Monoidal Topology
A Categorical Approach to Order, Metric and Topology

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To Horst Herrlich
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Monoidal Topology describes an active research area that, after many proposals throughout the past century on how to axiomatize “spaces” in terms of convergence, started to emerge at the beginning of the millennium. It provides a powerful unifying framework and theory for fundamental ordered, metric and topological structures. Inspired by the topological concept of filter convergence, its methods are lax-algebraic and categorical, with generalized notions of monoid recurring frequently as the fundamental building blocks of its key notions. Since the main components of this new area have to date been available only in a scattered array of research articles, the authors of this book hope that a self-contained and consistent introduction to the theory will serve a broad range of mathematicians, scientists and their graduate students with an interest in a modern treatment of the mathematical structures in question. With all essential elements from order and category theory provided in the book, it is assumed that the reader will appreciate a framework which highlights the power of equationally-defined algebraic structures as particularly important elements of the broader lax-algebraic context which, roughly speaking, replaces equalities by inequalities.

There are two principal roots to the theory presented in this book: Barr’s 1970 relational presentation of topological spaces which naturally extends Manes’ 1969 equational presentation of compact Hausdorff spaces as the Eilenberg–Moore algebras of the ultrafilter monad, and Lawvere’s 1973 description of metric spaces as (small individual) categories enriched over the extended non-negative real half-line. In hindsight it seems surprising that it took some thirty years until the two general parameters at play here were combined in a compatible fashion, given by a monad $T$ replacing the ultrafilter monad and a quantale (or, more generally, a monoidal closed category) $V$ replacing the half-line. Of course, when considered separately, these two pivotal papers triggered numerous important developments. Lawvere’s surprising discovery quickly became a cornerstone of enriched category theory, with his characterization of Cauchy completeness in purely enriched-categorical terms enjoying most of the attention, and Barr’s paper was followed by at least two major but quite distinct attempts to develop a general topologically-inspired theory using a lax-algebraic monad approach, by Manes [1974] and Burroni [1971]. However, the up-take of these articles in terms of follow-up work remained sporadic, perhaps because not many strikingly new applications beyond Barr’s work came to the fore, with one prominent exception: the inclusion of Lambek’s 1969 multicategories in addition to Barr’s topological spaces provides a powerful motivation for Burroni’s elegant setting.

In 2000 Bill Lawvere was the first to suggest (in a private communication to Walter Tholen)
that, in the same way as topological spaces generalize ordered sets, Lowen’s 1989 approach
spaces should be describable as generalized metric spaces “using $\mathcal{V}$-multicategories in a good
way” instead of just $\mathcal{V}$-categories, thus implicitly envisioning a merger of the parameters $T$
and $\mathcal{V}$. At about the same time, following a suggestion by George Janelidze, Clementino
and Hofmann 2003 gave a lax-algebraic description of approach spaces using a “numerical
extension” of the ultrafilter monad. Both suggestions set the stage for Clementino and
Tholen 2003 to develop a setting that combines the two parameters efficiently, especially
when the monoidal-closed category $\mathcal{V}$ is just a quantale. As emphasized in Clementino,
Hofmann, and Tholen [2004b], this setting suffices to capture ordered, metric and topological
structures. In a slightly relaxed form as presented in Seal [2005] it also permits to replace
ultrafilter convergence by filter convergence (and its “approach generalization”) for its key
applications, and it is this setting that has been adopted in this book.

When following a meeting in Barisciano (Italy) in 2006 the authors of this book began to
embark decisively on a project to give a self-contained presentation of the emerging theory,
the heterogeneous make-up of the group itself made it necessary to clearly document all
needed ingredients in a coherent fashion. Hence, this book contains:

• a “crash course” on order and category theory that highlights many aspects not readily
available in existing texts and of interest beyond its use for order, metric and topology;

• an in-depth presentation of the syntactical framework involving the monad $T$ and the
quantale $\mathcal{V}$ needed for a unified treatment of the principal target categories;

• and some novel applications leading to new insights even in the context of ordinary
topological spaces, with ample directions to additional or subsequent work that could
not be included in this book.

In acknowledging the valuable advice and contributions received from many colleagues,
we should highlight first some theses written on subjects pertaining to this book and to
various degrees influencing its development, including the PhD theses of Van Olmen 2005,
Schubert 2006, Cruttwell 2008 and Reis 2013, and the Master’s theses of Akhvlediani
2008 and Lucyshyn-Wright 2009. We are grateful especially to Christoph Schubert
and Andrei Akhvlediani, who respectively helped to transform Walter Tholen’s lecture
notes for courses given at the University of Bremen (Germany) in 2003 and at a workshop
organized by Francis Borceux at Haute Bodeux (Belgium) in 2007 into something legible
and digestible. Christoph was also an active contributor to the various meetings that the
group of authors held at the University of Antwerp until 2009, generously organized by
Eva Colebunders and Robert Lowen.

The long but surely incomplete list of names of colleagues who offered helpful comments at
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We dedicate this book to Horst Herrlich whose work and dedication to mathematics have had formative influence on all authors of this book.

D.H., G.J.S. and W.T.

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I Introduction

Robert Lowen, Walter Tholen

In this introductory chapter we explain, in largely non-technical terms, how monoids and their actions occur not only everywhere in algebra, but how they also provide a common framework for the ordered, metric, topological, or similar structures targeted in this book. This framework is categorical, both at a micro level, since individual spaces may be viewed as generalized small categories, and at a macro level, as we are providing a common setting and theory for the categories of all ordered sets, all metric spaces and all topological spaces—and many other categories.

While this Introduction uses some basic categorical terms, we actually provide all needed categorical language and theory in the next chapter, along with the basic terms about order, metric and topology, before we embark on presenting the common setting for our target categories. Many readers may therefore want to jump directly to Chapter III, using the Introduction just for motivation and Chapter II as a reference for terminology and notation.

1 The ubiquity of monoids and their actions

Nothing seems to be more benign in algebra than the notion of monoid, that is, of a set $M$ that comes with an associative binary operation $m : M \times M \to M$ and a neutral element, written as a nullary operation $e : 1 \to M$. If mentioned at all, normally the notion finds its way into an algebra course only as a brief precursor to the segment on group theory. However, with the advent of monoidal categories, as first studied by Bénabou [1963], Eilenberg and Kelly [1966], Mac Lane [1963], and others, came the realization that monoids and their actions occur everywhere in algebra, as the fundamental building blocks of more sophisticated structures. This book is about the extension of this realization from algebra to topology.

1.1 Monoids and their actions in algebra. Every algebraist of the past hundred years would subscribe to the claim that free algebras amongst all algebras of a prescribed type contain all the information needed to study these algebras in general. However, what “contain” means was made precise only during the second half of this period. First, there was the observation of the late 1950s (Godement [1958], Huber [1961]) that the endofunctor $T = GF$ induced by a pair $F \dashv G : A \to X$ of adjoint functors comes equipped with natural
transformation

\[ m : TT \to T \quad \text{and} \quad e : 1_X \to T \]

which, when we trade the cartesian product of sets and the singleton set 1 for functor composition and the identity functor on \( X \), respectively, are associative and neutral in an easily described diagrammatic sense. Hence, they make \( T \) a monoid in the monoidal category of all endofunctors on \( X \), that is, a monad on \( X \) ([Mac Lane, 1971]). If \( G \) is the underlying-set functor of an algebraic category, like the variety of groups, rings, or a particular type of algebras, the free structure \( TX \) on \( X \)-many generators is just a component of that monad.

On the question of how to recoup the other objects of the algebraic category from the monad they have induced, let us look at the easy example of actions of a fixed monoid \( M \) in \( \textbf{Set} \). Hence, our algebraic objects are simply sets \( X \) equipped with an action \( a : M \times X \to X \) making the diagrams

\[
\begin{array}{ccc}
M \times M \times X & \xrightarrow{1_M \times a} & M \times X \\
\downarrow \quad m \times 1_X & & \downarrow a \\
M \times X & \xrightarrow{a} & X
\end{array}
\quad \quad
\begin{array}{ccc}
X & \xrightarrow{(e, 1_X)} & M \times X \\
\downarrow 1_X & & \downarrow a \\
M \times X & \xrightarrow{a} & X
\end{array}
\]

commutative. Realizing that \( TX = M \times X \) is in fact the carrier of the free structure over \( X \), we may now rewrite these diagrams as

\[
\begin{array}{ccc}
TTX & \xrightarrow{T a} & TX \\
\downarrow m_X & & \downarrow a \\
TX & \xrightarrow{a} & X
\end{array}
\quad \quad
\begin{array}{ccc}
X & \xrightarrow{e_X} & TX \\
\downarrow 1_X & & \downarrow a \\
X & \xrightarrow{a} & X
\end{array}
\]

(1.1.i)

Using a similar presentation of the relevant morphisms, \( i.e. \), of the action-preserving or equivariant maps, Eilenberg and Moore [1965] realized that with every monad \( T = (T, m, e) \) on a category \( X \) (in lieu of \( \textbf{Set} \)) one may associate the category \( X^T \) whose objects are \( X \)-objects \( X \) equipped with a morphism \( a : TX \to X \) making the two diagrams (1.1.i) commutative. Furthermore, there is an adjunction \( F^T \dashv G^T : X^T \to X \) inducing \( T \), such that when \( T \) is induced by any adjunction \( F \dashv G : \mathcal{A} \to X \), there is a “comparison functor” \( K : \mathcal{A} \to X^T \) which, at least for \( X = \textbf{Set} \), measures the “degree of algebraicity” of \( \mathcal{A} \) over \( X \). In fact, for any variety of general algebras (with “arities” of operations allowed to be arbitrarily large, as long as the existence of free algebras is guaranteed), \( K \) is an equivalence of categories and therefore faithfully recoups the algebras from their monad. By contrast, an application of this procedure to the underlying set functors of categories of ordered sets or topological spaces in lieu of general algebras would just render the identity monad on \( \textbf{Set} \) whose Eilenberg–Moore category is \( \textbf{Set} \) itself, \( i.e. \), all structural information would be lost.

While all categories of general algebras allowing for free structures may be seen as categories of generalized monoid actions as just described, this fact by no means describes the full
extent of the ubiquity of monoids and their actions in algebra. For example, a unital ring $R$ is nothing but an abelian group $R$ equipped with homomorphisms

$$m : R \otimes R \to R \quad \text{and} \quad e : \mathbb{Z} \to R ,$$

which are associative and neutral in a quite obvious diagrammatic sense. Hence, when one trades the cartesian category $(\texttt{Set}, \times, 1)$ for the monoidal category $(\texttt{AbGrp}, \otimes, \mathbb{Z})$, monoids $R$ are simply rings, and their actions are precisely the left $R$-modules. This example, however, is just the tip of an iceberg which places the systematic use of monoidal structures, monoids, and their actions, at the core of post-modern algebra.

1.2 Orders and metrics as monoids and lax algebras. While trying to describe ordered sets via the monad induced by the forgetful functor to $\texttt{Set}$ is hopeless, since it induces just the identity monad on $\texttt{Set}$, a “monoidal perspective” on structures is nevertheless beneficial. First, departing from the notion of a monad, but trading endofunctors $T$ on a category $X$ for relations $a$ on a set $X$, one can express transitivity and reflexivity of $a$ by

$$a \cdot a \leq a \quad \text{and} \quad 1_X \leq a , \quad (1.2.i)$$

with $\leq$ to be read as set-theoretical inclusion if $a$ is presented as $a \subseteq X \times X$. Hence, with the morphisms $m : a \cdot a \to a$ and $e : 1_X \to a$ simply given by $\leq$, what we regard as the two indispensable requirements of an order $a$ on $X$, transitivity and reflexivity, are expressed by $a$ carrying the structure of a monoid in the monoidal category of endorelations of $X$\footnote{In this book, in order to avoid the proliferation of meaningless prefixes, we refer to what is usually called a preorder as an order, considering the much less-used antisymmetry axiom as an add-on separation condition whenever needed. In fact, with respect to the induced order topology, antisymmetry amounts to the T0-separation requirement.} (The fact that such a relation actually satisfies the equation $a \cdot a = a$ is of no particular concern at this point.) But it is also possible to consider an order $a$ on $X$ in its role as a structure on $X$ in the spirit of 1.1, as follows. Replacing $\texttt{Set}$ by the category $\texttt{Rel}$ of sets with relations as morphisms and choosing for $T$ the identity monad on the ordered category $\texttt{Rel}$, we see that the inequalities (1.2.i) are instances of lax versions of the Eilenberg–Moore requirements (1.1.i). Indeed, when formally replacing strict (“=”) by lax (“$\leq$”) commutativity in (1.1.i), we obtain

$$TTX \xrightarrow{Ta} TX \xrightarrow{m_X} TX \quad X \xrightarrow{e_X} TX \quad \xrightarrow{1_X} X$$

In doing so, we suppose that the ambient category $X$ (which is $\texttt{Rel}$ in the case at hand) is ordered, so that its hom-sets are ordered, compatibly with composition. Briefly: ordered sets are precisely the lax Eilenberg–Moore algebras of the identity monad on the ordered category $\texttt{Rel}$.\footnote{In this book, in order to avoid the proliferation of meaningless prefixes, we refer to what is usually called a preorder as an order, considering the much less-used antisymmetry axiom as an add-on separation condition whenever needed. In fact, with respect to the induced order topology, antisymmetry amounts to the T0-separation requirement.}
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Next, presenting relations $a$ on $X$ as functions $a : X \times X \to 2 = \{\bot \leq \top\}$ with at most two truth values, let us rewrite the transitivity and reflexivity requirements as

$$a(x, y) \land a(y, z) \leq a(x, z) \quad \text{and} \quad \top \leq a(x, x)$$

for all $x, y, z \in X$. In this way, there appears a striking formal similarity with what we regard as the two principal requirements of a metric $a : X \times X \to [0, \infty]$ on $X$, the triangle inequality and the 0-distance requirement for a point to itself:

$$a(x, y) + a(y, z) \geq a(x, z) \quad \text{and} \quad 0 \geq a(x, x).$$

Hence, the set $2$ with its natural order $\leq$ and its inherent structure $\land$ and $\top$ has been formally replaced by the extended real half-line $[0, \infty]$, ordered by the natural $\geq$, and structured by $+$ and $0$. Just as for orders, one can now interpret metrics as both monoids and lax Eilenberg–Moore algebras with respect to the identity monad, after extending the relational composition

$$(b \cdot a)(x, z) = \bigvee_{y \in Y} (a(x, y) \land b(y, z))$$

for $a : X \times Y \to 2$, $b : Y \times Z \to 2$ and all $x \in X$, $y \in Y$, by

$$(b \cdot a)(x, z) = \inf_{y \in Y} (a(x, y) + b(y, z))$$

for $a : X \times Y \to [0, \infty]$, $b : Y \times Z \to [0, \infty]$ and all $x \in X$, $y \in Y$.

The generalized framework encompassing both structures that we will use in this book is provided by a unital quantale $V$ in lieu of $2$ or $[0, \infty]$, that is: of a complete lattice equipped with a binary operation $\otimes$ (in lieu of $\land$ or $+$) respecting arbitrary joins in each variable, and a $\otimes$-neutral element $k$ (in lieu of $\top$ or $0$). The role of the monad $T$ that appears to be rather artificial in the presentation of ordered sets and metric spaces will become much more pronounced in the presentation of the structures discussed next.

1.3 Topological and approach spaces as monoids and lax algebras. In 1.2 we described ordered sets and metric spaces as lax algebras with respect to the identity monad on the category of relations and “numerical” relations, respectively. Taking a historical perspective, we can now indicate how topological spaces fit into this setting once we allow the identity monad to be traded for an arbitrary “lax monad”, and how the less-known approach spaces [Lowen, 1997] emerge as the natural hybrid of metric and topology in this context.

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2Similarly to the use of the term ordered set, in this book we refer to a distance function $a$ satisfying these two basic axioms as a metric, using additional attributes for the other commonly used requirements when needed, like finiteness, symmetry, and separation.

3While we use $\land$, $\lor$ to refer to infima and suprema in general, in order to avoid ambiguity arising from the “inversion of order” in $[0, \infty]$, we use sup and inf when denoting suprema and infima with respect to the natural order.
Although the axiomatization of topologies in terms of convergence, via filters or nets, has been pursued early on in the development of these structures since Hausdorff [1914], notably by Fréchet [1921] and others, the geometric intuition provided by the open-set and neighborhood perspective clearly dominates the way in which mathematicians perceive topological spaces. Nevertheless, the proof by Manes [1969] that compact Hausdorff spaces are precisely the Eilenberg–Moore algebras of the ultrafilter monad $\beta = (\beta, m, e)$ on Set could not be ignored, as it gives the ultimate explanation for why the category $\text{CompHaus}$ behaves in many ways just like algebraic categories do. (For example, just as in algebra, but unlike in the case of arbitrary topological spaces, the set-theoretic inverse of a bijective morphism in $\text{CompHaus}$ is automatically a morphism again.) In this description, a compact Hausdorff space is a set $X$ equipped with a map $a : \beta X \to X$ assigning to every ultrafilter $\chi$ on $X$ (what turns out to be) its point of convergence in $X$, requiring the two basic axioms of an Eilenberg–Moore algebra:

$$a(\beta a(X)) = a(m_X(X)) \quad \text{and} \quad a(e_X(x)) = x \quad (1.3.i)$$

for all $X \in \beta \beta X$ and $x \in X$; here the following ultrafilters on $X$ are used:

$$e_X(x) = \dot{x} = \{ A \subseteq X \mid x \in A \}$$

is the principal filter on $x$,

$$m_X(X) = \Sigma X = \{ A \subseteq X \mid \{ \chi \in \beta X \mid A \in \chi \} \in \Sigma \}$$

is the Kowalsky sum of $X$, and

$$\beta a(X) = a[X] = \{ A \subseteq X \mid \{ \chi \in \beta X \mid a(\chi) \in A \} \in \Sigma \}$$

is simply the image filter of $X$ under the map $a$. Writing $\chi \rightarrow y$ instead of $a(\chi) = y$ and $X \rightarrow y$ instead of $a[X] = y$, the conditions (1.3.i) take the more intuitive form

$$\exists y \in \beta X (X \rightarrow y \& y \rightarrow z) \iff \Sigma X \rightarrow z \quad \text{and} \quad \dot{x} \rightarrow x$$

for all $X \in \beta \beta X$ and $x \in X$. In fact, in the presence of the implication “$\implies$” in the displayed equivalence, the implication “$\iff$” comes for free (as $y = a[X]$ necessarily satisfies $y \rightarrow z$ when $\Sigma X \rightarrow z$), and conditions (1.3.i) take the form

$$X \rightarrow y \& y \rightarrow z \implies \Sigma X \rightarrow z \quad \text{and} \quad \dot{x} \rightarrow x \quad (1.3.ii)$$

for all $X \in \beta \beta X$, $y \in \beta X$, $x, z \in X$.

As Barr [1970] observed, if one allows $a$ to be an arbitrary relation between ultrafilters on $X$ and points of $X$, rather than a map, so that we are no longer assured that every ultrafilter has a point of convergence (compactness) and that there is at most one such point (Hausdorffness), then the relations $\rightarrow$ satisfying (1.3.ii) describe arbitrary topologies on $X$, with continuous maps characterized as convergence-preserving maps. Furthermore,
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given the striking similarity of (1.3.ii) with the transitivity and reflexivity conditions of an ordered set, it is not surprising that (1.3.ii) gives rise to the presentation of topological spaces as both monoids and lax algebras of the ultrafilter monad.

In this statement, however, we glossed over an important point: having the Set-functor \( \beta \), one knows what \( \beta a \) is when \( a \) is a map, but not necessarily when \( a \) is just a relation. While there is a fairly straightforward answer in the case at hand, in general we are confronted with the problem of having to extend a monad \( T = (T, m, e) \) on Set to Rel or, even more generally, to \( \mathcal{V}\)-Rel, the category of sets and \( \mathcal{V} \)-relations \( r : X \rightarrow Y \), given by functions \( r : X \times Y \rightarrow \mathcal{V} \). Although for our purposes it suffices that this extension be lax, i.e., quite far from being a genuine monad on \( \mathcal{V}\)-Rel, the study of the various needed methods of just laxly extending monads on Set to \( \mathcal{V}\)-Rel can be cumbersome and takes up significant space in this book.

The general framework that emerges as a common setting is therefore given by a unital (but not necessarily commutative) quantale \( (\mathcal{V}, \otimes, k) \) and a monad \( T = (T, m, e) \) on Set laxly extended to \( \mathcal{V}\)-Rel, with the lax extension usually denoted by \( \hat{T} : \mathcal{V}\)-Rel \( \rightarrow \mathcal{V}\)-Rel (although a given \( T \) may have several lax extensions). The lax algebras considered are sets \( X \) equipped with a \( \mathcal{V} \)-relation \( a : TX \rightarrow X \) satisfying the two basic axioms

\[
\hat{T}a(x, y) \otimes a(y, z) \leq a(m_X(x), z) \quad \text{and} \quad k \leq a(e_X(x), x)
\]

for all \( x \in TTX, y \in TX, z \in Z \). The lax algebras are to be considered generalized categories enriched in \( \mathcal{V} \), with the domain \( x \) of the hom-object \( a(x, y) \) not lying in \( X \) but in \( TX \). Furthermore, relational composition can be generalized to Kleisli convolution for \( \mathcal{V} \)-relations \( r : TX \times Y \rightarrow \mathcal{V}, s : T Y \times Z \rightarrow \mathcal{V} \) via

\[
(s \circ r)(x, z) = \bigvee_{x \in TTX} \bigvee_{y \in T Y} \hat{T}r(x, y) \otimes s(y, z)
\]

for all \( x \in TX, z \in Z \). The lax algebra axioms for \( (X, a) \) are then represented via the monoidal structures

\[
a \circ a \leq a \quad \text{and} \quad 1^X_X \leq a,
\]

where \( 1^X_X \) is neutral with respect to the Kleisli convolution.

In this general framework we have so far encountered the objects in the following table, displayed with the corresponding monad \( T \) and quantale \( \mathcal{V} \) (here, \( \mathcal{P}_+ = (([0, \infty], \geq), +, 0) \) is the extended non-negative real half-line):

<table>
<thead>
<tr>
<th></th>
<th>( T )</th>
<th>( \mathcal{V} )</th>
<th>( 2 )</th>
<th>( \mathcal{P}_+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>identity monad</td>
<td>ordered sets</td>
<td></td>
<td>metric spaces</td>
<td></td>
</tr>
<tr>
<td>ultrafilter monad</td>
<td>topological spaces</td>
<td></td>
<td></td>
<td>?</td>
</tr>
</tbody>
</table>

Fortunately, the field left blank is filled with a well-studied, but much less familiar structure, called approach space. It is perhaps easiest described in metric terms: an approach structure
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on a set \( X \) can be given by a point-set distance function \( \delta : X \times PX \to [0, \infty] \) satisfying suitable conditions. A metric space \((X, d)\) becomes an approach space via

\[
\delta(x, B) = \inf_{y \in B} d(y, x)
\]

for all \( x \in X, B \subseteq X \). When an approach space is presented as a lax algebra \((X, a)\) with \( a : \beta X \times X \to [0, \infty] \), one can think of the value \( a(\chi, y) \) as the distance that the point \( y \) is away from being a limit point of \( \chi \). Indeed, a topological space \( X \) has its approach structure given by

\[
a(\chi, y) = \begin{cases} 
0 & \text{if } \chi \to y, \\
\infty & \text{otherwise.}
\end{cases}
\]

As for topological spaces, the more categorical view of approach spaces in terms of convergence proves useful.

1.4 The case for convergence. A topology (of open sets) on a set \( X \) is most elegantly introduced as a subframe of the powerset \( X \), i.e., a collection of subsets of \( X \) closed under finite intersection and arbitrary union. Via complementation, a topology (of closed sets) is equivalently described as a collection closed under finite union and arbitrary intersection, and this simple tool of Boolean duality (switching between open and closed sets) proves to be very useful. There is, however, an unfortunate breakdown of this duality when it comes to morphisms. Although continuous maps are equivalently described by their inverse-image function preserving openness or closedness of subsets, the seemingly most important and natural subclasses of morphisms, namely those continuous maps whose image function preserve openness or closedness (open or closed continuous maps) behave very differently: while open maps are stable under pullback, closed maps are not; not even the subspace restriction \( f^{-1}B \to B \) of a closed maps \( f : X \to Y \) with \( B \subseteq Y \) will generally remain closed. Hence, as recognized by [Bourbaki 1989], more important than the closed maps are the proper maps, i.e., the morphisms \( f \) that are stably closed, so that every pullback of the map \( f \) is closed again, also characterized as the closed maps \( f \) with compact fibres.

While under the open- or closed-set perspective no immediate “symmetry” between open and proper maps becomes visible, their characterization in terms of ultrafilter convergence reveals a remarkable duality: a continuous map \( f : X \to Y \) is

- \textbf{open} if \( y \to f(x) \) (with \( x \in X \) and \( y \in \beta Y \)) implies \( y = f(\chi) \) with \( \chi \to x \) for some \( \chi \in \beta X \),

- \textbf{proper} if \( f(\chi) \to y \) (with \( \chi \in \beta X \) and \( y \in Y \)) implies \( y = f(x) \) with \( \chi \to x \) for some \( x \in X \).

In fact, once presented as lax homomorphisms between lax Eilenberg–Moore algebras with respect to the ultrafilter monad (laxly extended from \textbf{Set} to \textbf{Rel}), these two types of special
morphisms occur most naturally as the ones for which an inequality characterizing their continuity may be replaced by equality, \textit{i.e.}, by a strict homomorphism condition.

Another indicator why convergence provides a most useful complementary view of topological spaces is the following. For a set $X$ and maps $f_i : X \to Y_i$, $i \in I$, there is a “best” topology on $X$ making all $f_i$ continuous, often called “weak”, but “initial” in this book. Its description in terms of open sets is a bit cumbersome, as it is \textit{generated} by the sets $f_i^{-1}(B)$, $B \subseteq Y_i$ open, $i \in I$, whereas the characterization in terms of ultrafilter convergence is \textit{immediate}: $x \xrightarrow{\chi} x$ in $X$ precisely when $f_i[\chi] \xrightarrow{f(x)}$ for all $i \in I$. For example, when $X = \prod_{i \in I} Y_i$ with projections $f_i$, so that the topology on $X$ just described is the product topology, a proof of the Tychonoff Theorem (on the stability of compactness under products) becomes almost by necessity cumbersome when performed in the open-set environment, but is in fact a triviality in the convergence setting.

However, we stress the fact that the roles of open sets versus convergence relations are reversed in the dual situation, when one wants to describe the “best” (or “final”) topology on a set $Y$ with respect to given maps $f_i : X_i \to Y$ originating from topological spaces $X_i$, $i \in I$. Its description in terms of open sets is immediate, as $B \subseteq Y$ is declared open whenever all $f_i^{-1}(B)$ are open, whereas a characterization in terms of convergence involves a cumbersome generation process.

In conclusion, we regard the two perspectives not at all as mutually exclusive but rather as complementary to each other. Consequently, this book provides a number of results on topological and approach spaces which arise naturally from the general convergence perspective, but which are far from being obvious when expressed in the more classical open-set or point-set-distance language.

1.5 Filter convergence and Kleisli monoids. To what extent is it possible to trade ultrafilter convergence for filter convergence when presenting topological spaces as in \ref{1.3} or characterizing open and proper maps as in \ref{1.4}? In order to answer this question, it is useful to axiomatize topologies on a set $X$ in terms of maps $\nu : X \to FX$ into the set $FX$ of filters on $X$, to be thought of as assigning to each point its neighborhood filter. Ordering such maps pointwise by reverse inclusion and using the same notation as in \ref{1.3}, except that now $\circ$ denotes the \textit{Kleisli composition} rather than the Kleisli convolution, one obtains another (and, in fact, more elementary) monoidal characterization of topologies on a set $X$:

$$\nu \circ \nu \leq \nu \quad \text{and} \quad e_X \leq \nu ;$$

in pointwise terms, this reads as

$$\sum \nu[\nu(x)] \supseteq \nu(x) \quad \text{and} \quad \hat{x} \supseteq \nu(x)$$

for all $x \in X$. We say that topological spaces are represented as \textit{Kleisli monoids} $(X, \nu)$, or simply as $\mathcal{F}$-\textit{monoids}, since the filter monad $\mathcal{F} = (F, m, e)$ may be traded for any monad $\mathbb{T}$ on $\text{Set}$ such that the sets $TX$ carry a complete-lattice order, suitably compatible with the monad operations. As such a monad $\mathbb{T}$ may be characterized via a monad morphism
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\[ \tau : \mathcal{P} \to \mathcal{F}, \text{ with } \mathcal{P} \text{ the powerset monad, we call } \mathbb{T} \text{ power-enriched}. \]

The basic correspondence between filter convergence and neighborhood systems, given by

\[ f \xrightarrow{*} x \iff f \supseteq \nu(x), \]

may now be established at the level of a power-enriched monad \( \mathbb{T} \). With a suitable lax extension of \( \mathbb{T} \) to \( \text{Rel} \) it yields a presentation of \( \mathbb{T} \)-monoids as lax algebras. For \( \mathbb{T} = \mathcal{F} \) it tells us that, remarkably, the characterization (1.3.ii) of topological spaces remains valid if we trade ultrafilters for filters. This fact, although established by Pisani [1999] in slightly weaker form, remained unobserved until proved by Seal [2005]. All previous axiomatizations of the notions of topology in terms of filter convergence entailed redundancies.

The answer to our initial question is therefore affirmative with respect to the convergence presentation of topological space. Also the characterization of open maps given in [1.4] survives the filters-for-ultrafilters exchange, but that of proper maps does not. Hence, we must be cognizant of the fact that the notions introduced for lax algebras will in general depend on the parameters \( \mathbb{T} \) and \( \mathcal{V} \), not just on the category of lax algebras described by them, such as the category of topological spaces considered here.
2 Spaces as categories, and categories of spaces

It has been commonplace since the very beginning of category theory to regard individual ordered sets as categories: they are precisely the categories whose hom-sets have at most one element. By contrast, it was a very bold step for Lawvere [1973] to interpret the distance $a(x, y)$ in a metric space as hom$(x, y)$. To understand this interpretation, we first recall how ordinary categories fare in the context of orders and metrics as described in [1.2]. We then indicate how the consideration of individual ordered sets, metric spaces, topological spaces, and similar objects as small generalized categories leads to new insights and cross fertilization between different areas, as does the investigation of the properties of the category of all such small categories of a particular type.

2.1 Ordinary small categories. Replacing “truth values” (2-valued or $[0, \infty]$-valued) by arbitrary sets, for a given set $X$ of “objects” let us consider functions

$$a : X \times X \to \text{Set}.$$ 

$X$ is then the set of objects of a category with hom-sets $a(x, y)$ if there are families of maps

$$m_{X,Y,Z} : a(x, y) \times a(y, z) \to a(x, z) \quad \text{and} \quad e_X : 1 \to a(x, x)$$

satisfying the obvious associativity and neutrality conditions, expressible in terms of commutative diagrams. Hence, the notion of small category fits into the same structural pattern already observed for orders and metrics, where now the composition of functions $a : X \times Y \to \text{Set}$, $b : Y \times Z \to \text{Set}$ is given by

$$(b \cdot a)(x, z) = \coprod_{y \in Y} (a(x, y) \times b(y, z))$$

for all $x \in X$, $z \in Z$.

Briefly, if one allows the above-mentioned setting of a unital quantale $(V, \otimes, k)$ to be extended to that of a monoidal closed category, ordinary small categories occur as monoids or lax Eilenberg–Moore algebras of an identity monad when $V$ is taken to be $(\text{Set}, \times, 1)$. We note in passing that the presentation of ordinary small categories just given becomes perhaps more familiar when one exhibits functions $a : X \times X \to \text{Set}$ equivalently as directed graphs

$$E \xrightarrow{\text{domain}} X \xrightarrow{\text{codomain}}$$

(with a fixed set $X$ of vertices), where $a$ and $E$ determine each other via

$$E = \coprod_{x, y \in X} a(x, y) \quad \text{and} \quad a(x, y) = \{ f \in E \mid \text{domain}(f) = x, \text{codomain}(f) = y \}.$$ 

Hence, ordinary categories are monoids in the monoidal category of directed graphs, the tensor product (that is, composition) of which corresponds to the above composition of Set-valued functions.
While we have made clear now that the setting of a monoidal closed category \((V, \otimes, k)\) and the theory of categories enriched over \(V\) (so that their “hom-sets” and structural components live in \(V\) rather than \(\text{Set}\), see [Kelly, 1982]) provide the right environment for studying not only orders and metrics, but also categories themselves, ordinary or additive (with \(V = \text{AbGrp}\)), and much more, in this book we restrict ourselves to considering the highly simplified case of a quantale \((V, \otimes, k)\). This is sufficient for reaching the intended target categories, and it makes the theory “technically” simpler since the triviality of 2-cells (given by order in this case) makes all coherence issues disappear as all diagrams in \(V\) commute. Nevertheless, the categorical perspective of interpreting the entity \(a(x, y)\) as \(\text{hom}(x, y)\) turns out to be very useful even in this simplified situation, as we indicate next.

2.2 Considering a space as a category. A key tool of category theory is the \textit{Yoneda embedding}

\[ y : X \to \text{Set}^{X\text{op}}, \quad y \mapsto X(-, y) \]

which assigns to every object \(y\) of a category \(X\) its contravariant hom-functor \(X(-, y) = \text{hom}_X(-, y) : X\text{op} \to \text{Set}\). It fully embeds \(X\) into a category with all (small-indexed) colimits; moreover, it is dense, in the sense that every object of its codomain is, in a natural way, a colimit of representable functors, \textit{i.e.}, objects in the image of \(y\). In other words, \(\text{Set}^{X\text{op}}\) serves as a \textit{cocompletion} of \(X\). Other types of cocompletions of \(X\) may be found inside \(\text{Set}^{X\text{op}}\) through suitable closure processes, and this statement remains valid even when one moves from ordinary to enriched category theory, trading \(\text{Set}\) for a monoidal closed category \(V\).

For ordered sets, so that \(V = 2 = \{0 < 1\}\) is the two-chain, monotone maps \(X\text{op} \to 2\) correspond to down-closed subsets of the ordered set \(X\), so that

\[ y : X \to \text{Dn}X, \quad y \mapsto \downarrow y = \{x \in X \mid x \leq y\} \]

is the embedding of \(X\) into the \textit{sup-completion} of \(X\), \textit{i.e.}, the lattice with all suprema freely generated by \(X\).

For metric spaces, in the generality adopted in 1.2, it is natural to endow \(V = [0, \infty]\) with its “internal hom” given by the non-symmetric distance function

\[ \mu(v, w) = \begin{cases} w - v & \text{if } v \leq w < \infty, \\ 0 & \text{if } w \leq v, \\ \infty & \text{if } v < w = \infty, \end{cases} \]

the symmetrization of which gives the Euclidian metric suitably extended to \(\infty\). Dualization of a metric space \(X = (X, a)\) is as trivial as for ordered sets: \(X\text{op} = (X, a^\circ)\) with \(a^\circ(x, y) = a(y, x)\). Now \(y\) provides an isometric embedding into the space of all non-expansive maps \(X\text{op} \to [0, \infty]\), provided with the sup-metric. This space inherits various completeness properties from \([0, \infty]\), and inside of it one finds the \textit{Cauchy completion} of \(X\): for every Cauchy sequence \((x_n)_{n \in \mathbb{N}}\) in \(X\) one considers the non-expanding map

\[ \psi : X\text{op} \to [0, \infty], \quad x \mapsto \lim_{n \to \infty} a(x, x_n), \]
and the subspace formed by all such maps is the Cauchy completion of $X$. Amazingly, as first observed by Lawvere [1973], this construction may be performed for arbitrary $\mathcal{V}$-categories, since the Cauchy property of $(x_n)_{n \in \mathbb{N}}$ is fully characterized by an adjointness property of $\psi$ when viewed as a module, that is, as a generalized compatible relation.

It is now natural to ask whether such constructions may be performed for topological spaces, presented as lax algebras as in 1.3. The additional parameter given by the monad $T = (T, m, e)$ does in fact introduce a serious obstacle, which starts with trying to determine what $X^{\text{op}}$ should be: a simple switch of arguments of the structure $a$ of $X$ is no longer possible! It turns out that by changing carrier sets from $X$ to $TX$ when forming the dual, it is possible to develop a comprehensive completion theory in the general $(\mathbb{T}, \mathcal{V})$-context, with the Yoneda embedding providing the central tool also at this level of generality. For a topological space, among other constructions, the Yoneda embedding leads to its sobrification. While the core of this general completion theory, along with other advanced topics, will be presented only in [Clementino et al., 2014], many of the needed tools are presented in this book.

2.3 Moving to the large category of all spaces. The internally-defined property of completeness may be externally characterized within the category of all spaces of a particular type: Banaschewski and Bruns [1967] and Isbell [1964] respectively characterized completeness of ordered sets and metric spaces by injectivity. Remarkably this categorical characterization can be established in the general $(\mathbb{T}, \mathcal{V})$-context for a whole scheme of completeness notions. While this characterization may be seen as depending only on the category of lax algebras of a particular type, hence as independent of the parameters $\mathbb{T}$ and $\mathcal{V}$ presenting them, there is also an equational characterization of completeness, which uses these parameters in a substantial way. Indeed, within the context of (suitably defined) separated lax algebras, the cocomplete objects are precisely the Eilenberg–Moore algebras of a certain monad on the category of all lax algebras.

The characterization of equationally-defined objects as the injectives in a category of lax algebras or monoids is in fact a recurring theme in the book. For example, in Chapter IV we present general theorems that entail the identification of continuous lattices as the regular-injective objects in the category of $\mathbb{T}0$-spaces. Furthermore, the general context of lax algebras allows us to make precise the connection of the equationally-defined-versus-injective paradigm with the fundamental categorical notion of exponentiability. In the context of topological spaces it facilitates the formation of function spaces, and Day and Kelly [1970] identified the exponentiable objects as the core-compact spaces. When topological spaces are described as lax algebras $(X, a)$ by the inequalities (1.3.iii) which, with $\mathcal{V}$-relational composition, may be transcribed as

$$a \cdot \hat{T}a \leq a \cdot m_X \quad \text{and} \quad 1_X \leq a \cdot e_X,$$

then the core-compact spaces are precisely those that make the first of these two inequalities an equality. If in (2.3.i) one lets $T$ be the filter monad (with its Kleisli extension), rather
than the ultrafilter monad, those spaces which satisfy the multiplicative law (2.3.i) up to equality form again an important subclass of spaces, called observable realization spaces in this book, for which we give alternate characterizations in Chapter IV.

With these facts in mind, it is not surprising that the category of reflexive graphs, given by all pairs \((X,a)\) required only to satisfy the second of the two inequalities (2.3.i), form a quasitopos which contains the category of lax algebras as a reflexive subcategory, under mild conditions on the parameters \(T\) and \(V\). In fact, for \(T = 2\) and \(V = 2\), this extension is minimal, producing the category of pseudotopological spaces. In the general \((T,V)\)-context, however, the quasitopos hull of the category of lax algebras will form a proper subcategory of that of all reflexive graphs, which leads to the consideration of important intermediate categories (Chapter II).
3 Chapter highlights and dependencies

Chapter II gives a rapid introduction into ordered sets and category theory, to the extent needed in this book. It provides not only the notation, terminology and theory used in the main body of the book (starting with Chapter III), but emphasizes areas of importance in the sequel that may play a less prominent role in other introductory texts, such as monadic and topological functors. For enriched and higher-order category theory, we get by with a brief exposition of monoidal and of ordered categories.

While the presentation of topics is self-contained, the arguments provided are often quite compact and pitched at a level that requires a degree of mathematical maturity that may at times be challenging for a beginning graduate student. Some of the exercises at the end of each section should help to overcome these challenges. Others are complementary to the main body of the text and may be used later on.

Chapter III Sections 1–3 provide the first key notions, properties and examples of the theory and applications of lax algebras. Introduced under the name \((T, V)\)-category in 1.6 in order to stress their status as individual small generalized categories, they are alternatively called \((T, V)\)-algebras or \((T, V)\)-spaces, depending on whether we want to emphasize their algebraic or geometric-topological roles. It is important that the reader does not skip the preceding subsections 1.1–1.5 in which many of the syntactical tools pertaining to lax extensions of the monad \(T\) are being developed.

Topologicity of the resulting large category \((T, V)\)-Cat over \(Set\) is shown at the beginning of Section 3, followed by a discussion of the impact of change in the parameters, arising from morphisms \(S \to T\) and \(V \to W\).

As \((T, V)\)-Cat fails to be cartesian closed, a presentation of quasitopoi containing \((T, V)\)-Cat follows in Section 4 along with an introduction to the categorical tools on exponentiability of morphisms. For the role model \(\text{Top} \cong (\beta, 2)\)-Cat, the quasitopos extension \((T, V)\)-Gph (“\((T, V)\)-graphs”) of \((T, V)\)-Cat leads to the category of pseudotopological spaces. Section 5 gives a first demonstration of how the general theory feeds into applications and provides new insights. There is a key adjunction

\[(T, V)\text{-Cat} \quad \overset{\bot}{\longrightarrow} \quad (V\text{-Cat})^T\]

which compares \((T, V)\)-Cat with the Eilenberg–Moore category of \(T\) extended from \(Set\) to \(V\text{-Cat}\). In our role model, it relates \(\text{Top}\) with the category of ordered compact Hausdorff spaces and emphasizes the importance of the order

\[\chi \leq y \iff \forall A \subseteq X \text{ closed } (A \in \chi \implies A \in y) \iff \forall B \subseteq X \text{ open } (B \in y \implies B \in \chi)\]

on the set \(\beta X\) of ultrafilters on the set \(X\), for every topological space \(X\). This approach leads to the powerful notion of representable \((T, V)\)-category which, in the role model, entails core-compactness, i.e., exponentiability in the category \(\text{Top}\). While Sections 1–3 of
Chapter III are a necessary prerequisite for Chapters IV and V, the slightly more demanding Sections 4 and 5 will be used only sporadically.

Chapter IV provides powerful alternate descriptions of the category \((\mathbb{T}, \mathbb{V})\)-Cat, the most striking of which arises from the fact that the quantale \(\mathbb{V}\) and the monad \(\mathbb{T}\) on \(\text{Set}\) laxly extended to \(\mathbb{V}\)-Rel allow for the construction of a new monad \(\mathbb{Π} = \mathbb{Π}(\mathbb{T}, \mathbb{V})\) (read “Pi”) on \(\text{Set}\) laxly extended to \(\text{Rel} = 2\text{-Rel}\) such that

\[
(\mathbb{T}, \mathbb{V})\text{-Cat} = (\mathbb{Π}, 2\text{-Cat}) ,
\]

associativity of the Kleisli convolution granted. Consequently, \((\mathbb{T}, \mathbb{V})\)-categories may be presented equivalently as relational lax algebras, with respect to a power-enriched (see 1.5) monad \(\mathbb{Π}\). The fact that all relevant information provided by \(\mathbb{T}\) and \(\mathbb{V}\) can be encoded by a new monad \(\mathbb{Π}\) gives the parameter \(\mathbb{T}\) some prominence over \(\mathbb{V}\). This result, presented in Section 3 along with applications, including the relational description of approach spaces which initiated this research, needs some preparations from Sections 1 and 2 that are of independent interest.

