} = \varphi^{\dagger} \cdot (y_{P_{X}}^{\dagger})_{i}
\]
\[
\iff \varphi^{\dagger} \) satisfies (g).
\]

**Step 3.** \( f : (X, \overline{\varphi}) \rightarrow (Y, \overline{\psi}) \) is a continuous \( Q \)-functor if, and only if, \( f : (X, \varphi^{\dagger}) \rightarrow (Y, \psi^{\dagger}) \) satisfies (h). Indeed,

\[
f_{i} \cdot \overline{\varphi} \leq \overline{\psi} \cdot f_{i} \iff f_{i} \cdot \varphi^{\dagger} \cdot y_{P_{X}}^{\dagger} \cdot \ell_{i} \leq \psi^{\dagger} \cdot y_{P_{Y}}^{\dagger} \cdot f_{i}
\]
\[
\iff f_{i} \cdot \varphi^{\dagger} \cdot y_{P_{X}}^{\dagger} \leq \psi^{\dagger} \cdot \ell_{i} \cdot y_{P_{X}}^{\dagger}
\]
\[
\iff f_{i} \cdot \varphi^{\dagger} \leq \psi^{\dagger} \cdot \ell_{i},
\]

which completes the proof.
8. Distributive laws of $T$ over $P$ versus lax extensions of $T$ to $Q$-$Dist$

In this section, for an arbitrary 2-monad $T$ on $Q$-$Cat$, we outline the bijective correspondence between distributive laws of $T$ over $P$ and so-called lax extensions of $T$ to $Q$-$Dist$. The techniques adopted here generalize their discrete counterparts as given in [30].

Given a 2-functor $T : Q$-$Cat \to Q$-$Cat$, a lax extension of $T$ to $Q$-$Dist$ is a lax functor

$$\hat{T} : Q$-$Dist \to Q$-$Dist$$

that coincides with $T$ on objects and satisfies the extension condition (3) below. Explicitly, $\hat{T}$ is given by a family

$$(\hat{T}\varphi : TX \Rightarrow TY)_{\varphi \in Q$-$Dist(X,Y), \ X,Y \in \text{ob}(Q$-$Cat)} \quad (8.i)$$

of $Q$-distributors such that

1. $\varphi \leq \varphi' \implies \hat{T}\varphi \leq \hat{T}\varphi'$,
2. $\hat{T}\psi \circ \hat{T}\varphi \leq \hat{T}(\psi \circ \varphi)$,
3. $(Tf)_* \leq \hat{T}(f_*)$, $(Tf)_* \leq \hat{T}(f^*)$,

for all $Q$-distributors $\varphi, \varphi' : X \Rightarrow Y$, $\psi : Y \Rightarrow Z$ and $Q$-functors $f : X \to Y$.

It is useful to present the following equivalent conditions of (3), which can be proved analogously to their discrete versions in [30], by straightforward calculation:

8.1. Lemma. Given a family (8.i) of $Q$-distributors satisfying (1) and (2), the following conditions are equivalent when quantified over the variables occurring in them ($f : X \to Y$, $\varphi : Z \Rightarrow Y$, $\psi : Y \Rightarrow Z$):

1. $1^\tau_X \leq \hat{T}(1^\tau_X)$, $\hat{T}(f^* \circ \varphi) = (Tf)^* \circ \hat{T}\varphi$.
2. $1^\tau_X \leq \hat{T}(1^\tau_X)$, $\hat{T}(\psi \circ f_*) = \hat{T}\psi \circ (Tf)_*$.
3. $(Tf)_* \leq \hat{T}(f_*)$, $(Tf)_* \leq \hat{T}(f^*)$ (i.e., $\hat{T}$ satisfies (3)).

8.2. Proposition. Lax extensions of a 2-functor $T : Q$-$Cat \to Q$-$Cat$ to $Q$-$Dist$ correspond bijectively to lax natural transformations $TP \to PT$ satisfying the lax $P$-unit law the lax $P$-multiplication law.