Guided by the role model of the filter monad, for a power-enriched monad \(\mathbb{T}\) we describe in Section 1 the isomorphism

\[
(\mathbb{T}, 2\text{-Cat}) \cong \mathbb{T}\text{-Mon} ,
\]

presenting relational algebras as Kleisli monoids. In Section 2, taking the inclusion \(\mathcal{β} \rightarrow \mathcal{F}\) of ultrafilters into filters as the role model, for a suitable morphism \(\mathcal{S} \rightarrow \mathcal{T}\) of monads we present an isomorphism

\[
\mathbb{T}\text{-Mon} \cong (\mathcal{S}, 2\text{-Cat}) ,
\]

which provides the general framework for the identical description of topological spaces in terms of either filter or ultrafilter convergence. A \(\mathbb{V}\)-level generalization (in lieu of \(\mathbb{V} = 2\)) of this last isomorphism is also provided.

In the context of a morphism \(\mathcal{S} \rightarrow \mathcal{T}\) of power-enriched monads one can construct a monad \(\mathcal{T}'\) on \(\mathcal{S}\text{-Mon}\) which has the same Eilenberg–Moore category as \(\mathcal{T}\):

\[
\text{Set}^\mathcal{T} \cong (\mathcal{S}\text{-Mon})^{\mathcal{T}'} .
\]

For \(\mathcal{S} = \mathcal{T}\), the right-hand side category becomes isomorphic to a category of injective \(\mathcal{T}\)-monoids, as we show in Section 4. For \(\mathcal{T}\) the filter monad, so that \(\mathcal{T}\text{-Mon} \cong \text{Top}\), one obtains in particular the simultaneous description of continuous lattices as Eilenberg–Moore algebras and as injectives in \(\text{Top}\) (with respect to the class of initial morphisms).

Section 5 is devoted exclusively to the study of those topological spaces that, when presented as lax algebras with the filter monad, satisfy the multiplicative law (2.3.i) up to equality, for which alternative descriptions are given, and the continuous lattices amongst them are fully characterized.

Chapter V looks at \((\mathbb{T}, \mathbb{V})\)-categories as spaces and explores topological properties, such as separation, regularity, normality and compactness in \((\mathbb{T}, \mathbb{V})\text{-Cat}\) (Sections 1–4). Emphasis
is given to those properties which arise “naturally” in the \((\mathbb{T}, \mathcal{V})\)-setting, such as the symmetric descriptions of Hausdorff separation and compactness, or the symmetrically-described properties of properness and openness for morphisms, as already alluded to above in 1.4. There is also a much more hidden symmetry between normality and extremal disconnectedness.

Closure of the relevant properties under direct products (for compact objects or proper morphisms) is a prominent theme (including the Tychonoff Theorem), and so is the generalization of the Kuratowski–Mrówka Theorem characterizing compact spaces in the general \((\mathbb{T}, \mathcal{V})\)-context.

Section 4 gives an axiomatic categorical framework for treating some of these key properties in a most economical fashion, and Section 5 explores the notion of connectedness in extensive categories in general and in the \((\mathbb{T}, \mathcal{V})\)-context in particular.

The range of example categories is expanded beyond the realm of metric and topology. It includes multi-ordered sets, to be thought of as a “thin” version of Lambek’s multicategories [Lambek, 1969] (also known as colored operads) that are gaining considerable attention in algebraic topology.

Summary of chapter dependencies. The left column in the graph below indicates the principal stream of suggested reading, while the right column lists the order-theoretic and categorical prerequisites, as well as special topics that may be omitted initially.

![Diagram of chapter dependencies]

References to items occurring outside a current chapter are always preceded by the Roman numeral for the relevant chapter, but the chapter number is omitted for references within the same chapter.
A word about sets, classes, and choice. Without reference to any particular kind of set-theoretic foundations, in this book we distinguish between sets, classes and conglomerates, to be able to form the class of all sets and the conglomerate of all classes, leading us in particular to the category Set and the metacategory SET, respectively (as in II.2.1 and II.2.2). Classes whose elements may be labelled by a set are also called small; others are large or proper classes. We refer to the Notes on Chapter II for suggested further reading on this topic.

We frequently use the Axiom of Choice (guaranteeing that surjective maps of sets are retractions). In fact, key results (such as the equivalence of the open-set and the ultrafilter-convergence presentations of topologies) rely on it. We alert the reader to each new use of the Axiom of Choice by putting the symbol

\[ \Box \]

in the margin. The symbol merely indicates our use of Choice at the instance in question, without any affirmation that the use is actually essential.
II Monoidal Structures

Gavin J. Seal, Walter Tholen

This chapter provides a compactly written introduction into the order- and category-theoretic tools most commonly used throughout the remainder of the book. A newcomer to the subject may at times need to consult a standard reference on category theory, while the chapter may be skipped by the more advanced reader who might use it just as a reference point for notation and terminology.

It is hoped that all readers will appreciate the ubiquity of “monoidal structures” appearing in the text. We allude to them quite explicitly only in Sections 1, 3, and 4 but note that, after all, categories are generalized monoids.

1 Ordered sets

1.1 The cartesian structure of sets and its monoids. Any two sets $A, B$ may be “multiplied” in terms of their cartesian product

$$A \times B = \{(x, y) \mid x \in A, y \in B\},$$

and this multiplication extends to maps $f : A \to A', g : B \to B'$ via

$$f \times g : A \times B \to A' \times B', \quad (x, y) \mapsto (f(x), g(y)).$$

The cartesian product respects identity maps, since

$$1_A \times 1_B = 1_{A \times B},$$

as well as composition of maps, since for $f' : A' \to A'', g' : B' \to B''$, one has the middle-interchange law

$$(f' \times g') \cdot (f \times g) = (f' \cdot f) \times (g' \cdot g).$$

The cartesian product is associative “up to isomorphism”, and any one-element set $E = \{\star\}$ acts as a neutral element. More precisely, there are obvious natural bijections

$$\alpha_{A, B, C} : A \times (B \times C) \to (A \times B) \times C, \quad \lambda_A : E \times A \to A, \quad \rho_A : A \times E \to A.$$
satisfying the so-called \emph{coherence conditions}

\[
\lambda_E = \rho_E , \quad (\rho_A \times 1_B) \cdot \alpha_{A,E,B} = 1_A \times \lambda_B , \\
(\alpha_{A,B,C} \times 1_D) \cdot \alpha_{A,B,C,D} \cdot (1_A \times \alpha_{B,C,D}) = \alpha_{A,B,C,D} \cdot \alpha_{A,B,C,D} ,
\]

for all sets $A, B, C, D$. Moreover, the cartesian structure is \emph{symmetric}, since there is a natural bijection

\[
\sigma_{A,B} : A \times B \rightarrow B \times A
\]

with

\[
\sigma_{B,A} \cdot \sigma_{A,B} = 1_{A \times B} , \quad \rho_A = \lambda_A \cdot \sigma_{A,E} , \\
\alpha_{C,A,B} \cdot \sigma_{A \times B,C} \cdot \alpha_{A,B,C} = (\sigma_{A,C} \times 1_B) \cdot \alpha_{A,C,B} \cdot (1_A \times \sigma_{B,C}) .
\]

A \emph{monoid} $M$ (with respect to the cartesian structure of sets) is a set $M$ that comes with a binary and a nullary operation

\[
m : M \times M \rightarrow M , \quad e : E \rightarrow M
\]

that are associative and make $e = e(\ast)$ a neutral element of $M$; equivalently, the diagrams

\[
\begin{array}{ccc}
M \times (M \times M) & \xrightarrow{\alpha} & (M \times M) \times M & \xrightarrow{m \times 1} & M \times M \\
\downarrow 1 \times m & & \downarrow m & & \downarrow m \\
M \times M & \xrightarrow{m} & M
\end{array}
\quad
\begin{array}{ccc}
E \times M & \xrightarrow{e \times 1} & M \times M & \xrightarrow{1 \times e} & M \times E \\
\downarrow \lambda & & \downarrow m & & \downarrow \rho \\
M & \xrightarrow{m} & M
\end{array}
\]

commute. The monoid is \emph{commutative} if $m \cdot \sigma = m$. A \emph{homomorphism} $f : M \rightarrow N$ of monoids preserves both operations: $f \cdot m_M = m_N \cdot (f \times f)$ and $f \cdot e_M = e_N$.

\textbf{1.2 The compositional structure of relations.} A \emph{relation} $r$ from a set $X$ to a set $Y$ distinguishes those elements $x \in X$ and $y \in Y$ that are $r$-related; we write $x r y$ if $x$ is $r$-related to $y$. Hence, depending on whether we display $r$ as a subset, a two-valued function, or a multi-valued function via

\[
r \subseteq X \times Y , \quad r : X \times Y \rightarrow \{\text{true, false}\} , \quad r : X \rightarrow PY
\]

respectively, $x r y$ may be equivalently written as

\[
(x, y) \in r , \quad r(x, y) = \text{true} , \quad y \in r(x) ,
\]

where $PY$ denotes the powerset of $Y$. Writing $r : X \rightarrow Y$ when $r$ is a relation from $X$ to $Y$, we can “multiply” $r$ with $s : Y \rightarrow Z$ via ordinary relational composition:

\[
x (s \cdot r) z \iff \exists y \in Y (x r y \& y s z) .
\]
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Writing \( r \leq r' \) (with \( r' : X \to Y \)) when, equivalently,
\[
  r \subseteq r' , \quad \forall x \in X \forall y \in Y \ (r(x, y) \models r'(x, y)) , \quad \forall x \in X \ (r(x) \subseteq r'(x)) ,
\]
we see that the multiplication respects \( \leq \), since
\[
  r \leq r' , \ s \leq s' \implies s \cdot r \leq s' \cdot r' .
\] (1.2.i)

Moreover, relational composition is associative, so that
\[
  t \cdot (s \cdot r) = (t \cdot s) \cdot r
\]
when \( t : Z \to W \), and for the identity relation \( 1_X \) (with \( x \ 1_X \ x' \iff x = x' \)) one has
\[
  r \cdot 1_X = r = 1_Y \cdot r .
\]

Hence, comparing with 1.1, we observe that sets \( A, B, \ldots \) have been replaced by relations \( r, s, \ldots \) and the cartesian product by composition. While in 1.1 there is room for maps \( f, g \), here we have only \( \leq \) between relations, so that the middle-interchange law of 1.1 reduces to a mere property (1.2.i). The natural bijections \( \alpha, \lambda, \rho \) of 1.1 have become identities, but the multiplicative structure is no longer “symmetric”. However, for \( r : X \to Y \) one has the opposite (or dual) relation \( r^\circ : Y \to X \) with
\[
  y r^\circ x \iff x r y
\]
for all \( x \in X , \ y \in Y \), which satisfies
\[
  (s \cdot r)^\circ = r^\circ \cdot s^\circ , \quad (1_X)^\circ = 1_X , \quad (r^\circ)^\circ = r , \quad r \leq r' \implies r^\circ \leq (r')^\circ .
\]

Note that when \( r \) is the graph of a map \( f : X \to Y \) (so that \( x r y \iff f(x) = y \)), then \( r^\circ(y) = f^{-1}(y) \) is simply the fibre of \( f \) over \( y \in Y \). In what follows we make no notational distinction between a map and its graph.

1.3 Orders. An order on a set \( X \) is a relation \( a : X \to X \) that carries a monoid structure with respect to the compositional structure of relations; that is, \( a \) satisfies
\[
  a \cdot a \leq a , \quad 1_X \leq a .
\]

Hence, \( a \) is simply a transitive and reflexive relation on \( X \):
\[
  (x \leq y \ & \ y \leq z \implies x \leq z) , \quad x \leq x
\]
for all \( x, y, z \in X \), when we write \( x \leq y \) for \( x a y \). The order is

(1) separated if \( a \cap a^\circ = 1_X \) (so that \( x \leq y \ & \ y \leq x \implies x = y \));

(2) total if \( a \cup a^\circ = X \times X \) (so that \( x \leq y \) or \( y \leq x \), for all \( x, y \in X \)).
CHAPTER II. MONOIDAL STRUCTURES

In the literature, orders on $X$ are usually called preorders, and separated (that is, antisymmetric) orders are often called partial orders on $X$. In this book, an ordered set $X$ is simply a set $X$ equipped with an order, and $X$ is separated if the order is separated. If $a$ is an order on $X$ (respectively, a separated, or a total order), then so is $a^\circ$. A chain is a set with a separated total order. A map $f : X \to Y$ of ordered sets is monotone (or order-preserving) if

$$f \cdot a \leq b \cdot f,$$

where $a, b$ denote the orders on $X, Y$ respectively, and $f$ is identified with its graph; hence, if we write $\leq$ for both $a$ and $b$,

$$x \leq y \implies f(x) \leq f(y)$$

for all $x, y \in X$. If the implication “$\iff$” also holds, so that $a = f^\circ \cdot b \cdot f$, then $f$ is fully faithful. For an ordered set $X$, we write $X^{\text{op}}$ for the same set equipped with the opposite order; thus, when $f : X \to Y$ is monotone, so is $f^{\text{op}} : X^{\text{op}} \to Y^{\text{op}}$ (with $f^{\text{op}}(x) = f(x)$ for all $x \in X$).

Every relation $r : X \to X$ has an ordered hull $r$, which may be described as

$$r = \bigcup_{n \geq 0} r^n,$$

(where $r^0 := 1_X$, $r^{n+1} = r \cdot r^n$), and if $r$ is separated, so is $r$. For any order $a$ on $X$, $a \cap a^\circ$ is an equivalence relation on $X$ which, when $a$ is written as $\leq$, is denoted by $\simeq$, so that

$$x \simeq y \iff x \leq y \& y \leq x.$$

There is a least order $b$ on the quotient set $X/\simeq$ which makes the projection $p : X \to X/\simeq$ monotone, namely $b = p \cdot a \cdot p^\circ$, also described by

$$p(x) \leq p(y) \iff x \leq y.$$

The point of this construction is that $b$ is separated; we call $X/\simeq$ the separated reflection of $X$.

1.4 Modules. A relation $r : X \leftrightarrow Y$ between ordered sets is a module if $(\leq_Y) \cdot r \cdot (\leq_X) \leq r$, that is, if

$$x' \leq x \& x \, r \, y \& y \leq y' \implies x' \, r \, y'$$

for all $x, x', y, y' \in X$. Hence, the relation $r$ is a module if and only if the map $r : X^{\text{op}} \times Y \to \{\text{true, false}\}$ is monotone (where $X^{\text{op}} \times Y$ is ordered componentwise). Graphically, we indicate modularity of a relation $r : X \leftrightarrow Y$ by

$$r : X \leftrightarrow Y.$$

Every monotone map $f : X \to Y$ gives rise to the modules

$$f_* = (\leq_Y) \cdot f : X \leftrightarrow Y \quad \text{and} \quad f^* = f^\circ \cdot (\leq_Y) : Y \leftrightarrow X.$$
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that is,
\[ x f_s y \iff f(x) \leq y \quad \text{and} \quad y f^s x \iff y \leq f(x) \]

for all \( x \in X, \ y \in Y \). The following rules may be easily verified when \( g : Y \to Z \) is monotone:

1. \( 1^*_X = (1_X)_* = (\leq_X) \),
2. \( (g \cdot f)_* = g_* \cdot f_* \) and \( (g \cdot f)^* = f^* \cdot g^* \),
3. \( 1^*_X \leq f^* \cdot f_* \) and \( f_* \cdot f^* \leq 1^*_Y \).

Modularity is also closed under relational composition. Indeed, for modules \( r : X \leftrightarrow Y \) and \( s : Y \leftrightarrow Z \), one has

\[ (\leq_Z) \cdot (s \cdot r) \cdot (\leq_X) \leq (\leq_Z) \cdot s \cdot (\leq_Y) \cdot (\leq_Y) \cdot r \cdot (\leq_X) \leq s \cdot r , \]

so that \( s \cdot r : X \leftrightarrow Z \) is again a module.

1.5 Adjunctions. For ordered sets \( X, Y \), the set

\[ \text{Ord}(X, Y) = \{ f \mid f : X \to Y \text{ monotone} \} \]

is itself ordered pointwise by

\[ f \leq f' \iff \forall x \in X \ (f(x) \leq f'(x)) . \]

This order is preserved by composition on either side: whenever \( h : W \to X \) and \( k : Y \to Z \) are monotone, then

\[ f \leq f' \implies k \cdot f \cdot h \leq k \cdot f' \cdot h . \]

A monotone map \( g : Y \to X \) is called

1. a right adjoint if there is a monotone map \( f : X \to Y \) with \( 1_X \leq g \cdot f, f \cdot g \leq 1_Y \);
2. an equivalence if there is a monotone map \( f : X \to Y \) with \( 1_X \simeq g \cdot f, f \cdot g \simeq 1_Y \);
3. an isomorphism if there is a monotone map \( f : X \to Y \) with \( 1_X = g \cdot f, f \cdot g = 1_Y \).

By definition, one has the implications

\[ \text{isomorphism} \implies \text{equivalence} \implies \text{right adjoint}. \]

The map \( f \) occurring in the definition of right adjointness is, up to “\( \simeq \)”, uniquely determined by \( g \): if \( 1_X \leq g \cdot f' \) and \( f' \cdot g \leq 1_Y \), then

\[ f' = f' \cdot 1_X \leq f' \cdot g \cdot f \leq 1_Y \cdot f = f , \]
and dually \( f \leq f' \). If \( g \) is right adjoint, the corresponding \( f \) is called left adjoint to \( g \), and one writes
\[
f \dashv g.
\]
This terminology becomes more plausible when we consider the following fact:

1.5.1 Proposition. A map \( g : Y \to X \) (not assumed to be monotone a priori) is right adjoint if and only if there is a map \( f : X \to Y \) such that
\[
f(x) \leq y \iff x \leq g(y)
\]
for all \( x \in X, y \in Y \).

Proof. The necessity of the condition is obvious since \( x \leq g(y) \) implies \( f(x) \leq f \cdot g(y) \leq y \), and dually for \( " \Rightarrow " \). For its sufficiency, observe that \( f(x) \leq f(x) \) implies \( x \leq g \cdot f(x) \), and dually \( f \cdot g(y) \leq y \). The monotonicity of \( f \) follows, since \( x \leq x' \leq g \cdot f(x') \) yields \( f(x) \leq f(x') \), and likewise for \( g \).

Calling a pair \((f : X \to Y, g : Y \to X)\) of monotone maps an adjunction if \( f \) is left adjoint to \( g \), we see that \((g^{\text{op}} : Y^{\text{op}} \to X^{\text{op}}, f^{\text{op}} : X^{\text{op}} \to Y^{\text{op}})\) is also an adjunction. In other words, \( f \) is left adjoint (to \( g \)) if and only if \( f^{\text{op}} \) is right adjoint (with left adjoint \( g^{\text{op}})\):
\[
f \dashv g : Y \to X \iff g^{\text{op}} \dashv f^{\text{op}} : X^{\text{op}} \to Y^{\text{op}}.
\]
An adjunction \((f : X^{\text{op}} \to Y, g : Y \to X^{\text{op}})\) is often called a Galois correspondence between \( X \) and \( Y \).

One sees easily that the following statement holds:

1.5.2 Corollary. A right adjoint map \( g \) (with left adjoint \( f \)) is fully faithful if and only if \( f \cdot g \simeq 1_Y \).

The conjunction of this statement with its dual yields:

1.5.3 Corollary. Equivalences are given by those adjunctions for which both maps are fully faithful. In this case, each map serves as both a left and a right adjoint.

All properties for maps between ordered sets discussed so far (monotone, fully faithful, right adjoint, left adjoint, equivalence, isomorphism) are closed under composition.

1.6 Closure operations and closure spaces. For any adjunction \( f \dashv g : Y \to X \) one has
\[
f \cdot g \cdot f \simeq f \quad \text{and} \quad g \cdot f \cdot g \simeq g,
\]
since \( 1_X \leq g \cdot f \) and \( f \cdot g \leq 1_Y \) imply \( f \leq f \cdot g \cdot f = (f \cdot g) \cdot f \leq f \); hence the first equivalence holds, and the second follows by duality. Therefore, setting
\[
c := g \cdot f \quad \text{and} \quad d := f \cdot g,
\]
one obtains monotone maps $c : X \to X$, $d : Y \to Y$ with
\[ c \cdot c \simeq c, \quad 1_X \leq c \quad \text{and} \quad d \simeq d \cdot d, \quad d \leq 1_Y. \]
Any such map $c$ is called a \textit{closure operation} on $X$, and any such $d$ an \textit{interior operation} on $X$. Since $1_X \leq c$ and the monotonicity of $c$ imply $c \leq c \cdot c$, it suffices to ask $c$ to satisfy
\[ c \cdot c \leq c, \quad 1_X \leq c, \]
i.e., a closure operation on $X$ is nothing but an element of $\text{Ord}(X, X)$ that carries a monoid structure with respect to the compositional structure.

With $c$ and $d$ induced by $f \dashv g$ as above, one easily sees that $f$ and $g$ can be restricted to an equivalence between the subsets
\[ \text{Fix}(c) := \{ x \in X \mid c(x) \simeq x \} \quad \text{and} \quad \text{Fix}(d) := \{ y \in Y \mid d(y) \simeq y \}, \]
of $c$-closed and $d$-open elements (or simply closed and open elements), also referred to as \textit{fixpoints} of $c$ and $d$, respectively. The following diagram summarizes this situation:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{g} \\
\text{Fix}(c) & \xrightarrow{f} & \text{Fix}(d) \\
\end{array}
\]

Any closure operation $c$ on $X$ is induced by an adjunction whose left adjoint is defined on $X$:

\[
\begin{array}{ccc}
X & \xrightarrow{c} & \text{Fix}(c) \\
\end{array}
\]

A \textit{closure space} is a set $X$ which comes with a closure operation on the powerset $PX$, ordered by inclusion. A map $f : X \to Y$ is \textit{continuous} if
\[ f(c_X(A)) \subseteq c_Y(f(A)) \]
for all $A \subseteq X$. Since for any map $f : X \to Y$ one has an adjunction

\[
\begin{array}{ccc}
PX & \xrightarrow{f} & PY \\
\downarrow{f^{-1}(\cdot)} & & \downarrow{f^{-1}(\cdot)} \\
PY & \xrightarrow{f^{-1}(\cdot)} & PX \\
\end{array}
\]
given by image and preimage along $f$, the continuity condition is equivalently written as
\[ c_X(f^{-1}(B)) \subseteq f^{-1}(c_Y(B)) \]
for all $B \subseteq Y$. Via the order isomorphism $(-)^{\complement} : PX^{\text{op}} \to PX$ (which maps $A \in PX$ to its complement $A^{\complement} := X \setminus A$ in $X$), any closure operation $c$ on $PX$ corresponds to an interior operation $d$ on $PX$, and vice-versa:
\[ c(A)^{\complement} = d(A^{\complement}), \]
for all $A \subseteq X$. Therefore, there is a concept of interior space, equivalent to that of a closure space, and in this context, a map $f : X \to Y$ between interior spaces $(X, d_X)$ and $(Y, d_Y)$ is \textit{continuous} if
$$f^{-1}(d_Y(B)) \subseteq d_X(f^{-1}(B))$$
for all $B \subseteq Y$.

\textbf{1.7 Completeness.} For an element $x$ in an ordered set $X$, let
$$\downarrow_X x = \downarrow x = \{ y \in X \mid y \leq x \}$$
be the \textit{down-set} of $x$ in $X$. The \textit{down-closure} of $A \subseteq X$ is
$$\downarrow_X A = \downarrow A = \bigcup_{x \in A} \downarrow x ,$$
and $A$ is \textit{down-closed} (or a \textit{down-set}) if $\downarrow A = A$. There is a fully faithful map
$$\downarrow : X \to \text{Dn}X = \text{Fix}(\downarrow_X) = \{ A \subseteq X \mid \downarrow A = A \} ,$$
where the set of down-sets in $X$ is ordered by inclusion. The ordered set $X$ is \textit{complete} if and only if this map is right adjoint; equivalently, if there is a map $\bigvee_X = \bigvee : \text{Dn}X \to X$ which for every $A \in \text{Dn}X$ satisfies
$$\forall x \in X \ (\bigvee A \leq x \iff A \subseteq \downarrow x) . \quad \text{(1.7.i)}$$
Calling $x$ an \textit{upper bound} of $A$ in $X$ whenever $A \subseteq \downarrow x$, we may rephrase the characteristic property of the \textit{join} (or \textit{supremum}, or \textit{least upper bound}) $\bigvee A$ of $A$ more familiarly by:

1. $\bigvee A$ is an upper bound of $A$ in $X$ ("\implies" of (1.7.i)), and
2. if $x$ is an upper bound of $A$ in $X$, then $\bigvee A \leq x$ ("\impliedby" of (1.7.i)).

Of course, in general $\bigvee A$ is uniquely determined by $A$ only up to \"$\simeq$". Note also that our notion of completeness does not only give mere existence of $\bigvee A$, but comes with a given choice of $\bigvee A$ for every $A \in \text{Dn}X$. Finally, existence of suprema for arbitrary subsets $B \subseteq X$ (not necessarily down-closed) follows from that of down-closed subsets: since $B$ and $\downarrow B$ have the same upper bounds, one can put
$$\bigvee B = \bigvee \downarrow B ;$$
phrased more sophisticatedly, the adjunctions
$$\begin{array}{cccc}
X & \xleftarrow{\bigvee} & \text{Dn}X & \xrightarrow{\downarrow} & PX \\
\downarrow & \xleftarrow{\bigvee} & \xrightarrow{\downarrow} & \xrightarrow{\bigvee} & \xrightarrow{\downarrow} & \end{array}$$
compose!
Exploiting the adjunction $\bigvee_X \dashv \downarrow_X$ for $X^{\text{op}}$ in lieu of $X$, we obtain

$$X^{\text{op}} \xrightarrow{\downarrow X^{\text{op}}} \text{Dn}(X^{\text{op}}) \xleftarrow{\bigvee X^{\text{op}}} ,$$

and dualization of this adjunction yields, with

$$\text{Up } X := (\text{Dn}(X^{\text{op}}))^{\text{op}}, \quad \uparrow_X := (\downarrow X^{\text{op}})^{\text{op}}, \quad \wedge_X := (\bigvee X^{\text{op}})^{\text{op}},$$

the adjunction

$$X \xleftarrow{\wedge_X} \text{Up } X \xrightarrow{\uparrow_X} .$$

Note that, for $\uparrow_X$ to be monotone, $\text{Up } X$ is (unlike $\text{Dn } X$) ordered by reverse inclusion. The dual notions (like up-set, up-closure, up-closed, lower bound, meet, infimum, greatest lower bound) are all naturally describable in terms of this adjunction. For example, for $A \in \text{Up } X$, $\wedge A$ is characterized by

$$\forall x \in X \ (x \leq \wedge A \iff A \subseteq \uparrow x) .$$

Moreover, this adjunction exists (equivalently, $X^{\text{op}}$ is complete) precisely when $X$ is complete, since the meet of an (up-closed) set can be realized as the join of the set of its lower bounds; a more elegant argument is given in Corollary 1.8.4.

1.8 Adjointness criteria. A monotone map $f : X \rightarrow Y$ of ordered sets preserves the supremum $\bigvee A$ of $A \subseteq X$ if $f(\bigvee A)$ is a supremum of $f(A) = \{ f(x) \mid x \in A \}$ in $Y$. Moreover, $f$ is a sup-map if it preserves every existing supremum in $X$:

$$f(\bigvee A) \simeq \bigvee f(A)$$

whenever $\bigvee A$ exists. The dual notions are: preserves an infimum, inf-map. Sup-preserving maps are useful for detecting left adjoints:

1.8.1 Proposition. Every left adjoint map $f$ is a sup-map.

Proof. Indeed, if $\bigvee A$ exists, then

$$f(A) \subseteq \downarrow y \iff \forall x \in A \ (f(x) \leq y) \iff \forall x \in A \ (x \leq g(y)) \iff \bigvee A \leq g(y) \iff f(\bigvee A) \leq y$$

for all $y \in Y$. □
Dually, a right adjoint map is an inf-map. Being a sup-map (respectively, an inf-map) is not only a necessary condition for being left adjoint (respectively, right adjoint), but also sufficient, provided that the domain of the map is complete. More precisely:

**1.8.2 Proposition.** A monotone map \( f : X \to Y \) is left adjoint if and only if there is a map \( g : Y \to X \) such that for all \( y \in Y \)

\[
g(y) \simeq \bigvee \{ x \in X \mid f(x) \leq y \} ,
\]

and \( f \) preserves those suprema. Hence, when \( X \) is complete, the map \( g \) can be given as the composite map

\[
Y \xrightarrow{\downarrow} \text{D}nY \xrightarrow{f^{-1}(-)} \text{D}nX \xrightarrow{\bigvee} X .
\]

**Proof.** The condition (1.8.i) is clearly necessary since \( f \dashv g \) yields \( f^{-1}(\downarrow y) = \downarrow g(y) \), and \( \bigvee \downarrow g(y) \simeq g(y) \) for all \( y \in Y \). Conversely, existence of the join (1.8.i) gives \( x \leq g(y) \) whenever \( f(x) \leq y \), and its preservation by \( f \) yields \( f \cdot g(y) \leq y \), so that \( f(x) \leq y \) whenever \( x \leq g(y) \).

**1.8.3 Corollary.** When \( X \) is a complete ordered set, a map \( f : X \to Y \) is left adjoint if and only if \( f \) is a sup-map.

As an application, let us prove the following result (see 1.7):

**1.8.4 Corollary.**

1. \( X^{\text{op}} \) is complete when \( X \) is complete.

2. When \( Y \) is complete, a map \( g : Y \to X \) is right adjoint if and only if \( g \) is an inf-map.

**Proof.**

1. It suffices to show that \( \uparrow_X : X \to \text{Up} X \) is a sup-map. But

\[
\uparrow \bigvee A = \bigcap_{a \in A} \uparrow a
\]

for all \( A \subseteq X \) is just the defining property for suprema:

\[
\bigvee A \leq x \iff \forall a \in A \ (a \leq x) \iff A \subseteq \downarrow x .
\]

2. By (1) one may apply Corollary 1.8.3 with \( g^{\text{op}} \) in lieu of \( f \).
1. ORDERED SETS

1.9 Semilattices, lattices, frames, and topological spaces. For a separated ordered set \( X \), the map

\[
\downarrow_X : X \rightarrow \text{D}nX
\]

is an order-embedding, that is: the map is injective and fully faithful (separatedness is not essential, but is assumed for convenience). The set \( \text{D}nX \) is complete, with infima given by intersection. In particular, \( \text{D}nX \) with the binary operation \( \cap \) and the nullary operation \( X \) (largest element in \( \text{D}nX \)) is a commutative monoid. When do these operations restrict to \( X \) along \( \downarrow_X \)? That is, when do we have dotted maps making the diagrams commute? Precisely when all finite infima exist in \( X \), and then we must have

\[
m(x, y) = x \land y = \land \{x, y\}
\]

for all \( x, y \in X \), and \( e \) must be the largest element of \( X \): \( e = \land \emptyset \).

A meet-semilattice \( X \) is a separated ordered set with finite infima. A homomorphism of meet-semilattices preserves finite infima, that is, it preserves the binary \( \land \) and the largest element. Trading infima for suprema (hence, \( \land \) for \( \lor \) and largest for smallest), one obtains the notions of join-semilattice and homomorphism thereof.

Both meet- and join-semilattices have a common algebraic description: \((X, \land, \top)\) and \((X, \lor, \bot)\), with \( \top := \land \emptyset \) and \( \bot := \lor \emptyset \) the top and bottom elements (or maximum and minimum) of \( X \), respectively, are simply commutative monoids in which, under multiplicative notation, every element is idempotent, so that \( x \cdot x = x \) for all \( x \in X \). One calls such monoids \((X, \cdot, e)\) semilattices, since they may equivalently be considered as either a meet- or a join-semilattice, depending on whether one puts

\[
(x \leq y \iff x \cdot y = x) \quad \text{or} \quad (x \leq y \iff x \cdot y = y),
\]

in which case one obtains \( x \cdot y = x \land y \) or \( x \cdot y = x \lor y \) (see Exercise 1.11). Homomorphisms of such monoids are equivalently described as homomorphisms of meet- or join-semilattices.

A lattice is a separated ordered set \( X \) with finite infima and finite suprema. It may be equivalently described as a set \( X \) with binary operations \( \land, \lor \), and nullary operations \( \top, \bot \) such that both \((X, \land, \top)\), \((X, \lor, \bot)\) are commutative monoids such that

\[
x \land x = x = x \lor x, \quad x \land (x \lor y) = x = x \lor (x \land y).
\]

A homomorphism of lattices is a map that preserves the operations \( \land, \lor, \top, \bot \).

A frame is a complete meet-semilattice \( X \) such that, for all \( a \in X \),

\[
a \land (-) : X \rightarrow X
\]
is a sup-map. Hence, a frame is simply a complete lattice (that is, a complete separated ordered set) which satisfies the infinite distributive law
\[ a \land \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \land b_i . \]

In particular, each complete chain is a frame (see Exercise 1.E). A homomorphism of frames must be both a homomorphism of meet-semilattices, and a sup-map; i.e., it must preserve finite infima and arbitrary suprema.

For example, if \( X \) is an ordered set, then \( DnX \) is a frame, and every monotone map \( f : X \to Y \) induces a homomorphism of frames \( f^{-1}(-) : DnY \to DnX \) (see Exercise 1.K). This applies in particular when \( X \) is discrete, that is, when its order is given by equality, so that \( DnX = PX \).

A topology (of open sets) on \( X \) is nothing but a subframe of \( PX \), that is, a subset of \( PX \) that is closed under finite infima and arbitrary suprema, making \( X \) a topological space; the topology of \( X \) is usually denoted by \( \mathcal{O}X \). A map \( f : X \to Y \) of topological spaces is continuous if \( f^{-1}(-) : PY \to PX \) restricts to a map \( f^{-1}(-) : \mathcal{O}Y \to \mathcal{O}X \). A base for \( \mathcal{O}X \) is a subset \( \mathcal{B} \subseteq PX \) with \( \mathcal{O}X = \{ \bigcup C \mid C \subseteq \mathcal{B} \} \). Every topological space is a closure space, and the notions of continuity given here and in 1.6 are equivalent (Exercise 1.F). Every topological space can be endowed with its underlying (or induced) order
\[ x \leq y \iff \forall U \in \mathcal{O}X (y \in U \implies x \in U) . \]

This order is separated precisely when \( X \) is a T0-space. Every continuous map is monotone with respect to the underlying orders. The dual of this order is called the specialization order of a topological space.

1.10 Quantales. A quantale \( \mathcal{V} \) (more precisely: a unital quantale) is a complete lattice which carries a monoid structure with neutral element \( k \) (as in 1.1) such that, when the binary operation is denoted as a tensor \( \otimes \),
\[ a \otimes (-) : \mathcal{V} \to \mathcal{V} , \quad (-) \otimes b : \mathcal{V} \to \mathcal{V} \]
are sup-maps for all \( a, b \in \mathcal{V} \); hence the tensor distributes over suprema:
\[ a \otimes \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \otimes b_i) , \quad \bigvee_{i \in I} a_i \otimes b = \bigvee_{i \in I} (a_i \otimes b) . \]

A lax homomorphism of quantales \( f : \mathcal{V} \to \mathcal{W} \) is a monotone map satisfying
\[ f(a) \otimes f(b) \leq f(a \otimes b) , \quad l \leq f(k) \]
for all \( a, b \in \mathcal{V} \), and \( l \) the neutral element of \( \mathcal{W} \); monotonicity of \( f \) means equivalently lax preservation of joins, that is \( \bigvee f(A) \leq f(\bigvee A) \) for all \( A \subseteq X \). For \( f \) to be a homomorphism, these three inequalities must be identities. A quantale is commutative if it is commutative as a monoid. Every frame becomes a commutative quantale when we put \( \otimes = \land \) and let \( k \) be
the top element. In fact, frames are those commutative quantales \( V \) for which \( a \otimes a = a \) for all \( a \in V \), and \( k \) is the top element (see Exercise 1.1). If \( V \) and \( W \) are frames (considered as quantales), then a lax homomorphism of quantales \( f : V \to W \) is a homomorphism precisely when it is a sup-map. In a quantale \( V \), for every \( a \in V \), the sup-map \( a \otimes (-) \) is left adjoint to a map \( a \to (-) : V \to V \) which is uniquely determined by

\[
a \otimes v \leq b \iff v \leq a \to b
\]

for all \( v, b \in V \); hence

\[
a \to b = \bigvee \{ v \in V \mid a \otimes v \leq b \}
\]

Likewise, for all \( a \in V \), the sup-map \( (-) \otimes a \) is left adjoint to a map \( (-) \bullet a : V \to V \). In the case where \( V \) is commutative, \( a \to (-) \) and \( (-) \bullet a \) coincide, and either of the two notations may be used.

The following examples of commutative quantales are frequently used in this book. For examples of not necessarily commutative quantales, see Exercise 1.1.

1.10.1 Examples.

(1) The two-chain \( 2 = \{ \text{false} \models \text{true} \} = \{ \bot, \top \} \) with \( \otimes = \land \), \( k = \top \). Here, \( a \to b \) is the Boolean truth value of the implication \( a \to b \). More generally, we use this arrow notation for any frame considered as a quantale.

(2) The three-chain \( 3 = \{ \bot, k, \top \} \) is a complete chain, and therefore a frame (see Exercise 1.1). However, the quantale structure we will consider is given by choosing \( k \) to be the unit for the multiplication. In fact, the multiplication is now uniquely determined, and it is represented together with the right adjoints \( (-) \bullet a \), for \( a \in 3 \), in the following tables:

<table>
<thead>
<tr>
<th>( \otimes )</th>
<th>( \bot )</th>
<th>( k )</th>
<th>( \top )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bot )</td>
<td>( \bot )</td>
<td>( k )</td>
<td>( \top )</td>
</tr>
<tr>
<td>( k )</td>
<td>( \bot )</td>
<td>( k )</td>
<td>( \top )</td>
</tr>
<tr>
<td>( \top )</td>
<td>( \bot )</td>
<td>( \top )</td>
<td>( \top )</td>
</tr>
</tbody>
</table>

(3) Allowing for an interval of truth values, we consider the extended real half-line \([0, \infty)\) which is a complete lattice with respect to its natural order \( \leq \). But we reverse its order, so that \( 0 = \top \) is the top and \( \infty = \bot \) is the bottom element, and we consider it a quantale with \( \otimes \) given by addition extended via

\[
a + \infty = \infty + a = \infty
\]

for all \( a \in [0, \infty] \), and necessarily \( k = 0 = \top \). Briefly, we will write

\[
P_+ = ([0, \infty]^{\text{op}}, +, 0)
\]
When working with this quantale, the relation $\leq$ always refers to the natural order of $[0, \infty]$, and we use the symbols $\inf$ and $\sup$ when forming infima and suprema in $[0, \infty]$, while we use joins $\vee$ and meets $\wedge$ when forming these in $P_+$. Hence,

$$b \cdot a = b \ominus a := \inf \{ v \in [0, \infty] \mid b \leq a + v \} ,$$

so that $b \ominus a = b - a$ if $a \leq b < \infty$, and $b \ominus a = 0$ if $b \leq a$, while $b \ominus a = \infty$ if $a < b = \infty$.

(4) In the previous example, addition may be replaced by multiplication extended via

$$a \cdot \infty = \infty \cdot a = \infty$$

for all $a \in [0, \infty]$ (this definition is necessary, since the tensor must preserve the empty join, i.e., the bottom element $\infty$). Hence, we obtain the quantale

$$P_x = ([0, \infty]^\text{op}, \cdot, 1) .$$

Here,

$$b \cdot a = b \ominus a := \inf \{ v \in [0, \infty] \mid b \leq a \cdot v \} ,$$

so that $b \ominus a = \frac{b}{a}$ if $0 < a, b < \infty$, and $b \ominus 0 = \infty = \infty \ominus a$ if $0 < a, b$, while $0 \ominus a = 0 = b \ominus \infty$ for all $a, b \in [0, \infty]$.

(5) Since $[0, \infty]^\text{op}$ is (like $[0, \infty]$) a chain, it is a frame, and we may consider it a quantale with its meet operation (which, according to our conventions, is the max with respect to the natural order of $[0, \infty]$):

$$P_{\max} = ([0, \infty]^\text{op}, \max, 0) .$$

Here

$$a \rightarrow b = a \rightarrow b := \inf \{ v \in [0, \infty] \mid b \leq \max \{ a, v \} \} ,$$

which turns out to be $a \rightarrow b = 0$ if $b \leq a$, and $a \rightarrow b = b$ if $a < b$.

There is a sup-map $\iota : 2 \rightarrow [0, \infty]^\text{op}$, sending $\top$ to 0, and $\bot$ to $\infty$, which gives homomorphisms of quantales

$$\iota : 2 \rightarrow P_+ \quad \text{and} \quad \iota : 2 \rightarrow P_{\max} .$$

However,

$$\iota : 2 \rightarrow P_x$$

is only a lax homomorphism, since $\iota$ does not preserve the neutral elements of the respective monoid structures. This can be corrected if one replaces 2 by 3; then the sup-map

$$\kappa : 3 \rightarrow P_x$$

sending $\top$ to 0, $k$ to 1, and $\bot$ to $\infty$, is a homomorphism of quantales.
1. ORDERED SETS

1.11 Complete distributivity. As we saw in \[1.7\] completeness of the ordered set \(X\) is characterized by the existence of an adjunction

\[ \lor \dashv \downarrow : X \to \text{Dn}X. \]

We call a complete lattice \textit{completely distributive} if the left adjoint \(\lor\) has itself a left adjoint, \textit{i.e.}, if there is a map

\[ \downarrow : X \to \text{Dn}X \]

with

\[ \downarrow a \subseteq S \iff a \leq \lor S \]

for all \(a \in X, S \in \text{Dn}X\). Necessarily,

\[ \downarrow a = \bigcap\{S \in \text{Dn}X \mid a \leq \lor S\}, \]

so that when we write \(x \ll a\) (read as: \(x\) is \textit{totally below} \(a\)) instead of \(x \in \downarrow a\), this relation is given by

\[ x \ll a \iff \forall S \in \text{Dn}X \ (a \leq \lor S \implies x \in S). \]

Writing \(S = \downarrow A\) with \(A \subseteq X\), an equivalent characterization is given by

\[ x \ll a \iff \forall A \subseteq X \ (a \leq \lor A \implies \exists y \in A \ (x \leq y)). \]

Since \(\downarrow\) is a monotone map with \(1_X \leq \lor \downarrow\), one has for all \(a, b, x \in X\):

(1) if \(x \ll a \leq b\), then \(x \ll b\);

(2) \(a \leq \lor\{x \in X \mid x \ll a\}\).

In (2), one actually has equality (so that \(a = \lor \downarrow a\)): from \(a \leq \lor \downarrow a\) follows \(\downarrow a \subseteq \downarrow a\) by adjunction, so that \(\lor \downarrow a \leq a\). We also note that by the very definition of \(\ll\), every element in \(X\) is \(\ll\)-atomic in the sense that

\[ x \ll \lor S \implies x \in S \]

for all \(S \in \text{Dn}X\), so that in (2) the join is taken “only” over \(\ll\)-atomic elements. Keeping this in mind, we see that the existence of the relation \(\ll\) with properties (1) and (2) is characteristic of complete distributivity:

1.11.1 Proposition. If the complete lattice \(X\) allows for some relation \(\prec\) satisfying

(1') if \(x \prec a \leq b\), then \(x \prec b\), and

(2') \(a \leq \lor\{x \in X \mid x \text{ is } \prec\text{-atomic and } x \prec a\}\)

for all \(a, b, x \in X\), then \(X\) is completely distributive.
Proof. Indeed, assuming these conditions, and setting
\[ S_a := \downarrow \{ x \in X \mid x \text{ is } \prec \text{-atomic and } x \prec a \} , \]
one has \( a \leq \bigvee S_a \) by \((2')\), and whenever \( a \leq \bigvee S \) for some \( S \in \text{Dn} X \), then every \( x \in S_a \) satisfies \( x \prec \bigvee S \) by \((1')\), and therefore lies in \( S \). Consequently,
\[ S_a = \bigcap \{ S \in \text{Dn} X \mid a \leq \bigvee S \} = \downarrow a , \]
as desired. \( \Box \)

The complete lattices 2 and \([0, \infty]\) considered in Examples 1.10.1 are completely distributive, with the totally-below relation given by \((x \ll a \iff a = \text{true})\) and \((x \ll a \iff x < a)\) respectively. For every set \( X \), the ordered set \( \text{PX} \) is completely distributive; more generally, for every ordered set \( X \), the ordered set \( \text{Dn} X \) is completely distributive (see Exercise 1.O).

We note that \([0, \infty]^{\text{op}}\) is also completely distributive (with “totally-below” meaning “\(>\)”; in fact, one can prove that every chain is completely distributive and that \( X^{\text{op}} \) is completely distributive whenever \( X \) has that property (see [Wood, 2004]).

The term “completely distributive” still deserves some justification. By Corollary 1.8.3, the map \( \bigvee : \text{Dn} X \to X \) is right adjoint if it is an inf-map:
\[ \bigvee (\bigcap_{i \in I} S_i) = \bigwedge_{i \in I} \bigvee S_i \]
for all families of down-sets \( S_i \) in \( X \), \( i \in I \); equivalently, if
\[ \bigvee (\bigcap_{i \in I} \downarrow A_i) = \bigwedge_{i \in I} \bigvee A_i \] (1.11.i)
\( \circledast \) for all families of subsets \( A_i \) in \( X \), \( i \in I \). Now, assuming the Axiom of Choice, the latter identity may be written equivalently as
\[ \bigvee_{(a_i) \in \prod_{i \in I} A_i} \bigwedge_{i \in I} a_i = \bigwedge_{i \in I} \bigvee A_i . \] (1.11.ii)

Indeed, denoting by \( s, t \) the left sides of \((1.11.i)\), \((1.11.ii)\) respectively, \( t \leq s \) follows by noticing that \((a_i)_{i \in I} \in \prod_{i \in I} A_i \) yields \( \bigwedge_{i \in I} a_i \in \bigcap_{i \in I} \downarrow A_i \). Conversely, given \( x \in \bigcap_{i \in I} \downarrow A_i \), for every \( i \in I \), one has \( a_i \in A_i \) with \( x \leq a_i \); by the Axiom of Choice, this defines an element \((a_i)_{i \in I} \in \prod_{i \in I} A_i \) with \( x \leq \bigwedge_{i \in I} a_i \), and \( s \leq t \) follows.

Most authors define complete distributivity via \((1.11.ii)\), and reserve the name constructively completely distributive for the Choice-free notion given by \((1.11.i)\), that is, by the right adjointness of \( \bigvee \).

1.12 Directed sets, filters, and ideals. A subset \( A \subseteq Z \) of a separated ordered set \( Z \) is down-directed if every finite subset of \( A \) has a lower bound in \( A \); this just means that for all \( x, y \in A \), there is a \( z \in A \) with \( z \leq x, z \leq y \), and that \( A \neq \emptyset \). A subset \( A \subseteq Z \) is a filter in \( Z \) if \( A \) is down-directed and up-closed in \( Z \); a filter \( A \) is proper if \( A \neq Z \). When \( Z \)
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has finite infima, filters $A$ in $Z$ are up-closed meet-semilattices of $Z$; that is, the filters are those $A$ which satisfy

1. $x, y \in A \implies x \land y \in A$,
2. $\top \in A$, and
3. $x \in A, x \leq y \implies y \in A$

for all $x, y \in Z$. If $Z$ has a bottom element $\bot$, properness then means

4. $\bot /\notin A$.

Every down-directed set $A$ in $Z$ generates the filter $\uparrow A$ in $Z$, in which case $A$ is also called a filter base for $\uparrow A$. In particular, for every element $a \in Z$, one has the principal filter $\uparrow a$ in $Z$. The dual notions are those of up-directed set, ideal, proper ideal, ideal base and principal ideal.

We use these notions predominantly for $Z = PX$ (for some set $X$), ordered by inclusion. Hence, $a$ is a filter on the set $X$ if it is a filter in $PX$, that is, $a$ must satisfy

1'. $A, B \in a \implies A \cap B \in a$,
2'. $X \in a$,
3'. $A \in a, A \subseteq B \implies B \in a$,

and $a$ is proper when

4'. $\emptyset /\notin a$.

Here are some frequently used filter-generation procedures:

(a) For every map $f : X \to Y$ and every filter $a$ on $X$, one defines the image filter $f[a]$ on $Y$ by

$$f[a] = \uparrow \{ f(A) \mid A \in a \} = \{ B \subseteq Y \mid f^{-1}(B) \in a \}.$$  

For a filter $b$ on $Y$, one defines the inverse image $f^{-1}[b]$ by

$$f^{-1}[b] = \uparrow \{ f^{-1}(B) \mid B \in b \} = \{ A \subseteq X \mid \exists B \in b ( f^{-1}(B) \subseteq A) \} ,$$

but this filter on $X$ is proper only when $f^{-1}(B) \neq \emptyset$ for all $B \in b$. If $f$ is an inclusion map $X \hookrightarrow Y$, then

$$f^{-1}[b] = \{ X \cap B \mid B \in b \} \iff X \in b ;$$

when $X \in b$ one calls $b|_{X} := f^{-1}[b]$ the restriction of $b$ to $X$, and says that $b$ is a filter on $X$. In fact, $b$ is then determined by $b|_{X}$, since $\uparrow f[b|_{X}] = b$.  

(b) For every $A \subseteq X$, one has the following principal filter on $X$:

$$\hat{A} = \uparrow A = \{ B \subseteq X \mid A \subseteq B \}.$$ 

Obviously, for $f : X \to Y$, $f[\hat{A}] = \uparrow f(A)$.

(c) For a set $X$, let us consider a filter $\mathcal{A}$ on the set

$$FX = \{ a \mid a \text{ is a filter on } X \}.$$ 

Then the filtered sum (or Kowalsky sum) $\sum \mathcal{A}$, defined by

$$A \in \sum \mathcal{A} \iff A^F \in \mathcal{A},$$

for all $A \subseteq X$, where $A^F = \{ a \in FX \mid A \in a \}$, gives a filter on $X$. Hence, a set $A \subseteq X$ lies in $\sum \mathcal{A}$ precisely when the set of those filters on $X$ that are actually filters on $A$, lies in $\mathcal{A}$.

1.13 Ultrafilters. An ultrafilter $\chi$ on a set $X$ is a maximal element within the set of proper filters on $X$, ordered by inclusion; that is, $\chi$ is a proper filter on $X$ such that if $a$ is a proper filter on $X$ with $\chi \subseteq a$, then $\chi = a$. A handier characterization is the following:

1.13.1 Lemma. For a proper filter $\chi$ on $X$, the following statements are equivalent:

(i) $\chi$ is an ultrafilter on $X$;

(ii) for all $A, B \subseteq X$, if $A \cup B \in \chi$ then $A \in \chi$ or $B \in \chi$;

(iii) for every subset $A \subseteq X$, one has $A \in \chi$ or $A^c \in \chi$ (where $A^c = X \setminus A$ denotes the complement of $A$ in $X$).

Proof. (i) $\implies$ (ii): if $A \cup B \in \chi$ but $A \notin \chi$ (for some $A, B \subseteq X$), then $\chi \subseteq \uparrow \{ A \cap C \mid C \in \chi \}$, so by maximality of $\chi$, the right-hand side filter cannot be proper; hence, $A \cap C = \emptyset$ for some $C \in \chi$, and $(A \cup B) \cap C = B \cap C \in \chi$, so $B \cap C \subseteq B$ tells us that $B \in \chi$. (ii) $\implies$ (iii): Immediate from $X = A \cup A^c \in \chi$. (iii) $\implies$ (i): $\chi \subseteq a$ implies that there is an $A \in a$ with $A \notin \chi$; thus, $A^c \in \chi \subseteq a$ and $A \cap A^c = \emptyset \in a$ which therefore is not proper.

The filter generation processes described in 1.12 may be specialized to ultrafilters; more precisely:

(a) For a map $f : X \to Y$ and an ultrafilter $\chi$ on $X$, the image $f[\chi]$ is also an ultrafilter on $Y$. When $f : X \hookrightarrow Y$ is an inclusion map, and $y$ an ultrafilter on $Y$ with $X \in y$, then $y|_X$ is an ultrafilter on $X$.

(b) For every $x \in X$, the principal filter $\hat{x} = \uparrow \{ x \}$ is an ultrafilter on $X$. 
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(c) If \( X \) is an ultrafilter on the set
\[
\beta X = \{ \chi \mid \chi \text{ is an ultrafilter on } X \}
\]
then \( \sum X \) is an ultrafilter on \( X \).

For the creation of other ultrafilters, one resorts to the Axiom of Choice:

1.13.2 Proposition. Every proper filter \( a \) on \( X \) is contained in an ultrafilter \( \chi \) on \( X \).  

Proof. This is guaranteed by an easy application of Zorn’s Lemma, see Exercise 1.P.

In fact, this statement can be used to formulate a formally finer assertion:

1.13.3 Corollary. For a filter \( b \) and a proper filter \( a \) on \( X \) such that \( a \subsetneq b \), there is an ultrafilter \( \chi \) on \( X \) with \( a \subseteq \chi \) but \( b \nsubseteq \chi \).

Proof. Indeed, for some \( B \in b \) with \( B \notin a \), one considers the filter
\[
a' = \uparrow\{ B^c \cap A \mid A \in a \},
\]
which is proper since \( B^c \cap A = \emptyset \) would imply \( A \subseteq B \in a \). So there is an ultrafilter \( \chi \) containing \( a' \), and therefore also \( a \); as \( B^c \in \chi \), we must have \( B \notin \chi \).

As an important consequence, we obtain:

1.13.4 Corollary. Every filter \( a \) on \( X \) is the intersection of all ultrafilters on \( X \) containing \( a \).

Proof. We must show that every filter \( a \in FX \) may be obtained as
\[
a = \bigcap \{ \chi \in \beta X \mid a \subseteq \chi \}.
\]
This equality holds trivially when \( \emptyset \in a \), and the inclusion “\( \subseteq \)" is also immediate. Moreover, when \( a \) is proper, so is the filter \( b \) obtained on the right-hand side; therefore, if \( a \subsetneq b \), there exists an ultrafilter \( \chi \) with \( a \subseteq \chi \) but \( b \nsubseteq \chi \), contradicting the definition of \( b \).

One obtains alternatively the following result.

1.13.5 Corollary. For a proper filter \( a \) and an ideal \( j \) on \( X \) such that \( a \cap j = \emptyset \), there is an ultrafilter \( \chi \) with \( a \subseteq \chi \) and \( \chi \cap j = \emptyset \).

Proof. Since \( a \) is an up-set, the fact that \( a \) is disjoint from \( j \) translates as \( A \not\subseteq J \) for all \( A \in a \) and \( J \in j \), or equivalently as \( A \cap J^c \neq \emptyset \) for all \( A \in a, J \in j \). Thus, \( b := \{ A \cap J^c \mid A \in a, J \in j \} \) is a proper filter containing \( a \), and Proposition 1.13.2 yields the existence of an ultrafilter \( \chi \) with \( b \subseteq \chi \), and consequently \( a \subseteq \chi \). If there was \( J \in j \cap \chi \), one would conclude \( J^c \notin \chi \), a contradiction.
1.14 Natural and ordinal numbers. We end this introductory section with some foundational remarks. A natural numbers object for sets is a set \( N \) with a distinguished element \( 0 \) and a map \( s : N \to N \) such that, for any set \( X \) equipped with a map \( t : X \to X \) and an element \( a \in X \), there is a unique map \( f \) making the diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{s} & N \\
\downarrow f & & \downarrow f \\
& a \xrightarrow{t} & X
\end{array}
\]

commutative. Such a set \( N \) must necessarily have the form

\[
N = \{0\} \cup s(N),
\]

and \( N = (N, s, 0) \) is uniquely determined up to a unique compatible bijection. Briefly, \( N \) is characterized by the requirement to allow for inductive definitions, via

\[
f(0) = a, \quad f(s(n)) = t(f(n)) \quad (n \in N).
\]

Alternatively, in categorical language (as introduced in II.2.7) \( N \) is initial amongst all general algebras with one nullary and one unary operation and no other requirements. Defining the sets \( N_n \) recursively by

\[
N_0 = \{0\}, \quad N_{s(n)} = N_n \cup \{s(n)\} \quad (n \in N),
\]

one may define the natural order on \( N \) by

\[
n \leq m \iff N_n \subseteq N_m.
\]

It is the only order that makes \( N \) a chain with \( n \leq s(n) \) for all \( n \in N \). Let us now assume that \( X \) in [1.14.a] is ordered and \( t \) is pointed, in the sense that \( x \leq t(x) \) for all \( x \in X \). Then the recursively defined function \( f \) is monotone. Consequently, \( N \) is also initial amongst ordered general algebras with a nullary and unary operation that is pointed. No such object exists within the realm of finite sets or algebras. In other words, the inductive condition may also be seen as an infinity axiom.

Missing a top element, \( N \) fails to be complete. It is therefore natural to ask: is there a separated complete ordered set \( O \) with a distinguished element \( 0 \in O \) and a pointed operation \( s : O \to O \) such that, for any separated complete ordered set \( X \) with a distinguished element \( a \in X \) and pointed operation \( t : O \to O \), there is a unique sup-map \( f : O \to X \) with \( f(0) = a \) and \( t(f(\alpha)) = f(s(\alpha)) \) for all \( \alpha \in O \)? Naively, such set should have the form

\[
O = \{0, s(0), s(s(0)), \ldots, \omega = \sup_{n<\omega} n, s(\omega), s(s(\omega)), \ldots\},
\]

with a considerably increased degree of uncertainty of how to “continue” the description of its elements, in comparison to the description of those of \( N \). In fact, just like the fact that a
natural numbers object may not be found within the realm of finite sets, the desired object $O$ would be “too big” to be a set. But regardless of which foundational requirements one may adopt to govern the use of “sets”, it is reasonable to require the existence of a class $O$ with a separated complete order (so that it has all set-indexed suprema), a distinguished element 0 and a pointed unary operation that is initial amongst all classes $X$ structured by $a$ and $t$ in the same way; in other words, such that maps $f : O \to X$ may be uniquely defined by ordinal recursion, via
\[
    f(0) = a, \quad f(s(\alpha)) = t(f(\alpha)), \quad f(\sup_{i \in I} \alpha_i) = \sup_{i \in I} f(\alpha_i)
\]
for all $\alpha, \alpha_i \in O$, $i \in I$, and set $I$.

We will apply this ordinal recursion principle only twice in this book, for the proof of the main result of III.4.2 and for Proposition V.4.4.9, with rather limited impact on other results.

**Exercises**

1.A Universal property of the separated reflection. Every monotone map $f : X \to Y$ of ordered sets, with $Y$ separated, factors through the separated reflection $p : X \to X/\simeq$ by a uniquely determined monotone map $g : X/\simeq \to Y$, so that $f = g \cdot p$.

1.B Adjunctions and fully faithful maps. For an adjunction $f \dashv g : Y \to X$, the map $g$ is fully faithful if and only if $f \cdot g \simeq 1_Y$. If $h \dashv f \dashv g$, then $g$ is fully faithful if and only if $h$ is fully faithful.

1.C Adjunctions for free. For a module $r : X \to Y$ and an ordered set $Z$, denote by $r_Z$ the map
\[
    \text{Mod}(Z, X) \to \text{Mod}(Z, Y), \quad t \mapsto r \cdot t,
\]
where $\text{Mod}(Z, X)$ denotes the ordered set of modules $Z \to X$. With a module $s : Y \to X$, show
\[
    \forall Z \quad (r_Z \dashv s_Z) \iff 1_X^* \leq s \cdot r \& r \cdot s \leq 1_Y^*.
\]
Conclude that for every monotone map $f : X \to Y$ and every ordered set $Z$, one has an adjunction $(f_s)_Z \dashv (f^*_s)_Z$.

1.D Closure operations on complete ordered sets. For every closure operation $c$ on an ordered set $X$, there is an adjunction $e \dashv j : \text{Fix}(c) \to X$. If $X$ is complete, so is $\text{Fix}(c)$, with infima in $\text{Fix}(c)$ formed as in $X$ while the supremum of $A \subseteq \text{Fix}(c)$ in $\text{Fix}(c)$ is given by $c(\bigvee A)$.

1.E Complete chains are frames. For elements $a, b$ in a chain $X$, put
\[
a \to b := \begin{cases} 
    \top & \text{if } a \leq b, \text{ and} \\
    b & \text{otherwise.}
\end{cases}
\]
Then

\[ x \land a \leq b \iff x \leq (a \rightarrow b). \]

Hence, every complete chain is a frame.

1.F **Topological spaces as closure spaces.** Every topological space \( X \) becomes a closure space via its Kuratowski closure operation:

\[
\overline{A} = \bigcap\{B \subseteq X \mid A \subseteq B, B^\circ \in \mathcal{O}X\} = \{x \in X \mid \forall U \in \mathcal{O}X \ (x \in U \implies A \cap U \neq \emptyset)\}
\]

for all \( A \subseteq X \). The map \( (\overline{\_}) : PX \to PX \) is not only a closure operation, but is also finitely additive:

\[
\overline{A \cup B} = \overline{A} \cup \overline{B}, \quad \emptyset = \emptyset
\]

for all \( A, B \subseteq X \). Hence, the map \( (\overline{\_}) \) is a monoid homomorphism of the join-semilattice \( PX \). A map \( f : X \to Y \) between topological spaces is continuous if and only if it is continuous as a map of closure spaces. The underlying order may then be written as

\[ x \leq y \iff y \in \{x\}. \]

Of course, since closure spaces and interior spaces are equivalent concepts, every topological space also becomes an interior space (whose interior operation preserves finite intersections).

1.G **Closure spaces as topological spaces.** For a set \( X \), any finitely additive closure operation \( c \) on \( PX \) defines the topology

\[
\{A \subseteq X \mid c(A^\complement) = A^\complement\}
\]

on \( X \), and this establishes a bijective correspondence between topologies on \( X \) and finitely additive closure operations on \( PX \). The bijective correspondence between topologies on \( X \) and interior operations on \( PX \) that preserve finite intersections is even more immediate, as it avoids the use of \( (\overline{\_})^\complement \).

1.H **A simple non-linear quantale.** The diamond lattice \( 2^2 = \{\bot, u, v, \top\} \), with \( u \) and \( v \) two incomparable elements, is a frame, and as such is obviously isomorphic to the powerset of 2. Compute \( a \rightarrow b \) for all \( a, b \in 2^2 \).

1.I **Lax homomorphisms from the extended real half-line.** There are uniquely determined monotone maps \( o, p : [0, \infty]^\text{op} \to 2 \) with

\[ o \dashv \iota \dashv p, \]

where \( \iota(\bot) = \infty \) and \( \iota(\top) = 0 \). For the “optimist’s map”, one has \( o(v) = \top \) for all \( v < \infty \), while for the “pessimist’s map”, one has \( p(v) = \top \) only when \( v = 0 \). When \( [0, \infty]^\text{op} \) is considered as a quantale \( P_+, P_x \), or \( P_{\text{max}} \), which of these maps turn into lax homomorphisms, and among these, which are in fact homomorphisms?
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1.J Right and left adjoints to $DnX \hookrightarrow PX$. For an ordered set $X$, the inclusion $DnX \hookrightarrow PX$ has a left adjoint given by the down-closure $\downarrow$, and a right adjoint given by the \textit{down-interior} $\downarrow^\circ$, where

$$\downarrow^\circ A = \{x \in X \mid \downarrow x \subseteq A\}$$

for $A \subseteq X$. In particular, $DnX$ is a complete lattice with meets and joins formed as in $PX$.

1.K Right and left adjoints to $f^{-1}(-) : DnY \to DnX$. For a monotone map $f : X \to Y$ of ordered sets, the map $f^{-1}(-) : DnY \to DnX$ has a left adjoint $f_!$ given by $f_!(A) = \downarrow f(A)$, and a right adjoint $f^*$ given by $f^*(A) = \downarrow^\circ \bigvee \{B \subseteq Y \mid f^{-1}(B) \subseteq A\}$ for all $A \subseteq X$. In particular, $f^{-1}(-)$ is a homomorphism of frames.

1.L Semilattices as monoids, frames as quantales. Let $X$ be a separated ordered set which is also a (multiplicatively written) commutative monoid in which the multiplication is monotone in each variable. If every element in the monoid $X$ is idempotent, and the neutral element is the top element, then the ordered set $X$ is a meet-semilattice: the order and the monoid structure determine each other via

$$(x \leq y \iff x \cdot y = x), \quad \text{and} \quad x \cdot y = x \land y.$$  

In particular, for a commutative quantale $(\mathcal{V}, \otimes, k)$ with $k = \top$ and $v \otimes v = v$ for all $v \in \mathcal{V}$, one has $\otimes = \land$.

1.M Quantales arising from monoids. For a monoid $M = (M, m, e)$, the powerset $PM$ becomes a monoid with

$$B \cdot A = \{y \cdot x \mid x \in A, y \in B\} \quad (A, B \subseteq M)$$

and neutral element $\{e\}$. Ordered by inclusion, $PM$ is a quantale which is commutative precisely when $M$ is commutative. For a homomorphism $f : M \to N$ of monoids, $f(-) : PM \to PN$ is a homomorphism of quantales.

1.N Complete lattices as frames. A complete lattice $X$ is a frame if and only if $\vee : DnX \to X$ is a meet-semilattice homomorphism. Every completely distributive lattice is a frame.

1.O Complete distributivity of $DnX$. For every ordered set $X$, $DnX$ is completely distributive. In particular, the powerset $PX$ of a set $X$ is completely distributive.

\textit{Hint.} The join map of $DnX$ coincides with $(\downarrow^\circ)^{^{-1}}(-) : DnDnX \to DnX$ (where $\downarrow : X \to DnX$), and by Exercise 1.K it has the left adjoint $(\downarrow^\circ)^{^{-1}}$.

1.P Zorn’s Lemma and ultrafilters. Zorn’s Lemma, which is in fact \textit{equivalent} to the Axiom \circ of Choice (for an elementary account, see for example [Davey and Priestley, 1990]), states:

If $X$ is a separated ordered set and every chain $A \subseteq X$ has an upper bound in $X$, then $X$ has a \textit{maximal} element, that is, an element $x \in X$ such that if $y \in X$ satisfies $x \leq y$, then $x = y$. 

As a consequence, every proper filter \( a \) on a set \( X \) is contained in an ultrafilter \( \chi \) on \( X \).

1.Q  **Convergence of sequences.** Given a sequence \( s : \mathbb{N} \to X \) in \( X \) (also denoted by \( s = (x_n)_{n \in \mathbb{N}} \)), one can consider the sets \( S_n := \{ x_m \in X \mid n \leq m \} \), and the filter associated to \( s \)

\[
\langle s \rangle := \uparrow_{PX}\{ S_n \mid n \in \mathbb{N} \}.
\]

If \( X \) is a topological space (with topology \( O_X \)), then the neighborhood filter at \( x \in X \) is

\[
\nu(x) := \uparrow_{PX}\{ U \in O_X \mid x \in U \}.
\]

A neighborhood of \( x \in X \) is an element of \( \nu(x) \). Show that the open sets of \( X \) are exactly the subsets \( V \) that are neighborhoods of each of their points, and that a sequence \( s \) converges to a point \( x, \) that is,

\[
\forall V \in \nu(x) \exists n \in \mathbb{N} \forall m \geq n (x_m \in V),
\]

if and only if \( \langle s \rangle \supseteq \nu(x) \). Similarly, if \( s' = (x_n)_{n \in \mathbb{N}} \) is a subsequence of \( s \) (that is, \( \langle s' \rangle \supseteq \langle s \rangle \)), and \( s \) converges to \( x \), then \( s' \) also converges to \( x \). Finally, the constant sequence \( s_x : \mathbb{N} \to X \) at \( x \) (so that \( s_x(\mathbb{N}) = \{ x \} \)) converges to \( x \) for any topology on \( X \).

1.R  **Continuity and sequences.** The image of a sequence \( s : \mathbb{N} \to X \) under a map \( f : X \to Y \) is the sequence \( f \cdot s : \mathbb{N} \to Y \), or equivalently the sequence \( (f(x_n))_{n \in \mathbb{N}} \) (where \( s = (x_n)_{n \in \mathbb{N}} \)). The filter associated to \( f \cdot s \) is then the image filter of \( \langle s \rangle \):

\[
f[\langle s \rangle] = \langle f \cdot s \rangle.
\]

If \( f : X \to Y \) is a continuous map between topological spaces \( X \) and \( Y \) and a sequence \( s \) converges to \( x \in X \) (in symbols \( \langle s \rangle \to x \)), then \( f \cdot s \) converges to \( f(x) \):

\[
\langle s \rangle \to x \implies f[\langle s \rangle] \to f(x);
\]

in other words, continuous maps preserve convergence of sequences.

1.S  **Sequences do not suffice.** In general, topological concepts are not faithfully represented by convergence of sequences. For example, consider the real line \( \mathbb{R} \) equipped with the topology given by complements of countable subsets:

\[
O_{\mathbb{R}} := \{ U \subseteq \mathbb{R} \mid \overline{U}^c \text{ is countable} \}.
\]

In this topology, a sequence \( s = (x_n)_{n \in \mathbb{N}} \) converges to a point \( x \) if and only if \( s \) is eventually constant:

\[
\langle s \rangle \to x \iff \exists n \in \mathbb{N} \forall m \geq n (x_m = x).
\]

The identity map \( 1_{\mathbb{R}} : (\mathbb{R}, O_{\mathbb{R}}) \to (\mathbb{R}, P_{\mathbb{R}}) \) preserves convergence of sequences, but is not continuous (the converse statement is true however, see Exercise 1.R).
2 Categories and adjunctions

2.1 Categories. A small category \( \mathcal{C} \) is given by a set \( \text{ob}\mathcal{C} \) of objects of \( \mathcal{C} \), a map \( \text{hom}_\mathcal{C} \) which assigns to each pair of objects \((A, B)\) a set \( \text{hom}_\mathcal{C}(A, B) \), called their hom-set and more briefly written as \( \mathcal{C}(A, B) \), as well as composition and identity operations:

\[
\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C) \quad \{\star\} \to \mathcal{C}(A, A)
\]

\[
(f, g) \mapsto g \cdot f \quad \star \mapsto 1_A
\]

subject to the associativity and right and left identity laws

\[
h \cdot (g \cdot f) = (h \cdot g) \cdot f, \quad f \cdot 1_A = f = 1_B \cdot f
\]

for all \( f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C), h \in \mathcal{C}(C, D), A, B, C, D \in \text{ob}\mathcal{C} \). Given objects \( A, B \), one writes

\[
f : A \to B
\]

instead of \( f \in \mathcal{C}(A, B) \), and calls \( A = \text{dom} f \) the domain, and \( B = \text{cod} f \) the codomain of the morphism (or arrow) \( f \) in \( \mathcal{C} \), with the understanding that two morphisms \( f, g \) can coincide only if

\[
\text{dom} f = \text{dom} g \quad \text{and} \quad \text{cod} f = \text{cod} g.
\]

The morphism \( f : A \to B \) in \( \mathcal{C} \) is an isomorphism of \( \mathcal{C} \) if \( f' \cdot f = 1_A \) and \( f \cdot f' = 1_B \) for a (necessarily uniquely determined) morphism \( f' : B \to A \) in \( \mathcal{C} \). One has the self-explanatory rules

\[
(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}, \quad 1_A^{-1} = 1_A, \quad (f^{-1})^{-1} = f.
\]

An equivalence relation on \( \text{ob}\mathcal{C} \) is defined by writing \( A \cong B \) if there is an isomorphism \( f : A \to B \) in \( \mathcal{C} \). An order on \( \text{ob}\mathcal{C} \) is obtained by setting \( A \leq B \) if there exists a morphism \( f : A \to B \). In this case, \( A \cong B \) implies \( A \simeq B \) (meaning \( A \leq B \) and \( B \leq A \), as in 1.3), but the converse is not true in general.

In Section 1 we encountered two important types of small categories. First, every monoid \( M \) becomes a one-object category \( \mathcal{C} \) when one puts \( \mathcal{C}(\star, \star) = M \) (with \( \star \) denoting the only object of \( \mathcal{C} \)) and interprets monoid operation as composition. In fact, monoids can be thought of as precisely the one-object small categories, and small categories are simply “multi-object monoids”. Second, every ordered set \( X \) becomes a small category \( \mathcal{C} \) with \( \text{ob}\mathcal{C} = X \) and \( \mathcal{C}(x, y) = \{\star\} \) if \( x \leq y \) in \( X \), and \( \mathcal{C}(x, y) = \emptyset \) otherwise; the composition and identity operations appear as the transitivity and reflexivity properties in \( X \). Note that in this case, the order on \( \mathcal{C} \) defined above coincides with that of \( X \). This shows that ordered sets can be thought of as precisely those small categories for which the hom map is \( \{\emptyset, \{\star\}\}\)-valued, and small categories are simply sets with “multi-valued structured orders”.

Arbitrary categories are defined just like small categories, but one allows \( \text{ob}\mathcal{C} \) and \( \mathcal{C}(A, B) \) to be classes rather than just sets (for all \( A, B \in \text{ob}\mathcal{C} \)). The understanding here is that a
set is a particular class, and that we are allowed to form the class of all sets (see I.3). For every category \( C \), the opposite or dual category \( C^{\text{op}} \) of \( C \) is given by

\[
\text{ob } C^{\text{op}} = \text{ob } C, \quad C^{\text{op}}(A, B) = C(B, A), \quad \text{and} \quad g \cdot_{C^{\text{op}}} f = f \cdot_C g.
\]

Here are some examples of categories that we encountered (implicitly) in Section I:

- **Set**: the category whose objects are sets, and whose morphisms are maps with ordinary composition of maps.
- **Rel**: objects are sets, but morphisms are relations with ordinary relational composition.
- **Ord**: ordered sets, with monotone maps and ordinary map composition.
- **Mod**: ordered sets, with modules and ordinary relational composition.
- **Ord_{sep}**: separated ordered sets, with monotone maps and, as in all the following examples, ordinary map composition. A more common name for this category is \( \text{PoSet} \), which stands for “partially ordered sets”.
- **Sup**: complete lattices, with sup-maps.
- **Inf**: complete lattices, with inf-maps.
- **Mon**: monoids, with monoid homomorphisms.
- **SLat**: semilattices, with semilattice homomorphisms.
- **Lat**: lattices, with lattice homomorphisms.
- **Frm**: frames, with frame homomorphisms.
- **Qnt**: quantales, with quantale homomorphisms.
- **Top**: topological spaces, with continuous maps.
- **Cls**: closure spaces, with continuous maps.
- **Int**: interior spaces, with continuous maps.

### 2.2 Functors.

A functor \( F : C \to D \) from a category \( C \) to a category \( D \) is given by functions

\[
F : \text{ob } C \to \text{ob } D, \quad \text{and} \quad F_{A,B} : C(A, B) \to D(FA, FB)
\]

for all \( A, B \in \text{ob } C \), with (when writing \( Ff = F_{A,B}(f) \))

\[
F(g \cdot f) = Fg \cdot Ff, \quad F1_A = 1_{FA}
\]

for all \( f : A \to B, g : B \to C \) in \( C \). There is an obvious identity functor \( 1_C : C \to C \), and the composition \( GF \) of two functors \( F : C \to D \) and \( G : D \to E \) is defined by using
composition of functions. The functor $F$ is \textit{faithful} if all functions $F_{A,B}$ are injective, and it is \textit{full} if these functions are surjective. \textit{Fully faithful} functors (that is, functors that are both full and faithful) typically arise when forming a \textit{full subcategory} $D$ of a category $C$, so that $\text{ob} D \subseteq \text{ob} C$, $D(A,B) = C(A,B)$ for all $A,B \in \text{ob} D$, and composition in $D$ is as in $C$; then the \textit{inclusion functor} $F : D \hookrightarrow C$ is full and faithful. Here, the functor $F$ has the additional property that its object function is injective, which makes $F$ a \textit{full embedding}. Small categories and functors form a \textit{category $\text{Cat}$}. With suitable foundational provisions, we may also form the \textit{metacategory $\text{CAT}$} of all categories and functors.

Functors between monoids (considered as categories) are precisely monoid homomorphisms, and functors between ordered sets are precisely monotone maps; in this last case, the “fully faithful” terminology coincides with the one introduced in 1.3 for ordered sets. Many functors also arise as \textit{forgetful functors}, forgetting properties or structures of objects and morphisms, such as in the sequence

$$Qnt \rightarrow \text{Sup} \rightarrow \text{Ord} \rightarrow \text{Set}.$$

Some other functors that we implicitly encountered in Section 1 include:

- The \textit{covariant powerset functor} $P : \text{Set} \rightarrow \text{Set}$, with $Pf(A) = f(A)$ for $f : X \rightarrow Y$, $A \subseteq X$.

- The \textit{contravariant powerset functor} $P^\bullet : \text{Set}^{\text{op}} \rightarrow \text{Set}$, with $P^\bullet f(B) = f^{-1}(B)$ for $f : X \rightarrow Y$, $B \subseteq Y$.

- The \textit{down-set functor} $Dn : \text{Ord} \rightarrow \text{Ord}$, with $Dnf(A) = \downarrow f(A)$ for $f : X \rightarrow Y$, $A \subseteq X$ (see 1.7).

- The \textit{open-set functor} $\mathcal{O} : \text{Top}^{\text{op}} \rightarrow \text{Frm}$ (see 1.9).

- The \textit{underlying-order functor} $\text{Top} \rightarrow \text{Ord}$ (see 1.9).

- The \textit{Kuratowski-closure functor} $\text{Top} \rightarrow \text{Cls}$ (see Exercise 1.F).

- The functor $\text{Cls} \rightarrow \text{Int}$ arising from two-fold complementation (see 1.6), which is an isomorphism in $\text{CAT}$.

- The \textit{dualization functor} $(-)^{\text{op}} : \text{Ord} \rightarrow \text{Ord}$ (see 1.3). Since $(X^{\text{op}})^{\text{op}} = X$, this functor is an isomorphism (in $\text{CAT}$), and extends to an isomorphism $(-)^{\text{op}} : \text{CAT} \rightarrow \text{CAT}$ (see 2.1). It maps a functor $F : C \rightarrow D$ to $F^{\text{op}} : C^{\text{op}} \rightarrow D^{\text{op}}$, where $F^{\text{op}}f = Ff$. A restriction of the dualization functor yields an isomorphism $\text{Inf} \rightarrow \text{Sup}$ in $\text{CAT}$.

- The \textit{adjunction functor} $\text{Sup}^{\text{op}} \rightarrow \text{Inf}$ which maps objects identically and assigns to a sup-map its right adjoint (see Corollary 1.8.3); again this is an isomorphism in $\text{CAT}$. Composition with the isomorphism $\text{Inf} \rightarrow \text{Sup}$ shows that $\text{Sup}$ is \textit{self-dual}: $\text{Sup}^{\text{op}} \cong \text{Sup}$; likewise for $\text{Inf}$. 


The module functors \((-)_*: \text{Ord} \to \text{Mod}\) and \((-)^\ast: \text{Ord}^{\text{op}} \to \text{Mod}\) which map objects identically and assign to a monotone map \(f\) the induced modules \(f_*\) and \(f^\ast\), respectively.

For every object \(A\) in a category \(C\), one has the covariant hom-functor of \(A\)

\[ C(A, -): C \to \text{SET} \]

(where the objects of the metacategory \(\text{SET}\) are classes) with

\[ C(A, -)(B) := C(A, B), \]
\[ C(A, -)(g) := C(A, g): C(A, B) \to C(A, C), \quad f \mapsto g \cdot f \]

for all \(f: A \to B, g: B \to C\) in \(C\). Similarly, one has the contravariant hom-functor of \(A\)

\[ C(-, A) = C^{\text{op}}(A, -): C^{\text{op}} \to \text{SET} \]

These functors take value in \(\text{Set}\) if \(C\) is locally small, i.e., if all \(C(A, B)\) are sets. Note that when \(C\) is a monoid \(M\) seen as a one-object category, then the hom-functor \(M(*) , -): M \to \text{Set}\) has as values those maps \(M \to M\) which are the “left translations by \(a\)” \((a \in M)\); with a suitable codomain restriction, \(M(*) , -)\) is in most algebra books referred to as the Cayley representation of \(M\), although \(M\) is normally assumed to be a group (that is, a monoid in which every element is invertible). When \(C\) is an ordered set \(X\) and \(x \in X\), then \(X(x, -)\) is two-valued and therefore the characteristic function of a subset of \(X\), namely \(\uparrow_X x\).

2.3 Natural transformations. Let \(F, G: C \to D\) be functors. A natural transformation \(\alpha: F \to G\), also depicted by

\[ C \xrightarrow{\begin{array}{c} F \\ \downarrow \alpha \\ \downarrow G \end{array}} D, \]

is given by a family of morphisms \(\alpha_A: FA \to GA\) in \(D\) (with \(A\) running through \(\text{ob} C\)), making the naturality diagrams

\[
\begin{align*}
FA & \xrightarrow{Ff} FB \\
\downarrow \alpha_A & \quad \quad \quad \quad \quad \downarrow \alpha_B \\
GA & \xrightarrow{Gf} GB
\end{align*}
\]

commute for all \(f: A \to B\) in \(C\); hence, writing \(\alpha_f: FA \to GB\) for the diagonal morphism, we have

\[ \alpha_f = Gf \cdot \alpha_A = \alpha_B \cdot Ff . \]

For example, forming singleton sets is a natural transformation \(1_{\text{set}} \to P\) (where \(P\) is the powerset functor), and forming unions is a natural transformation \(PP \to P\). The down-set function defines a natural transformation \(\downarrow: \text{1}_{\text{Ord}} \to \text{Dn}\), and when we restrict \(\text{Dn}\) to a functor \(\text{Dn}: \text{Sup} \to \text{Sup}\), there is a natural transformation \(\lor: \text{Dn} \to \text{1}_{\text{Sup}}\).
2. CATEGORIES AND ADJUNCTIONS

Vertical pasting of naturality diagrams defines the \textit{vertical composition}

\[ \beta \cdot \alpha : F \to H \quad \text{with} \quad (\beta \cdot \alpha)_A = \beta_A \cdot \alpha_A \quad (A \in \text{ob } C), \]

for \( \alpha : F \to G, \beta : G \to H, \) and \( F, G, H : C \to D; \) one also has the identity transformation

\[ 1_F : F \to F \quad \text{with} \quad (1_F)_A = 1_{FA} \quad (A \in \text{ob } C). \]

The vertical composition is trivially associative, and makes the functors from \( C \) to \( D \) the objects of the \textit{functor category} \( \mathfrak{D}^C \) whose morphisms are natural transformations. (We note that unless \( C \) is small, \( \mathfrak{D}^C \) is actually a metacategory.) For functors \( S : B \to C \) and \( T : D \to E, \) one has the \textit{whiskering functors}

\[ (-)S : \mathfrak{D}^C \to \mathfrak{D}^B \quad \text{and} \quad T(-) : \mathfrak{D}^C \to \mathfrak{D}^E \]

which map natural transformations via

\[ (\alpha S)_B = \alpha_{SB}, \quad (T\alpha)_A = T\alpha_A \quad (B \in \text{ob } B, A \in \text{ob } C), \]

as suggested by

\[ B \xrightarrow{S} C \xrightarrow{F} D \xrightarrow{T} E. \]

These whiskering functors are of course induced by either \( \text{CAT}(-, D) \) or \( \text{CAT}(C, -) \) applied to \( S : B \to C \) or \( T : D \to E, \) respectively. In fact, whiskering by a functor (from the left or the right) is just an instance of the \textit{horizontal composition} for natural transformations

\[ C \xrightarrow{\gamma \circ \alpha} D \xrightarrow{\gamma} E \]

defined by

\[ \gamma \circ \alpha : KF \to LG, \quad (\gamma \circ \alpha)_A = \gamma_{\alpha_A} \quad (A \in \text{ob } C). \]

For a morphism \( f \) in \( C, \) one obtains

\[ (\gamma \circ \alpha)_f = \gamma_{\alpha_f}. \]

Moreover, since

\[ (\gamma \circ \alpha)_A = L\alpha_A \cdot \gamma_{FA} = \gamma_{GA} \cdot K\alpha_A \]

\[ = (L\alpha)_A \cdot (\gamma F)_A = (\gamma G)_A \cdot (K\alpha)_A, \]

the natural transformation \( \gamma \circ \alpha \) is completely determined by the values of the whiskering functors on \( \alpha \) and \( \gamma: \)

\[ \gamma \circ \alpha = L\alpha \cdot \gamma F = \gamma G \cdot K\alpha. \]
Equivalently, since
\[ \gamma \circ \alpha = (1_L \circ \alpha) \cdot (\gamma \circ 1_F) = (\gamma \circ 1_G) \cdot (1_K \circ \alpha) , \quad (2.3.i) \]
the horizontal composition is determined by the special instances of composition with identity transformations. The equality (2.3.i) is itself a special instance of the middle-interchange law; that is, given a commutative diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{\gamma} & D \\
\downarrow^F & & \downarrow^K \\
H & \xrightarrow{\alpha} & E \\
\downarrow^\beta & & \downarrow^\delta \\
M & \xrightarrow{\gamma} & L
\end{array}
\]

one has
\[ (\delta \cdot \gamma) \circ (\beta \cdot \alpha) = (\delta \circ \beta) \cdot (\gamma \circ \alpha) . \]

The horizontal composition is associative, and \( \alpha \circ 1_{1_C} = \alpha = 1_{1_D} \circ \alpha \). With \( \alpha : F \to G \), we also have a natural transformation \( \alpha^{op} : G^{op} \to F^{op} \), where \( (\alpha^{op})_A = \alpha_A \) is an arrow in \( D^{op} \), for all \( A \in \text{ob} \ C \).

Finally, \( \alpha \) is a natural isomorphism (that is, an isomorphism in \( D^C \)) if and only if every morphism \( \alpha_A \) is an isomorphism in \( D \), and then one has \( (\alpha^{-1})_A = (\alpha_A)^{-1} \), briefly written as \( \alpha_A^{-1} \).