Proof. Step 1. For each $\lambda : TP \to PT$ satisfying (a), (b) and (c), $\Phi(\lambda) := \hat{T}(\hat{T}\varphi)_\lambda$ with $\hat{T}\varphi := \lambda_X : T\varphi X \Rightarrow PX$ is a lax extension of $T$ to $Q$-$Dist$. 

$$\Phi(\lambda) = \hat{T} : Q$-$Dist(X,Y) \to Q$-$Dist(TX,TY)$$

$$\begin{array}{ccc}
Y & \xrightarrow{\varphi} & PX \\
\downarrow & & \downarrow \varphi \\
TX & \xrightarrow{\lambda_X} & PTX \\
\downarrow \varphi & & \downarrow \varphi \\
TPX
\end{array}$$
Indeed, (1) follows immediately from the 2-functoriality of $T$. For (2), just note that

$$
\xymatrix{
\hat{T}_\psi \circ \hat{T}_\varphi = y_X^j \cdot (\hat{T}_\varphi)^* \cdot \hat{T}_\psi & (\text{Lemma } 2.12(1)) \\
= y_X^j \cdot (\lambda_X) \cdot (\hat{T}_\varphi)^* \cdot \lambda_Y \cdot \hat{T}_\psi & (\lambda \text{ satisfies (a)}) \\
\leq y_X^j \cdot (\lambda_X) \cdot \lambda_P X \cdot T(\varphi_1) \cdot \hat{T}_\psi & (\lambda \text{ satisfies (c)}) \\
= \lambda_X \cdot T(y_X^j) \cdot T(\varphi_1) \cdot \hat{T}_\psi & (\text{Lemma } 2.12(1)) \\
\leq \lambda_X \cdot T(\psi \circ \varphi). &}

For (3), it suffices to check Lemma 8.1(i). Since $\lambda$ satisfies (b), it follows easily that

$$
\xymatrix{
\hat{T}_X^* = y_X \leq \lambda_X \cdot T(y_X) = \lambda_X \cdot T1_X = \hat{T}(1_X^*). &}
$$

For the second identity, Lemma 2.12(1) implies

$$
\hat{T}(f^* \circ \varphi) = \lambda_X \cdot T(f^* \circ \varphi) = \lambda_X \cdot T(\varphi) \cdot Tf = \hat{T}_\varphi \cdot Tf = (Tf)^* \circ \hat{T}_\varphi.
$$

**Step 2.** For every lax extension $\hat{T}$ of $T$, $\Psi(\hat{T}) := \lambda = (\lambda_X)_X$ with

$$
\lambda_X := \hat{T} \varepsilon_X = T(y_X)^* : TPX \tr{PTX}
$$

is a lax natural transformation satisfying the $P$-unit law and the $P$-multiplication law.

(a) $(Tf)! \cdot \lambda_X \leq \lambda_Y \cdot T(f_!)$ for all $Q$-functors $f : X \tr{Y}$. Indeed,

$$
\xymatrix{
(Tf)! \cdot \lambda_X = \hat{T}(y_X)^* \circ (Tf)^* & (\text{Lemma } 2.12(1)) \\
\leq \hat{T}(y_X)^* \circ (Tf)^* \circ \hat{T}(1_X^*) & (\text{Lemma } 8.1(i)) \\
= \hat{T}(y_X)^* \circ \hat{T}(f^*) & (\text{Lemma } 8.1(i)) \\
\leq \hat{T}(y_X)^* \circ f^* & (\hat{T} \text{ satisfies (2)}) \\
\leq \hat{T}((f_!)^* \circ (y_Y)^*) & (\text{Lemma } 2.11(4)) \\
= (Tf_!)^* \circ \hat{T}(y_Y)^* & (\text{Lemma } 8.1(i)) \\
= \lambda_Y \cdot T(f_!). & (\text{Lemma } 2.12(1))
$$

(b) $y_X \leq \lambda_X \cdot Ty_X$. Indeed,

$$
\xymatrix{
y_X = 1_X^* \leq \hat{T}(1_X^*) & (\text{Lemma } 8.1(i)) \\
= \hat{T}(y_X^* \circ (y_X)^*) & (y_X \text{ is fully faithful}) \\
= (Ty_X)^* \circ \hat{T}(y_X)^* & (\text{Lemma } 8.1(i)) \\
= \lambda_X \cdot Ty_X. & (\text{Lemma } 2.12(1))
$$

(c) $s_{TX} \cdot (\lambda_X)! \cdot \lambda_{P_X} \leq \lambda_X \cdot Ts_X$. Indeed,
\[
\begin{align*}
s_{TX} \cdot (\lambda_X)! \cdot \lambda_{P_X} &= \check{T}(y_{PX}) \circ \check{T}(y_X) \\
&\leq \check{T}((y_{PX}) \circ (y_X)) \\
&= \check{T}(((y_X)_! \circ (y_X)_*)) \\
&= \check{T}((y_X)_! \circ (y_X)_*) \\
&= (T y_X^*) \circ \check{T}(y_X) \\
&= \lambda_X \cdot Ts_X.
\end{align*}
\]