**2.4 The Yoneda embedding.** For every morphism \( g : B \to C \) in a category \( C \), one has a natural transformation
\[ C(-, g) : C(-, B) \to C(-, C) \]
of contravariant hom-functors. When \( C \) is locally small, there is therefore a functor
\[ y : C \to \hat{C} = \text{Set}^{C^{op}} , \quad g \mapsto y(g) = C(-, g) . \]
This functor is faithful, since \( C(-, g) = C(-, h) \) (for \( g, h : B \to C \)) yields \( g = C(B, g)(1_B) = C(B, h)(1_B) = h \). Moreover, on objects \( y \) is one-to-one: if \( yB = yC \), then \( 1_B \in C(B, B) = C(B, C) \) and \( B = \text{cod} 1_B = C \). It is less obvious that \( y \) is also full, so that
\[ y : C \to \hat{C} = \text{Set}^{C^{op}} \]
is a full embedding, called the Yoneda embedding. As a preparation for that, we first prove the Yoneda Lemma:

**2.4.1 Lemma.** For every object \( A \) of a locally small category \( C \), and every functor \( F : C^{op} \to \text{Set} \), the map
\[ \eta : \hat{C}(yA, F) \to FA , \quad \alpha \mapsto \alpha_A(1_A) \]
is bijective.
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Proof. Routine verification shows that the map
\[ \delta : FA \to \hat{C}(yA, F) , \]
with \( \delta(a)_B : \mathcal{C}(B, A) \to FB \) sending \( f \) to \( Ff(a) \), for all \( a \in FA, B \in \text{ob} \mathcal{C} \), is well-defined and inverse to \( \eta \).

In the special case where \( F = yB \) for \( B \in \text{ob} \mathcal{C} \), the map
\[ \delta : \mathcal{C}(A, B) \to \hat{C}(yA, yB) \]
is precisely the hom-map \( y_{A,B} \) of the functor \( y \). Hence:

2.4.2 Corollary. The functor \( y : \mathcal{C} \to \hat{C} \) is a full embedding.

Consequently, every locally small category may be considered as a full subcategory of a functor category over \( \text{Set} \).

As we saw at the end of 2.2 in the dual situation, when \( \mathcal{C} \) is an ordered set \( X \) and \( x \in X \), the functor \( y(x) = X(-, x) \) may be identified with the down-set \( \downarrow_X x \). But the map \( \downarrow_X : X \to DnX \) is injective only when \( X \) is separated, and in that case \( \downarrow_X \) is just a codomain restriction of \( y \).

2.5 Adjunctions. In generalization of the notion introduced in 1.5, one calls a functor \( G : A \to X \) right adjoint if there is a functor \( F : X \to A \) and natural transformations \( \eta : 1_X \to GF \) and \( \varepsilon : FG \to 1_A \) satisfying the triangular identities
\[ G\varepsilon \cdot \eta = 1_G \quad \text{and} \quad \varepsilon F \cdot F\eta = 1_F. \]

These data are, up to isomorphism, uniquely determined by \( G \). Indeed, if \( F', \eta', \varepsilon' \) satisfy the corresponding conditions, then
\[ \alpha = (F \xrightarrow{F\eta'} GF \xrightarrow{\varepsilon'F} F') \quad \text{and} \quad \beta = (F' \xrightarrow{F'\eta} FG \xrightarrow{F'\varepsilon} F) \]
are inverse to each other, as one shows by repeated applications of the middle-interchange law; for example:
\[ \beta \cdot \alpha = \varepsilon'F \cdot F'\eta \cdot \varepsilon F' \cdot F\eta' \]
\[ = \varepsilon'F \cdot \varepsilon F'GF \cdot F'F\eta \cdot F\eta' \]
\[ = \varepsilon F \cdot F'\varepsilon' F \cdot F\eta'GF \cdot F\eta \]
\[ = \varepsilon F \cdot 1_{FGF} \cdot F\eta \]
\[ = 1_F \]
(middle-interchange)
(middle-interchange, twice)
(triangular identity)
(triangular identity).

Furthermore, \( \eta' = G\alpha \cdot \eta \) and \( \varepsilon' = \varepsilon \cdot \beta G \). One calls \( F \) left adjoint to \( G \), and \( \eta \) is the unit, while \( \varepsilon \) is the counit of the adjunction, denoted by
\[ F \xrightarrow{\eta} G : A \to X. \]
A functor $F$ is left adjoint to $G : A \to X$ if and only if $G^{\text{op}}$ is left adjoint to $F^{\text{op}} : X^{\text{op}} \to A^{\text{op}}$; more precisely,

$$F \xrightarrow{\eta} G \iff \quad G^{\text{op}} \xleftarrow{\eta^{\text{op}}} F^{\text{op}}.$$ 

### 2.5.1 Examples.

1. Adjunctions between ordered sets (considered as categories) are precisely those described in 1.5.

2. The forgetful functor $G : \text{Mon} \to \text{Set}$ is right adjoint. Its left adjoint $F$ is the free-monoid functor which, for a set $X$ may be constructed as the set of “words over the alphabet” $X$

$$FX = X^* = \bigcup_{n \geq 0} X^n,$$

with concatenation as multiplication; the map $\eta_X : X \to X^*$ (for a set $X$) considers an element of $X$ as a one-letter word, and the homomorphism $\varepsilon_A : A^* \to A$ (for a monoid $A$) sends words over $A$ to the actual product of its letters in $A$.

3. The forgetful functor $\text{Ord} \to \text{Set}$ has a left adjoint that provides each set with the discrete order (given by the graph of $1_X$); similarly for $\text{Top} \to \text{Set}$ whose left adjoint provides a set $X$ with the discrete topology. Both functors have also right adjoints which provide a set $X$ with the respective indiscrete (or chaotic) structure.

4. For a fixed set $A$, the hom-functor $\text{Set}(A, -) : \text{Set} \to \text{Set}$ has a left adjoint, namely

$$(-) \times A : \text{Set} \to \text{Set}.$$  

Writing $B^A = \text{Set}(A, B)$, the counit $\varepsilon$ is given by the evaluation maps

$$\varepsilon_B : B^A \times A \to B, \quad (f, x) \mapsto f(x),$$

while the unit sends every element $x$ of a set $X$ to the section

$$\eta_X(x) : A \to \times X, \quad a \mapsto (x, a) .$$

5. Considering the powerset $PX$ of a set $X$ as a complete lattice (ordered by inclusion) makes $P : \text{Set} \to \text{Sup}$ left adjoint to the forgetful functor $U : \text{Sup} \to \text{Set}$. The unit is described by the singleton maps $\eta_X : X \to PX$ that send $x$ to $\{x\}$, and the components of the counit $\varepsilon_A : PA \to A$ (for a complete lattice $A$) are simply given by $\vee$.

6. The contravariant powerset functor $P^* : \text{Set}^{\text{op}} \to \text{Set}$ is self-adjoint, that is, $(P^*)^{\text{op}} : \text{Set} \to \text{Set}^{\text{op}}$ is its left adjoint, as shown by Proposition 2.5.2 below, and by the natural bijection

$$X \xrightarrow{f} PY \quad \iff \quad PY \xleftarrow{g} Y.$$
where \( f \) and \( g \) determine each other via \( (x \in g(y) \iff y \in f(x)) \) for all \( x \in X, y \in Y \). Of course, this bijection stems from the self-dual category \( \text{Rel} \) for which \( \text{Rel}(X,Y) \cong \text{Rel}(Y,X) \).

### 2.5.2 Proposition

A functor \( G : \mathcal{A} \rightarrow \mathcal{X} \) is right adjoint if and only if there are a map \( F : \text{ob} \mathcal{X} \rightarrow \text{ob} \mathcal{A} \) and an \( \text{ob} \mathcal{X} \)-indexed family of natural isomorphisms

\[
\phi_X : \mathcal{A}(FX, -) \rightarrow \mathcal{X}(X,G-) = \mathcal{X}(X, -)G .
\]

This last condition yields bijections

\[
\phi_{X,A} : \mathcal{A}(FX, A) \rightarrow \mathcal{X}(X, GA)
\]

for all \( X \in \text{ob} \mathcal{X}, A \in \text{ob} \mathcal{A} \) that are “natural in \( A \)”; casually, one writes

\[
\begin{array}{ccc}
FX & \xrightarrow{g} & A \\
\downarrow & & \downarrow \\
X & \xrightarrow{F} & GA
\end{array}
\]

with \( F \) appearing on the left, and \( G \) on the right. Note that one necessarily obtains that these bijections are also “natural in \( X \)”. Any pair of arrows \((g, f)\) related to each other by the bijection \( \phi_{X,A} \) are called mates to each other under the adjunction.

**Proof.** If \( F \xrightarrow{\eta} G \), then \( \phi_{X,A} \) is defined by

\[
\phi_{X,A}(g) = Gg \cdot \eta_X
\]

for all \( g : FX \rightarrow A \) in \( \mathcal{A} \), with its inverse given by

\[
(f : X \rightarrow GA) \mapsto \varepsilon_A : Ff .
\]

Conversely, having the bijections \( \phi_{X,A} \) (natural in \( A \)), one obtains morphisms

\[
\eta_X = \phi_{X,FX}(1FX) , \quad \varepsilon_A = \phi^{-1}_{GX,A}(1GA) .
\]

Then \( F \), already defined on objects, can be made a functor via

\[
(f : X \rightarrow Y) \xrightarrow{F} (\phi^{-1}_{X,FY}(\eta_Y \cdot f) : FX \rightarrow FY) ,
\]

such that \( \eta \) and \( \varepsilon \) become natural transformations satisfying the triangular identities. \( \square \)

### 2.5.3 Example

Considering \( 2 = \{ \bot, \top \} \) successively as a set, an ordered set, a meet-semilattice, a lattice, or a frame, one obtains right adjoint functors

\[
\mathcal{C}(-, 2) : \mathcal{C}^{\text{op}} \rightarrow \mathcal{S}et ,
\]

with \( \mathcal{C} = \text{Set}, \text{Ord}, \text{SLat}, \text{Lat}, \) or \( \text{Frm} \), respectively, with left adjoint \( X \mapsto PX \). Of course, for \( \mathcal{C} = \text{Set} \), \( \mathcal{C}(-, 2) \cong P^* \) is isomorphic to the contravariant powerset functor of Example 2.5.1(6), and the needed natural isomorphisms

\[
\text{Set}(X, \mathcal{C}(Y, 2)) \cong \mathcal{C}(Y, PX)
\]

(with \( X \in \text{ob} \mathcal{S}et, Y \in \text{ob} \mathcal{C} \)) are just restrictions of those described in Example 2.5.1(6).
Returning to Proposition 2.5.2, we note that naturality of \( \phi_{X,A} \) in \( A \) gives, for \( g : FX \to A \) in \( A \), the commutative diagram

\[
\begin{array}{ccc}
A(FX, FX) & \xrightarrow{A(FX, g)} & A(FX, A) \\
\downarrow{\phi_{X,FX}} & & \downarrow{\phi_{X,A}} \\
X(X, GFX) & \xrightarrow{X(X,Gg)} & X(X, GA)
\end{array}
\]

so that with \( \eta_X = \phi_{X,FX}(1_{FX}) \), one has

\[
\phi_{X,A}(g) = \phi_{X,A} \cdot A(FX, g)(1_{FX}) = X(X, Gg) \cdot \phi_{X,FX}(1_{FX}) = Gg \cdot \eta_X;
\]

in other words, the formula for \( \phi_{X,A} \) in terms of the unit \( \eta \) is forced upon us by naturality in \( A \). The condition that \( \phi_{X,A} \) is bijective now says:

for all \( f : X \to GA \) in \( X \) (with \( A \in \text{ob} A \)), there is precisely one morphism \( g : FX \to A \) in \( A \) with \( Gg \cdot \eta_X = f \).

For a functor \( G : A \to X \) and an object \( X \) in \( X \), a morphism \( u : X \to GU \) in \( X \) with an object \( U \) in \( A \) is called a \( G \)-universal arrow for \( X \) if it satisfies the universal property [2.5.i], with \((U, u)\) in lieu of \((FX, \eta_X)\). This property allows us to define \( \phi_{X,A} \) as above whenever we have a chosen \( G \)-universal arrow for every \( X \in \text{ob} X \). As a consequence, we get:

2.5.4 Theorem. A functor \( G : A \to X \) is right adjoint if and only if there exist a map \( F : \text{ob} X \to \text{ob} A \) and an \( \text{ob} X \)-indexed family of \( G \)-universal arrows \( \eta_X : X \to GFX \). There is then a unique way of making \( F \) a functor such that \( \eta \) becomes a natural transformation.

Of course, \( F \) will be left adjoint to \( G \) with unit \( \eta \). There is a dual way of describing adjunctions in terms of their counit: for any functor \( F : X \to A \) and object \( A \) in \( A \), a morphism \( \varepsilon_A : FGA \to A \) in \( A \) together with an object \( GA \) is an \( F \)-couniversal arrow for \( A \) if, as an arrow in \( A^{\text{op}} \), it is \( F^{\text{op}} \)-universal, that is:

for all \( g : FX \to A \) in \( A \) (with \( X \in \text{ob} X \)), there is precisely one morphism \( f : X \to GA \) in \( X \) with \( \varepsilon_A \cdot Ff = g \).
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Usually, one also refers to (2.5.2) as a universal property, although, strictly speaking, it is a “couniversal” property. In summary, one obtains from Theorem 2.5.4 the following statement by dualization:

2.5.4 op Theorem. A functor $F : X \to A$ is left adjoint if and only if there exist a map $G : \text{ob} A \to \text{ob} X$ and an $\text{ob} A$-indexed family of $F$-couniversal arrows $\varepsilon_X : FGX \to X$. There is then a unique way of making $G$ a functor such that $\varepsilon$ becomes a natural transformation.

Proof. Apply Theorem 2.5.4 with $F \text{op}$ in lieu of $G$ (recall from 2.5 that $F \dashv G \iff G \text{op} \dashv F \text{op}$).

In what follows, we rarely formulate explicitly such dual statements.

Adjunctions compose; more precisely, direct calculation based on the definitions shows:

2.5.5 Proposition. If $F \eta \varepsilon G : A \to X$ and $H \delta \gamma J : C \to A$, then $HF \alpha \beta GJ : C \to X$, with

$$
\alpha = (1_X \xrightarrow{\eta} GF \xrightarrow{G\gamma F} GJHF) \quad \beta = (HFGJ \xrightarrow{H\varepsilon J} HJ \xrightarrow{\delta} 1_C).
$$

2.5.6 Corollary. If $F \dashv G : A \to X$, $H \dashv J : C \to A$ and $L \dashv GJ$, then $L \cong HF$.

2.6 Reflective subcategories, equivalence of categories. Whether a right adjoint functor is fully faithful is detected by the counit:

2.6.1 Proposition. For an adjunction $F \eta \varepsilon G : A \to X$, the functor $G$ is fully faithful if and only if $\varepsilon$ is a natural isomorphism. Dually, $F$ is fully faithful if and only if $\eta$ is a natural isomorphism.

Proof. Since, in the notations of 2.5, for all $g : A \to B$ in $A$,

$$
\phi_{GA,B} \cdot A(\varepsilon_A, B)(g) = G(g \cdot \varepsilon_A) \cdot \eta_{GA} = Gg,
$$

the following diagram commutes for all $A, B \in \text{ob} A$:

$$
\begin{array}{ccc}
A(FGA, B) & \xrightarrow{A(\varepsilon_A, B)} & A(A, B) \\
\phi_{GA,B} & & \downarrow \quad G_{A,B} \\
& \xrightarrow{GA,B} & A(GB)
\end{array}
$$

Since $\phi_{GA,B}$ is bijective, one therefore has that for all $A, B \in \text{ob} A$ the map $G_{A,B}$ is bijective precisely when every $\varepsilon_A$ is an isomorphism in $A$. \qed
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A full subcategory \( \mathbf{A} \) of a category \( \mathbf{X} \) is **reflective** in \( \mathbf{X} \) if the inclusion functor \( J : \mathbf{A} \hookrightarrow \mathbf{X} \) is right adjoint. A left adjoint \( R : \mathbf{X} \to \mathbf{A} \) to \( J \) is called a **reflector** of \( \mathbf{X} \) onto \( \mathbf{A} \), and the unit \( \rho : 1_\mathbf{X} \to JR \) gives the \( \mathbf{A} \)-**reflections** \( \rho_X : \mathbf{X} \to RX \) that are characterized as \( J \)-universal arrows for \( X \). By Theorem 2.5.4, \( \mathbf{A} \) is reflective in \( \mathbf{X} \) if and only if for every object in \( \mathbf{X} \), there is a chosen \( \mathbf{A} \)-reflection. By Proposition 2.6.1, the counit of the adjunction is an isomorphism.

An example of a reflective subcategory encountered in Section 1 is \( \text{Ord}_{\text{sep}} \) in \( \text{Ord} \), with the separated reflections of \( \mathbf{I} \) forming the unit. For a reflective subcategory \( \mathbf{A} \) of \( \mathbf{X} \) with unit \( \rho \), an object \( X \) in \( \mathbf{X} \) is isomorphic to an object in \( \mathbf{A} \) if and only if \( \rho_X \) is an isomorphism. Hence, when \( \mathbf{A} \) is **replete** in \( \mathbf{X} \), that is, when \( X \cong A \in \text{ob} \mathbf{A} \) implies \( X \in \text{ob} \mathbf{A} \), an object of \( \mathbf{X} \) lies in \( \mathbf{A} \) if and only if its \( \mathbf{A} \)-reflection is an isomorphism.

A full subcategory \( \mathbf{A} \) is **coreflective** in \( \mathbf{X} \) if \( \mathbf{A}^{\text{op}} \) is reflective in \( \mathbf{X}^{\text{op}} \); equivalently, if the inclusion functor has a right adjoint, which is the **coreflector** of \( \mathbf{X} \) onto \( \mathbf{A} \). The counit of the adjunction gives the \( \mathbf{A} \)-**coreflections**.

A functor \( G : \mathbf{A} \to \mathbf{X} \) is an **equivalence** of categories if there is a functor \( F : \mathbf{X} \to \mathbf{A} \) with \( 1_\mathbf{X} \cong GF \) and \( FG \cong 1_\mathbf{A} \). A category \( \mathbf{A} \) is **equivalent** to a category \( \mathbf{X} \) if there is an equivalence \( \mathbf{A} \to \mathbf{X} \). If \( G \) is an equivalence, then it is fully faithful (see the proof of the next proposition), and therefore **reflects isomorphisms**, that is, if \( Gf \) is an isomorphism then so is \( f \) (Exercise 2.A). In fact, these last two notions may be used to verify that \( G \) is an equivalence:

**2.6.2 Proposition.** The following are equivalent for a functor \( G : \mathbf{A} \to \mathbf{X} \):

(i) \( G \) is an equivalence;

(ii) \( G \) reflects isomorphisms and has a fully faithful left adjoint;

(iii) \( G \) is fully faithful and there are a map \( F : \text{ob} \mathbf{X} \to \text{ob} \mathbf{A} \) and an \( \text{ob} \mathbf{X} \)-indexed family of isomorphisms \( \eta_X : X \to GFX \ (X \in \text{ob} \mathbf{X}) \).

**Proof.** (i) \( \implies \) (iii): The hypothesis yields faithfulness of \( F \) and \( G \). For fullness of \( G \), consider \( f : GA \to GB \) in \( \mathbf{X} \) (with \( A,B \in \text{ob} \mathbf{A} \)). In order to show \( Gg = f \) for \( g := \varepsilon_B \cdot Ff \cdot \varepsilon_A^{-1} \) (with \( \varepsilon : FG \to 1_\mathbf{A} \) some isomorphism), it suffices to verify \( FGg = Ff \). But this follows from

\[
\varepsilon_B \cdot Fg \cdot \varepsilon_A^{-1} = g \cdot \varepsilon_A \cdot \varepsilon_A^{-1} = \varepsilon_B \cdot Ff \cdot \varepsilon_A^{-1}.
\]

(iii) \( \implies \) (ii): As a fully faithful functor, \( G \) reflects isomorphisms, and the isomorphism \( \eta_X \) is easily seen to be \( G \)-universal arrow for \( X \in \text{ob} \mathbf{X} \). Hence (b) follows from Theorem 2.5.4 and Proposition 2.6.1.

(ii) \( \implies \) (i): By Proposition 2.6.1, one has \( F \dashv G \) with the unit \( \eta \) an isomorphism. Since \( G\varepsilon \cdot \eta G = 1_G \), the natural transformation \( G\varepsilon \) is an isomorphism, and so is \( \varepsilon \) whenever \( G \) reflects isomorphisms.

**2.6.3 Corollary.** If \( G : \mathbf{A} \to \mathbf{X} \) is fully faithful and right adjoint, then \( \mathbf{A} \) is equivalent to a replete reflective subcategory \( \overline{\mathbf{A}} \) of \( \mathbf{X} \).
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Proof. Take $\overline{A}$ to be the full subcategory of objects $X$ in $X$ for which $\eta_X$ is an isomorphism, with $\eta$ the unit of an adjunction $F \dashv G$. \hfill \Box

2.7 Initial and terminal objects, comma categories. Let $1 = \{ \ast \}$ be the one-object category whose only morphism is $1_\ast$. For every category $C$ there is then a unique functor $! : C \to 1$. A right adjoint to this functor is given by an object $T$ in $C$ such that

for all objects $X$ in $C$, there is precisely one morphism $X \to T$ in $C$.

Any such object $T$ is called terminal in $C$. As the value of a right adjoint, it is unique up to isomorphism (a fact that is easily established directly): if $S, T$ are both terminal in $C$, then $S \cong T$; moreover, for any isomorphic objects $S, T$ in $C$, one is terminal if the other is. A terminal object in a category is often denoted by 1.

An object $I$ is initial in $C$ if it is terminal in $C^{\text{op}}$, so that

for all objects $X$ in $C$, there is precisely one morphism $I \to X$ in $C$.

Equivalently, $I$ is initial if the functor $1 \to C$ with $\ast \mapsto I$ is left adjoint to the functor $!$. An initial object in a category is often denoted by 0.

The empty set is initial in $\text{Set}$, and every one-element set is terminal; the same holds for $\text{Ord}$, $\text{Top}$, and $\text{Cat}$, for example. In $\text{Mon}$, the trivial (one-element) monoids are both initial and terminal; such objects are called zero-objects. An individual monoid $M$, considered as a one-object category, can have an initial or terminal object only if it is trivial. In an ordered set $X$ considered as a category, a terminal object is given by a top element, and an initial object by a bottom element.

The process of defining initial and terminal objects via adjunctions may be reversed, as follows. For a functor $G : A \to X$ and an object $X$ in $X$, define the comma category $[X \downarrow G]$ to have as objects pairs $(A, f)$ with $A \in \text{ob} A$, and $f : X \to GA$ in $X$; a morphism $h : (A, f) \to (B, g)$ in $(X \downarrow G)$ is given by a morphism $h : A \to B$ in $A$ that makes the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & GA \\
\downarrow & & \searrow^{Gh} \\
& X & \xrightarrow{g} GB
\end{array}
\]

commute. Then

a $G$-universal arrow for $X$ is simply an initial object in $(X \downarrow G)$,

and Theorem 2.5.4 states that $G$ is right adjoint if and only if there are chosen initial objects in every comma category $(X \downarrow G)$ (with $X$ running through $\text{ob} X$).

For a functor $F : X \to A$, and $A \in \text{ob} A$, one also has the comma category

$[F \downarrow A] := (A \downarrow F^{\text{op}})^{\text{op}}$
whose morphisms $h : (X, f) \to (Y, g)$ are depicted by

$$
\begin{array}{c}
FX \xrightarrow{Fh} FY \\
\downarrow f \quad \quad \downarrow g \\
A
\end{array}
$$

Its terminal objects are precisely the $F$-couniversal arrows for $A$. In the special case where $F = 1_X$, one calls

$$X/A := (1_X \downarrow A)$$

the \textit{slice of $X$ over $A$}. Note that $X/A$ always has a terminal object given by $1_A$:

$$
\begin{array}{c}
X \xrightarrow{f} \cdots \cdots \xrightarrow{f} A \\
\downarrow f \quad \quad \downarrow 1_A \\
A
\end{array}
$$

We also denote the \textit{slice $(A \downarrow 1_X)$ of $X$ under $A$} by $A/X$.

\section*{2.8 Limits.} A functor $D : J \to C$ is also called a \textit{diagram (of shape $J$)} in $C$. If $J$ is \textit{discrete}, so that for all objects $i, j$ in $J$,

$$J(i, j) := \begin{cases} 
\{1_i\} & \text{if } i = j \\
\emptyset & \text{otherwise,}
\end{cases}$$

then $D$ is simply a family $(A_i)_{i \in \text{ob} J}$ of objects in $C$. If $J$ has precisely two objects, and two non-identical arrows, as in

$$a \xrightarrow{x} b ,$$

then a diagram of shape $J$ is given by a pair of (not necessarily distinct) parallel arrows in $C$.

For every object $A$ in $C$ and any $J$, one has the \textit{constant diagram} of shape $J$ in $C$ given by

$$\Delta A : J \to C , \quad i \mapsto A , \quad (\delta : i \to j) \mapsto 1_A .$$

For a diagram $D : J \to C$, any natural transformation $\alpha : \Delta A \to D$ is called a \textit{cone} over $D$ in $C$ with vertex $A$. Naturality of $\alpha$ amounts to the property that for all $\delta : i \to j$ in $J$, the diagram

$$
\begin{array}{c}
A \xrightarrow{\alpha_i} Di \\
\downarrow \alpha_j \quad \quad \downarrow D\delta \\
A \xrightarrow{\alpha_j} Dj
\end{array}
$$
commutes in $C$. Hence, a cone over $D$ with vertex $A$ is simply an object $(A, \alpha)$ in the comma category $(\Delta \downarrow D)$, where $D$ is considered as an object of $C^J$, and where $\Delta$ has been made into a functor

$$\Delta : C \to C^J, \quad (f : A \to B) \mapsto (\Delta f : \Delta A \to \Delta B),$$

with the natural transformation $\Delta f$ defined by $(\Delta f)_i = f$ for all $i \in \text{ob} \, J$.

A limit of $D : J \to C$ is a terminal cone over $D$ in $C$, or equivalently, a $\Delta$-couniversal arrow for $D \in \text{ob} \, C^J$. Hence, a limit of $D$ is a cone $\lambda : \Delta L \to D$ with $L \in \text{ob} \, C$ such that

for every cone $\alpha : \Delta A \to D$ with $A \in \text{ob} \, C$, there is a unique morphism $f : A \to L$ in $C$ with $\lambda \cdot \Delta f = \alpha$, that is: $\lambda_i \cdot f = \alpha_i$ for all $i \in \text{ob} \, J$.

Limits are uniquely determined by $D$, up to isomorphism, so that when $(A, \alpha)$ is also a limit of $D$, the unique morphism $f : A \to L$ is an isomorphism in $C$. We write $L \cong \lim D$ for any object that serves as the vertex of a limit for $D$ in $C$.

Important instances of limits include: products, equalizers, and pullbacks (as well as certain special pullbacks), which we now proceed to describe.

**Products.** A product of a family of objects $(A_i)_{i \in I}$ in $C$ is a limit of the diagram $J \to C$, $i \mapsto A_i$, with $J$ discrete and $\text{ob} \, J = I$. Hence, a product is an object $P$ in $C$ together with a family of morphisms $p_i : P \to A_i$ ($i \in I$) such that

for every family $f_i : B \to A_i$ ($i \in I$) of morphisms in $C$, there is a unique morphism $f : B \to P$ with $p_i \cdot f = f_i$ ($i \in I$); this morphism is usually written

$$f = \langle f_i \rangle_{i \in I}.$$ 

Any object $P$ that serves as a product for $(A_i)_{i \in I}$ is denoted by $\prod_{i \in I} A_i$, and comes with projections $p_i$ ($i \in I$). Note that $I = \emptyset$ is permitted, in which case $P$ is simply a terminal object of $C$. The case $I = \{\star\}$ is trivial: a limit of the one-object family $A$ can be given by $A$, with projection $1_A$. The product of a pair $(A_1, A_2)$ of objects is denoted by $A_1 \times A_2$, with projections $p_1, p_2$ whose universal property may be depicted by:

$$A_1 \leftarrow p_1 A_1 \times A_2 \overrightarrow{p_2} A_2.$$
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More casually, we may write:

\[
\begin{array}{c}
B \rightarrow A_1, B \rightarrow A_2 \\
B \rightarrow A_1 \times A_2
\end{array}
\]

In \textbf{Set}, small-indexed products can be constructed as cartesian products. In many \emph{concrete categories} (categories admitting a faithful functor to \textbf{Set}) like \textbf{Ord} and \textbf{Mon}, the cartesian product is provided with an appropriate structure that makes the projections morphisms of the category; see Propositions 5.8.3 and 3.3.1 respectively for specifications of this statement in two general contexts. In an ordered set \(X\) regarded as a category, a product of a family \((a_i)_{i \in I}\) of elements in \(X\) is simply an infimum of \(\{a_i \mid i \in I\}\) in \(X\).

\textbf{Equalizers.} An equalizer of a pair \((f, g : A \rightarrow B)\) of “parallel” morphisms in \(C\) is a limit \((E, \lambda)\) of the corresponding diagram \(D\) of shape \(\{ a \xrightarrow{y} b \}\) in \(C\), with \(Dx = f, Dy = g\). Since \(\lambda_b = Dx \cdot \lambda_a = Dy \cdot \lambda_a\), an equalizer of \((f, g)\) is given by a morphism \(u : E \rightarrow A\) with \(f \cdot u = g \cdot u\) (namely, \(u = \lambda_a\)) such that

for every morphism \(h : C \rightarrow A\) with \(f \cdot h = g \cdot h\), there is a unique morphism \(t : C \rightarrow E\) with \(u \cdot t = h\).

\[
\begin{array}{ccc}
E & \xrightarrow{u} & A \\
\downarrow{t} & & \downarrow{f} \quad \downarrow{g} \\
C & \rightarrow & B
\end{array}
\]

We sometimes write \(u \cong \text{equ}(f, g)\) in this case. If \(f = g\), then \(u\) can be taken to be \(1_A\); conversely, if \(\text{equ}(f, g)\) exists and is an isomorphism, then \(f = g\).

In \textbf{Set}, an equalizer \(\text{equ}(f, g)\) can be realized as the inclusion map

\[
E = \{x \in A \mid f(x) = g(x)\} \hookrightarrow A,
\]

and in concrete categories like \textbf{Ord} and \textbf{Mon}, one can endow \(E\) with an obvious structure inherited from \(A\). An ordered set \(X\) (regarded as a category) trivially has equalizers, since any two parallel arrows are equal. In a group regarded as a one-object category, an equalizer of two elements \(x, y\) can exist only if \(x = y\).

\textbf{Pullbacks.} A pair of morphisms \(f : A \rightarrow C, g : B \rightarrow C\) with common codomain can be viewed as a diagram \(D\) of shape \(J\), where \(J\) is a three-object category with two non-identical arrows, as in

\[
a \xrightarrow{x} c \leftarrow y \rightarrow b.
\]

A pullback of \((f, g)\) in \(C\) is a limit \((P, \lambda)\) of \(D\) in \(C\). Since \(\lambda_c = Dx \cdot \lambda_a = Dy \cdot \lambda_b\), a pullback of \((f, g)\) is given by a pair \((p_1 : P \rightarrow A, p_2 : P \rightarrow B)\) with \(f \cdot p_1 = g \cdot p_2\) (namely, \(p_1 = \lambda_a, p_2 = \lambda_b\)), such that

for all morphisms \(h_1 : X \rightarrow A, h_2 : X \rightarrow B\) with \(f \cdot h_1 = g \cdot h_2\), there is a unique \(t : X \rightarrow P\) with \(p_1 \cdot t = h_1, p_2 \cdot t = h_2\). The morphism \(t\) is often denoted by
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\langle h_1, h_2 \rangle$, and the square in the following diagram is called a pullback diagram:

\[ \begin{array}{ccc}
X & \xrightarrow{h_2} & B \\
\downarrow{t} & & \downarrow{g} \\
P & \xrightarrow{p_2} & B \\
\downarrow{p_1} & & \downarrow{g} \\
A & \xrightarrow{f} & C
\end{array} \]

One often writes (somewhat casually) $P \cong A \times_C B$ and calls $P$ a fibred product of $A$ and $B$ over $C$, thus unduly neglecting the morphisms $f$ and $g$. But pullbacks are in fact products, since a pullback of $(f, g)$ in $C$ is nothing but the product of the pair of objects $(A, f), (B, g)$ in the comma category $C/C$. In case $C = 1$ is terminal in $C$, one has $A \times_C B \cong A \times B$, i.e., the pullback is a product in $C$. Multiple pullbacks are simply products in $C/C$ of families of objects in $C$ (that is, of families of morphisms in $C$ with common codomain $C$).

In $\text{Set}$, a pullback of $(f : A \to C, g : B \to C)$ can be constructed as

\[ P = \{(x, y) \in A \times B \mid f(x) = g(y)\} , \]

with $p_1, p_2$ the restrictions of the product projections $\pi_1, \pi_2$. Hence, $P$ is obtained as the equalizer of $f \cdot \pi_1, g \cdot \pi_2 : A \times B \to C$. This construction works in general categories, see Corollary 2.10.2 below.

**Special pullbacks.** A kernel pair of a morphism $f : A \to B$ in $C$ is a pullback of $(f, f)$; hence, it is a pair $(p_1, p_2 : K \to A)$ of morphisms in $C$ with $f \cdot p_1 = f \cdot p_2$ satisfying the universal property of a pullback diagram. A morphism $f : A \to B$ in $C$ is a monomorphism (or is monic) if and only if $f \cdot x = f \cdot y$ implies $x = y$ whenever $x : X \to A$; equivalently, if $(1_A, 1_A)$ serves as a kernel pair, or more generally, if $f$ has a kernel pair $(p_1, p_2)$ with $p_1 = p_2$. For a monomorphism $f$, any commutative diagram

\[ \begin{array}{ccc}
C & \xrightarrow{h} & A \\
\downarrow{1_C} & & \downarrow{f} \\
C & \xrightarrow{g} & B
\end{array} \]

is a pullback diagram. Composites of monomorphisms are monic, and monomorphisms are stable under pullback in the following sense: for every pullback diagram

\[ \begin{array}{ccc}
P & \xrightarrow{g'} & A \\
\downarrow{f'} & & \downarrow{f} \\
C & \xrightarrow{g} & B
\end{array} \]
if \( f \) is monic, so is \( f' \). One calls \( f' \) a pullback of \( f \) along \( g \), alluding to the fact that in \( \text{Set} \), when \( f : A \hookrightarrow B \) is a subset inclusion, \( f' \) can be taken to be \( g^{-1}(A) \hookrightarrow C \).

A frequently used property of pullback diagrams is:

**2.8.1 Proposition.** If in a commutative diagram in \( \mathcal{C} \) of the form

\[
\begin{array}{ccc}
\cdot & \rightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \rightarrow & \cdot
\end{array}
\]

the outer rectangle and the right square are pullback diagrams, then so is the left square.

**Proof.** The statement follows by routine diagram chasing. \( \square \)

**2.9 Colimits.** A pair \((K, \gamma)\) is a colimit of the diagram \( D : J \rightarrow \mathcal{C} \) in a category \( \mathcal{C} \) if \((K, \gamma^{\text{op}})\) is a limit of \( D^{\text{op}} : J^{\text{op}} \rightarrow \mathcal{C}^{\text{op}} \) in \( \mathcal{C}^{\text{op}} \). Hence, \( \gamma : D \rightarrow \Delta K \) is a cocone over \( D \) with vertex \( K \in \text{ob} \mathcal{C} \) such that for every cocone \( \alpha : D \rightarrow \Delta A \) with \( A \in \text{ob} \mathcal{C} \), there is a unique morphism \( f : K \rightarrow A \) in \( \mathcal{C} \) with \( \Delta f \cdot \gamma = \alpha \), that is: \( f \cdot \gamma_i = \alpha_i \) for all \( i \in \text{ob} J \).

One writes \( K \cong \text{colim} \ D \) in this case. Dually to products, equalizers, pullbacks, kernel pairs, and monomorphisms (or morphisms that are monic), one obtains coproducts, coequalizers, pushouts, cokernel pairs, and epimorphisms (or morphisms that are epic), respectively. A coproduct \((A_i)_{i \in I} \) in \( \mathcal{C} \) is denoted by \( \bigsqcup_{i \in I} A_i \), together with injections \( k_i \) \((i \in I) \). For \( I = \emptyset \), the empty coproduct \( \bigsqcup_{i \in I} A_i \cong 0 \) is an initial object of \( \mathcal{C} \). In the binary case \( I = \{1, 2\} \), the coproduct is normally denoted by \( A_1 + A_2 \), and its universal property visualized by:

\[
\begin{array}{ccc}
A_1 & \rightarrow & A_1 + A_2 \\
k_1 & \downarrow & k_2 \\
A_1 & \rightarrow & A_1 + A_2
\end{array}
\]

As for products, we often write \( f = \langle f_1, f_2 \rangle \) for the arrow given by \( f_1, f_2 \). The universal properties of coequalizers and pushouts may be displayed as follows:

\[
\begin{array}{ccc}
A & \rightarrow & B \\
g & \downarrow & \downarrow q \\
\downarrow h & & \downarrow t \\
C & \rightarrow & Q \\
\end{array}
\] and \[
\begin{array}{ccc}
C & \rightarrow & B \\
g & \downarrow & \downarrow q_2 \\
\downarrow f & & \downarrow \gamma_t \\
A & \rightarrow & Q \\
\end{array}
\]
In \( \text{Set} \), a coproduct of \( (A_i)_{i \in I} \) may be constructed as its disjoint union \( \bigcup_{i \in I} A_i \times \{i\} \). A coequalizer of \( f, g : A \rightarrow B \) may be constructed as the projection onto \( Q = B/\sim \), where \( \sim \) is the least equivalence relation on \( B \) with \( f(x) \sim g(x) \) for all \( x \in A \). For the pushout of \( (f : C \rightarrow A, g : C \rightarrow B) \), form the coproduct \( A + B \) with injections \( k_1, k_2 \) and then the coequalizer of \( k_1 \cdot f, k_2 \cdot g \). In \( \text{Ord} \), and \( \text{Top} \) for example, one may “lift” these constructions from \( \text{Set} \) by putting appropriate structures on them (see the dual statement of Proposition 5.8.3 further on). In an ordered set \( X \) considered as a category, coproducts are given by suprema, and coequalizers are, like equalizers, trivial.

2.10 Construction of limits and colimits. For categories \( C \) and \( J \), one says that \( C \) is \( J \)-complete if the functor \( \Delta : C \rightarrow C^J \) has a right adjoint, that is, if all diagrams of shape \( J \) have a chosen limit in \( C \); the right adjoint of \( \Delta \) is usually denoted by \( \text{lim} \). A category \( C \) is finitely complete if \( C \) is \( J \)-complete for every finite category \( J \) (categories for which \( \text{ob} \ J \) and all hom-sets of \( J \) are finite), and \( C \) is small-complete if \( C \) is \( J \)-complete for every small category \( J \). A category \( C \) has products if \( C \) is \( J \)-complete for every small discrete category \( J \), and \( C \) has finite products if this holds for every finite discrete category \( J \). Similarly, \( \{ \cdot \rightarrow \cdot \} \)-completeness and \( \{ \cdot \rightarrow \cdot \leftarrow \cdot \} \)-completeness are phrased as “has equalizers”, “has pullbacks”, and so on. The dual notions are those of \( J \)-cocompleteness (with \( \text{colim} \) denoting the left adjoint of \( \Delta \)), finite cocompleteness, small-cocompleteness, has coproducts, has finite coproducts, has coequalizers, has pushouts, etc.

General limits can be constructed in terms of products and equalizers, as follows. Given \( D : J \rightarrow C \), one forms the product \( P = \prod_{i \in \text{ob} \ J} D_i \) with projections \( p_i \). Since in general the \( p_i \) do not form a cone (so that generally \( D\delta \cdot p_j \neq p_k \) for \( \delta : j \rightarrow k \) in \( J \)), its existence granted one also forms the product

\[
Q = \prod_{\delta \in \text{mor} \ J} D(\text{cod} \ \delta) \quad \text{with projections } q_\delta,
\]

where \( \text{mor} \ J = \bigcup_{i, k \in \text{ob} \ J} J(j, k) \) (with the hom-sets of \( J \) assumed to be disjoint, without loss of generality). With

\[
f = \langle p_{\text{cod} \ \delta} \rangle_{\delta \in \text{mor} \ J}, \quad g = \langle D\delta \cdot p_{\text{dom} \ \delta} \rangle_{\delta \in \text{mor} \ J},
\]

one obtains for every \( \delta : j \rightarrow k \) in \( J \) the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{f} & Q \\
p_j \downarrow & \swarrow g & \downarrow q_\delta \\
Dj & \xrightarrow{\delta} & Dk
\end{array}
\]

with \( q_\delta \cdot f = p_k \), \( q_\delta \cdot g = D\delta \cdot p_j \) (but generally \( D\delta \cdot p_j \neq p_k \)). Now, for all morphisms \( h : A \rightarrow P \) in \( C \),

\[
f \cdot h = g \cdot h \iff \forall (\delta : j \rightarrow k) \text{ in } J \ (D\delta \cdot (p_j \cdot h) = p_k \cdot h).
\]
Hence, when \( h \) satisfies \( f \cdot h = g \cdot h \), then \( \alpha_i = p_i \cdot h \) \((i \in \text{ob } J)\) defines a cone \( \alpha : \Delta A \to D \); and conversely, for every cone \( \alpha : \Delta A \to D \), the morphism \( h = \langle \alpha_i \rangle_{i \in \text{ob } J} \) satisfies \( f \cdot h = g \cdot h \). Consequently, an equalizer \( h \) of \((f, g)\) gives a limit cone over \( D \), and a limit cone over \( D \) gives an equalizer of \((f, g)\). This sketches the proof of the following result:

**2.10.1 Theorem.** A category \( C \) is \( J \)-complete if it has equalizers as well as \( \text{ob } J \)- and \( \text{mor } J \)-indexed products. Hence, \( C \) is small-complete if and only if \( C \) has products and equalizers.

**2.10.2 Corollary.** The following conditions are equivalent for a category \( C \):

(i) \( C \) is finitely complete;

(ii) \( C \) has finite products and equalizers;

(iii) \( C \) has pullbacks and a terminal object.

*Proof.* The equivalence (i) \( \iff \) (ii) follows from the Theorem. (ii) \( \implies \) (iii): A terminal object exists as a product of the empty family of objects in \( C \). To obtain the pullback \((P, p_1, p_2)\) of \((f : A \to C, g : B \to C)\), let \( u \cong \text{equ}(f \cdot \pi_1, g \cdot \pi_2) \), with \( \pi_1, \pi_2 \) the projections of \( A \times B \), and let \( p_1 = \pi_1 \cdot u \), \( p_2 = \pi_2 \cdot u \):

\[
\begin{array}{ccc}
P & \xrightarrow{p_2} & B \\
\downarrow{u} & & \downarrow{g} \\
A \times B & \xrightarrow{\pi_2} & C .
\end{array}
\]

(iii) \( \implies \) (ii): Pullbacks over 1 give binary products; \( n \)-ary products can be constructed from binary products via \( A_1 \times A_2 \times A_3 \cong (A_1 \times A_2) \times A_3 \), etc. An equalizer \( u \) of \( f, g : A \to B \) can be obtained as a pullback of \( \delta = \langle 1_B, 1_B \rangle \) along \( \langle f, g \rangle \):

\[
\begin{array}{ccc}
E & \xrightarrow{\delta} & B \\
\downarrow{u} & & \downarrow{} \\
A & \xrightarrow{\langle f, g \rangle} & B \times B .
\end{array}
\]

**2.10.3 Example.** According to Theorem 2.10.1 a limit \((L, \lambda)\) of \( D : J \to \text{Set} \) with \( J \) small is obtained as

\[
L = \{(x_i)_{i \in \text{ob } J} \in \prod_{i \in \text{ob } J} D_i \mid \forall \delta : j \to k \in J \text{ (} D\delta(x_j) = x_k \text{)} \},
\]

with \( \lambda_i : L \to D_i \) the restriction of the product projection.
For the colimit \((K, \gamma)\) of \(D : J \to \mathbf{Set}\) with \(J\) small, one considers dually the least equivalence relation \(\sim\) on the disjoint union \(\coprod_{i \in \text{ob} J} Di = \bigcup_{i \in \text{ob} J} Di \times \{i\}\) such that for all \(\delta : j \to k\) in \(J\) and \(x \in Dj\),
\[(x, j) \sim (D\delta(x), k) .\]

Then \(K = \coprod_{i \in \text{ob} J} Di/\sim\), and \(\gamma_i(x)\) is the equivalence class of \((x, i)\).

There is an important special case when this equivalence relation has an easy description, namely when \(J\) is an up-directed ordered set \(I\), considered as a category \(J\), with \(\text{ob} J = I\). Then the diagram \(D\) is simply a family of sets and maps \(f_{j,k} : X_j \to X_k\) (for \(j \leq k\) in \(I\)), with
\[f_{k,l} \cdot f_{j,k} = f_{j,l}\text{ if } j \leq k \leq l , \quad f_{j,j} = 1_{X_j} .\]

Now \(\sim\) is given by
\[(x, j) \sim (y, k) \iff \exists l \geq j, k (f_{j,l}(x) = f_{k,l}(y)) .\]

Classically, in any category \(C\), colimits of diagrams in \(C\) of shape \(J\), where \(J\) is an up-directed ordered set, are called direct limits, and limits of shape \(J^{\text{op}}\) are called inverse limits in \(C\); we prefer to call them directed colimits and directed limits, respectively.

### 2.11 Preservation and reflection of limits and colimits.

Let \(F : C \to D\) be a functor and \((L, \lambda)\) a limit of \(D : J \to C\) in \(C\). One says that \(F\) preserves this limit if \((FL, F\lambda)\) is a limit of \(FD\) in \(D\). The functor \(F\) preserves \(J\)-limits (or is \(J\)-continuous) if \(F\) preserves all existing limits in \(C\) of diagrams of shape \(J\); if \(C\) is \(J\)-complete, this property may be casually communicated by
\[F \lim \cong \lim F .\]

The functor \(F\) reflects \(J\)-limits if for every cone \(\lambda : \Delta L \to D\) (with \(D : J \to C\)) such that \((FL, F\lambda)\) is a limit of \(FD\) in \(D\), then \((L, \lambda)\) is a limit of \(D\) in \(C\). This language is applied in various special contexts; for example, \(F\) preserves monomorphisms if \(Fm\) is monic in \(D\) whenever \(m\) is monic in \(C\), and \(F\) reflects monomorphisms if \(m\) is monic whenever \(Fm\) is monic. This terminology is used similarly for colimits. For example, \(J\)-cocontinuity is preservation of \(J\)-colimits.

### 2.11.1 Proposition.

1. A right adjoint functor preserves all (existing) limits, and a left adjoint all colimits.

2. A fully faithful functor reflects all (existing) limits and colimits.

3. A replete reflective subcategory of a \(J\)-complete category is \(J\)-complete. Also, a replete reflective subcategory of a \(J\)-cocomplete category is \(J\)-cocomplete.
Proof. For (1), if \( F \dashv G : \mathcal{A} \to \mathcal{X} \), and \( L \cong \text{lim} \, D \), the following natural correspondences sketch the proof of \( GL \cong \text{lim} \, GD \):

\[
\begin{align*}
\Delta X & \to GD \\
\Delta FX & \to D \\
FX & \to L \\
X & \to GL .
\end{align*}
\]

For (2), if \( G : \mathcal{A} \to \mathcal{X} \) is full and faithful and \( GL \cong \text{lim} \, GD \), one may sketch the proof for \( L \cong \text{lim} \, D \) as follows:

\[
\begin{align*}
\Delta A & \to D \\
\Delta GA & \to GD \\
GA & \to GL \\
A & \to L .
\end{align*}
\]

For the completeness part of (3), we show the stronger statement that \( \mathcal{A} \) is closed under limits in \( \mathcal{X} \), that is: whenever \( (L, \lambda) \) is a limit of \( JD \) in \( \mathcal{X} \) (with \( J : \mathcal{A} \to \mathcal{X} \)), then \( L \in \text{ob} \, \mathcal{A} \). For this, it suffices to show that \( \rho_L \) is an isomorphism, with \( \rho_L : L \to RL = JRL \) the \( \mathcal{A} \)-reflection of \( L \). But the inverse of \( \rho_L \) may be found as follows:

\[
\begin{align*}
\Delta L & \xrightarrow{\lambda} JD \\
\Delta RL & \to D \\
\Delta JRL & \to JD \\
JRL & \xrightarrow{\rho_L^{-1}} L .
\end{align*}
\]

For the cocompleteness part of (3), if \( (K, \gamma) \) is a colimit of \( JD \) in \( \mathcal{X} \), then \( (RK, \Delta \rho_K \cdot \gamma) \) is a colimit of \( D \) in \( \mathcal{A} \):

\[
\begin{align*}
D & \to \Delta A \\
JD & \to \Delta JA \\
K & \to JA \\
RK & \to A .
\end{align*}
\]

Hence, a left adjoint \( \text{colim}_A \) to \( \Delta_A : \mathcal{A} \to \mathcal{A}^I \) may be written as the composite functor

\[
\mathcal{A}^I \xrightarrow{J(-)} \mathcal{X}^I \xrightarrow{\text{colim}_X} \mathcal{X} \xrightarrow{R} \mathcal{A} .
\]

In addition to (3), note that while the inclusion functor \( J \) preserves all limits, it generally fails to preserve colimits. Nonetheless, existence in \( \mathcal{A} \) of colimits of a specified shape is guaranteed by their existence in \( \mathcal{X} \).

Limits and colimits in a functor category \( \mathcal{D}^\mathcal{C} \) can be formed “pointwise” in \( \mathcal{D} \). More precisely, for every \( A \in \text{ob} \, \mathcal{C} \), one has the evaluation functor

\[
\text{Ev}_A : \mathcal{D}^\mathcal{C} \to \mathcal{D} , \quad F \mapsto FA , \quad \alpha \mapsto \alpha_A .
\]
A limit \((L, \lambda)\) of \(D : J \to D^C\) can be formed by letting \((LA, \lambda_A)\) be a limit of \(\text{Ev}_A D\) in \(D\), if the latter exist for all \(A \in \text{ob } C\); the same construction holds for colimits. This way one proves:

**2.11.2 Proposition.** The functor category \(D^C\) is \(J\)-complete if \(D\) is \(J\)-complete, and then the evaluation functors are \(J\)-continuous. Dually, \(D^C\) is \(J\)-cocomplete if \(D\) is \(J\)-cocomplete, in which case the evaluation functors are \(J\)-cocontinuous.

**2.11.3 Corollary.** Every hom-functor \(C(A, -) : C \to \text{Set}\) of a locally small category \(C\) preserves all existing limits, and so does the Yoneda embedding \(y : C \to \text{Set}^{C^{\text{op}}}\).

*Proof.* When \((L, \lambda)\) is a limit of \(D\) in \(C\), the isomorphism \(C(A, L) \cong \lim C(A, D(-))\) follows from:

\[
\begin{align*}
\Delta X \xrightarrow{\alpha} C(A, D(-)) \\
\Delta A \xrightarrow{\alpha_A} D(x \in X) \\
A \xrightarrow{f_X} L(x \in X) \quad (L \cong \lim D) \\
X \xrightarrow{f_X} C(A, L).
\end{align*}
\]

To show that \(yL \cong \lim yD\) in \(\text{Set}^{C^{\text{op}}}\), it suffices to show that \(\text{Ev}_A yL \cong \lim \text{Ev}_A yD\) in \(\text{Set}\) for all \(A \in \text{ob } C\). But \(\text{Ev}_A y = C(A, -)\), which preserves the limit, as we just saw. \(\square\)

Note that since \(\text{Set}\) is small-complete and small-cocomplete, every functor category of \(\text{Set}\) is too. Hence, every locally small category is fully embedded into a small-complete and small-cocomplete category by its Yoneda embedding.

**2.12 Adjoint Functor Theorem.** Preservation of limits by a functor \(G : A \to X\) is a necessary condition for its right adjointness (see Proposition 2.11.1). For ordered sets, completeness of \(A\) also makes this condition sufficient (see Corollary 1.8.3). For general categories however, small-completeness of \(A\) and preservation of limits by \(G\) do not generally suffice to make \(G\) right adjoint (see Exercise 2.0). Here is a two-step procedure on how to construct a \(G\)-universal arrow for an object \(X\) in \(X\).

**Step 1.** A \(G\)-solution set for \(X\) is a set \(L\) of objects in \(A\) with the property that for all \(f : X \to GA\) \((A \in \text{ob } A)\), there exist \(L \in L\), \(e : X \to GL\) in \(X\), and \(h : L \to A\) in \(A\) with \(Gh \cdot e = f\). If \(X\) is locally small and \(A\) has products which are preserved by \(G\), then there exists a weakly \(G\)-universal arrow for \(X\), consisting of an object \(V\) in \(A\) and a morphism \(v : X \to GV\) in \(X\) such that for all \(f : X \to GA\) \((A \in \text{ob } A)\), there is a (not necessarily unique) morphism \(g : V \to A\) with \(Gg \cdot v = f\): indeed, simply take

\[V = \prod_{L \in L} L^{X(X,GL)}, \quad v = \langle e \rangle_{e \in X(X,GL), L \in L},\]

with \(L^{X(X,GL)} = \prod_{X(X,GL)} L\), and with existence of \(v\) guaranteed by the product preservation by \(G\).
Step 2. If \(A\) is locally small and has generalized equalizers (that is, limits for non-empty sets—not just pairs—of parallel morphisms) that are preserved by \(G\), then existence of a weakly \(G\)-universal arrow \((V, v)\) for \(X\) forces existence of a \(G\)-universal arrow \((U, u)\) for \(X\), as follows: let \(U\) be the limit of the diagram

\[
\{ t \in A(V, V) \mid Gt \cdot v = v \} \longrightarrow A,
\]

with limit projection \(k\); hence, \(k : U \to V\) is universal with respect to the property \(t \cdot k = k\) for all \(t\) with \(Gt \cdot v = v\). Limit preservation by \(G\) gives the morphism \(u : X \to GU\) with \(Gk \cdot u = v\). Weak universality of \((V, v)\) gives weak universality of \((U, u)\). Furthermore, if \(a, b : U \to A\) in \(A\) satisfy \(Ga \cdot u = Gb \cdot u\), consider the equalizer \(c : C \to U\) of \(a, b\), and obtain \(w : X \to GC\) with \(Gc \cdot w = u\) from the limit preservation by \(G\). Weak universality of \((V, v)\) gives \(d : V \to C\) with \(Gd \cdot v = w\):

\[
\begin{array}{ccc}
X & \xrightarrow{v} & GV \\
\downarrow w & & \downarrow \downarrow \downarrow Gd \\
GC & \xrightarrow{Gc} & GU \\
& \xrightarrow{Gk} & \downarrow \downarrow \downarrow GA \\
\end{array}
\]

Since \(G(k \cdot c \cdot d) \cdot v = G(k \cdot c) \cdot w = Gk \cdot u = v\), one has \(k \cdot c \cdot d \cdot k = k\) by definition of \(k\), and since \(k\) is monic (by the uniqueness part of the limit property), one obtains \(c \cdot d \cdot k = 1_U\). Finally, since \(a \cdot c = b \cdot c\), this implies \(a = b\).

Hence, we have proved the non-trivial part of the following result:

2.12.1 Theorem. A functor \(G : A \to X\) of locally small categories with \(A\) small-complete is right adjoint if and only if

1. for every object \(X\) in \(X\) one has a chosen \(G\)-solution set, and
2. \(G\) preserves small limits.

Proof. For the necessity of the conditions, if \(F \dashv G\), then \(\{FX\}\) is a \(G\)-solution set for \(X \in \text{ob}X\), and limit preservation by \(G\) follows from Proposition 2.11.1.

2.12.2 Example. With \(\text{Grp}\) denoting the category of groups, let us prove the existence of a left adjoint to the forgetful functor \(U : \text{Grp} \to \text{Set}\), using Theorem 2.12.1. For a set \(X\) and a map \(f : X \to G\) into a group \(G\) such that \(f(X)\) generates \(G\), the cardinality of \(G\) cannot exceed the cardinality of \((X + 1)^\mathbb{N}\) (since there is a surjection \((X + 1)^\mathbb{N} \to G\)).

Hence, as a \(U\)-solution set for \(X\), one may choose a representative system of non-isomorphic groups whose cardinality does not exceed the cardinality of \((X + 1)^\mathbb{N}\). Since small limits in \(\text{Grp}\) may be formed by taking the limit in \(\text{Set}\), and providing it with a group structure, small-limit preservation by \(U\) is clear. (Another argument for limit preservation would be \(U \cong \text{Grp}(\mathbb{Z}, -)\).) Hence, Theorem 2.12.1 guarantees existence of the free-group functor \(F\).
The argument just given extends to any algebraic structure defined by a set of operations and equations between them. That is, groups may be traded in this example for rings, $R$-modules, $R$-algebras, etc.

2.13 Kan Extensions. For functors $S : A \to B$ and $T : A \to X$, the right Kan extension of $T$ along $S$ is a $F$-couniversal arrow for $T$, where $F$ is the whiskering functor

$$F = (-)S : X^B \to X^A, \quad \alpha \mapsto \alpha S.$$ 

Hence, the right Kan extension is given by a functor $K : B \to X$ and a natural transformation $\kappa : KS \to T$ such that

for all $\alpha : QS \to T$ in $A$ (with $Q : B \to X$), there is precisely one natural transformation $\sigma : Q \to K$ with $\kappa \cdot \sigma S = \alpha$.

$$\begin{array}{ccc}
K & \xrightarrow{\kappa} & T \\
\downarrow{\sigma} & \rotatebox{90}{$\Rightarrow$} & \downarrow{\alpha} \\
QS & \xrightarrow{\sigma S} & T \\
\end{array}$$

It is common to use the notation

$$K \cong \text{Ran}_S T.$$ 

By Theorem 2.5.4, if the right Kan extension of all functors $T : A \to X$ along a fixed $S$ exist, then $(-)S$ has a right adjoint. However, the Kan extension is useful without the existence of the entire right adjoint.

In the previous definition, one can factor in the categories to obtain the following picture for the right Kan extension $(K, \kappa)$ of $T$ along $S$:

$$\begin{array}{ccc}
B & \xleftarrow{\sigma} & Q \\
\downarrow{\kappa} & \rotatebox{90}{$\Leftarrow$} & \downarrow{\alpha} \\
A & \xrightarrow{T} & X \\
\end{array}$$

The terminology is explained by the case where $A$ is a full subcategory of $B$, as $K$ then does indeed extend $T$ along the full embedding $S$. This extension is made to the right because one has a natural bijection

$$X^A(QS, T) \cong X^B(Q, \text{Ran}_S T).$$

To construct the right Kan extension, consider an object $B$ in $B$. For any $A$ in $A$ admitting a $B$-morphism $f : B \to SA$, one expects to obtain an $X$-morphism $Kf : KB \to TA$. In
fact, for any commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{f} & f' \\
\downarrow & & \downarrow \\
SA & \xrightarrow{Sh} & SA'
\end{array}
\]

in \( B \), one wants a similar commutative diagram in \( X \) where \( B \) is replaced by \( KB \) and \( Sh \) by \( Th \). Taking into consideration the couniversal property of the right Kan extension, one defines

\[
KB \cong \lim D
\]

where \( D : (B \downarrow S) \to X \) is the diagram sending an object \( f : B \to SA \) to \( TA \), and a morphism \( h : (A, f) \to (A', f') \) to \( Th : TA \to TA' \):

\[
\begin{array}{ccc}
B & \xrightarrow{f} & f' \\
\downarrow & & \downarrow \\
SA & \xrightarrow{Sh} & SA' \\
\end{array} \quad \mapsto \quad 
\begin{array}{ccc}
TA & \xrightarrow{Th} & TA'
\end{array}
\]

2.13.1 Theorem. Suppose that functors \( S : A \to B \) and \( T : A \to X \) are given such that the diagram \( D : (B \downarrow S) \to X \) has a limit for all objects \( B \) of \( B \). Then the right Kan extension \((K, \kappa)\) of \( T \) along \( S \) exists. When \( S \) is full and faithful, the natural transformation \( \kappa \) is an isomorphism.

Proof. The previous construction defined on objects of \( B \) extends to yield a functor \( K : B \to X \) via the couniversal property of limits. The component at \( A \in A \) of the natural transformation \( \kappa : KS \to T \) is then obtained as the component \( \lambda_{1SA} : KSA \to TA \) at \( 1_{SA} \) of the limit cone of \( D : (SA \downarrow S) \to X \). When \( S \) is full and faithful, every object \( SA \to SB \) of the comma category \( (SA \downarrow S) \) can be written as \( Sf \) for a unique \( A \)-morphism \( f : A \to B \); hence, \( 1_{SA} \) is an initial object in this category, and the construction of \( KSA \) yields an object isomorphic to \( TA \). If the right Kan extension \((K, \kappa)\) of \( T \) along \( S \) is constructed as in the Theorem, so that

\[
KB \cong \lim((B \downarrow S) \xrightarrow{\to} A \xrightarrow{T} X)
\]

for all \( B \in \text{ob} B \), then the extension is called pointwise.

The left Kan extension

\[
K \cong \text{Lan}_S T : B \to X
\]

of \( T : A \to X \) along \( S : A \to B \) is defined dually: \( K^{op} \) is the right Kan extension of \( T^{op} \) along \( S^{op} \), so that \( K \) comes with a \((-)S\)-universal natural transformation \( \kappa : T \to KS \). This is briefly summarized in

\[
\begin{array}{ccc}
B & \xrightarrow{K} & Q \\
\downarrow & & \downarrow \\
S & \xrightarrow{\sigma} & A \\
\downarrow & \xleftarrow{\kappa} & \downarrow \\
A & \xrightarrow{T} & X
\end{array}
\]
Again, the extension is pointwise if

\[ KB \cong \text{colim}( (S \downarrow B) \longrightarrow A \longrightarrow X ) \]

for all objects \( B \) in \( B \), with the colimit cone constructed dually to 2.13.1.

2.14 Dense functors. A functor \( J : A \rightarrow X \) is dense if \((1_X,1_J)\) is a pointwise left Kan extension of \( J \) along \( J \); that is, if for all \( X \in \text{ob} X \), one has \( \text{colim} D_X \cong X \), with

\[ D_X = ( (J \downarrow X) \longrightarrow A \longrightarrow X ) \]

and the colimit cocone given by

\[ \lambda_{(A,f)} = f : JA \rightarrow X \]

for all \((A,f) \in \text{ob}(J \downarrow X)\). Arbitrary cocones

\[ \alpha : D_X \rightarrow \Delta Y \]

correspond bijectively to natural transformations

\[ \gamma : X(J-,X) \rightarrow X(J-,Y) \]

via

\[ \gamma_A(f) = \alpha_{(A,f)} \]

for \( A \in \text{ob} A \), \( f : JA \rightarrow X \) in \( X \), naturally in \( Y \in \text{ob} X \).

2.14.1 Proposition. For a functor \( J : A \rightarrow X \) of locally small categories, the following statements are equivalent:

(i) \( J \) is dense;

(ii) the maps \( X(X,Y) \rightarrow \hat{A}(X(J-,X),X(J-,Y)) \), \( f \mapsto X(J-,f) \) are bijective for all \( X,Y \in \text{ob} X \);

(iii) the functor \( \hat{J} : \hat{X} \rightarrow \hat{A} \), \( \phi \mapsto \phi J^{\text{op}} \) is fully faithful.

Proof. Condition (i) means that there is a natural bijective correspondence

\[ X \longrightarrow Y \]
\[ D_X \longrightarrow \Delta Y \]

and condition (ii) means that there is a natural bijective correspondence

\[ X \longrightarrow Y \]
\[ X(J-,X) \longrightarrow X(J-,Y) \]

By the introductory observations, (i) and (ii) are therefore equivalent. Since \( X(X,Y) \cong \hat{X}(X(-,X),X(-,Y)) \) by the Yoneda Lemma, (ii) and (iii) are equivalent as well. \( \square \)
2.14.2 Theorem. For every locally small category \( C \), the Yoneda embedding is dense.

Proof. The Yoneda Lemma facilitates a natural bijective correspondence

\[ F \cong \hat{C}(y-, F) \]

for all \( F : C^{\text{op}} \to \text{Set} \), with \( y : C \to \hat{C} = \text{Set}^{C^{\text{op}}} \) denoting the Yoneda embedding. Hence, for \( F, G \in \text{ob}\,\hat{C} \), there is a natural bijective correspondence

\[ F \to G \]
\[ \hat{C}(y-, F) \to \hat{C}(y-, G) \],

showing that \( y \) satisfies condition (ii) of Proposition 2.14.1.

2.14.3 Corollary. Every functor \( F : C^{\text{op}} \to \text{Set} \) is the colimit of the diagram

\[ (1 \downarrow F)^{\text{op}} \cong \text{el}(F)^{\text{op}} \longrightarrow C \xrightarrow{y} \hat{C} \]

(with 1 the terminal object in \( \text{Set} \) and \( \text{el}(F) \) the category of “elements of \( F \)”, see Exercise 2.P).

Proof. By the Yoneda Lemma, \((1 \downarrow F)^{\text{op}} \cong (y \downarrow F)\), and the diagram becomes the same as the one used in the defining property of density of \( y \).

The Yoneda embedding of a locally small category \( C \) therefore embeds \( C \) into the category \( \hat{C} \) with all small colimits, and every object in \( \hat{C} \) is naturally a colimit of objects originating from \( C \).

Exercises

2.A Isomorphisms and functors. If \( f : A \to B, g : B \to C \) are isomorphisms in a category \( C \), then so are \( g \cdot f \) and \( f^{-1} \), and \((g \cdot f)^{-1} = f^{-1} \cdot g^{-1}, (f^{-1})^{-1} = f \). Every functor \( F : C \to D \) preserves isomorphisms but does not generally reflect them: if \( f \) is an isomorphism, then so is \( Ff \), but not conversely in general; as an example, consider the forgetful functor \( \text{Ord} \to \text{Set} \). However, if \( F \) is fully faithful, then \( F \) reflects isomorphisms.

2.B Monomorphisms, epimorphisms and functors. Consider \( f : A \to B, g : B \to C \) in \( C \). If \( f \) and \( g \) are monic, then so is \( g \cdot f \); if \( g \cdot f \) is monic, then so is \( f \). Establish the dual statements for epimorphisms. If \( F : C \to D \) preserves kernel pairs, then \( F \) preserves monomorphisms; in particular, every right adjoint functor and every hom-functor does so. A faithful functor reflects both monomorphisms and epimorphisms. The monomorphisms of \( \text{Set} \) are precisely the injective maps, and the epimorphisms are the surjective maps. Not every epimorphism in \( \text{Mon} \) is surjective, and a morphism that is both monic and epic need not be an isomorphism.
2. CATEGORIES AND ADJUNCTIONS

2.C Split monomorphisms and split epimorphisms. A morphism \( f : A \to B \) in \( C \) is a \textit{split monomorphism} (or a \textit{section}) if \( g \cdot f = 1_A \) for some \( g : B \to A \) in \( C \); the dual notion is that of \textit{split epimorphism} (or \textit{retraction}). Every split monomorphism is monic, and it is an isomorphism precisely when it is epic. Composites of split epimorphisms are split epic, and split epimorphisms are stable under pullback. Dualize. In \( \text{Set} \), every monomorphism \( \emptyset \neq A \to B \) splits, and to say that every epimorphism splits is equivalent to the Axiom of Choice.

2.D Regular epimorphisms. A morphism \( f : A \to B \) in \( C \) is a \textit{regular epimorphism} in \( C \) if for any \( g : A \to C \) with the property that \( f \cdot x = f \cdot y \) always implies \( g \cdot x = g \cdot y \), one has a uniquely determined morphism \( h : B \to C \) with \( h \cdot f = g \). Every split epimorphism is regular, and every regular epimorphism is in fact an epimorphism. If \( C \) has kernel pairs, then \( f \) is a regular epimorphism if and only if \( f \) is the coequalizer of some pair of morphisms \( p, q : K \to A \); in fact, \( p, q \) may always be chosen to be the kernel pair of \( f \). Dualize to \textit{regular monomorphisms}.

2.E Full or faithful right adjoints. If \( F \eta \varepsilon : A \to X \), then \( G \) is faithful if and only if \( \varepsilon_A \) is epic for all \( A \in \operatorname{ob} A \), and \( G \) is full if and only if \( \varepsilon_A \) is a split monomorphism for all \( A \in \operatorname{ob} A \).

2.F Limits and colimits in slices of categories. For any object \( A \) of a category \( C \), the comma category \( C / A \) is finitely complete when \( C \) has fibred products. If \( C \) is \( J \)-cocomplete, so is \( C / A \), and the forgetful functor \( C / A \to C \), \( (X, f) \mapsto X \), is \( J \)-cocontinuous.

2.G Products and coproducts in \( \text{Rel} \). Coproducts in \( \text{Rel} \) are formed as in \( \text{Set} \), and also serve as products. In the category \( \text{AbGrp} \) of abelian groups, finite coproducts and finite products may be formed using the same abelian group.

2.H Cartesian structure of categories with finite products. In a category with finite products, establish natural isomorphisms \( \alpha, \lambda, \rho, \sigma \) satisfying the same coherence conditions as in 1.1.

2.I Products as down-directed limits of finite products. For a family \( (A_i)_{i \in I} \) of objects in a category \( C \) with finite products, put \( A_F = \prod_{i \in F} A_i \) for every finite set \( F \subseteq I \). If \( G \subseteq F \), there is then an obvious morphism \( A_F \to A_G \). With \( F \) the set of finite subsets of \( I \) ordered by inclusion, this defines a diagram \( D : \mathcal{F}^{\text{op}} \to C \), and the limit of \( D \) exists in \( C \) if and only if the product \( \prod_{i \in I} A_i \) exists, and then both limits coincide up to isomorphism.

2.J Hom-functors. The metacategory \( \text{CAT} \) has products and coproducts. For every category \( C \), there is a functor \( C : C^{\text{op}} \times C \to \text{SET} \). Does \( C \) preserve products or coproducts?

2.K Completeness of \( \text{Ord} \) and \( \text{Top} \). The categories \( \text{Ord} \) and \( \text{Top} \) are both small-complete and small-cocomplete. The respective forgetful functors to \( \text{Set} \) preserve all existing limits and colimits.

2.L Representable functors. A functor \( H : C \to \text{Set} \) (with \( C \) locally small) is \textit{representable} if it is isomorphic to a hom-functor \( C(A, -) \). If \( H \) is right adjoint, then it is representable.
(Hint. If \( F \dashv H \), then \( \text{Set}(1, H(-)) \cong C(F1, -) \).) Conversely, a representable functor \( H \cong C(A, -) \) is right adjoint if and only if for every set \( X \) the coproduct \( X \cdot A := \coprod_{x \in X} A_x \) with \( A_x = A \) for all \( x \in X \) exists in \( C \).

2.M **Equivalence of categories, irrelevance of injectivity on objects.** Equivalence of categories is a reflexive, symmetric, and transitive property. A category equivalent to a \( J \)-complete category is \( J \)-complete itself. Every functor \( F : C \to D \) can be factored as

\[
F = (C \xrightarrow{\tilde{F}} \tilde{D} \xrightarrow{U} D)
\]

with \( \tilde{F} \) injective on objects, and \( U \) an equivalence of categories. (Hint. When \( C \neq \emptyset \), let \( \text{ob} \tilde{D} = \text{ob} C \times \text{ob} D \) and \( \tilde{D}((A, B), (A', B')) = D(B, B') \).) Moreover, \( \tilde{F} \) is full or faithful precisely when \( F \) has the respective property. If \( F \) is faithful, then \( C \) is isomorphic to a subcategory \( \tilde{C} \) of \( \tilde{D} \), so that there are inclusion functions both for objects and hom-sets forming a functor \( \tilde{C} \to \tilde{D} \).

2.N **Formal adjointness criterion.** A functor \( G : A \to X \) is right adjoint if and only if for every object \( X \) in \( X \), the (not necessarily small) diagram

\[
D_X : (X \downarrow G) \to A, \quad (A, f) \mapsto A,
\]

has a chosen limit in \( A \) that is preserved by \( G \).

Hint. \( FX \cong \lim D_X \).

2.O **Limit preservation does not suffice for right adjointness.** Let \( S \) be the class of all sets ordered by set inclusion. Then every non-empty subclass has a least element, but there is no top element. Hence, when considered as a category, \( S^{\text{op}} \) is small-complete, and the only functor \( S^{\text{op}} \to 1 \) trivially preserves all limits but has no left adjoint.

2.P **The element construction.** For a functor \( H : C \to \text{Set} \) (with \( C \) locally small), one defines the category \( \text{el}(H) \) of “elements of \( H \)” to have objects \( (A, x) \) with \( A \in \text{ob} C \), \( x \in HA \); a morphism \( f : (A, x) \to (B, y) \) is a \( C \)-morphism \( f : A \to B \) with \( Hf(x) = y \). Then \( H \) is representable (see Exercise 2.L) if and only if \( \text{el}(H) \) has an initial object.

2.Q **Connectedness, colimits in \text{Set} revisited.** A category \( C \) is connected if \( C \neq \emptyset \) and, whenever \( C = A + B \) in \( \text{CAT} \), then \( A = \emptyset \) or \( B = \emptyset \); equivalently, if \( C \neq \emptyset \) and, for any \( A, B \in \text{ob} C \), there is a finite “zigzag” \( A \to \cdots \to \cdot \to B \) of morphisms in \( C \). Every category is a (not necessarily small) coproduct of its connected components (that is, its maximal connected full subcategories). Show also that the colimit of a small diagram \( D : J \to \text{Set} \) can be taken to be the set of connected components of the category \( \text{el}(D) \) (see Exercise 2.P).

2.R **General comma categories.** For functors \( G : A \to X, \ H : C \to X \), consider the category \( (G \downarrow H) \) whose objects \( (A, f, C) \) are given by \( A \in \text{ob} A, \ C \in \text{ob} C \) and \( f : GA \to HC \) in \( X \);
a morphism \((h, j) : (A, f, C) \rightarrow (B, g, D)\) is formed by morphisms \(h : A \rightarrow B\) in \(A\) and \(j : C \rightarrow D\) in \(C\) that make

\[
\begin{array}{ccc}
GA & \xrightarrow{Gh} & GB \\
f & \downarrow & g \\
HC & \xrightarrow{Hj} & HD
\end{array}
\]

commute in \(X\). Composition is such that the projections

\[
P : (G \downarrow H) \rightarrow A, \quad (h, j) \mapsto h,
\]

\[
Q : (G \downarrow H) \rightarrow C, \quad (h, j) \mapsto j
\]

become functors, and

\[
\kappa : GP \rightarrow HQ, \quad \kappa_{(A, f, B)} = f
\]

a natural transformation. Show that, given any functors \(R : K \rightarrow A, S : K \rightarrow C\) and a natural transformation \(\phi : GR \rightarrow HS\), there is a unique functor \(T : K \rightarrow (G \downarrow H)\) with

\[
PT = R, \quad QT = S, \quad \kappa T = \phi.
\]

Observe also that the comma categories discussed in \[2.7\] are all special cases of the general type introduced here, up to isomorphism.

2.S Testing limits via Yoneda. A cone \(\lambda : \Delta A \rightarrow D\) in a locally small category \(C\) is a limit if and only if for all objects \(X\) in \(C\), the cone \(X(X, \lambda_{(-)}) : \Delta X(X, A) \rightarrow X(X, D\)) is a limit in \(Set\).

2.T Quantales freely generated by monoids. Show that the forgetful functor \(\text{Qnt} \rightarrow \text{Mon}\) has a left adjoint (see Exercise [1.M]).
3 Monads

3.1 Monads and adjunctions. A monad \( \mathbb{T} = (T, m, e) \) on a category \( X \) is given by a functor \( T : X \to X \) and two natural transformations, the multiplication and unit of the monad

\[
m : TT \to T, \quad e : 1_X \to T,
\]
satisfying the multiplication law and the right and left unit laws

\[
m \cdot mT = m \cdot Tm, \quad m \cdot eT = 1_T = m \cdot Te;
\]
equivalently, these equalities mean that the diagrams

\[
\begin{array}{ccc}
TTT & \xrightarrow{Tm} & TT \\
mT & \downarrow m & \downarrow m \\
TT & \xrightarrow{m} & T
\end{array}
\quad \quad
\begin{array}{ccc}
T & \xrightarrow{eT} & TT & \xleftarrow{Te} & T \\
1_T & \downarrow m & \downarrow 1_T & \downarrow m & \downarrow 1_T
\end{array}
\]

commute. A morphism of monads \( \alpha : S \to \mathbb{T} \) (where \( S = (S, n, d) \)) is a natural transformation \( \alpha : S \to T \) that preserves the monad structure:

\[
\alpha \cdot n = m \cdot (\alpha \circ \alpha), \quad \alpha \cdot d = e.
\]

Hence, comparing with the definition of a monoid given in 1.1 we observe that the set \( M \) has been replaced by a functor, the cartesian product by horizontal composition, and maps by natural transformations. Since the natural isomorphisms \( \alpha, \lambda, \) and \( \beta \) are identities, the coherence conditions are immediately verified, so that a monad is nothing but an object of the functor category \( X^X \) that carries a monoid structure with respect to the (horizontal) compositional structure.

Any adjunction \( F \xleftarrow{\eta} G : A \to X \) gives rise to the associated monad (or induced monad) \( \mathbb{T} = (GF, G\varepsilon, F, \eta) \) on \( X \). Indeed, by whiskering the identities \( \varepsilon \cdot \varepsilon FG = \varepsilon \cdot FG\varepsilon, G\varepsilon \cdot \eta G = 1_G, \) and \( \varepsilon F \cdot F\eta = 1_F \) appropriately by \( F \) on the right and \( G \) on the left, we obtain the corresponding multiplication and unit laws

\[
G\varepsilon F \cdot G\varepsilon FGF = G\varepsilon F \cdot GFG\varepsilon F, \quad G\varepsilon F \cdot \eta GF = 1_{GF} = G\varepsilon F \cdot GF\eta.
\]

In fact, any monad can be obtained as a monad associated to an adjunction, although different adjunctions may yield the same monad. The “largest” of these adjunctions is described in 3.2 while the “smallest” is treated in 3.6.

3.1.1 Examples.

(1) A closure operation on an ordered set \( X \) is a monad on \( X \) (considered as a category), and the closure operation \( c = g \cdot f \) associated to an adjunction of ordered sets \( f \dashv g : X \to Y \) is also the associated monad.
(2) The list monad (or the free-monoid monad) \( L = (L = GF, G \varepsilon F, \eta) \) on \( \text{Set} \) is the monad associated to the adjunction \( F \dashv G : \text{Mon} \to \text{Set} \) described in 2.5.1. Similarly, from Example 2.12.2, one obtains the free-group monad, the free-abelian-group monad, the free-ring monad, etc.

(3) The covariant powerset functor \( P : \text{Set} \to \text{Set} \), together with the union \( m_X : PPX \to PX \) and singleton \( e_X : X \to PX \) maps
\[
m_X(A) = \bigcup A, \quad e_X(x) = \{x\},
\]
for all \( x \in X, A \in P PX \), form the powerset monad \( P = (P, m, e) \).

(4) The contravariant powerset adjunction \( (P^*)^{op} \dashv P^* : \text{Set}^{op} \to \text{Set} \) (see Example 2.5.1(6)) has the double-powerset monad \( P^2 = (P^2 = P^*(P^*)^{op}, m, e) \) as its associated monad on \( \text{Set} \), with \( P^2f(x) = f[x] \) defined by
\[
B \in f[x] \iff f^{-1}(B) \in x
\]
for \( f : X \to Y \), and all \( B \in PY, \chi \in P^2X \). The components \( m_X : P^2P^2X \to P^2X \) are defined as in the filtered sum construction (see 1.12):
\[
m_X(A) = \sum A \quad (A \subseteq PPX),
\]
with \( (A \in \sum A \iff \{a \subseteq PX \mid A \in a\} \in A) \) for all \( A \subseteq X \), and the components \( e_X : X \to P^2X \) are given by the principal filter
\[
e_X(x) = \hat{x} \quad (x \in X).
\]

(5) The double-powerset functor \( P^2 : \text{Set} \to \text{Set} \), along with its monad structure can be restricted to the
\[
\cdot \text{up-set functor } U \text{ with } UX = \{a \subseteq PX \mid \uparrow_{PX} a = a\};
\]
\[
\cdot \text{filter functor } F \text{ with } FX = \{a \subseteq PX \mid a \text{ a filter on } X\};
\]
\[
\cdot \text{ultrafilter functor } \beta \text{ with } \beta X = \{a \subseteq PX \mid a \text{ an ultrafilter on } X\};
\]
\[
\cdot \text{identity functor } 1_{\text{Set}} \text{ with } 1_{\text{Set}}X = X \cong \{\hat{x} \mid x \in X\},
\]
and we obtain the monads \( \mathbb{U} = (U, m, e), \mathbb{F} = (F, m, e), \mathbb{B} = (\beta, m, e), \mathbb{I} = (1_{\text{Set}}, 1, 1) \). These monads can be thought of as being induced by the adjunctions
\[
\text{Set} \xrightarrow{\text{C}^{op}} \xleftarrow{\text{C}(-, 2)} \text{Set}^{op}
\]
of Example 2.5.3 with successively \( \text{C} = \text{Ord}, \text{SLat}, \text{Lat}, \) and \( \text{Frm} \). We obtain a chain of monad morphisms
\[
\mathbb{I} \to \mathbb{B} \to \mathbb{F} \to \mathbb{U} \to P^2,
\]
all given objectwise by inclusion maps.

Note that there is also for each set \( X \) a map \( \alpha_X : PX \to FX \) sending a subset \( A \) of \( X \) to the principal filter \( \hat{A} \), and extending to a monad morphism \( \alpha : \mathbb{P} \to \mathbb{F} \).
(6) On every category $X$, there is a unique monad morphism from the identity monad $I = (1_X, 1, 1)$ to any monad $T = (T, m, e)$ on $X$, and it is given by the unit $e : 1_X \to T$ of $T$. Hence, the identity monad is an initial object in the metacategory $\text{MND}_X$ of monads on $X$, with monad morphisms.

3.2 The Eilenberg–Moore category. Given a monad $T = (T, m, e)$ on $X$, a $T$-algebra (or Eilenberg-Moore algebra) is a pair $(X, a)$, where $X$ is an object of $X$, and the structure morphism $a : TX \to X$ satisfies

$$a \cdot Ta = a \cdot m_X \quad \text{and} \quad 1_X = a \cdot e_X,$$

diagrammatically:

$$
\begin{array}{c}
TTX \xrightarrow{Ta} TX \\
\downarrow^{m_X} \quad \downarrow^{a} \\
TX \xrightarrow{a} X
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{e_X} TX \\
\downarrow^{1_X} \quad \downarrow^{a} \\
X
\end{array}
$$

A $T$-homomorphism $f : (X, a) \to (Y, b)$ is an $X$-morphism $f : X \to Y$ such that

$$f \cdot a = b \cdot Tf ,$$

which amounts to commutativity of the diagram

$$
\begin{array}{c}
TX \xrightarrow{Tf} TY \\
\downarrow^{a} \quad \downarrow^{b} \\
X \xrightarrow{f} Y
\end{array}
$$

The category of $T$-algebras and $T$-homomorphisms is denoted by $\text{X}^T$ and is also called the Eilenberg–Moore category of $T$.

The forgetful functor $G^T : \text{X}^T \to \text{X}$ has a left adjoint $F^T : \text{X} \to \text{X}^T$:

$$X \mapsto (TX, m_X) , \quad (f : X \to Y) \mapsto (Tf : TX \to TY) ,$$

where $(TX, m_X)$ is the free $T$-algebra on $X$. The unit $\eta^T : 1_X \to G^TF^T = T$ of this adjunction is $e$, and the counit is described by its components as

$$\varepsilon^T_{(X, a)} = a : (TX, m_X) \to (X, a) .$$

Since the monad associated to $F^T \dashv G^T$ gives back the original $T = (T, m, e)$, any monad may be obtained via an adjunction. Among such adjunctions, $F^T \dashv G^T$ is characterized as follows:
3.2.1 Proposition. If $\mathbb{T} = (T, m, e)$ is the monad associated to $F \xrightarrow{\eta} G : A \to X$, then there exists a unique functor $K : A \to X^T$ making both the inner and outer triangles of the following diagram commute:

$\begin{array}{ccc}
A & \xrightarrow{K} & X^T \\
\downarrow{F} & & \downarrow{G} \\
X & \xrightarrow{G^T} & T \end{array}$  \hspace{1cm} (3.2.i)

Proof. The functor $K : A \to X^T$ defined on objects and morphisms by

$$A \mapsto (GA, G\varepsilon_A), \quad (f : A \to B) \mapsto (Gf : GA \to GB)$$

makes both triangles in (3.2.i) commute. If $K' : A \to X^T$ is another such functor, commutativity of (3.2.i) yields $K'A = (GA, a)$ (for a certain $\mathbb{T}$-algebra structure $a : TGA \to GA$) and $K'f = Gf$; since $K'\varepsilon_A = G\varepsilon_A : K'FGA \to K'A$ is an $X^T$-morphism, the following diagram commutes:

$\begin{array}{ccc}
TGA & \xrightarrow{T\etaGA} & TGA \\
\downarrow{T\varepsilon TGA} & & \downarrow{G\varepsilon TGA} \\
TTGA & \xrightarrow{TGA} & TGA \\
\downarrow{G\varepsilon FGA} & & \downarrow{a} \\
TGA & \xrightarrow{G\varepsilon_A} & GA
\end{array}$

which tells us that $a = G\varepsilon_A$. $\square$

A right adjoint functor $G : A \to X$ is monadic if the comparison functor $K : A \to X^T$ of the previous proposition is an equivalence, and $G$ is strictly monadic if $K$ is an isomorphism; one then says that $A$ is monadic over $X$, or strictly monadic over $X$, respectively. A monadic functor $G : A \to X$ can be considered as a measure of the “algebraic character” of $A$ over $X$, where the Eilenberg–Moore category $X^T$ represents the “algebraic part” of $A$.

3.2.2 Examples.

(1) The forgetful functor $G : \text{Mon} \to \text{Set}$ is monadic; in fact, the comparison functor is even an isomorphism:

$$\text{Set}^k \cong \text{Mon}.$$  

Similarly, the Eilenberg–Moore category of the free-group monad is isomorphic to $\text{Grp}$, that of the free-abelian-group monad is isomorphic to $\text{AbGrp}$, etc.