(Lemma 2.12(1))

Step 3. $\Phi$ and $\Psi$ are inverses to each other. For each $\lambda : TP \rightarrow PT$, $\Psi \Phi(\lambda) = \lambda$ since
\[
(\Psi \Phi(\lambda))_X = \check{\Phi}(\lambda)(y_X)_* = \lambda_X \cdot \check{T}(y_X)_* = \lambda_X \cdot T1_{PX} = \lambda_X.
\]
Conversely, for every lax extension $\hat{T}$, one has
\[
(\Phi \Psi(\hat{T}))(\varphi)_X = \check{\Phi}(y_X)_* \cdot \check{T} \varphi = \check{(T \varphi)^*} \circ \check{T}(y_X)_* = \check{\varphi^*} \circ (y_X)_* = \check{T} \varphi,
\]
where the last three equalities follow respectively from Lemmas 2.12(1), 8.1(i) and 2.10(4).

For a 2-monad $T = (T, m, e)$ on $Q-Cat$, a lax extension $\hat{T}$ of $T$ to $Q-Dist$ becomes a lax extension of the 2-monad $T$ if it further satisfies
\[
(4) \varphi \circ e_Y^* \leq e_Y^* \circ \hat{T} \varphi,
\]
\[
(5) \hat{T} \hat{T} \varphi \circ m_Y^* \leq m_Y^* \circ \hat{T} \varphi
\]
for all $Q$-distributors $\varphi : X \rightarrow Y$. By adjunction, (4) and (5) may be equivalently expressed as
\[
(4') (e_Y)_* \circ \varphi \leq \hat{T} \varphi \circ (e_X)_*,
\]
\[
(5') (m_Y)_* \circ \hat{T} \varphi \leq \hat{T} \varphi \circ (m_X)_*.
\]

8.3. **Theorem.** Lax extensions of a 2-monad $T = (T, m, e)$ on $Q-Cat$ to $Q-Dist$ correspond bijectively to distributive laws of $T$ over $P$.

**Proof.** With Proposition 8.2 at hand, it suffices to prove

- $\hat{T}$ satisfies (4) $\iff$ $\lambda$ satisfies (d), and
- $\hat{T}$ satisfies (5) $\iff$ $\lambda$ satisfies (e)
for every lax extension \( \hat{T} \) of the 2-functor \( T \) and \( \lambda = \Psi(\hat{T}) \) with \( \lambda_X = \hat{T}(y_X)_* : TPX \rightarrow PTX \).

First, (\( \hat{T} \) satisfies (4) \( \iff \) \( \lambda \) satisfies (d)). Since Lemma 2.12(1) and the naturality of \( e \) imply

\[
(e_X)! \cdot \hat{T}\varphi = \varphi \circ e_X^* \quad \text{and} \quad \lambda_X \cdot e_{PX} \cdot \hat{T}\varphi = \lambda_X \cdot T\hat{T}\varphi \cdot e_Y = T\varphi \cdot e_Y = e_Y^* \circ T\varphi
\]

for all \( \varphi : X \rightarrow Y \), it follows that

\[
(e_X)! \leq \lambda_X \cdot e_{PX} \iff \forall \varphi : X \rightarrow Y : (e_X)! \cdot \hat{T}\varphi \leq \lambda_X \cdot e_{PX} \cdot \hat{T}\varphi
\]

\[
\iff \forall \varphi : X \rightarrow Y : \varphi \circ e_X^* \leq e_Y^* \circ T\varphi.
\]

Second, (\( \hat{T} \) satisfies (5) \( \iff \) \( \lambda \) satisfies (e)). Similarly as above, one has

\[
(m_X)! \cdot \lambda_{TX} \cdot T\lambda_X \cdot TT\hat{T}\varphi = (m_X)! \cdot \lambda_{TX} \cdot T\hat{T}\varphi = (m_X)! \cdot T\hat{T}\varphi = T\hat{T}\varphi \circ m_X^*
\]

and

\[
\lambda_X \cdot m_{PX} \cdot TT\hat{T}\varphi = \lambda_X \cdot T\hat{T}\varphi \cdot m_Y = \hat{T}\varphi \cdot m_Y = m_Y^* \circ \hat{T}\varphi
\]

by Lemma 2.12(1) and the naturality of \( m \). Consequently,

\[
(m_X)! \cdot \lambda_{TX} \cdot T\lambda_X \leq \lambda_X \cdot m_{PX}
\]

\[
\iff \forall \varphi : X \rightarrow Y : (m_X)! \cdot \lambda_{TX} \cdot T\lambda_X \cdot TT\hat{T}\varphi \leq \lambda_X \cdot m_{PX} \cdot TT\hat{T}\varphi
\]

\[
\iff \forall \varphi : X \rightarrow Y : \hat{T}\hat{T}\varphi \circ m_X^* \leq m_Y^* \circ \hat{T}\varphi.
\]