(2) For the powerset monad $\mathbb{P}$, the structure map $a : PX \to X$ of a $\mathbb{P}$-algebra $(X, a)$ defines a complete order on $X$ via

$$\bigvee A := a(A),$$

$\forall A := a(A),$
and every \( P \)-homomorphism \( f : (X, a) \to (Y, b) \) is then a sup-map. On the other hand, when \( X \) is a complete lattice, the map \( \lor : PX \to X \) is a \( P \)-algebra structure, and sup-maps between complete lattices become \( P \)-homomorphisms. Therefore, there is an isomorphism:

\[
\text{Set}^P \cong \text{Sup} ,
\]

and the forgetful functor \( \text{Sup} \to \text{Set} \) is strictly monadic.

(3) Neither of the forgetful functors \( \text{Ord} \to \text{Set} \) or \( \text{Top} \to \text{Set} \) is monadic, since the Eilenberg–Moore category of each of these is equivalent to \( \text{Set} \).

(4) Any full and faithful right adjoint functor is monadic (see also Exercise 3.3.1).

3.3 Limits in the Eilenberg–Moore category. The forgetful functor \( G^T : \mathcal{X}^T \to \mathcal{X} \) is very well-behaved with respect to limits. Because \( G^T \) is right adjoint, it naturally preserves all the limits that exist in \( \mathcal{X}^T \) (Proposition 2.11.1). It also reflects limits; in fact, it does better than that.

One says that a functor \( F : \mathcal{C} \to \mathcal{D} \) creates \( J \)-limits if for every diagram \( D : J \to \mathcal{C} \) such that \((A, \alpha)\) is a limit of \( FD \) in \( \mathcal{D} \), there is a unique pair \((L, \lambda)\) consisting of an object \( L \) of \( \mathcal{C} \) and a cone \( \lambda : \Delta L \to D \) satisfying \((FL, F\lambda) = (A, \alpha)\), and moreover, \((L, \lambda)\) is a limit of \( D \). In particular, if \( F \) creates \( J \)-limits, then it reflects them. The dual notion is that of creation of \( J \)-colimits.

3.3.1 Proposition. The forgetful functor \( G^\mathcal{T} : \mathcal{X}^\mathcal{T} \to \mathcal{X} \) creates all limits.

Proof. Let \( D : J \to \mathcal{X}^\mathcal{T} \) be a diagram and \((X, \lambda)\) a limit of \( G^T D \) in \( \mathcal{X} \). There is then a unique morphism \( a : TX \to X \) with

\[
\lambda \cdot \Delta a = G^T \varepsilon^\mathcal{T} D \cdot T\lambda .
\]

All that needs to be shown is that \((X, a)\) is a \( \mathcal{T} \)-algebra, and that \(((X, a), \lambda)\) serves as a limit for \( D \) in \( \mathcal{X}^\mathcal{T} \). For the \( \mathcal{T} \)-algebra laws, observe that

\[
\lambda \cdot \Delta(a \cdot e_X) = G^T \varepsilon^\mathcal{T} D \cdot T\lambda \cdot \Delta e_X \\
= G^T \varepsilon^\mathcal{T} D \cdot e G^T D \cdot \lambda \quad \text{(naturality of } e = \eta^\mathcal{T}) \\
= \lambda ,
\]

and

\[
\lambda \cdot \Delta(a \cdot m_X) = G^T \varepsilon^\mathcal{T} D \cdot T\lambda \cdot \Delta m_X \\
= G^T \varepsilon^\mathcal{T} D \cdot m G^T D \cdot TT\lambda \quad \text{(naturality of } m = G^T \varepsilon^\mathcal{T} F^\mathcal{T}) \\
= G^T \varepsilon^\mathcal{T} D \cdot TG^\mathcal{T} \varepsilon^\mathcal{T} D \cdot TT\lambda \quad \text{(naturality of } \varepsilon^\mathcal{T}) \\
= G^T \varepsilon^\mathcal{T} D \cdot T\lambda \cdot \Delta Ta \\
= \lambda \cdot \Delta(a \cdot Ta) .
\]
3. MONADS

Obviously, \( \lambda : \Delta(X, a) \to D \) is actually a cone in \( X^\mathcal{T} \), and for any cone \( \alpha : \Delta(Y, b) \to D \), one obtains a unique morphism \( f : Y \to X \) in \( X \) with

\[
\lambda \cdot \Delta f = G^\mathcal{T} \alpha .
\]

That \( f : (Y, b) \to (X, a) \) is indeed a \( \mathcal{T} \)-homomorphism follows from

\[
\lambda \cdot \Delta(f \cdot b) = G^\mathcal{T} \alpha \cdot \Delta G^\mathcal{T} \varepsilon_{(Y,b)} = G^\mathcal{T} \varepsilon_{\mathcal{T} D} \cdot T G^\mathcal{T} \alpha \text{ (naturality of } \varepsilon_{\mathcal{T}}) = G^\mathcal{T} \varepsilon_{\mathcal{T} D} \cdot T \lambda \cdot \Delta Tf = \lambda \cdot \Delta(a \cdot Tf) .
\]

3.3.2 Corollary. For a monadic functor \( G : A \to X \), if \( X \) is \( J \)-complete, then \( A \) is also \( J \)-complete, and \( G \) preserves and reflects \( J \)-limits.

Proof. This is an immediate consequence of the previous Proposition.

3.4 Beck’s monadicity criterion. In order to characterize monadic functors, let us first observe that for a monad \( \mathcal{T} \) on \( X \), and a \( \mathcal{T} \)-algebra \((X, a)\), the diagram

\[
TTX \xrightarrow{m_X} TX \xrightarrow{a} X
\]

(3.4.i)

in addition to satisfying \( a \cdot m_X = a \cdot Ta \) has the property that there are splittings \( \eta_X \) and \( \eta_{TX} \): in general, one calls a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow g & & \downarrow h \\
Y & \xrightarrow{h} & Z
\end{array}
\]

in \( X \) with \( h \cdot f = h \cdot g \) a split fork if there are morphisms \( i : Z \to Y \) and \( j : Y \to X \) (the aforementioned “splittings”) with \( h \cdot i = 1_Z \), \( f \cdot j = 1_Y \) and \( g \cdot j = i \cdot h \):

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & Z \\
\downarrow j & & \downarrow i \\
X & \xrightarrow{g} & Y \\
\downarrow f & & \downarrow h \\
Y & \xrightarrow{h} & Z
\end{array}
\]

In this case, \( h \) is a coequalizer of \( (f, g) \) which, trivially, is absolute, in the sense that it is preserved by every functor defined on \( X \).
The coequalizer diagram (3.4.i) has moreover the special property that it can be “lifted” along $G^\mathbb{T}$, that is

$$(TTX, m_{TX}) \xrightarrow{m_X} (TX, m_X) \xrightarrow{a} (X, a)$$  (3.4.ii)

is a coequalizer in $X^\mathbb{T}$, presenting the $\mathbb{T}$-algebra $(X, a)$ as a quotient of the free $\mathbb{T}$-algebra $F^\mathbb{T}X$. In particular, the parallel pair of morphisms $(m_X, Ta)$ in (3.4.ii) has the property that its $G^\mathbb{T}$-image is part of the split fork (3.4.i). This property turns out to be crucial, and for a functor $G : A \to X$ one therefore says that morphisms $s, t : A \to B$ in $A$ form a $G$-split pair $(s, t)$ if $f = Gs, g = Gt$ belong to a split fork in $X$; similarly, $(s, t)$ is a $G$-absolute pair if $(Gs, Gt)$ has a coequalizer in $X$ that is absolute. Obviously, every $G$-split pair is $G$-absolute.

3.4.1 Theorem (Beck’s Monadicity Criterion). A functor $G : A \to X$ is monadic if and only if the following conditions hold:

1. $G$ is right adjoint;
2. $G$ reflects isomorphisms;
3. $A$ has coequalizers of $G$-split pairs, and $G$ preserves them.

Condition (2) may be replaced by:

(2’) $G$ reflects coequalizers of $G$-split pairs.

The criterion remains valid if “$G$-split” is replaced by “$G$-absolute” everywhere.

Proof. For the necessity of the conditions, even in the formally stronger “$G$-absolute” version, we remark for (2) that isomorphisms are 1-limits (then see 3.3.1), and refer to Exercise 3.D for (3) and (2’). For their sufficiency, consider the adjunction $F \xleftarrow{\eta} G : A \to X$ with its induced monad $\mathbb{T} = (T, m, e)$. Since $G = G^\mathbb{T}K$, the comparison functor $K$ also reflects isomorphisms, and because of Propositions 2.6.2 and 2.6.1 it suffices to establish an adjunction $L \xrightarrow{\kappa} K : A \to X^\mathbb{T}$ with $\kappa$ an isomorphism.

For a $\mathbb{T}$-algebra $(X, a)$, the $G$-image of the pair $(\varepsilon_{FX}, Fa)$ is $(m_X, Ta)$. Hence, condition (3) guarantees the existence of a coequalizer diagram

$$FGFX \xrightarrow{\varepsilon_{FX}} FX \xrightarrow{\gamma(X,a)} L(X, a)$$  (3.4.iii)

in $A$ that is preserved by $G = G^\mathbb{T}K$. Since $G^\mathbb{T}$ reflects such coequalizers, $K$ preserves the coequalizers (3.4.iii) as well. Consequently, the unique morphism $\kappa(X,a)$ given by the
coequalizer property of (3.4.iii) and making the following diagram commute, must actually be an isomorphism:

\[
\begin{array}{ccc}
F^2TX & \xrightarrow{m_X} & F^2X \\
\downarrow & & \downarrow \\
KFTX & \xrightarrow{K\gamma(X,a)} & KFX \\
& \xrightarrow{K\epsilon(X,a)} & KL(X,a)
\end{array}
\]

That \(\kappa_{(X,a)}\) is \(K\)-universal is easily seen with the natural correspondence:

\[
(X,a) \xrightarrow{f} KA \\
L(X,a) \xrightarrow{g} A
\]

for all \(A \in \text{ob} \mathcal{A}\), which arises from the coequalizer property of (3.4.iii), as shown by

\[
\begin{array}{ccc}
FGFX & \xrightarrow{\gamma_{(X,a)}} & L(X,a) \\
\downarrow & & \downarrow \\
FGA & \xrightarrow{\epsilon_A} & A
\end{array}
\]

This completes the proof that (1)–(3) are sufficient for monadicity of \(G\). If (2) is replaced by (2'), one should observe that the last diagram exhibits the counit \(\lambda_A = g : LKA \rightarrow A\) if we let \((X,a) = KA\) and \(f = 1_{KA}\). In this case, \(G\) maps the bottom row of the diagram to a coequalizer diagram (trivial case of (3.4.i)), so under assumption (2') the bottom row is already a coequalizer in \(\mathcal{A}\), and the “comparison” morphism \(\lambda_A = g\) must be an isomorphism.

\[\square\]

3.4.2 Example. The contravariant powerset functor \(P^\bullet : \text{Set}^{op} \rightarrow \text{Set}\) (see 2.2) is monadic. It certainly is right adjoint (see Example 2.5.1(6)) and reflects isomorphisms. Moreover, one easily verifies that \(P^\bullet\) transforms an equalizer diagram

\[
E \xhookrightarrow{f} X \xrightarrow{g} Y
\]

in \(\text{Set}\) into a coequalizer diagram

\[
PY \xrightarrow{f^{-1}(-)} PX \xrightarrow{(-) \cap E} PE
\]

in \(\text{Set}\), provided that \(f^{-1}(-) = P^\bullet f\) is surjective (that is, if \(f\) is injective). Hence, the Beck Criterion applies.
3.5 Duskin’s monadicity criterion. Coequalizers are often difficult to construct explicitly, so the third condition in Theorem 3.4.1 can be delicate to handle. In certain categories however, coequalizers can be replaced by other, more practical structures. For example, in Set a kernel pair \((r, r') : R \to X\) of a map \(f : X \to Y\) yields the equivalence relation \(R \subseteq X \times X\):

\[\forall x, y \in X \ ( (x, y) \in R \iff f(x) = f(y)) ,\]

and a coequalizer of \((r, r')\) is then simply given by the projection \(\pi : X \to X/R\). In fact, every equivalence relation in Set is the kernel pair of a split epimorphism (namely the projection \(\pi\)), see Exercise 2.C.

For a functor \(G : A \to X\), a \(G\)-kernel pair is a pair \((f, g : A \to B)\) such that there is a diagram

\[\begin{array}{ccc}
X & \xrightarrow{Gf} & Y \xrightarrow{h} Z \\
\downarrow{Gg} & & \downarrow{h} \\
X & = & Z
\end{array} \tag{3.5.i}\]

where \((Gf, Gg)\) is the kernel pair of \(h\) in \(X\). In the case where \(h\) is a split epimorphism, this diagram becomes a split fork, so that \(h\) is a coequalizer of \((Gf, Gg)\) (Exercise 3.C).

A joint kernel pair of \((f, g : A \to B)\) in a category \(A\) is a pair \((s, s' : S \to A)\) with \(f \cdot s = f \cdot s', g \cdot s = g \cdot s'\), and such that if there is a pair \((t, t' : L \to A)\) with \(f \cdot t = f \cdot t', g \cdot t = g \cdot t'\), then there exists a unique \(A\)-morphism \(u : L \to S\) with \(s \cdot u = t\) and \(s' \cdot u = t'\):

\[\begin{array}{ccc}
S & \xrightarrow{s} & A \xrightarrow{f} B \\
\downarrow{u} & & \downarrow{g} \\
L & \xrightarrow{t} & B \\
\downarrow{t'} & & \\
& & \end{array}\]

Since a joint kernel pair of \((f, f)\) is simply a kernel pair of \(f\), a category that has joint kernel pairs also has kernel pairs. Whenever the product \(B \times B\) exists, the joint kernel pair of \((f, g)\) is simply the kernel pair of \(\langle f, g \rangle : A \to B \times B\).

3.5.1 Theorem (Duskin’s Monadicity Criterion). Suppose that \(A\) has joint kernel pairs and that \(X\) has kernel pairs of split epimorphisms. Then a functor \(G : A \to X\) is monadic if and only if the following conditions hold:

1. \(G\) is right adjoint;
2. \(G\) reflects isomorphisms;
3. Every \(G\)-kernel pair of a split epimorphism has a coequalizer that is preserved by \(G\).

Proof. The necessity of these conditions is immediate from Theorem 3.4.1. Using this same result, the sufficiency obviously follows if one proves that every \(G\)-split pair

\[\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \\
\end{array}\]
has a coequalizer that is preserved by $G$. Given the hypothesis (3), it is enough to construct a $G$-kernel pair $(f', g')$ of a split epimorphism whose coequalizer is the coequalizer of $(f, g)$, and such that this property is preserved by $G$. Thus, consider the kernel pair $(r, r')$ of $g$ and the joint kernel pair $(s, s')$ of $(f \cdot r, f \cdot r')$; in $X$, we obtain a diagram

$$
\begin{array}{cccccc}
GS & \xrightarrow{Gr} & GR & \xrightarrow{k} & K \\
\downarrow{Gr'} & & \downarrow{k'} & & \\
GA & \xrightarrow{Gf} & GB & \xrightarrow{h} & Z
\end{array}
$$

whose bottom row is a split fork with splittings $i : Z \to GB$ and $j : GB \to GA$ (so that $h \cdot i = 1_Z$, $Gf \cdot j = 1_{GB}$ and $Gg \cdot j = i \cdot h$), and where $(k, k')$ is the kernel pair of $h$. Since

$$
h \cdot (Gf \cdot Gr) = h \cdot Gg \cdot Gr = h \cdot Gg \cdot Gr' = h \cdot (Gf \cdot Gr'),
$$

there is a uniquely determined $X$-morphism $v : GR \to K$ with $k \cdot v = Gf \cdot Gr$ and $k' \cdot v = Gf \cdot Gr'$. This morphism is split epic: as $G$ is right adjoint, it preserves kernel pairs, so

$$
Gg \cdot (j \cdot k) = i \cdot h \cdot k = i \cdot h \cdot k' = Gg \cdot (j \cdot k')
$$

yields the existence of $w : K \to GR$ with $j \cdot k = Gr \cdot w$ and $j \cdot k' = Gr' \cdot w$; hence,

$$
j \cdot k \cdot v \cdot w = j \cdot Gf \cdot Gr \cdot w = j \cdot Gf \cdot j \cdot k = j \cdot k
$$

implies $k \cdot v \cdot w = k$ and similarly $k' \cdot v \cdot w = k'$, so $v \cdot w = 1_K$ by the universal property of $(k, k')$. Right adjointness of $G$ also makes $(Gs, Gs')$ into a joint kernel pair of $(Gf \cdot Gr, Gf \cdot Gr')$, and it follows that $(Gs, Gs')$ is a kernel pair of the split epimorphism $v$. By hypothesis, $(s, s')$ has a coequalizer $u : R \to Q$ that is preserved by $G$, and since $v : GR \to K$ is also a coequalizer of $(Gs, Gs')$ (Exercise 3.C), there is an isomorphism $\phi : K \to GQ$ with $Gu = \phi \cdot v$. By definition of the joint kernel pair $(s, s')$, the universal property of $u$ yields $A$-morphisms $q, q' : Q \to B$ such that $q \cdot u = f \cdot r$ and $q' \cdot u = f \cdot r'$:

$$
\begin{array}{cccccc}
S & \xrightarrow{s} & R & \xrightarrow{u} & Q \\
\downarrow{r} & & \downarrow{r'} & & \\
A & \xrightarrow{f} & B & \xrightarrow{q} & C
\end{array}
$$

One has $k \cdot v = G(f \cdot r) = Gq \cdot Gu = Gq \cdot \phi \cdot v$ so that $k = Gq \cdot \phi$ and similarly $k' = Gq' \cdot \phi$ because $v$ is epic. Hence, $(Gq, Gq')$ is a kernel pair of the split epimorphism $h$, and by hypothesis, $(q, q')$ has a coequalizer $c : B \to C$ that is preserved by $G$. Thus, one obtains a split fork

$$
\begin{array}{cccc}
GA & \xrightarrow{Gf} & GB & \xrightarrow{Gc} & GC
\end{array}
$$
and $Gc$ is a coequalizer of $(Gf, Gg)$. To see that $c$ is a coequalizer of $(f, g)$, consider an $A$-morphism $d : B \to D$ such that $d \cdot f = d \cdot g$. Since

$$d \cdot q \cdot u = d \cdot f \cdot r = d \cdot g \cdot r = d \cdot g \cdot r' = d \cdot q' \cdot u$$

and $u$—being a coequalizer—is epic, one has $d \cdot q = d \cdot q'$ and the universal property of $(C, c)$ follows.

3.6 The Kleisli category. The objects of the Kleisli category $X_T$ associated to the monad $T = (T, m, e)$ on $X$ are the objects of $X$, and a morphism $f : X \to Y$ in $X_T$ is simply an $X$-morphism $f : X \to TY$. The Kleisli composition of $f : X \to Y$ and $g : Y \to Z$ in $X_T$, is defined via the composition in $X$ as

$$g \circ f := m_Z \cdot Tg \cdot f.$$ 

The identity $1_X : X \to X$ in this category is just the component $e_X : X \to TX$ of the unit $e$.

There is a functor $G_T : X_T \to X$ defined on objects and morphisms by

$$X \mapsto TX \quad (f : X \to Y) \mapsto (m_Y \cdot Tf : TX \to TY).$$

As for the Eilenberg–Moore category, this functor has a left adjoint $F_T : X \to X_T$:

$$X \mapsto X, \quad (f : X \to Y) \mapsto (e_Y \cdot f : X \to Y).$$

The unit $\eta_T : 1_X \to G_T F_T = T$ of this adjunction is $e$, and the components of the counit $\varepsilon_T : F_T G_T \to 1_{X_T}$ are simply the morphisms $1_{TX} : TX \to TX$ in $X$. Also in this case, the monad associated to this adjunction gives back the original monad $\mathbb{T} = (T, m, e)$, and the Kleisli category may be characterized dually to $X_T$:

3.6.1 Proposition. If $\mathbb{T} = (T, m, e)$ is the monad associated to $F \xrightarrow{\eta} G : A \to X$, then there exists a unique functor $L : X_T \to A$ making both the inner and outer triangles of the following diagram commute:

$$\begin{array}{ccc}
X_T & \xrightarrow{L} & A \\
\downarrow{G_T} & & \downarrow{F} \\
F_T & \xleftarrow{\varepsilon_T} & X.
\end{array}$$

Moreover, $L : X_T \to A$ is full and faithful.

Proof. The functor $L : X_T \to A$ maps objects like $F$ does, and one has $Lf = \varepsilon_{FY} \cdot Ff$ on morphisms $f : X \to Y$. Uniqueness follows from the universal property of $\eta$, as does the fact that $L$ is full and faithful. \qed
3.6.2 Example. For $\mathbb{P} = \mathbb{P}$ the powerset monad, a Set$_\mathbb{P}$-morphism from $X$ to $Y$ is just a relation $r : X \to PY$, while the Kleisli composition is the ordinary relational composition of $1.2$

$$s \cdot r = s \circ r,$$

for any relations $r : X \to PY$, $s : Y \to PZ$. Therefore,

$$\text{Set}_\mathbb{P} = \text{Rel}.$$

3.7 Kleisli triples. There is an alternative presentation of monads which turns out to be very practical in verifying that given data describe a monad when there is no obvious adjunction inducing it. A Kleisli triple $(T, (-)^T, e)$ on a category $X$ consists of

1. a function $T : \text{ob } X \to \text{ob } X$ sending $X$ to $TX$,
2. an extension operation $(-)^T$ sending a morphism $f : X \to TY$ to a morphism $f^T : TX \to TY$,
3. a morphism $e_X : X \to TX$ for each $X \in \text{ob } X$,

subject to

$$(g^T \cdot f)^T = g^T \cdot f^T, \quad e_X^T = 1_{TX}, \quad f^T \cdot e_X = f$$

(3.7.i) for all $X \in \text{ob } X$, $f : X \to TY$, and $g : Y \to TZ$. One can set

$$g \circ f := g^T \cdot f,$$

so the previous conditions are equivalent to requiring that this “Kleisli composition” $\circ$ is associative, and that $e_X$ acts as an identity. A Kleisli triple morphism $\alpha : (S, (-)^S, d) \to (T, (-)^T, e)$ is given by a family of morphisms $\alpha_X : SX \to TX$ in $X$ (with $X$ running through $\text{ob } X$) such that

$$\alpha_Y \cdot f^S = (\alpha_Y \cdot f)^T \cdot \alpha_X, \quad \alpha_X \cdot d_X = e_X$$

for all $f : X \to SY$. That is, a Kleisli triple morphism is a family of morphisms that preserve the Kleisli composition together with its unit.

A Kleisli triple $(T, (-)^T, e)$ gives rise to a monad $\mathbb{T} = (T, m, e)$ on $X$ simply by setting

$$Tf := (e_Y \cdot f)^T, \quad m_X := (1_{TX})^T,$$

for all $X$-morphisms $f : X \to Y$. The facts that $T$ defines a functor, $e$ and $m$ are natural transformations, and that the multiplication and unit laws are verified, all follow from simple manipulations of the equalities (3.7.i). Similarly, a Kleisli triple morphism $\alpha : (S, (-)^S, d) \to (T, (-)^T, e)$ yields a natural transformation $\alpha : S \to T$, which turns out to be a morphism $\alpha : S \to \mathbb{T}$ between the corresponding monads.
Conversely, given a monad $\mathbb{T} = (T, m, e)$ on $X$, one obtains a Kleisli triple $(T, (-)^T, e)$ via

$$f^T := m_Y \cdot Tf ,$$

for all $X$-morphisms $f : X \rightarrow Y$, and the conditions readily follow by using naturality of $e$ and $m$, as well as their multiplication and unit laws. A monad morphism $\alpha : S \rightarrow \mathbb{T}$ also yields a morphism $\alpha : (S, (-)^S, d) \rightarrow (T, (-)^T, e)$ between the corresponding Kleisli triples.

Moreover, the passages from a Kleisli triple to a monad and from a monad to a Kleisli triple are mutually inverse, so that both definitions describe the same structure on $X$ (and the two definitions of Kleisli composition correspond).

**3.8 Distributive laws, liftings, and composite monads.** Given monads $\mathbb{S} = (S, n, d)$ and $\mathbb{T} = (T, m, e)$ on $X$, one can consider the functor $ST : X \rightarrow X$ together with the natural transformation $d \circ e : 1_X \rightarrow ST$, but there is no obvious choice in general for a natural transformation $w : STST \rightarrow ST$ making $ST = (ST, w, d \circ e)$ into a monad. A first step towards solving this problem is the introduction of a distributive law of $T$ over $S$, that is, a natural transformation $\delta : TS \rightarrow ST$ making the following diagrams commute:

$$
\begin{array}{ccc}
TSS & \xrightarrow{\delta S} & STS \\
\downarrow{Tn} & & \downarrow{nT} \\
TS & \xrightarrow{\delta} & ST \\
\downarrow{mS} & & \downarrow{Sm} \\
TTS & \xrightarrow{T\delta} & TST \\
\end{array}
\hspace{1cm}
\begin{array}{ccc}
TS & \xrightarrow{\delta} & ST \\
\downarrow{Td} & & \downarrow{dT} \\
S & \xrightarrow{Se} & S \\
\end{array}
\hspace{1cm}
\tag{3.8.i}
$$

Such a $\delta$ allows for a lifting of the monad $\mathbb{S}$ on $X$ through $G^T : X^T \rightarrow X$, that is, there exists a monad $\tilde{\mathbb{S}} = (\tilde{S}, \tilde{n}, \tilde{d})$ on $X^T$ such that

$$G^T \tilde{S} = SG^T , \quad G^T \tilde{n} = nG^T , \quad G^T \tilde{d} = dG^\mathbb{T} .$$

The first of these conditions may be used to identify the domains and codomains of the natural transformations in the last two equalities. The latter state that the underlying $X$-morphisms of $\tilde{n}_{(X,a)}$, $\tilde{d}_{(X,a)}$ are $n_X$, $d_X$, respectively, for any $\mathbb{T}$-algebra $(X, a)$, so the multiplication and unit laws of $\tilde{S}$ are automatically satisfied. Therefore, a lifting of $\mathbb{S}$ through $G^\mathbb{T} : X^T \rightarrow X$ is simply provided by a functor $\tilde{S} : X^T \rightarrow X^\mathbb{T}$ making

$$
\begin{array}{ccc}
X^T & \xrightarrow{\tilde{S}} & X^\mathbb{T} \\
\downarrow{G^T} & & \downarrow{G^\mathbb{T}} \\
X & \xrightarrow{S} & X \\
\end{array}
$$
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commute, and such that \( n_X : \tilde{S}\tilde{S}(X,a) \to \tilde{S}(X,a) \), \( d_X : (X,a) \to \tilde{S}(X,a) \) are \( X^T \)-morphisms for all \((X,a) \in \text{ob}(X^T)\). Thus, if \( \delta : TS \to ST \) is a distributive law of \( T \) over \( S \), a lifting \( \tilde{S} : X^T \to X^T \) is obtained via

\[
(X,a) \mapsto (SX, Sa \cdot \delta_X), \quad (f : X \to Y) \mapsto (Sf : SX \to SY).
\]

The fact that the components of \( \tilde{n} : \tilde{S}\tilde{S} \to \tilde{S} \) and \( \tilde{d} : 1_{X^T} \to \tilde{S} \) have underlying \( X^T \)-morphisms then follows directly from commutativity of the diagrams in (3.8.i).

In turn, a lifting of \( S \) through \( G^T : X^T \to X \) yields the following composite adjunction

\[
(X^T) \xleftarrow{\tilde{G}^S} X^T \xrightarrow{\tilde{G}^T} X.
\]

Since \( ST = SG^TF^T = G^T\tilde{S}F^T = G^T\tilde{G}^S F^S F^T \), the monad on \( X \) associated to this adjunction is

\[ S\mathbb{T} = (ST, w, d \circ e), \]

where \( w = G^T\tilde{G}^S \tilde{S} \tilde{G}^T \tilde{S} F^S F^T \cdot \tilde{G}^T G^S F^S F^T \cdot \tilde{G}^T \tilde{S} \tilde{G}^S F^S F^T \) by Proposition 2.5.5 and 3.1 so that

\[ w = G^T\tilde{n} F^T \cdot G^T\tilde{S} \tilde{e} T \tilde{S} F^T = nT \cdot SG^T \tilde{n} F^T \tilde{S} F^T. \]

By combining this last expression of \( w \) with the lifting conditions, one easily verifies that the following equalities also hold:

1. \( w \cdot SeST = nT; \)
2. \( w \cdot STdT = Sm; \)
3. \( w \cdot STSm = Sm \cdot wT; \)
4. \( w \cdot nTST = nT \cdot Sw. \)

More generally, we say that a monad \( S\mathbb{T} = (ST, w, d \circ e) \) is a composite of \( S \) and \( \mathbb{T} \) if its multiplication \( w \) satisfies (1)–(4). These conditions are useful in proofs, but there is a more elegant and practical presentation, as the next Lemma shows. Given monads \( S = (S, n, d) \), \( \mathbb{T} = (T, m, e) \) on \( X \), one says that a monad \( S\mathbb{T} = (ST, w, d \circ e) \) satisfies the middle unit law if the following diagram commutes:

\[
\begin{array}{ccc}
ST & \xleftarrow{1_{ST}} & ST \\
\downarrow{SeodT} & & \downarrow{w} \\
STST & \xrightarrow{w} & ST
\end{array}
\]

that is, if \( w \cdot (Se \circ dT) = 1_{ST}. \)

3.8.1 Lemma. For monads \( S = (S, n, d) \), \( \mathbb{T} = (T, m, e) \), and \( S\mathbb{T} = (ST, w, d \circ e) \) on \( X \), the following are equivalent:

(i) \( S\mathbb{T} \) is a composite monad;
(ii) the natural transformations $Se : S \to ST$ and $dT : T \to ST$ are monad morphisms, and $S\mathbb{T}$ satisfies the middle unit law.

Proof. The proof that (i) \implies (ii) follows from routine verifications. To show (ii) \implies (i), we must verify the four composite monad conditions. For (1), observe that

$$
\begin{align*}
  w \cdot SeST &= w \cdot (STw \cdot STSTdT \cdot STSeT) \cdot SeST & \text{(middle unit law)} \\
  &= w \cdot wST \cdot STSTdT \cdot STSeT \cdot SeST & \text{(multiplication law)} \\
  &= w \cdot STdT \cdot wT \cdot STSeT \cdot SeST & \text{(naturality of } w) \\
  &= w \cdot STdT \cdot SeT \cdot nT & \text{(Se monad morphism)} \\
  &= nT & \text{(middle unit law)}.
\end{align*}
$$

Condition (2) is proved in a similar way, by inserting the middle unit law $wST \cdot SeSTST \cdot SdTST = 1_{STST}$ into $w \cdot STdT$, and proceeding as above. To see (3), we write

$$
\begin{align*}
  w \cdot STSm &= w \cdot (STw \cdot STSTdT) & \text{(ST applied to (2))} \\
  &= w \cdot wST \cdot STSTdT & \text{(multiplication law)} \\
  &= w \cdot STdT \cdot wT & \text{(naturality of } w) \\
  &= Sm \cdot wT & \text{(by (2) again)}.
\end{align*}
$$

The last condition is proved similarly by exploiting (1) instead of (2). \hfill \Box

Given a monad $S\mathbb{T} = (ST, w, d \circ e)$ on $X$ that is a composite of $S = (S, n, d)$ and $\mathbb{T} = (T, m, e)$, there is a natural transformation

$$
\delta := (TS \xrightarrow{dT \circ Se} STST \xrightarrow{w} ST),
$$

which turns out to be a distributive law of $\mathbb{T}$ over $S$ (as can be checked routinely).

As these constructions suggest, distributive laws, liftings, and composite monads are equivalent concepts:

3.8.2 Proposition. For monads $S$ and $\mathbb{T}$ on $X$, there is a bijective correspondence between:

(i) distributive laws $\delta$ of $\mathbb{T}$ over $S$;

(ii) liftings $\tilde{S}$ of $S$ through $G^\mathbb{T} : X^\mathbb{T} \to X$;

(iii) monads $S\mathbb{T} = (ST, w, d \circ e)$ that are composites of $S$ and $\mathbb{T}$. 

Proof. The previous discussion has already shown that these concepts are related as in:

\[
\begin{align*}
(i) & \quad \Downarrow \quad \Downarrow \\
(iii) & \quad \Longleftrightarrow (ii) 
\end{align*}
\]

To verify the statement, it suffices to prove that any path of length three in this diagram is the identity. We check this for liftings, the other verifications being similar. By definition, a lifting is entirely determined by its behavior on the structure morphisms of \(X^T\)-objects \((X,a)\). Let us denote by \((S\tilde{X},a')\) the \(\tilde{S}\)-image of \((X,a)\), and by \(a''\) the structure corresponding to the path \((ii) \implies (iii) \implies (i) \implies (ii)\). By using that \(\tilde{S}\varepsilon^T_{(X,a)} : (STX,m'X) \to (SX,a')\) is an \(X^T\)-morphism, one observes

\[
a'' = Sa \cdot n_{TX} \cdot S(G^T\varepsilon^T_{STX} \cdot TS\varepsilon_X) \cdot d_{TSX}
\]

\[
= Sa \cdot n_{TX} \cdot d_{STX} \cdot G^T\varepsilon^T_{(STX,m'X)} \cdot TS\varepsilon_X \quad \text{(naturality of } d) 
\]

\[
= Sa \cdot m'_X \cdot TS\varepsilon_X \quad \text{(unit law for } S) 
\]

\[
= a' \cdot TSa \cdot TS\varepsilon_X 
\]

\[
= a'.
\]

3.8.3 Corollary. The monads \(ST\) that are composites of \(S = (S,n,d)\) and \(T = (T,m,e)\)
are exactly those of the form

\[
ST = (ST, (n \circ m) \cdot S\delta T, d \circ e),
\]

where \(\delta\) is a distributive law of \(T\) over \(S\). There are also monad morphisms

\[
Se : S \to ST, \quad dT : T \to ST.
\]

Proof. By the Proposition, any composite monad comes from a distributive law, and the resulting multiplication \(w\) is then easily seen to be \((n \circ m) \cdot S\delta T\). The fact that \(Se\) and \(dT\) are monad morphisms follows by a routine verification. \(\square\)

3.8.4 Proposition. For a composite monad \(ST\) and the corresponding lifting \(\tilde{S}\) (see Proposition 3.8.2), the comparison functor \(K : (X^T)^{\tilde{S}} \to X^{ST}\) is an isomorphism.

Proof. The comparison functor sends an object \(((X,t),\tilde{s})\) of \((X^T)^{\tilde{S}}\) to the \(ST\)-algebra \((X,\tilde{s} \cdot St)\), and its inverse maps \((X,a)\) to \(((X,a \cdot d_{TX}),a \cdot Se_X)\). \(\square\)
3.8.5 Examples.

(1) For any monad $T$ on $X$, there are trivial distributive laws $1_T : 1_X T \to T1_X$ of the identity monad $\mathbb{1}$ over $T$, and $1_T : T1_X \to 1_X T$ of $\mathbb{1}$ over $T$.

(2) A distributive law $\delta : LG \to GL$ of the free-monoid monad $L$ over the free-abelian-group monad $G$ (Example 3.1.1(2)) is obtained by sending an element of $LGX$ of the form
\[
\left(\sum_{i_1=1}^{n_1} \alpha_{1,i_1} \cdot x_{1,i_1}, \ldots, \sum_{i_k=1}^{n_k} \alpha_{k,i_k} \cdot x_{k,i_k}\right)
\]
to the element of $GLX$ given by
\[
\sum_{i_1,j_2,\ldots,j_k} \alpha_{1,i_1} \alpha_{2,j_2} \cdots \alpha_{k,i_k} \cdot (x_{1,i_1}, x_{2,j_2}, \ldots, x_{k,i_k})
\]
The resulting category of $GL$-algebras is the category $\text{Rng}$ of unital rings and their homomorphisms (rings also occur as monoids in the category of abelian groups, see Example 4.2.1(2)).

From this example, distributive laws may be seen as a generalization of the usual notion of distributivity (of multiplication over addition) that holds in rings.

3.9 Distributive laws and extensions. For monads $S$ and $T$ on $X$, an extension of $T$ along $F_S : X \to X_S$ is a monad $\hat{T} = (\hat{T}, \hat{m}, \hat{e})$ on $X_S$ satisfying
\[
F_ST = \hat{T}F_S, \quad F_Sm = \hat{m}F_S, \quad F_Se = \hat{e}F_S.
\]
As is the case for liftings, the first condition insures that the last two equalities make sense. Moreover, since $F_S$ is identical on objects, one has $\hat{m}_X = F_Sm_X$, $\hat{e}_X = F_Se_X$ for all $X$-objects $X$, so the multiplication and unit laws are immediate. An extension of $T$ is therefore given by a functor $\hat{T} : X_S \to X_S$ making
\[
\begin{array}{ccc}
X_S & \xrightarrow{T} & X_S \\
F_S & \downarrow & F_S \\
X & \xrightarrow{T} & X
\end{array}
\]
commute, and such that $\hat{m}_X = F_Sm_X : TT X \to TX$, $\hat{e}_X = F_Se_X : X \to TX$ ($X \in \text{ob}X$) become the components of natural transformations $\hat{m} : \hat{T}\hat{T} \to \hat{T}$, $\hat{e} : 1_{X_S} \to \hat{T}$.
In the presence of a distributive law $\delta$ of $\mathbb{1}$ over $S$, an extension $\hat{T} : X_S \to X_S$ is obtained via
\[
X \mapsto TX \quad (f : X \to SY) \mapsto (\delta_Y \cdot Tf : TX \to STY)
\]
Functionality of $\hat{T}$ follows from naturality of $\delta$, and commutativity of the lower diagrams in (3.8.i) of 3.8. To verify naturality of $\hat{m}$ and $\hat{e}$ in $X_S$, one uses the other two diagrams.
An extension of $\mathbb{T}$ along $F_S : X \to X_S$ yields the following composite adjunction

$$(X_S)\mathbf{\hat{T}} \xymatrix{ & X_S \ar[r]^{G_\mathbb{T}} & X \ar[l]_{F_\mathbb{T}}}.$$

One can then consider the monad $S\mathbb{T} = (ST, w, d)$ as being induced by $F_\mathbb{T} F_S \dashv G_S G_\mathbb{T}$, with

$$w = Sm \cdot G_S \mathbf{\hat{T}} \varepsilon_S F_S T.$$

The fact that this monad is a composite of $S$ and $\mathbb{T}$ may then be checked directly. By 3.8, we can get back a distributive law from an extension, and we have:

**3.9.1 Proposition.** For monads $S$ and $\mathbb{T}$ on $X$, there is a bijective correspondence between:

(i) distributive laws $\delta$ of $\mathbb{T}$ over $S$;

(ii) extensions $\mathbf{\hat{T}}$ of $\mathbb{T}$ along $F_S : X \to X_S$.

*Proof.* An easy verification shows that when an extension is obtained via a distributive law, the induced composite monad is the same as in Proposition 3.8.2, so we only need to check that the path (ii) $\Rightarrow$ (i) $\Rightarrow$ (ii) returns the same extension. We denote by $\mathbf{\hat{T}}$ the original extension, by $\mathbf{\hat{T}}'$ the new one, and recall that $\varepsilon_{SY} = 1_{SY}$. If $f : X \to SY$ is an $X_S$-morphism, then

$$\mathbf{\hat{T}}' f = Sm_Y \cdot G_S \mathbf{\hat{T}} 1_{STY} \cdot G_S F_S T G_S e_Y \cdot d_{TSY} \cdot T f$$

$$= Sm_Y \cdot G_S \mathbf{\hat{T}}(1_{STY} \circ F_S G_S e_Y) \cdot d_{TSY} \cdot T f \quad \text{(functoriality of $\mathbf{\hat{T}}$)}$$

$$= Sm_Y \cdot G_S \mathbf{\hat{T}}(F_S e_Y \circ 1_{SY}) \cdot d_{TSY} \cdot T f \quad \text{(naturality of $\varepsilon_S$)}$$

$$= Sm_Y \cdot G_S F_S T e_Y \cdot G_S \mathbf{\hat{T}} 1_{SY} \cdot d_{TSY} \cdot T f \quad \text{($\mathbf{\hat{T}}$ an extension)}$$

$$= n_{TY} \cdot S \mathbf{\hat{T}} 1_{SY} \cdot d_{TSY} \cdot T f \quad \text{(unit law of $\mathbb{T}$)}$$

$$= n_{TY} \cdot d_{STY} \cdot \mathbf{\hat{T}} 1_{SY} \cdot T f \quad \text{(naturality of $d$)}$$

$$= \mathbf{\hat{T}} 1_{SY} \circ F_S T f \quad \text{(unit law of $S$)}$$

$$= \mathbf{\hat{T}} (1_{SY} \circ F_S f) \quad \text{($\mathbf{\hat{T}}$ an extension)}$$

$$= \mathbf{\hat{T}} f.$$

**3.9.2 Proposition.** For a composite monad $S\mathbb{T}$ and the corresponding extension $\mathbf{\hat{T}}$, the functor $L : (X_S)\mathbf{\hat{T}} \to X_{S\mathbb{T}}$ of Proposition 3.6.1 is the identity.

*Proof.* The statement is immediate once the definition of the composition in $(X_S)\mathbf{\hat{T}}$ has been unravelled. \qed
Exercises

3.A Trivial monads on \( \text{Set} \). An object \((X, a)\) of the category \( \text{Set}^\mathbb{T} \) is trivial if \( X \) is either the empty set \( \emptyset \) or a singleton \( 1 = \{\ast\} \). If there exists at least one non-trivial \( \mathbb{T} \)-algebra, then the unit \( e_X : X \to TX \) is injective for all sets \( X \). The only examples of trivial monads (monads that admit only trivial Eilenberg–Moore algebras) are the terminal monad \( 1 \) in \( \text{MND}_{\text{Set}} \), whose functor sends all sets \( X \) to 1, and the monad whose functor sends the empty set to itself and all other sets to 1.

Hint. \((TX, m_X)\) is a \( \mathbb{T} \)-algebra for any monad \( \mathbb{T} \).

3.B \( M \)-actions and equivariant maps. Any monoid \((M, m, e)\) (as in 1.1) yields a monad \( M = (M \times (-), m, e) \) on \( \text{Set} \), where \( m_X(a, (b, x)) = (m(a, b), x) \) and \( e_X(x) = (e, x) \). The Eilenberg–Moore category \( \text{Set}^M \) of this monad is the category \( M\text{-Set} \) of \( M \)-actions and equivariant maps which is isomorphic to the functor category \( \text{Set}^M \), with \( M \) considered as a one-object category. Every monoid homomorphism \( f : M \to N \) yields a monad morphism \( M \to N \); in fact, this operation describes a functor \( \text{Mon} \to \text{MND}_{\text{Set}} \).

3.C Split forks and coequalizers. A fork in a category \( X \) is a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} \quad & & \downarrow{h} \\
\quad & & Z
\end{array}
\]

such that \( h \cdot f = h \cdot g \). Show that a fork splits if and only if \( h \) is a coequalizer of \( (f, g) \) and there is \( j : Y \to X \) with \( f \cdot j = 1_Y \) and \( g \cdot j \cdot f = g \cdot j \cdot g \). If \( h \) is a split epimorphism and \( (f, g) \) is its kernel pair, then the fork splits.