A strict extension of \( T : \mathcal{Q}\text{-Cat} \rightarrow \mathcal{Q}\text{-Cat} \) is a 2-functor

\[
\hat{T} : \mathcal{Q}\text{-Dist} \rightarrow \mathcal{Q}\text{-Dist}
\]

that coincides with \( T \) on objects and satisfies

(3*) \( \hat{T}(f^* \circ \varphi) = (Tf)^* \circ \hat{T}\varphi \)

for all \( f : X \rightarrow Y \), \( \varphi : Z \rightarrow Y \). It is moreover a strict extension of the 2-monad \( T = (T, m, e) \) on \( \mathcal{Q}\text{-Cat} \) if

(4*) \( \varphi \circ e_X^* = e_Y^* \circ \hat{T}\varphi \),

(5*) \( \hat{T}\hat{T}\varphi \circ m_X^* = m_Y^* \circ \hat{T}\varphi \)

for all \( \varphi : X \rightarrow Y \). In other words, a lax extension \( \hat{T} \) of \( T \) is strict if all the inequalities in (2), Lemma 8.1(i), (4) and (5) are equalities. From the above proofs one immediately sees that strict extensions of \( T \) to \( \mathcal{Q}\text{-Dist} \) correspond bijectively to strict distributive laws of \( T \) over \( \mathcal{P} \).

For a lax extension \( \hat{T} \) of \( T \) we can now define:
8.4. Definition. A \((T, Q)\)-category \((X, \alpha)\) consists of a \(Q\)-category \(X\) and a \(Q\)-distributor \(\alpha: X \longrightarrow TX\) satisfying the lax unit and lax multiplication laws:

\[
1_X^* \leq e_X^* \circ \alpha \quad \text{and} \quad \hat{T} \alpha \circ \alpha \leq m_X^* \circ \alpha.
\]

A \((T, Q)\)-functor \(f: (X, \alpha) \longrightarrow (Y, \beta)\) is a \(Q\)-functor \(f: X \longrightarrow Y\) with

\[
\alpha \circ f^* \leq (Tf)^* \circ \beta.
\]

\((T, Q)\)-categories and \((T, Q)\)-functors constitute a 2-category \((T, Q)\)-\textbf{Cat}, and we write \((T, \hat{T}, Q)\)-\textbf{Cat} to stress the dependency on the chosen extension \(\hat{T}\) if there is any danger of ambiguity.

8.5. Theorem. If \(\lambda\) and \(\hat{T}\) are related by the correspondence of Theorem 8.3, then

\[(\lambda, Q)\text{-Alg} \simeq (T, \hat{T}, Q)\text{-Cat}\]

Proof. For any \(Q\)-category \(X\), as one already has

\[Q\text{-Dist}(X, TX) \simeq Q\text{-Cat}(TX, PX)\]

with the isomorphism given by

\[
(\alpha: X \longrightarrow TX) \mapsto (\widetilde{\alpha}: TX \longrightarrow PX),
\]

in order to establish a bijection between \((T, Q)\)-category structures on \(X\) and \((\lambda, Q)\)-algebra structures on \(X\), it suffices to prove

- \(1_X^* \leq e_X^* \circ \alpha \iff y_X \leq \widetilde{\alpha} \cdot e_X\), and
- \(\hat{T} \alpha \circ \alpha \leq m_X^* \circ \alpha \iff y_X^* \cdot \alpha \cdot \lambda_X \cdot T\widetilde{\alpha} \leq \widetilde{\alpha} \cdot m_X\),

for all \(Q\)-distributors \(\alpha: X \longrightarrow TX\). Indeed, the first equivalence is easy since \(\widetilde{1}_X = y_X\) and \(\widetilde{e}_X \circ \alpha = \widetilde{\alpha} \cdot e_X\) by Lemma 2.12(1). For the second equivalence, just note that \(\widetilde{m}_X \circ \alpha = \widetilde{\alpha} \cdot m_X\) and

\[
\begin{align*}
\hat{T} \alpha \circ \alpha &= \hat{T}(\widetilde{\alpha}^* \circ (y_X)_*) \circ \alpha \\
&= (T \widetilde{\alpha})^* \circ \hat{T}(y_X)_* \circ \alpha \\
&= \hat{T}(y_X)_* \circ \alpha \circ T\widetilde{\alpha} \\
&= y_X^* \cdot \alpha \cdot \lambda_X \cdot T\widetilde{\alpha},
\end{align*}
\]