3.D Creation of colimits by \( G^\mathbb{T} \). If \( \mathbb{T} \) is a monad on \( X \), the functor \( G^\mathbb{T} : X^\mathbb{T} \to X \) creates all colimits that exist in \( X \) and that are preserved by \( T \) and \( TT \). Therefore, \( G^\mathbb{T} \) creates coequalizers of \( G \)-absolute pairs.

3.E Full- and faithfulness of the comparison functor. The comparison functor \( K : \text{A} \to X^\mathbb{T} \) (see Proposition 3.2.1) is full and faithful if and only if \( \varepsilon_A \) is a regular epimorphism for all \( A \in \text{ob A} \); in that case, \( \varepsilon_A \) is coequalizer of \( (\varepsilon_{FGA}, FG\varepsilon_A) \) for all \( A \in \text{ob A} \).

3.F Strict monadicity criterion. For a functor \( G : \text{A} \to X \), the following conditions are equivalent:

(i) \( G \) is strictly monadic;

(ii) \( G \) is right adjoint and creates coequalizers of \( G \)-absolute pairs;

(iii) \( G \) is right adjoint and creates coequalizers of \( G \)-split pairs.

3.G Eilenberg–Moore algebras via Kleisli triples. If \( (T, (-)^\mathbb{T}, e) \) is a Kleisli triple on a category \( X \), then the Eilenberg–Moore algebras associated to the monad \( \mathbb{T} \) are those pairs...
(X, a) with X ∈ ob(X) and a : TX → X an X-morphism such that
\[ \forall f, g ∈ X(Y, TX) \ (a · f = a · g \implies a · f^T = a · g^T) \quad \text{and} \quad a · e_X = 1_X. \]

3.H Functor liftings and monad morphisms. Let \( F \xleftarrow{\varepsilon} G : A \to X \) be an adjunction with associated monad \( \mathcal{T} = (T, m, e) \) on X. For a monad \( S = (S, n, d) \) on Y, a lifting of a functor \( R : X \to Y \) through \( (G, G^S) \) is a functor \( \tilde{R} : A \to Y^S \) that makes the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\tilde{R}} & Y^S \\
\downarrow G & & \downarrow G^S \\
X & \xrightarrow{R} & Y
\end{array}
\]

commute. Equivalently, a lifting of \( R \) through \( (G, G^S) \) can be given by a natural transformation \( \alpha : SR \to RT \) satisfying
\[
Rm · \alpha T · S\alpha = \alpha · nR \quad \text{and} \quad Re = \alpha · dR. \quad (3.9.i)
\]

Indeed, such an \( \alpha \) yields a functor \( \tilde{R} : A \to Y^S \) via
\[
\tilde{R}A := (RGA, RG\varepsilon_A · \alpha_A),
\]
and conversely, a lifting \( \tilde{R} \) of \( R \) through \( (G, G^S) \) returns a natural transformation \( \alpha \) satisfying \( (3.9.i) \) via
\[
\alpha_X := \hat{m}_X · SRe_X,
\]
where \( \hat{m}_X : SRTX \to RTX \) denotes the Y-morphism defined by \( \tilde{R}FX = (RTX, \hat{m}_X) \).

As a consequence, liftings of \( 1_X : X \to X \) through \( (G, G^S) \) are in one-to-one correspondence with monad morphisms \( \alpha : S \to \mathcal{T} \). A lifting of a monad \( S \) along \( G^T \) (in the sense of \( 3.8 \)) provides a lifting of the underlying functor \( S : X \to X \) through \( (G^T, G^T) \).

3.I Functor extensions and monad morphisms. Consider an adjunction \( F \xleftarrow{\varepsilon} G : A \to X \) with associated monad \( \mathcal{T} = (T, m, e) \) on X. For a monad \( S = (S, n, d) \) on Y, an extension of a functor \( R : Y \to X \) along \( (F_S, F) \) is a functor \( \hat{R} : Y_S \to A \) that makes the following diagram commute:

\[
\begin{array}{ccc}
Y_S & \xrightarrow{\hat{R}} & A \\
\downarrow F_S & & \downarrow F \\
Y & \xrightarrow{R} & X
\end{array}
\]

Extensions of \( R \) along \( (F_S, F) \) are in one-to-one correspondence with natural transformations \( \alpha : RS \to TR \) satisfying
\[
mR · T\alpha · \alpha S = \alpha · Rn \quad \text{and} \quad eR = \alpha · Rd. \quad (3.9.ii)
\]
Indeed, given such an $\alpha$, one obtains a functor $\hat{R} : Y \to A$ defined on $Y$-objects $Y$ by $\hat{R}Y = FRY$, and on $Y$-morphisms $f : X \to SY$ by

$$\hat{R}f := \varepsilon_{FRY} \cdot F(\alpha_Y \cdot Rf)$$

conversely, an extension $\hat{R}$ of $R$ along $(F_S, F)$ returns a natural transformation $\alpha$ satisfying (3.9.ii) via

$$\alpha_Y := G\hat{R}1_{SY} \cdot e_{RSY}.$$  

In particular, an extension of $1_X : X \to A$ along $(F, S)$ is equivalently described by a monad morphism $\alpha : S \to T$, and an extension of a monad $S$ along $F_S$ (in the sense of 3.9) yields an extension of $T : X \to X$ along $(F_S, F_S)$. In fact, for monads $S$ and $T$ on $X$, a distributive law $\alpha$ of $T$ over $S$ is equivalent to a natural transformation $\alpha$ inducing both an extension of $T : X \to X$ along $(F, S)$ and a lifting of $S : X \to X$ through $(G^T, G^S)$ (as in Exercise 3.H).

3.J Distributive laws as monoids. A lifting $\alpha : SR \to RT$ of a functor $R : X \to A$ through $(G^T, G^S)$, where $S = (S, n, d)$ is a monad on $A$ and $T = (T, m, e)$ a monad on $X$ (see Exercise 3.H), can be seen as an arrow $(R, \alpha) : S \to T$. Composition with $(L, \beta) : T \to U$ is given via

$$(L, \beta) \circ (R, \alpha) := (RL, R\beta \cdot \alpha L).$$

Liftings from $S$ to $T$ form the objects of a metacategory $MNDLFT(S, T)$, whose morphisms $\lambda : (R, \alpha) \to (R', \alpha')$ are natural transformations $\lambda : R \to R'$ satisfying $\alpha' \cdot S\lambda = \lambda T \cdot \alpha$. In the same way that a monad is a monoid in a functor category (see 3.1), a distributive law $\delta$ of $T$ over $S$ is equivalently described as a monoid in $MNDLFT(T, T)$ given by the lifting $(S, \delta)$ together with two morphisms

$$n : (S, \delta) \circ (S, \delta) \to (S, \delta) \quad \text{and} \quad d : (1_X, 1_T) \to (S, \delta).$$

A similar procedure using functor extensions (see Exercise 3.I) leads to the description of a distributive law $\delta$ of $T$ over $S$ as a monoid $((T, \delta), m, e)$ in $MNDEXT(S, S)$, where $MNDEXT(S, T)$ denotes the metacategory of extensions from $S$ to $T$.


(1) Show that the following assertions are equivalent for an adjunction $F \xrightarrow{\eta} G : A \to X$

with induced monad $T$:

(a) for every object $A$ in $A$, $\varepsilon_A$ is a coequalizer of $\varepsilon_{FGA}$, $FG\varepsilon_A$;
(b) for every object $A$ in $A$, $\varepsilon_A$ is a regular epimorphism;
(c) the comparison functor $K : A \to X^T$ is full and faithful.

Adjunctions satisfying these equivalent conditions are said to be of descent type, in which case $G$ is called premonadic.
(2) Consider a commutative triangle of functors

\[
\begin{array}{ccc}
A & \xrightarrow{K} & B \\
\downarrow{G} & & \downarrow{J} \\
X & \xleftarrow{\delta} & B
\end{array}
\]

with \(G, J\) right adjoint, so that one has \(F \xrightarrow{\eta} G\), \(H \xrightarrow{\delta} J\), and \(J\) premonadic. With \(\mu : H \to KF\) determined by \(J\mu \cdot \delta = \eta\), set

\[
\alpha := FJ\gamma, \quad \beta := \varepsilon FJ \cdot FJ\mu J.
\]

Prove the equivalence of the following assertions for \(B \in \text{ob} B\):

(a) there exists a \(K\)-universal arrow for \(B\);
(b) there exists a coequalizer of \(\alpha_B, \beta_B\) in \(A\).

(3) For functors \(K : A \to B, J : B \to X\) with \(J\) premonadic and coequalizers existing in \(A\), the functor \(K\) has a left adjoint if and only if \(JK\) has a left adjoint.

3.L Cocompleteness of the Eilenberg–Moore category. For a monadic functor \(G : A \to X\), if \(A\) has coequalizers and \(X\) is \(J\)-cocomplete, then \(A\) is also \(J\)-cocomplete. In particular, for a monad \(T\) on a category \(X\) with coproducts, \(X^T\) is small-cocomplete whenever it has coequalizers.

Hint. Consider the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A^J \\
\downarrow{G} & & \downarrow{G^J} \\
X & \xleftarrow{\Delta} & X^J
\end{array}
\]

and apply 3.K(2).

3.M Idempotent monads. A monad \(T = (T, m, e)\) on \(X\) is idempotent if \(m : TT \to T\) is an isomorphism. For any adjunction

\[
F \xrightarrow{\eta} G : A \to X
\]

inducing \(T\), the following statements are equivalent:

(i) \(T\) is idempotent;
(ii) \(T\eta = \eta T\);
(iii) any one of \(G\varepsilon, \eta G, \varepsilon F, F\eta\) is an isomorphism;
(iv) for all $\mathcal{T}$-algebras $(X, a)$, the structure $a$ is an $X$-isomorphism;

(v) $G^\mathcal{T} : X^\mathcal{T} \to X$ restricts to an isomorphism of $X^\mathcal{T}$ with the full subcategory of $X$ containing those $X$ with $\eta_X$ an isomorphism;

(vi) $FG^\mathcal{T}$ is left adjoint to the comparison functor $K : A \to X^\mathcal{T}$ and the unit $\kappa$ of this adjunction satisfies $G^\mathcal{T}\kappa = \eta G^\mathcal{T}$.

These conditions hold in particular when $G$ or $F$ is full and faithful.
4 Monoidal and ordered categories

4.1 Monoidal categories. A monoidal category is a category $\mathcal{C}$ together with

1. a functor $(-) \otimes (-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
2. a distinguished object $E$ in $\mathcal{C}$,
3. natural isomorphisms
   
   \[ \alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C, \quad \lambda_A : E \otimes A \to A, \quad \rho_A : A \otimes E \to A \]

making the following diagrams commute for all objects $A, B, C, D$:

\[
\begin{array}{c}
A \otimes (B \otimes (C \otimes D)) \xrightarrow{\alpha_{A,B,C,D}} (A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha_{A,B,C,D}} ((A \otimes B) \otimes C) \otimes D \\
\downarrow_{1_A \otimes \alpha_{B,C,D}} \downarrow \alpha_{A,B,C} \\
A \otimes ((B \otimes C) \otimes D) \xrightarrow{\alpha_{A,B,C,D}} (A \otimes (B \otimes C)) \otimes D \\
\end{array}
\]

and satisfying $\lambda_E = \rho_E$.

The monoidal category $\mathcal{C}$ is symmetric if there are also

4. natural isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$ with $\sigma_{B,A} \cdot \sigma_{A,B} = 1_{A \otimes B}$ such that the following diagrams commute:

\[
\begin{array}{c}
A \otimes (B \otimes C) \xrightarrow{\alpha_{A,B,C}} (A \otimes B) \otimes C \xrightarrow{\sigma_{A,B,C}} C \otimes (A \otimes B) \\
\downarrow_{1_A \otimes \sigma_{B,C}} \downarrow \alpha_{A,C,B} \\
A \otimes (C \otimes B) \xrightarrow{\alpha_{A,C,B}} (A \otimes C) \otimes B \xrightarrow{\sigma_{A,C,B} \otimes 1_B} (C \otimes A) \otimes B \\
E \otimes A \xrightarrow{\sigma_{E,A}} A \otimes E \\
\end{array}
\]

Since $\lambda_E = \rho_E$, the commutative triangle gives $\sigma_{E,E} = 1_{E \otimes E}$.

This somewhat cumbersome definition is best illustrated in terms of examples.

4.1.1 Examples.

1. $\textbf{Set}$, with its cartesian structure $\otimes = \times$ (see 1.1) is symmetric monoidal. In fact, every category $\mathcal{C}$ with finite products can be considered a symmetric monoidal category (see Exercise 2.H).
(2) The prototype of a symmetric monoidal category is the category $\text{AbGrp}$ of abelian groups with its usual tensor product. (For abelian groups $A, B$, the abelian group $A \otimes B$ comes with a bilinear map $\otimes : A \times B \to A \otimes B$ such that any bilinear map $f : A \times B \to C$ factors as $f = g \cdot \otimes$, with a uniquely determined homomorphism $g : A \otimes B \to C$.) One puts $E = \mathbb{Z}$ and produces $\alpha, \lambda, \rho, \sigma$ with the universal property of the tensor product. More generally, such a construction works for the category $[\text{Mod}_R]$ of $R$-modules, where $R$ is a commutative unital ring.

(3) $\text{Sup}$ has a symmetric monoidal structure which may be constructed analogously to the tensor product of $\text{AbGrp}$. Indeed, for $X, Y, Z \in \text{ob} \; \text{Sup}$, call a mapping $f : X \times Y \to Z$ a bimorphism if $f(x, -) : Y \to Z$ and $f(-, y) : X \to Z$ are morphisms of $\text{Sup}$ for all $x \in X, y \in Y$. Then one can construct the universal bimorphism $\otimes : X \times Y \to X \otimes Y$ (so that every bimorphism $f$ factors as $f = g \cdot \otimes$, with a unique sup-map $g : X \otimes Y \to Z$) similarly to the tensor product of abelian groups: on the free sup-lattice $P(X \times Y)$ (see Example 2.5.1(5)), consider the least compatible congruence relation $\sim$ that makes the composite map

$$X \times Y \xrightarrow{\text{proj}} P(X \times Y) \xrightarrow{\text{proj}} P(X \times Y)/\sim$$

a bimorphism; this composite map is universal, hence $X \otimes Y = P(X \times Y)/\sim$.

Many of the monoidal categories considered in this book are strict, that is: $\alpha, \lambda, \rho$ can all be taken to be identity morphisms, so that the coherence conditions (expressed by the first two commutative diagrams) are trivially satisfied. These strict monoidal categories, however, typically fail to be symmetric:

(4) For a set $X$, consider the ordered set $\text{Rel}(X, X)$ of relations on $X$, seen as a category. Relational composition defines a strict monoidal structure on $\text{Rel}(X, X)$ (see 1.2).

(5) More generally, let $\mathcal{V}$ be an ordered set, considered as a category. $\mathcal{V}$ becomes strict monoidal precisely when it carries a monoid structure whose multiplication $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is monotone. In particular, every quantale $\mathcal{V}$ is a strict monoidal category.

(6) For a category $\mathcal{C}$ the functor category $\mathcal{C}^\mathcal{C}$ becomes a strict monoidal metacategory, with composition of functors as tensor product.

(7) If $\mathcal{C}$ is a monoidal category, then so is $\mathcal{C}^\text{op}$ (and similarly for a symmetric monoidal category $\mathcal{C}$).

A homomorphism of monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ which comes with natural isomorphisms

$$\delta_{A, B} : FA \otimes_D FB \to F(A \otimes \mathcal{C} B), \quad \varepsilon : E_D \to FE_{\mathcal{C}}$$
for all \( A, B \in \text{ob} \, C \), which commute with the natural isomorphisms defining the structure of the monoidal categories \( C \) and \( D \):

\[
\begin{align*}
FA \otimes (FB \otimes FC) & \xrightarrow{\alpha} (FA \otimes FB) \otimes FC \\
1 \otimes \delta & \quad \quad \delta \otimes 1
\end{align*}
\]

\[
\begin{array}{ccc}
FA \otimes F(B \otimes C) & F(A \otimes B) \otimes FC \\
\delta & \quad \quad \delta
\end{array}
\]

\[
\begin{array}{ccc}
F(A \otimes (B \otimes C)) & F((A \otimes B) \otimes C) \\
F\alpha & \\
\end{array}
\]

Observe that naturality of \( \delta \) makes \( F \) commute with \( \otimes \), up to isomorphism, not just for objects but also for morphisms:

\[
\begin{align*}
FA \otimes FB & \xrightarrow{Ff \otimes Fg} FC \otimes FD \\
\delta & \quad \quad \delta
\end{align*}
\]

\[
\begin{array}{ccc}
F(A \otimes B) & F(C \otimes D) \\
F(f \otimes g) & \\
\end{array}
\]

4.2 Monoids. Let \( C \) be a monoidal category as in 4.1. A monoid \( M \) in \( C \) is a \( C \)-object together with two morphisms

\[
m : M \otimes M \to M , \quad e : E \to M
\]

such that the diagrams

\[
\begin{array}{ccc}
M \otimes (M \otimes M) & (M \otimes M) \otimes M & M \otimes M \\
\alpha_{M,M,M} & m \otimes 1_M & m
\end{array}
\]

\[
\begin{array}{ccc}
E \otimes M & M \otimes M & M \otimes E \\
\lambda_M & m & \rho_M
\end{array}
\]

commute. A homomorphism of monoids \( f : (M, m, e) \to (N, n, d) \) is a \( C \)-morphism making the diagrams

\[
\begin{array}{ccc}
M \otimes M & N \otimes N \\
f & n & e
\end{array}
\]

\[
\begin{array}{ccc}
M & N \\
f & e & d
\end{array}
\]
CHAPTER II. MONOIDAL STRUCTURES

commute. The resulting category is denoted by $\text{Mon}_C$. A comonoid $M$ in $C$ is simply a monoid $M$ in $C^{\text{op}}$.

4.2.1 Examples.

(1) For $\text{Set}$ with its cartesian structure, $\text{Mon}_\text{Set} = \text{Mon}$.

(2) A unital ring $R$ is an abelian group that is also a monoid in which the distributive laws hold, that is, the multiplication $R \times R \to R$ is $\mathbb{Z}$-bilinear and is therefore equivalently described as a homomorphism $R \otimes R \to R$. Hence, unital rings are precisely the monoids in $\text{AbGrp}$ (with its usual tensor product), and $\text{Mon}_{\text{AbGrp}} = \text{Rng}$, the category of unital rings and their homomorphisms.

(3) A quantale $V$ is a complete lattice with a monoid operation $\otimes : V \times V \to V$ that preserves suprema in each variable; with the tensor product in $\text{Sup}$ (see Example 4.1.1(3)), the monoid operation may equivalently be considered a morphism $V \otimes V \to V$ in $\text{Sup}$. This way, one shows $\text{Mon}_{\text{Sup}} = \text{Qnt}$.

(4) Monoids in $\text{Rel}(X, X)$ (see Example 4.1.1(4)) are precisely orders on the set $X$ (see also 1.3).

(5) A monoid in a quantale $V$ (see Example 4.1.1(5)) is simply an idempotent element $v \in V$ (so that $v \otimes v = v$) with $k \leq v$. A comonoid in $V$ is an idempotent element $v \in V$ with $v \leq k$. A frame is thus an example of a commutative quantale in which every element is a comonoid.

(6) For a category $C$, monoids of the monoidal category $C^C$ (see Example 4.1.1(6)) are precisely monads on $C$, and homomorphisms of monoids are precisely morphisms of monads (see 3.1).

(7) A topology on a set $X$ may be defined by a monoid in $\text{SLat}(PX, PX)$ (see Exercise 1.F), where $PX$ is considered as a join-semilattice.

4.3 Actions. Let $C$ be a monoidal category, and $M = (M, m, e)$ a monoid in $C$. A left $M$-action (or simply an $M$-action) is an object $A$ in $C$ that comes with a $C$-morphism

$$a : M \otimes A \to A$$

such that (with the notations of 4.1 and 4.2) the following diagrams commute:

$$
\begin{array}{ccc}
M \otimes (M \otimes A) & \xrightarrow{1_M \otimes a} & M \otimes A \\
(m \otimes 1_A) \cdot \alpha_{M,M,A} & \downarrow & \downarrow a \\
M \otimes A & \xrightarrow{a} & A
\end{array}
$$

$$
\begin{array}{ccc}
E \otimes A & \xrightarrow{e \otimes 1_M} & M \otimes A \\
E \otimes A & \xrightarrow{\lambda_A} & A
\end{array}
$$
A C-morphism $f : A \to B$ between $M$-actions $(A, a)$ and $(B, b)$ is **equivariant** if

$$
\begin{array}{ccc}
M \otimes A & \xrightarrow{1_M \otimes f} & M \otimes B \\
\downarrow{a} & & \downarrow{b} \\
A & \xrightarrow{f} & B
\end{array}
$$

commutes. Similarly, a **right $M$-action** is an object $A$ in $C$ with a $C$-morphism $a : A \otimes M \to A$ making the corresponding diagrams commute.

Any monoid $(M, m, e)$ in a monoidal category $C$ gives rise to a monad $\mathbb{M} = (M \otimes (-), m, e)$ on $C$, where

$$
\overline{m}_A = (m \otimes 1_A) \cdot \alpha_{M,M,A} \quad \text{and} \quad \overline{e}_A = (e \otimes 1_A) \cdot \lambda_A^{-1}
$$

for all $A \in \text{ob} C$ (using the notations of 4.2). The Eilenberg–Moore category $C^\mathbb{M}$ of this monad is the category of $M$-actions and equivariant $C$-morphisms.

**4.3.1 Examples.**

1. If $C = \text{Set}$ the category of $M$-actions is the category $M\text{-Set}$ of $M$-actions and equivariant maps (see Exercise 3.B).

2. The monoidal structure of $\text{AbGrp}$ is given by the tensor product over $\mathbb{Z}$ (Example 4.1.1(2)), and a monoid $R$ in $\text{AbGrp}$ is a ring (Example 4.2.1(2)). Hence, $\text{AbGrp}^R$ (with $\mathbb{R}$ the monad induced by the monoid $R$) is the usual category of left $R$-modules.

3. Given a quantale $\mathcal{V} = (V, \otimes, k)$, that is, a monoid in $\text{Sup}$ (Example 4.2.1(3)), the category $\text{Sup}^\mathcal{V}$ is described as follows. A $\mathcal{V}$-action $X$ in $\text{Sup}$ is a complete lattice $X$ together with a bimorphism $(-) \cdot (-) : \mathcal{V} \times X \to X$ in $\text{Sup}$ (Example 4.1.1(3)) such that $$(u \otimes v) \cdot x = u \cdot (v \cdot x), \quad k \cdot x = x,$$ for all $v \in \mathcal{V}$, $x \in X$, and a sup-map $f : X \to Y$ is equivariant whenever $$f(v \cdot x) = v \cdot f(x)$$ for all $v \in \mathcal{V}$, $x \in X$. Objects in $\text{Sup}^\mathcal{V}$ are also known as $\mathcal{V}$-modules (see [Kruml and Paseka, 2008]), but here we reserve this name for a generalization of the term introduced in 1.4 (see 1.3).

**4.4 Monoidal closed categories.** An object $A$ in a monoidal category $C$ (as in 4.1) is $\otimes$-**exponentiable** if the functors $A \otimes (-), (-) \otimes A : C \to C$

have right adjoints $A \dashv (-), (-) \dashv A : C \to C$,
respectively, that is: if for all objects \(B, C\) in \(C\) there are bijections
\[
\frac{A \otimes B \to C}{B \to (A \leftrightarrow C)} \quad \frac{B \otimes A \to C}{B \to (C \leftrightarrow A)}
\]
which are natural in \(B\). The functor \(A \to (\_\_\_)\) is the right internal hom-functor of \(A\), and \(\_\_\_ \leftrightarrow A\) is the left internal hom-functor. Obviously, when \(C\) is symmetric, \((A \to (\_\_\_)) \cong ((\_\_\_) \leftrightarrow A)\), and there is no need for distinction between right and left, so either of the two notations may be used. The monoidal category \(C\) is closed if every object is \(\otimes\)-exponentiable.

If the monoidal structure of \(C\) is the cartesian structure of \(C\) (given by finite direct products), then one says exponentiable instead of \(\times\)-exponentiable and usually writes
\[
B^A
\]
for the internal hom-object \(A \to B \cong B \leftrightarrow A\), also called an exponential, and one says that \(C\) is cartesian closed if it is closed.

4.4.1 Examples.

1. \textbf{Set}, \textbf{Ord}, and \textbf{Cat} are cartesian closed. For \textbf{Set}, see Example 2.5.1(4). Essentially the same proof can be used to prove cartesian closure of \textbf{Ord}, with
\[
B^A = \text{Ord}(A, B)
\]
provided with its pointwise order. In \textbf{Cat}, the functor category \(B^A\) serves as internal hom-object, see Exercise 4.A.

2. \textbf{AbGrp}, and more generally \textbf{Mod}_R (for a commutative unital ring \(R\)) are monoidal closed, with
\[
A \leftrightarrow B = \text{hom}_R(A, B) = \text{Mod}_R(A, B)
\]

3. \textbf{Sup} is monoidal closed, with
\[
X \rightarrow Y = \text{Sup}(X, Y)
\]
provided with the pointwise structure.

4. \textbf{Rel}(\(X, X\)), considered as a monoidal category as in Example 4.1.1(4), is monoidal closed, and one has
\[
\begin{align*}
x (r \leftrightarrow s) y & \iff \forall z \in X (y r z \implies x s z), \\
x (s \leftrightarrow r) y & \iff \forall z \in X (z r x \implies z s y),
\end{align*}
\]
for all \(x, y \in X\).
(5) The internal hom-objects in a quantale $\mathcal{V}$ have been described in 1.10; hence $\mathcal{V}$ is monoidal closed. As mentioned in Example 4.1.1(5), any ordered monoid (that is, any ordered set with a monotone monoid operation) can be considered a monoidal category. In particular, every monoid $\mathcal{V}$, provided with the discrete order, is a monoidal category, and it is closed precisely when it is a group. In that case, for $u, v \in \mathcal{V}$, one has

$$u \dashv v = u^{-1}v, \quad v \sqcup u = vu^{-1},$$

which provides some justification for the internal hom notation in general.

(6) For a category $\mathcal{C}$, the monoidal category $\mathcal{C}$ of Example 4.1.1(6) is rarely closed. For example, when $\mathcal{C}$ is a small discrete category, $\mathcal{C}$ is the monoid $\text{Set}(X, X)$ with $X = \text{ob} \mathcal{C}$ which, according to the previous example, is monoidal closed only when it is a group, that is, when $|X| \leq 1$.

4.4.2 Proposition. In a locally small monoidal closed category $\mathcal{C}$, one has natural isomorphisms

1. $(A \otimes B) \rightarrow C \cong B \rightarrow (A \rightarrow C);
2. C \leftarrow (A \otimes B) \cong (C \leftarrow B) \leftarrow A;
3. (A \rightarrow C) \rightarrow B \cong A \rightarrow (C \rightarrow B).

If $\mathcal{C}$ is cartesian closed, then $\mathcal{C}^{A \times B} \cong (\mathcal{C}^{A})^{B} \cong (\mathcal{C}^{B})^{A}$.

Proof. For all objects $X$, one has

$$X \rightarrow ((A \otimes B) \rightarrow C)$$

$$\xrightarrow{(A \otimes B) \otimes X \rightarrow C}$$

$$\xrightarrow{A \otimes (B \otimes X) \rightarrow C}$$

$$\xrightarrow{B \otimes X \rightarrow (A \rightarrow C)}$$

$$\xrightarrow{X \rightarrow (B \rightarrow (A \rightarrow C))} ,$$

which implies (1) (because the Yoneda embedding reflects isomorphisms, see Corollary 2.4.2 and Exercise 2.A). Point (2) is proved analogously to (1). For (3), we note

$$X \rightarrow ((A \rightarrow C) \leftrightarrow B)$$

$$\xrightarrow{X \otimes B \rightarrow (A \rightarrow C)}$$

$$\xrightarrow{A \otimes (X \otimes B) \rightarrow C}$$

$$\xrightarrow{(A \otimes X) \otimes B \rightarrow C}$$

$$\xrightarrow{A \otimes X \rightarrow (C \rightleftharpoons B)}$$

$$\xrightarrow{X \rightarrow (A \rightleftharpoons (C \rightleftharpoons B))} .$$

$\square$
Given an object $C$ of a monoidal closed category $C$, there is a functor
\[ (\_ \mapsto C) : C^{\text{op}} \to C , \quad (f : A \to B) \mapsto (\_ \mapsto 1_C : (B \to C) \to (A \to C)) , \]
with $f \mapsto 1_C$ the unique $C$-morphism that makes the diagram commute (here $ev^A : A \otimes (A \otimes (\_)) \to 1_C$ denotes the counit of the adjunction $A \otimes (\_) \dashv (\_)^A$). One can naturally define a functor $C \mapsto (\_)$ that will be isomorphic to $(\_ \mapsto C)$ whenever $C$ is symmetric. In this case, the functor is also right adjoint:

**4.4.3 Proposition.** For a symmetric monoidal closed category $C$, the functor $(\_ \mapsto C) : C^{\text{op}} \to C$ is self-adjoint (for all $C$-objects $C$).

**Proof.** By couniversality of the component at $C$ of the counit $ev^A \cdot C : (A \mapsto C) \otimes (\_ \mapsto C) \mapsto 1_C$, there is a unique $C$-morphism $u_A : A \to ((A \mapsto C) \otimes C)$ making the following diagram commute:

\[
\begin{array}{ccc}
(A \mapsto C) \otimes ((A \mapsto C) \otimes C) & \xrightarrow{ev^A \cdot C} & C \\
1_A \otimes u_A & \downarrow & \\
(A \mapsto C) \otimes A & \xrightarrow{ev^A \cdot C \cdot \sigma_{A \mapsto C,A}} & (A \mapsto C) \otimes A
\end{array}
\]

The morphism $u_A$ is a $((\_ \mapsto C))$-universal arrow for $A$. Indeed, given a $C$-morphism $f : A \to (B \mapsto C)$, we define $g : B \to (A \mapsto C)$ as the unique $C$-morphism such that

\[
\begin{array}{ccc}
A \otimes (\_ \mapsto C) & \xrightarrow{ev^A} & C \\
1_A \otimes g & \downarrow & \\
A \otimes B & \xrightarrow{ev^B \cdot (1_B \otimes f)} & \sigma_{A \mapsto B}
\end{array}
\]

commutes; since

\[
\begin{align*}
\ev_C^B \cdot (1_B \otimes ((g \mapsto 1_C) \cdot u_A)) & = \ev_C^B \cdot (1_B \otimes (g \mapsto 1_C)) \cdot (1_B \otimes u_A) \\
& = \ev_C^A \cdot (g \otimes 1_{(A \mapsto C)}) \cdot (1_B \otimes u_A) \\
& = \ev_C^A \cdot (1_{A \mapsto C} \otimes u_A) \cdot (g \otimes 1_A) \\
& = \ev_C^A \cdot \sigma_{A \mapsto C,A} \cdot (g \otimes 1_A) \\
& = \ev_C^A \cdot (1_A \otimes g) \cdot \sigma_{B,A} \\
& = \ev_C^B \cdot (1_B \otimes f)
\end{align*}
\]

(definition of $g \mapsto 1_C$)

(definition of $u_A$)

(definition of $g$),
one can conclude that \((g \circ 1_C) \cdot u_A = f\) by couniversality of \(ev_B^C\). The displayed identities also show that \(g\) is uniquely determined, so that \(u_A\) is universal as claimed. To see that 

\(((−) \to C)^{\text{op}} : C \to C^{\text{op}}\) is the required left adjoint, it suffices to verify that the morphisms \(u_A\) (with \(A \in C\)) form a natural transformation \(u : 1_C \to (((−) \to B) \to B)\) (Theorem 2.5.4); by couniversality of \(ev_B^C\), this follows from the identities

\[
ev_B^C \cdot (1_B \otimes (u_B \cdot f)) = ev_B^C \cdot \sigma_B \cdot (1_B \otimes \otimes) \cdot (\sigma_B \cdot f)
\]

(4.5.1) Examples. 

(1) \(\text{Rel}\) is an ordered category, with \(\text{Rel}(X, Y)\) ordered by inclusion for all sets \(X, Y\).

(2) \(\text{Mod}\) is, like \(\text{Rel}\), an ordered category.

(3) \(\text{Ord}\) is an ordered category, with \(\text{Ord}(X, Y)\) carrying the pointwise order.
(4) \textbf{Sup} is, like \textbf{Ord}, an ordered category.

(5) Every category can be considered an ordered category when provided with the discrete order.

Note that for any object $A$ of an ordered category $C$, $C(A, A)$ can be considered a strict monoidal category, with the tensor product given by composition in $C$ (see Examples 4.1.1(4), (5), and (6)).

If $C$ is an ordered category, then its dual $C^{\text{op}}$ (obtained by “turning arrows around”) is also ordered. Moreover, one can form the conjugate ordered category $C^{\text{co}}$ that leaves the arrows intact but turns around the order:

$$C^{\text{co}}(A, B) = (C(A, B), \geq) = (C(A, B))^{\text{op}}.$$ 

There is only one combination of the two dualization processes, since

$$C^{\text{co} \text{op}} = C^{\text{op} \text{co}}.$$ 

4.6 \textbf{Lax functors, pseudo-functors, 2-functors, and their transformations.} A \textit{lax functor} $F : C \to D$ of ordered categories is given by functions

$$F : \text{ob } C \to \text{ob } D \quad \text{and} \quad F_{A, B} : C(A, B) \to D(FA, FB)$$

for all $A, B \in \text{ob } C$ (with $F_{A, B}$ usually written as $F$), such that

(1) $F_{A, B}$ is monotone;

(2) $Fg \cdot Ff \leq F(f \cdot g)$;

(3) $1_{FA} \leq F1_A$;

for all $A, B, C \in \text{ob } C$, $f : A \to B$, $g : B \to C$ in $C$. The lax functor $F$ is a \textit{pseudo-functor} if the inequalities “$\leq$” in (2) and (3) may be replaced by “$\simeq$” (see 1.3), and $F$ is a \textit{2-functor} if they may be replaced by “$=$”.

An \textit{oplax functor} $F : C \to D$ must satisfy condition (1) and, instead of (2) and (3),

(2*) $F(f \cdot g) \leq Ff \cdot Fg$;

(3*) $F1_A \leq 1_{FA}$.

Hence, pseudo-functors are precisely the simultaneously lax and oplax functors.

This terminology coincides with the one used in the more general context of 2-categories (see Borceux 1994a). If one thinks of $C$ and $D$ as ORD-enriched categories, then 2-functors are precisely ORD-functors (see Kelly 1982). For ordinary categories, considered as discrete
ordered categories, all the functor notions just introduced coincide with the ordinary notion of functor.

For lax or oplax functors $F, G : C \to D$, a *lax transformation* $\alpha : F \to G$ is given by an ob $C$-indexed family of $D$-morphisms $\alpha_A : FA \to GA$ such that for all $f : A \to B$ in $C$,

\[ (4) \quad Gf \cdot \alpha_A \leq \alpha_B \cdot Ff, \]

while an *oplax transformation* satisfies

\[ (4^*) \quad \alpha_B \cdot Ff \leq Gf \cdot \alpha_A \]

instead. A *pseudo-natural transformation* is a simultaneously lax and oplax transformation, and the prefix “pseudo” may be dropped if the order of the hom-class of $D$ is separated.

Lax transformations can be *vertically composed* and lead to a lax transformation again, via $(\beta \cdot \alpha)_A = \beta_A \cdot \alpha_A$ (for $\alpha : F \to G$, $\beta : G \to H$); oplax transformations compose similarly. *Horizontal composition* is more delicate. Here we only mention that for a lax or oplax functor $S : B \to C$ and a lax transformation $\alpha : F \to G$, one obtains a lax transformation $\alpha S : FS \to GS$; one can proceed analogously with oplax transformations. However, “whiskering” from the other side with $T : D \to E$ requires a pseudo-functor in order to obtain from $\alpha$ again a lax or oplax transformation $T\alpha : TF \to TG$.

4.7 Maps. A morphism $f : A \to B$ in an ordered category $C$ is a *map* if there is a morphism $g : B \to A$ in $C$ with

\[ 1_A \leq g \cdot f \quad \text{and} \quad f \cdot g \leq 1_B ; \]

one writes $f \dashv g$ in this situation, and calls $f$ the *left adjoint* and $g$ the *right adjoint* of the adjunction.

For $C = \text{Ord}$, being a map means being left adjoint (see [1.5]). The map terminology is however better motivated by the example $C = \text{Rel}$: a relation $r : A \to B$ of sets $A, B$ is a map in $\text{Rel}$ precisely when it is the graph of a morphism $r : A \to B$ in $\text{Set}$. Indeed, the existence of a relation $s : B \to A$ with

\[ 1_A \leq s \cdot r \quad \text{means} \quad \forall x \in A \exists y \in B \ (x r y \ & \ y s x) , \]

while

\[ r \cdot s \leq 1_B \quad \text{means} \quad \forall x \in A \forall y, y' \in B \ (y s x \ & \ x r y' \implies y = y') . \]

Hence, the expression $(r(x) = y \iff x r y)$ defines a unique $\text{Set}$-map $r : A \to B$. One can also easily see that necessarily $s = r^\circ$ (see Exercise [4.B]).

Of course, in any ordered category (or more generally, in a 2-category), a right adjoint $g$ of a map $f$ as above is uniquely determined up to “≃”; we often write $f^*$ (or $f^\sim$) for $g$:

\[ 1_A \leq f^* \cdot f \quad \text{and} \quad f \cdot f^* \leq 1_B . \]
This notation is coherent with the fact that for a monotone map \( f : A \to B \) of ordered sets one has \( f_* \dashv f^* \) in \( \text{Mod} \) (see \[1.4\]). Adjunctions in \( C \) can be detected by “hom-ing” into \( \text{ORD} \):

4.7.1 Proposition. For morphisms \( f : A \to B \) and \( g : B \to A \) in an ordered category \( C \), one has \( f \dashv g \) if and only if \( C(T,f) \dashv C(T,g) \) in \( \text{ORD} \) for all \( T \in \text{ob} \; C \).

Proof. \( C(T,f) \dashv C(T,g) \) means (by Proposition \[1.5.1\]) that

\[
C(T,f)(x) \leq y \iff x \leq C(T,g)(y)
\]

for all \( x \in C(T,A), \; y \in C(T,B) \); that is:

\[
f \cdot x \leq y \iff x \leq g \cdot y .
\]

But this equivalence (for all \( x,y \)) means exactly \( f \dashv g \) in \( C \) (for which one may re-employ the proof of Proposition \[1.5.1\]). \( \square \)

4.8 Quantaloids. A quantaloid is a category \( C \) with each hom-class being a complete lattice (although not necessarily small), and with composition preserving suprema on either side:

\[
g \cdot \left( \bigvee_{i \in I} f_i \right) = \bigvee_{i \in I} (g \cdot f_i) , \quad \left( \bigvee_{i \in I} g_i \right) \cdot f = \bigvee_{i \in I} (g_i \cdot f)
\]

for all \( C \)-morphisms \( f, f_i : A \to B, \; g, g_i : B \to C \) \((i \in I)\). A quantaloid is therefore an ordered category. However, \( C \) is not just \( \text{ORD} \)-enriched, but \( \text{SUP} \)-enriched (where \( \text{SUP} \) denotes the metacategory of sup-complete classes and sup-maps). A homomorphism of quantaloids \( F : C \to D \) is simply a functor that preserves suprema:

\[
F\left( \bigvee_{i \in I} f_i \right) = \bigvee_{i \in I} Ff_i .
\]

4.8.1 Examples.

1. The ordered categories \( \text{Rel} \) and \( \text{Sup} \) are quantaloids, but \( \text{Ord} \) is not. An abstract category provided with the discrete order is not a quantaloid either, unless it is an ordered class (so that its hom-classes are at most singleton sets).

2. Quantales are precisely the one-object quantaloids. Indeed, one-object quantaloids have a monoid structure that can be considered as the tensor product of the quantale (see \[1.10\]). A homomorphism of one-object quantaloids is precisely a homomorphism of quantales.

3. For every quantaloid \( C \), its opposite category \( C^{\text{op}} \) is a quantaloid as well, but \( C^{\text{co}} \) is generally not so: composition will preserve infima from either side, but not suprema in general.
A biproduct of a family \((A_i)_{i \in I}\) of objects in a quantaloid \(C\) is given by an object \(P\) together with morphisms
\[
P \xrightarrow{p_i} A_i \quad (i \in I)
\]
such that
\[
p_i \cdot e_j = \begin{cases} 1_{A_i} & \text{if } i = j \\ \bot & \text{otherwise} \end{cases}
\]
and
\[
\bigvee_{i \in I} e_i \cdot p_i = 1_P .
\]
The terminology explains itself with the following statement.

4.8.2 Proposition. For a biproduct \((P, p_i, e_i)_{i \in I}\) of \((A_i)_{i \in I}\) in a quantaloid, \((P, p_i)_{i \in I}\) is a product and \((P, e_i)_{i \in I}\) is a coproduct of \((A_i)_{i \in I}\).

Proof. For \(f_i : B \to A_i, (i \in I)\), \(f := \bigvee_{i \in I} e_i \cdot f_i\) is the only morphism with \(p_i \cdot f = f_i (i \in I)\). Likewise, given \(g_i : A_i \to B (i \in I)\), \(g := \bigvee_{i \in I} g_i \cdot p_i\) is the only morphism with \(g \cdot e_i = g_i (i \in I)\).

4.8.3 Example. In \(\text{Rel}\), the biproduct of \((A_i)_{i \in I}\) can be taken to be the coproduct (that is, the disjoint union in \(\text{Set}\)) \((e_i : A_i \to C)_{i \in I}\) with \(p_i = e_i^\circ (i \in I)\).

For a morphism \(f : A \to B\) in a quantaloid \(C\), the sup-map
\[
C(C, f) = f \cdot (-) : C(C, A) \to C(C, B)
\]
has, for all objects \(C\), a right adjoint, denoted by \(f \dashv (-)\) and characterized by
\[
\frac{f \cdot g \leq h}{g \leq f \dashv h} \quad \frac{A \xrightarrow{f} B}{g \xleftarrow{\leq} h} \quad \frac{\downarrow g}{\text{C}}
\]
In this situation, \(g\) is called a lifting of \(h\) along \(f\). Dually, for \(f : B \to A\) in \(C\), the sup-maps
\[
C(f, C) = (-) \cdot f : C(A, C) \to C(B, C)
\]
have right adjoints, denoted by \((-) \dashv f\) and characterized by
\[
\frac{g \cdot f \leq h}{g \leq h \dashv f} \quad \frac{B \xrightarrow{f} A}{g \xleftarrow{\geq} h} \quad \frac{\downarrow g}{\text{C}}
\]
One calls \(g\) an extension of \(h\) along \(f\).

For \(A = B = C\), the notations used here coincide with those introduced in 4.4 when the monoidal category is taken to be \(C(A, A)\).
Expanding on Proposition 4.7.1, one can prove the following.