(Lemma 2.10(4)

\[
\begin{align*}
\alpha \circ f^* \leq (Tf)^* \circ \beta &\iff f \cdot \widetilde{\alpha} = \alpha \circ f^* \leq (Tf)^* \circ \beta = \beta \cdot Tf
\end{align*}
\]

by Lemma 2.12(1).
8.6. Example.

(1) For the identity 2-monad $\mathbb{I}$ on $\mathbf{Q-Cat}$, the identity 2-functor on $\mathbf{Q-Dist}$ is a strict extension of $\mathbb{I}$ and it is easy to see $(\mathbb{I}, \mathbf{Q})$-$\mathbf{Cat} \cong \mathbf{Mon}(\mathbf{Q-Dist})$.

(2) The distributive law $\lambda$ of $\mathbb{P}$ over itself described in Theorem 4.1 corresponds to the lax extension $\hat{\mathbb{P}}$ of $\mathbb{P}$ (which is strict as an extension of $\mathbb{P}$) with

$$\hat{\mathbb{P}} \varphi := \varphi^{\circ \ast} : \mathbb{P}X \rightrightarrows \mathbb{P}Y$$

for $\varphi : X \rightrightarrows Y$. From Theorem 4.2 one soon knows $(\mathbb{P}, \mathbf{Q})$-$\mathbf{Cat} \cong \mathbf{Q-Cls}$.

(3) The strict distributive law $\lambda^\dagger$ of $\mathbb{P}^\dagger$ over $\mathbb{P}$ given in Theorem 5.1 determines the strict extension $\check{\mathbb{P}}$ of $\mathbb{P}^\dagger$ with

$$\check{\mathbb{P}} \varphi := (\varphi^{\oplus})^\ast : \mathbb{P}^\dagger X \rightrightarrows \mathbb{P}^\dagger Y.$$ 

Theorem 5.3 shows that $(\mathbb{P}^\dagger, \mathbf{Q})$-$\mathbf{Cat} \cong \mathbf{Mon}(\mathbf{Q-Dist})$.

(4) Theorem 6.1 gives the distributive law $\Lambda$ of $\mathbb{P}^\dagger \mathbb{P}$ over $\mathbb{P}$ that corresponds to the lax extension $\hat{\mathbb{P}^\dagger \mathbb{P}}$ of $\mathbb{P}^\dagger \mathbb{P}$ (which is strict as an extension of $\mathbb{P}^\dagger \mathbb{P}$) with

$$\hat{\mathbb{P}^\dagger \mathbb{P}} \varphi := \hat{\mathbb{P}} \check{\mathbb{P}} \varphi = ((\varphi^{\oplus})^\ast)^\circ \ast = \varphi^{\oplus^\dagger^\ast} : \mathbb{P}^\dagger \mathbb{P}X \rightrightarrows \mathbb{P}^\dagger \mathbb{P}Y.$$ 

From Theorem 6.3 one has $(\mathbb{P}^\dagger \mathbb{P}, \mathbf{Q})$-$\mathbf{Cat} \cong \mathbf{Q-Int}$.

(5) The distributive law $\Lambda^\dagger$ of $\mathbb{P}^\dagger \mathbb{P}$ over $\mathbb{P}$ (see Theorem 7.1) is related to the lax extension $\hat{\mathbb{P}^\dagger \mathbb{P}}$ of $\mathbb{P}^\dagger \mathbb{P}$ (which is strict as an extension of $\mathbb{P}^\dagger \mathbb{P}$) with

$$\hat{\mathbb{P}^\dagger \mathbb{P}} \varphi := \hat{\mathbb{P}^\dagger} \check{\mathbb{P}} \varphi = (\varphi^{\circ \oplus})^\ast = (\varphi^{\oplus^\dagger})^\ast : \mathbb{P}^\dagger \mathbb{P}X \rightrightarrows \mathbb{P}^\dagger \mathbb{P}Y.$$ 

Theorem 7.2 shows that $(\mathbb{P}^\dagger \mathbb{P}, \mathbf{Q})$-$\mathbf{Cat} \cong \mathbf{Q-Cls}$.

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