**4.8.4 Proposition.** For morphisms \( f : A \to B \) and \( g : B \to A \) in a quantaloid \( C \), the following are equivalent:

(i) \( f \dashv g \) in \( C \);

(ii) \( C(T, f) \dashv C(T, g) \) in \( \text{SUP} \) for all \( T \in \text{ob} \ C \);

(iii) \( C(g, T) \dashv C(f, T) \) in \( \text{SUP} \) for all \( T \in \text{ob} \ C \);

(iv) \( C(T, g) = f \cdot (-) \) in \( \text{SUP} \) for all \( T \in \text{ob} \ C \);

(v) \( C(f, T) = (-) \cdot g \) in \( \text{SUP} \) for all \( T \in \text{ob} \ C \).

**Proof.** Note that, when deriving (i) from any of (ii)–(v), the choices \( T = A \) and \( T = B \) suffice. Further details are left to the reader. \( \square \)

**4.9 Kock–Zöberlein monads.** A monad \( \mathbb{T} = (T, m, e) \) on an ordered category \( C \) is of **Kock–Zöberlein type** (or simply **Kock–Zöberlein**) or **lax idempotent** if \( T \) is a 2-functor, and for every \( C \)-object \( X \) there is a chain of adjunctions:

\[
T e_X \dashv m_X \dashv e_{T X}.
\]

This condition can be replaced by any of the following equivalent expressions:

\[
\forall X \in \text{ob} \ C \ (T e_X \leq e_{T X}) \iff \forall X \in \text{ob} \ C \ (T e_X \dashv m_X) \iff \forall X \in \text{ob} \ C \ (m_X \dashv e_{T X}).
\]

Indeed, if any one of the previous conditions holds, then the two others can be obtained by using the following implications:

\[
\begin{align*}
T e_X \leq e_{T X} & \Rightarrow T e_X \cdot m_X = m_{T X} \cdot T T e_X \leq m_{T X} \cdot T e_{T X} = 1_{T X}, \\
T e_{T X} \cdot m_{T X} \leq 1_{TTT X} & \Rightarrow 1_{TT X} = T m_X \cdot T e_{T X} \cdot m_{T X} \cdot e_{T T X} \leq T m_X \cdot e_{T T X} = e_{T X} \cdot m_X, \\
1_{T T X} \leq e_{T X} \cdot m_X & \Rightarrow T e_X \leq e_{T X} \cdot m_X \cdot T e_X = e_{T X}.
\end{align*}
\]

The study of the Eilenberg–Moore algebras is facilitated in the case of a Kock–Zöberlein monad: if \( a : TX \to X \) is a \( C \)-morphism with \( a \cdot e_X \simeq 1_X \), then one has

\[
1_{T X} \simeq T a \cdot T e_X \leq T a \cdot e_{T X} = e_X \cdot a,
\]

so that \( a \vdash e_X \). The adjunctions \( T a \vdash T e_X \), \( a \vdash e_X \) and \( m_X \vdash e_{T X} \) yield

\[
a \cdot T a \vdash T e_X \cdot e_X \quad \text{and} \quad a \cdot m_X \vdash e_{T X} \cdot e_X
\]

and therefore \( a \cdot T a \simeq a \cdot m_X \) because each side is a left adjoint of \( T e_X \cdot e_X = e_{T X} \cdot e_X \).

**4.9.1 Proposition.** Let \( \mathbb{T} = (T, m, e) \) be a monad on an ordered category \( C \). The following are equivalent when \( T \) is a 2-functor:

(i) \( \mathbb{T} \) is of Kock–Zöberlein type;
(ii) any \( C \)-morphism \( a : TX \to X \) with \( a \cdot e_X \simeq 1_X \) is left adjoint to \( e_X \) and satisfies \( a \cdot T \cdot a \simeq a \cdot m_X \).

**Proof.** The proof of (i) \( \implies \) (ii) is given in the previous discussion. For (ii) \( \implies \) (i), notice that if (ii) holds, then \( m_X \cdot e_{TX} = 1_{TX} \) implies \( m_X \dashv e_{TX} \), so that \( T \) is Kock–Zöberlein. \( \square \)

**4.9.2 Corollary.** The Eilenberg–Moore category \( C^T \) of a Kock–Zöberlein monad \( T = (T, m, e) \) on a separated ordered category \( C \) may be described as follows: objects are those \( C \)-objects \( X \) for which \( e_X \) is a section; morphisms are those \( C \)-morphisms that commute in the usual way with the retractions of the unit.

**Proof.** When the order on the hom-sets of \( C \) is separated, \( \simeq \) becomes an equality in Proposition [4.9.1] and left adjoints are uniquely determined. \( \square \)

**4.9.3 Example.** The down-set functor \( D_n : \text{Ord} \to \text{Ord} \) (see [2.2]) with the union maps \( V_{D_n X} = V : D_n D_n X \to D_n X \) (where \( V A = \bigcup_{A \in A} A \) for all \( A \in D_n D_n X \)) and the down-set maps \( \downarrow_X : X \to D_n X \) form the down-set monad

\[
D_n = (D_n, V_{D_n}, \downarrow)
\]

on \( \text{Ord} \). As \( V_{D_n X} \) is left adjoint to the down-set map \( \downarrow_{D_n X} \) and the down-set functor is monotone, the monad \( D_n \) is of Kock–Zöberlein type. The up-set monad \( U_p = (U_p, \Lambda_{U_p}, \uparrow) \) is dual Kock–Zöberlein (that is, of dual Kock–Zöberlein type or colax idempotent), as it is Kock–Zöberlein with the hom-sets equipped with their dual order (the infimum map is right adjoint to the up-set one); note that because \( U_p X \) is ordered by reverse inclusion (see [1.7]), the map \( \Lambda_{U_p X} : U_p U_p X \to U_p X \) is given by set-theoretic union.

The down-set and the up-set monads on \( \text{Ord} \) restrict to separated ordered sets, so Proposition [4.9.1] and [1.7] yield 2-isomorphisms

\[
\text{Ord}_{\text{sep}}^{D_n} \simeq \text{Sup} \text{ and } \text{Ord}_{\text{sep}}^{U_p} \simeq \text{Inf} .
\]

**4.10 Enriched categories.** A locally small ordered category \( C \) has its hom-sets and composition operation live in the cartesian closed category \( \text{Ord} \). More generally, one may consider any monoidal category \( V \) instead of \( \text{Ord} \) and, keeping the notations of [1.7], define a \( V \)-category \( C \) to be given by a class \( \text{ob} C \) of objects, a hom-object \( C(A, B) \in \text{ob} V \) for each pair \( A, B \in \text{ob} C \), composition and identity operations in \( V \)

\[
m_{A,B,C} : C(A, B) \otimes C(B, C) \to C(A, C) \text{ and } e_A : E \to C(A, A) ,
\]
for all $A, B, C \in \text{ob } C$, subject to the associativity and identity laws which require that the diagrams

$$C(A, B) \otimes (C(B, C) \otimes C(C, D)) \xrightarrow{\alpha} (C(A, B) \otimes C(B, C)) \otimes C(C, D) \xrightarrow{\text{m} \otimes 1} C(A, C) \otimes C(C, D)$$

$$C(A, B) \otimes C(B, D) \xrightarrow{\text{m}} C(A, D)$$

$$E \otimes C(A, B) \xrightarrow{e \otimes 1} C(A, A) \otimes C(A, B)$$

$$C(A, B) \otimes C(B, B) \xleftarrow{1 \otimes e} C(A, B) \otimes E$$

commute in $V$.

For $V = \text{Set}$ and $V = \text{Ord}$, the definition of a $V$-category returns the ordinary notions of locally small category and of locally small ordered category, respectively. A 2-category is simply a CAT-category, where again the monoidal structure is given by the cartesian product. Starting with Chapter II a major part of this book is devoted to the study of $V$-categories when $V$ is merely a quantale, and generalizations thereof which involve the consideration of a monad on Set. If $V$ is $\text{AbGrp}$, the prototype of a monoidal category, then $V$-categories are called additive categories.

The general theory of $V$-categories can be found in [Kelly, 1982]. For the reader’s convenience, we mention here the definitions of $V$-functors and $V$-natural transformations. A $V$-functor $F : C \to D$ of $V$-categories $C, D$ is given by a function $F : \text{ob } C \to \text{ob } D$ and morphisms

$$F_{A,B} : C(A, B) \to D(FA, FB)$$

in $V$ for all $A, B \in \text{ob } C$, subject to the commutativity of the diagrams

$$C(A, B) \otimes C(B, C) \xrightarrow{F \otimes F} D(FA, FB) \otimes D(FB, FC)$$

$$C(A, C) \xrightarrow{F} D(FA, FC)$$

$$E \xleftarrow{e} D(FA, GA)$$

For $V$-functors $F, G : C \to D$, a $V$-natural transformation $\tau : F \to G$ is given by morphisms

$$\tau_A : E \to D(FA, GA)$$
in \( V \) such that the diagrams

\[
\begin{array}{ccc}
C(A, B) \otimes E & \xrightarrow{F \otimes B} & D(FA, FB) \otimes D(FB, GB) \\
\rho^{-1} & & m \\
E \otimes C(A, B) & \xrightarrow{TA \otimes G} & D(FA, GA) \otimes D(GA, GB) \\
\lambda^{-1} & & m
\end{array}
\]

commute. Again, for \( V = \text{Set} \), one obtains the ordinary categorical notions, and for \( V = \text{Ord} \), a \( V \)-functor is a 2-functor of ordered categories (see [4.6]), while a \( V \)-natural transformation is simply a natural transformation.

**Exercises**

4.A **Cartesian closedness of \( \text{Cat} \).** In \( \text{Cat} \), the functor category \( B^A \) serves as internal hom-object.

4.B **Relational adjoints to maps.** In \( \text{Rel} \), if \( r : A \to B \) is a map and \( s : B \to A \) is its right adjoint, then one has \( s = r^o \).

4.C **Tensor commutes with coproducts.** In a closed monoidal category, one has the rules

\[
A \otimes \prod_{i \in I} B_i \cong \prod_{i \in I} A \otimes B_i , \quad A \otimes 0 \cong 0 ,
\]

whenever the coproduct on the left exists. Likewise,

\[
A \to \prod_{i \in I} B_i \cong \prod_{i \in I} (A \to B_i) , \quad (A \to 1) \cong 1 ,
\]

whenever the product on the left exists. Thus, in a cartesian closed category, one has in particular

\[
(A \times B)^C \cong A^C \times B^C , \quad 1^C \cong 1 .
\]

4.D **Further isomorphisms for the internal hom-functors.** If \( C \) is a monoidal closed category with \((\_)_C : \text{C}^{\text{op}} \to \text{C} \) right adjoint (for example, if \( C \) is symmetric monoidal, see Proposition 4.4.3), then for all objects \( A, B, C \), there are natural isomorphisms

\[
(E \to A) \cong A \quad \text{and} \quad \prod_{i \in I} (A_i \to B) \cong \left( \prod_{i \in I} A_i \right) \to B .
\]

In the case where \( B \to (\_)_C : \text{C}^{\text{op}} \to \text{C} \) is right adjoint, one has for all objects \( A, B, C \),

\[
(A \to E) \cong A \quad \text{and} \quad \prod_{i \in I} (B \to A_i) \cong B \to \prod_{i \in I} A_i .
\]
4.E Adjunctions and equivalent maps. If \( f \dashv g : B \to A \) and \( f' \dashv g' : B \to A \) in an ordered category, then \( f \leq f' \) if and only if \( g' \leq g \). In particular, if \( f \leq f' \), and \( g \leq g' \), then \( f \simeq f' \) and \( g \simeq g' \).

4.F Trivial extensions and liftings. For a morphism \( f : A \to B \) in a quantaloid, one has
\[
1_B \otimes f = f \quad \text{and} \quad f \otimes 1_A = f.
\]

4.G Adjoints in quantaloids. A morphism \( f : A \to B \) in a quantaloid has a right adjoint \( g \) if and only if \( (f \otimes 1_B) \cdot f = (f \otimes f) \); in this case, \( g = (f \otimes 1_B) \). Dually, \( g : B \to A \) has a left adjoint \( f \) if and only if \( g \cdot (1_B \otimes g) = (g \otimes g) \), and in this case, \( f = (1_B \otimes g) \).

Moreover, one has the following rules:

1. \( h \cdot (\phi \otimes \psi) = (h \cdot \phi) \otimes \psi \) if \( h \) is right adjoint;
2. \( (\phi \otimes \psi) \cdot f = \phi \otimes (g \cdot \psi) \) if \( f \dashv g \);
3. \( (\phi \cdot p) \otimes \psi = \phi \otimes (\psi \cdot q) \) if \( p \dashv q \).

4.H Existence of biproducts. For a family \( (A_i)_{i \in I} \) of objects in a quantaloid \( C \), the following are equivalent:

(i) the biproduct of \( (A_i)_{i \in I} \) exists;
(ii) the product of \( (A_i)_{i \in I} \) exists;
(iii) the coproduct of \( (A_i)_{i \in I} \) exists.

4.I \( V \) as a \( V \)-category. For \( V \) a symmetric monoidal closed category, and all objects \( A, B, C \) in \( V \), there are morphisms
\[
(A \to B) \otimes (B \to C) \to (A \to C) \quad \text{and} \quad E \to (A \to A)
\]
which render \( V \) a \( V \)-category, with the same object class and \( (A \to B) \) as the internal hom.

Hint. For the first morphism, construct \( A \otimes (A \to B) \to B \), and tensor with \( (B \to C) \).


1. Let \( O \) be a set. A directed graph \( G = (M, d : M \to O, c : M \to O) \) over \( O \) (the vertices of \( G \)) is given by a set \( M \) (the edges of \( G \)) and maps \( d, c \) (which assign to an edge its domain and codomain). A morphism \( f : (M, d, c) \to (N, b, a) \) in the category \( \text{Gph}(O) \) of graphs over \( O \) is a map \( f : M \to N \) with \( b \cdot f = d, a \cdot f = c \). Show that \( \text{Gph}(O) \) becomes a monoidal category when one puts
\[
(M, d, c) \otimes (N, b, a) = (M \times_O N, d \cdot b', a \cdot c')
\]
using the pullback diagram

\[
\begin{array}{ccc}
M \times_O N & \xrightarrow{e'} & N \\
\downarrow{b'} & & \downarrow{b} \\
M & \xrightarrow{c} & O
\end{array}
\]

However, the tensor product fails to be symmetric.

(2) Show that the category of monoids in $\text{Gph}(O)$ is equivalent to the category $\text{Cat}(O)$ of small categories with object set $O$ and functors mapping $O$ identically.
5 Factorizations, fibrations and topological functors

5.1 Factorization systems for morphisms. We denote respectively by $\text{Iso}_C$, $\text{Epi}_C$, and $\text{Mono}_C$ the classes of all isomorphisms, epimorphisms, and monomorphisms in $C$. An orthogonal factorization system for morphisms (or simply a factorization system) in a category $C$ is given by a pair $(\mathcal{E}, \mathcal{M})$ of morphism classes in $C$ such that:

1. $\text{Iso}_C \cdot \mathcal{E} \subseteq \mathcal{E}$, $\mathcal{M} \cdot \text{Iso}_C \subseteq \mathcal{M}$: that is, $\mathcal{E}$ and $\mathcal{M}$ are closed under composition with isomorphisms from the left and the right, respectively;

2. $\text{mor}_C = \mathcal{M} \cdot \mathcal{E}$: every morphism factors into an $\mathcal{E}$-morphism followed by an $\mathcal{M}$-morphism; in fact, we tacitly assume that there is a fixed choice for these factorizations;

3. $\mathcal{E} \perp \mathcal{M}$: every $\mathcal{E}$-morphism $e$ is orthogonal to every $\mathcal{M}$-morphism $m$, so that for every commutative solid-arrow square

\[
\begin{array}{ccc}
\bullet & \xrightarrow{w} & \bullet \\
\text{e} & \downarrow & \text{v} \\
\downarrow & \text{u} & \downarrow \\
\bullet & \xrightarrow{m} & \bullet
\end{array}
\]

there is a unique morphism $w$ with $w \cdot e = u$, $m \cdot w = v$; one writes $e \perp m$ in this situation.

The system $(\mathcal{E}, \mathcal{M})$ is called proper if $\mathcal{E} \subseteq \text{Epi}_C$ and $\mathcal{M} \subseteq \text{Mono}_C$; in that case, any $w$ with $w \cdot e = u$ or $m \cdot w = v$ (where $m \cdot u = v \cdot e$) satisfies both equations and is trivially uniquely determined. The notion of factorization system is selfdual in the sense that $(\mathcal{M}, \mathcal{E})$ is a factorization system in $C^{\text{op}}$ if $(\mathcal{E}, \mathcal{M})$ is one in $C$ (and the same is true for proper factorization systems).

An immediate consequence of the unique diagonalization property (3) is that $(\mathcal{E}, \mathcal{M})$-factorizations of a morphism are unique, up to a unique isomorphism: if $m \cdot e = m' \cdot e'$ with $e, e' \in \mathcal{E}$, $m, m' \in \mathcal{M}$, then $j \cdot e = e'$, $m' \cdot j = m$ for a unique isomorphism $j$. (This fact follows also from (1) and (2) below.)

5.1.1 Proposition.

1. For a factorization system $(\mathcal{E}, \mathcal{M})$ in $C$ one has:

\[
\mathcal{E} = \uparrow \mathcal{M} := \{ e \in \text{mor}_C \mid \forall m \in \mathcal{M} : e \perp m \} ,
\]

\[
\mathcal{M} = \mathcal{E}^\perp := \{ m \in \text{mor}_C \mid \forall e \in \mathcal{E} : e \perp m \} .
\]

In particular, $\mathcal{E}$ and $\mathcal{M}$ determine each other uniquely, and

\[
\mathcal{E} \cap \mathcal{M} = \text{Iso}_C .
\]

2. Any class $\mathcal{M} = \mathcal{E}^\perp$ (for some $\mathcal{E} \subseteq \text{mor}_C$) satisfies:

   a. $\text{Iso}_C \subseteq \mathcal{M}$ and $\mathcal{M} \cdot \mathcal{M} = \mathcal{M}$ ($\mathcal{M}$ is closed under composition);
(b) if \( g \cdot f, g \in \mathcal{M} \), then \( f \in \mathcal{M} \) (\( \mathcal{M} \) is weakly left-cancellable);

(c) \( \mathcal{M} \) is stable under pullbacks (see 2.8), so that for any pullback diagram

\[
\begin{array}{ccc}
& g' & \\
\downarrow & & \downarrow \\
f' & f & \\
\downarrow & & \downarrow \\
g & & \\
\end{array}
\]

\( f \in \mathcal{M} \) implies \( f' \in \mathcal{M} \);

(d) \( \mathcal{M} \) is stable under multiple pullbacks, so that when \( (A,f) \) is a product of \( (A_i,f_i) \) in \( \mathcal{C}/\mathcal{B} \), one has \( f \in \mathcal{M} \) whenever all \( f_i \in \mathcal{M} \);

(e) \( \mathcal{M} \) is closed under limits, so that when \( \mu : D \to E \) is a natural transformation that is componentwise in \( \mathcal{M} \) (with \( D,E : J \to \mathcal{C} \)), the induced morphism \( \text{lim} \mu : \text{lim} D \to \text{lim} E \) also lies in \( \mathcal{M} \) (provided that the needed limits exist); in particular, \( \mathcal{M} \) is closed under products:

\[ \forall i \in I \ (m_i \in \mathcal{M}) \implies \prod_{i \in I} m_i \in \mathcal{M} . \]

Any class \( \mathcal{E} = \perp \mathcal{M} \) (for some \( \mathcal{M} \subseteq \text{mor} \mathcal{C} \)) satisfies the properties dual to (a)–(e).

(3) A factorization system \((\mathcal{E}, \mathcal{M})\) in \( \mathcal{C} \) with \( \mathcal{E} \subseteq \text{Epi} \mathcal{C} \) satisfies:

(a) \( \text{ExtMono} \mathcal{C} \subseteq \mathcal{M} \), where a monomorphism \( f \) is an extremal monomorphism in \( \mathcal{C} \) if \( f = g \cdot e \) with \( e \) epic only if \( e \) is an isomorphism;

(b) \( \text{SplitMono} \mathcal{C} \subseteq \mathcal{M} \) (see Exercise 2.C);

(c) \( g \cdot f \in \mathcal{M} \) implies \( f \in \mathcal{M} \) (\( \mathcal{M} \) is left-cancellable).

Conversely, if \( \mathcal{C} \) has finite products, any of (a), (b), (c) implies \( \mathcal{E} \subseteq \text{Epi} \mathcal{C} \). The dual assertions hold as well.

Proof. (1), (2) are shown by standard arguments. (It is less standard to show that properties (c) and (d) in (2) actually follow from (e); see [Dikranjan and Tholen 1995].) (3) First assume \( \mathcal{E} \subseteq \text{Epi} \mathcal{C} \). For (a), \((\mathcal{E}, \mathcal{M})\)-factoring \( f \in \text{ExtMono} \mathcal{C} \) gives \( f = m \cdot e \) with \( e \) epic, so that \( e \) must be an isomorphism and \( f \in \mathcal{M} \). (b) follows trivially since \( \text{SplitMono} \mathcal{C} \subseteq \text{ExtMono} \mathcal{C} \). For (c), let \( g \cdot f \in \mathcal{M} \). In order to show \( f \in \mathcal{M} = \mathcal{E}^\perp \), assume \( f \cdot u = v \cdot e \) with \( e \in \mathcal{E} \). Since \( (g \cdot f) \cdot u = (g \cdot v) \cdot e \) with \( g \cdot f \in \mathcal{M} \) there is \( w \) with \( w \cdot e = u \). Thus, \( \mathcal{E} \subseteq \text{Epi} \mathcal{C} \) implies that \( w \) is unique and also satisfies \( f \cdot w = v \).

Conversely, in order to show \( \mathcal{E} \subseteq \text{Epi} \mathcal{C} \) under hypotheses (a), (b), or (c), assume \( e \in \mathcal{E} \) and \( f \cdot e = g \cdot e \). With \( A = \text{cod} f = \text{dom} f, h = (f,g), \delta = (\langle 1_A, 1_A \rangle) \) one obtains the commutative diagram

\[
\begin{array}{ccc}
f \cdot e & \rightarrow & A \\
\downarrow & & \downarrow \\
e & \rightarrow & \delta \\
\downarrow & & \\
h & \rightarrow & A \times A .
\end{array}
\]
Since \( p_1 \cdot \delta = 1_A \in \mathcal{M} \) (with \( p_1, p_2 \) the projections of \( A \times A \)), \( \delta \in \mathcal{M} \) follows under either hypothesis (a), (b) or (c), so that \( e \perp \delta \) gives \( w \) with \( \delta \cdot w = h \), hence \( f = p_1 \cdot h = w = p_2 \cdot h = g \). \( \square \)

Because of (1) one calls \( \mathcal{E} \) a \textit{left factorization class} and \( \mathcal{M} \) a \textit{right factorization class} if they belong to a factorization system \( (\mathcal{E}, \mathcal{M}) \), and then \( \mathcal{E} \) is the \textit{left companion} of \( \mathcal{M} \) and \( \mathcal{M} \) the \textit{right companion} of \( \mathcal{E} \). Of course, one talks about extremal epimorphisms in the situation dual to (3)(a).

\[ \begin{array}{ccc}
X & \xrightarrow{X/\sim} & Y \\
\xrightarrow{f} & \xrightarrow{\text{reg. epi}} & \xrightarrow{\text{mono}} \\
\end{array} \quad \begin{array}{ccc}
X & \xrightarrow{\text{epi}} & f(X) \\
\xrightarrow{f} & \xrightarrow{\text{reg. mono}} & \xrightarrow{\text{mono}} \\
\end{array} \]

where \( \sim \) is the equivalence relation induced by \( f \).

For a class \( \mathcal{E} \) of morphisms in \( \mathcal{C} \), a subcategory \( \mathcal{B} \) of \( \mathcal{C} \) is called \( \mathcal{E} \)-\textit{reflective} if all \( \mathcal{B} \)-reflections lie in \( \mathcal{E} \); accordingly, \( \text{epi-reflective}, \ \text{regular epi-reflective}, \ \text{etc.} \), refers to the cases \( \mathcal{E} = \{ \text{epimorphisms} \}, \ \{ \text{regular epimorphisms} \}, \ \text{etc.} \) \( \text{Bireflective} \) means \( \text{epi-reflective} \) and \( \text{mono-reflective} \). The dual notions are \( \mathcal{M} \)-\textit{coreflective}, \( \text{mono-coreflective}, \ \text{regular mono-coreflective}, \ \text{bicoreflective}, \ \text{etc.} \)

5.1.3 Proposition. For a factorization system \( (\mathcal{E}, \mathcal{M}) \) in \( \mathcal{C} \), a replete reflective subcategory \( \mathcal{B} \) of \( \mathcal{C} \) is \( \mathcal{E} \)-\textit{reflective} if and only if for all \( m : A \to B \) in \( \mathcal{M} \), one has \( A \in \text{ob} \mathcal{B} \) when \( B \in \text{ob} \mathcal{B} \).

Proof. For the necessity of the condition, factor \( m : A \to B \) in \( \mathcal{M} \) with \( B \in \text{ob} \mathcal{B} \) through the \( \mathcal{B} \)-reflection \( \rho_A : A \to RA \) and obtain \( t : RA \to A \) with \( t \cdot \rho_A = 1_A \) from \( \rho_A \perp m \). Now, \( (\rho_A \cdot t) \cdot \rho_A = \rho_A \) given \( \rho_A \cdot t = 1_{RA} \), by universality of \( \rho_A \). For the sufficiency of the condition, consider an \( (\mathcal{E}, \mathcal{M}) \)-factorization \( \rho_A : (A \xrightarrow{e} B \xrightarrow{n} RA) \) of the \( \mathcal{B} \)-reflection of \( A \in \text{ob} \mathcal{C} \). Since \( B \in \text{ob} \mathcal{B} \), there is \( s : RA \to B \) with \( s \cdot \rho_A = e \), hence \( (n \cdot s) \cdot \rho_A = \rho_A \) and \( n \cdot s = 1_{RA} \). Now \( e \perp n \) implies \( s \cdot n = 1_B \). \( \square \)
5.2 Subobjects, images and inverse images. For a class \( M \subseteq \text{Mono}_C \) containing all isomorphisms and being closed under composition with them, and for every \( A \in \text{ob} \, C \), one forms the full subcategory

\[
\text{sub} \, A = \text{sub}_M \, A = M / A
\]

of the comma category \( C / A \), given by the objects \( m : M \to A \) with \( m \in M \). (If \( M \) is closed under composition and is weakly left-cancellable, one may also consider \( M / A \) as a comma category of the category \( M \) with \( \text{ob} \, M = \text{ob} \, C \).) Of course \( \text{sub} \, A \) is just an ordered class (with \( m \leq n \) if \( n \cdot l = m \) for some morphism \( l \)). Its objects are called \( M \)-subobjects (or simply subobjects) of \( A \). Most authors reserve this term for isomorphism classes of objects in \( M / A \). The category \( C \) is said to have inverse \( M \)-images (or inverse images) if for all \( f : A \to B \) in \( C \) and all \( n : N \to B \) in \( M \) there is a pullback diagram

\[
f^{-1}(N) \quad \begin{array}{c} \downarrow n \\ \downarrow f^{-1}(n) \end{array} \quad N \quad \begin{array}{c} f' \\ f \end{array} \quad B
\]

with \( f^{-1}(n) \) in \( M \) (for convenience, the choice of such a diagram is supposed to be fixed). One then has the inverse-image functor

\[
f^{-1}(-) : \text{sub} \, B \to \text{sub} \, A .
\]

5.2.1 Proposition. Let \( C \) have inverse \( M \)-images. Then \( M \) is a right factorization class of \( C \) if and only if \( M \) is closed under composition and \( f^{-1}(-) \) has a left adjoint \( f(-) \), for every morphism \( f \) in \( C \). In that case, the \( M \)-image \( f(m) \) of \( m : M \to A \) in \( M \) under \( f : A \to B \) may be constructed by the \((E, M)\)-factorization of \( f \cdot m \) (with \( E \) the left companion of \( M \)):

\[
\begin{array}{c}
M \quad \begin{array}{c} \in \mathcal{E} \\ m \end{array} \quad f(M) \\
\downarrow \\
A \quad \begin{array}{c} f \\ f(m) \end{array} \quad B
\end{array}
\]

One has:

(1) \( m \leq f^{-1}(f(m)) \) for all \( m \in \text{sub} \, A \), with \( m \cong f^{-1}(f(m)) \) when \( f \in \mathcal{M} \);

(2) \( f \) lies in \( \mathcal{E} \) if and only if \( f(1_A) \cong 1_B \);

(3) \( f(f^{-1}(n)) \leq n \) for all \( n \in \text{sub} \, B \), while \( n \cong f(f^{-1}(n)) \) for all \( n \) precisely when \( f \) lies \( M \)-hereditarily in \( \mathcal{E} \), that is, when \( f' \in \mathcal{E} \) in every pullback diagram (5.2.1) with \( n \in \mathcal{M} \).

Proof. All verifications are routine, except the proof that \( M \) is a right factorization class when \( M \cdot M \subseteq M \) and there is \( f(-) \dashv f^{-1}(-) \) for all \( f \). To establish a factorization of \( f \),
one puts \( m := f(1_A) \) and lets \( e \) be the upper horizontal composite arrow in

\[
\begin{array}{c}
A \xrightarrow{1_A} C \\
\downarrow f^{-1}(f(1_A)) \quad \downarrow f(1_A) \\
A \xrightarrow{f} B
\end{array}
\]

The universal property of the adjunction amounts to the fact that for every commutative solid-arrow diagram

\[
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow e \quad \downarrow w & & \downarrow n \\
C & \xrightarrow{m} & B
\end{array}
\]

with \( n \in \mathcal{M} \) there is a unique “diagonal” \( w \). Applying the same factorization procedure to \( e \) in lieu of \( f \) gives \( e = m' \cdot e' \) and the above solid arrow diagram with \( n = m \cdot m' \) and \( u = e' \). Since \( m \cdot m' \in \mathcal{M} \), by hypothesis one obtains \( w \) with \( m \cdot m' \cdot w = m \), hence \( m' \cdot w = 1_C \). Consequently, \( m' \) is an isomorphism so that \( e \) must have the universal property symbolized by

\[
\begin{array}{ccc}
A & \xrightarrow{e} & C \\
\downarrow & & \downarrow n \\
C & \xrightarrow{1_C} & C
\end{array}
\]

This is sufficient to verify \( e \in \perp \mathcal{M} \), since the search for a diagonal for an arbitrary square

\[
\begin{array}{ccc}
\cdot & \xrightarrow{u} & \cdot \\
\downarrow e \quad \downarrow v & & \downarrow n \\
\cdot & \xrightarrow{v} & \cdot
\end{array}
\]

can be reduced to the case \( v = 1 \) by pulling \( n \) back along \( v \).

It follows that the image and inverse-image functors associated with a factorization system \((\mathcal{E}, \mathcal{M})\) with \( \mathcal{M} \subseteq \text{Mono} \mathcal{C} \) preserve existing joins and meets, respectively:

5.2.2 Corollary.

\[
f(\bigvee_{i \in I} m_i) \simeq \bigvee_{i \in I} f(m_i) \; \; , \; \; f^{-1}(\bigwedge_{i \in I} m_i) \simeq \bigwedge_{i \in I} f^{-1}(m_i) \; .
\]

Proof. Left adjoints preserve colimits, and right adjoints limits.

5.3 Factorization systems for sinks and sources. A source in a category \( \mathcal{C} \) is simply a discrete cone in \( \mathcal{C} \); hence, it is given by an object \( A \) and a family \((g_i : A \to B_i)_{i \in I}\) of morphisms in \( \mathcal{C} \) (\( I \) may be empty and is not required to be small); the dual notion is sink.
One says that the sink \((f_i : A_i \to B)_{i \in I}\) is *orthogonal* to the morphism \(m : C \to D\) if for every solid-arrow diagram

\[
\begin{array}{c}
A_i \xrightarrow{u_i} C \\
\downarrow f_i \quad \downarrow m \\
B \xrightarrow{v} D
\end{array}
\quad (i \in I)
\]

there is a unique morphism \(w\) with \(w \cdot f_i = u_i\) \((i \in I)\) and \(m \cdot v = v\); one writes \((f_i)_{i \in I} \perp m\) in this case. A *factorization system for sinks* (also called an *orthogonal factorization system for sinks*) is a pair \((\mathcal{E}, \mathcal{M})\) consisting of a collection \(\mathcal{E}\) of sinks and a class \(\mathcal{M}\) of morphisms in \(C\) such that:

1. \(\mathcal{E}\) and \(\mathcal{M}\) are closed under composition with isomorphisms from the left and right, respectively;
2. every sink in \(C\) factors into an \(\mathcal{E}\)-sink followed by an \(\mathcal{M}\)-morphism;
3. \(\mathcal{E} \perp \mathcal{M}\).

We can leave it to the reader to formulate the appropriate sink generalizations of the statements of Proposition 5.1.1. For example, closure of \(\mathcal{E}\) under composition means that when all \((f_{i,j} : A_{ij} \to B_i)_{j \in J_i}\) and \((g_i : B_i \to C)_{i \in I}\) lie in \(\mathcal{E}\), the composite sink \((g_i \cdot f_{i,j})_{i \in I, j \in J_i}\) also lies in \(\mathcal{E}\). Conversely, weak right-cancellation of \(\mathcal{E}\) stipulates that when the composite sink and all \((f_{i,j})_{j \in J_i}\) lie in \(\mathcal{E}\), then \((g_i)_{i \in I}\) also lies in \(\mathcal{E}\).

Of course, a factorization system \((\mathcal{E}, \mathcal{M})\) for sinks gives in particular a factorization system \((\mathcal{E}, \text{mor}\ C)\) for sinks in any category \(C\) containing morphisms that are not monic, as we show next:

**5.3.1 Lemma.** For a factorization system \((\mathcal{E}, \mathcal{M})\) for sinks, one has \(\mathcal{M} \subseteq \text{Mono}\ C\).

**Proof.** For \(f : A \to B\) in \(\mathcal{M}\), assume \(f \cdot x = f \cdot y = h\) with \(x, y : D \to A\), and consider the constant sink \((h)_{i \in I}\), with \(I := \text{ob}(C/A)\). We obtain an \((\mathcal{E}, \mathcal{M})\)-factorization \(h = m \cdot e_i \quad (i \in I)\) and consider

\[
J := \{w \mid \forall i \in I : w \cdot e_i \in \{x, y\}\}.
\]

The diagrams

\[
\begin{array}{ccc}
D & \xrightarrow{u_i} & A \\
\downarrow e_i & & \downarrow f \\
C & \xrightarrow{m} & B
\end{array}
\]

show \(J \neq \emptyset\) (take \(u_i := x\) for all \(i\)); therefore, the inclusion map \(\tau : J \hookrightarrow I\) has a retraction \(\sigma : I \to J\) (so that \(\sigma \cdot \tau = 1_I\)). For \(i \in I\), consider

\[
u_i := \begin{cases} 
  x & \text{if } \sigma(i) \cdot e_i = y \\
  y & \text{if } \sigma(i) \cdot e_i = x,
\end{cases}
\]
and obtain $w \in J$ as above. Then, with $\sigma(i_0) = w$, one has

$$\sigma(i_0) \cdot e_{i_0} = x \iff \sigma(i_0) \cdot e_{i_0} = y,$$

which is possible only if $x = y$.

5.3.2 Theorem. A class of morphisms $\mathcal{M}$ belongs to a factorization system $(\mathcal{E}, \mathcal{M})$ for sinks in $\mathcal{C}$ if and only if:

1. $\text{Iso}\mathcal{C} \subseteq \mathcal{M} \subseteq \text{Mono}\mathcal{C}$ and $\mathcal{M} \cdot \mathcal{M} \subseteq \mathcal{M}$,
2. $\mathcal{C}$ has inverse $\mathcal{M}$-images,
3. $\mathcal{C}$ has $\mathcal{M}$-intersections, that is, multiple pullbacks of families of $\mathcal{M}$-morphisms exist in $\mathcal{C}$, and $\mathcal{M}$ is stable under them.

Proof. The Lemma takes care of the only delicate part in the proof of the necessity of condition (1). For the necessity of (2), let $f : A \to B$ be in $\mathcal{C}$, $n : N \to B$ in $\mathcal{M}$ and consider the class

$$J := \{ k \in \mathcal{M} \mid \exists x_k : f \cdot k = n \cdot x_k \}.$$

The sink $(k)_{k \in J}$ has an $(\mathcal{E}, \mathcal{M})$-factorization $m \cdot e_k = k$, with $m \in \mathcal{M}$ and $(e_k)_{k \in J}$ in $\mathcal{E}$. Since $(e_k) \perp n$, one finds $f'$ making

$$\begin{array}{ccc}
\varepsilon_k & \xrightarrow{x_k} & N \\
\downarrow k & & \downarrow n \\
A & \xrightarrow{f} & B
\end{array}$$

commute for all $k \in J$. Using the fact that we have in particular an $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms, one easily verifies that the square of the diagram is a pullback. Condition (3) is shown similarly.

For the sufficiency of conditions (1)–(3), let us consider a sink $(f_i : A_i \to B)_{i \in I}$, and form

$$J := \{ k \in \mathcal{M} \mid \forall i \in I \exists x_{i,k} : k \cdot x_{i,k} = f_i \}.$$

The universal property of the multiple pullback

$$m = \bigwedge_{k \in J} k$$

shows $m \in J$, that is: there are morphisms $e_i$ with $f_i = m \cdot e_i$ ($i \in I$). It remains to be shown that $(e_i) \perp t$ for all $t \in \mathcal{M}$. But if

$$\begin{array}{ccc}
A_i & \xrightarrow{u_i} & T \\
\varepsilon_i & \downarrow & \downarrow t \\
C & \xrightarrow{v} & D
\end{array}$$
for all \( i \in I \), then \( t' = v^{-1}(t) \) is a factor of each \( e_i \), so that \( m \cdot t' \in \mathcal{M} \) is a factor of each \( f_i \). Consequently, \( m \cdot t' \in J \), and one has that \( t' \) is an isomorphism by construction of \( m \). Hence, \( w = v' \cdot (t')^{-1} \) is the needed diagonal for the square above, with \( v' \) the pullback of \( v \) along \( t \).

With \( \mathcal{M} \subseteq \text{Mono} \mathcal{C} \), a category \( \mathcal{C} \) is \( \mathcal{M} \)-wellpowered if, for every \( A \in \text{ob} \mathcal{C} \), the separated reflection of \( \text{sub}_A \mathcal{M}A \) (see 1.3) is small; that is, every object has only a set of isomorphism types of \( \mathcal{M} \)-subobjects. The prefix \( \mathcal{M} \) is omitted when \( \mathcal{M} = \text{Mono} \mathcal{C} \).

5.3.3 Corollary. Let \( \mathcal{C} \) be small-complete and \( \mathcal{M} \)-wellpowered, for a class \( \mathcal{M} \) with \( \text{Iso} \mathcal{C} \subseteq \mathcal{M} \subseteq \text{Mono} \mathcal{C} \). The following are equivalent:

(i) \( \mathcal{M} \) is a right factorization class for morphisms;

(ii) \( \mathcal{M} \) is a right factorization class for sinks;

(iii) \( \mathcal{M} \) is closed under composition, and stable under pullbacks, as well as under multiple pullbacks.

Proof. The equivalence follows from Proposition 5.1.1 and the Theorem.

A pair \((\mathcal{E}, \mathcal{M})\) is a factorization system for sources in \( \mathcal{C} \) if \((\mathcal{M}, \mathcal{E})\) is a factorization system for sinks in \( \mathcal{C}^{\text{op}} \). Similarly, \( \mathcal{C} \) is \( \mathcal{E} \)-cowellpowered if \( \mathcal{C}^{\text{op}} \) is \( \mathcal{E} \)-wellpowered. We leave the formulation of the statements dual to 5.3.1–5.3.3 to the reader.

For \( \mathcal{C} \) any of the categories \( \text{Set} \), \( \text{Mon} \), \( \text{Grp} \), \( \text{Ord} \), \( \text{Sup} \), and \( \text{Top} \), \( \text{Mono} \mathcal{C} \) is not just a right factorization class for morphisms, but also for sinks. For \( \mathcal{C} = \text{Set} \), the left companion \( \mathcal{E} \) contains precisely the jointly surjective sinks \( (f_i : A_i \to B)_{i \in I} \), so that \( B = \bigcup_{i \in I} f_i(A_i) \). In \( \text{Mon} \), \( \text{Grp} \), and \( \text{Sup} \), \( B \) is only generated by \( \bigcup_{i \in I} f_i(A_i) \) for \( (f_i)_{i \in I} \in \mathcal{E} \). In \( \text{Ord} \) and \( \text{Top} \), sinks in \( \mathcal{E} \) are still jointly surjective but must carry the appropriate structure that will be discussed more generally in Examples 5.6.1.

Epi \( \mathcal{C} \) is a left factorization class for sources, with an easily described companion \( \mathcal{M} \) for \( \mathcal{C} = \text{Set} \) or \( \text{Grp} \): \( \mathcal{M} \) contains precisely the point-separating sources \( (f_i : A \to B_i)_{i \in I} \), so that for \( x, y : X \to A \), one has \( x = y \) when \( f_i \cdot x = f_i \cdot y \) for all \( i \in I \). In a general category, sources with this property are called mono-sources (the dual notion is that of an epi-sink).

In \( \text{Ord} \) and \( \text{Top} \), the domain \( A \) must carry the appropriate structure, the description of which will appear more generally in Examples 5.6.1.

5.4 Closure operators. Let \( \mathcal{M} \subseteq \text{Mono} \mathcal{C} \) be a right factorization class, hence part of a factorization system \((\mathcal{E}, \mathcal{M})\) for morphisms. As in 5.2, for every object \( A \) in \( \mathcal{C} \), we consider the ordered class \( \text{sub} A = \mathcal{M}/A \) and have, for every morphism \( f : A \to B \), the image function

\[ f(-) : \text{sub} A \to \text{sub} B \]
(constructed as in Proposition 5.2.1) which, if $C$ has inverse $\mathcal{M}$-images, is left adjoint to $f^{-1}(-)$. An $\mathcal{M}$-closure operator (or simply closure operator) on $C$ is a family

$$c = (c_A : \text{sub } A \to \text{sub } A)_{A \in \text{ob } C}$$

of functions which are

1. extensive: $m \leq c_A(m)$,
2. monotone: if $m \leq m'$, then $c_A(m) \leq c_A(m')$,
3. and which satisfy the continuity condition: $f(c_A(m)) \leq c_B(f(m))$,

for all $f : A \to B$ in $C$ and $m, m' \in \text{sub } A$. In the presence of inverse images, it is easy to see that the continuity condition may be equivalently formulated as

$$c_A(f^{-1}(n)) \leq f^{-1}(c_B(n)),$$

for all $f : A \to B$ in $C$ and $n \in \text{sub } B$. Extensivity yields for every $m : M \to A$ in $\mathcal{M}$ a factorization

$$\begin{array}{ccc}
M & \xrightarrow{m} & A \\
\downarrow{j_m} & & \downarrow{c_A(m)} \\
\downarrow{c_A(M)} & & \downarrow{c_A(m)}
\end{array}$$

with $c(m) = c_A(m) \in \mathcal{M}$ and a uniquely determined morphism $j_m \in \mathcal{M}$. Monotonicity and continuity ensure that the passage from $m$ to its $c$-closure $c_A(m)$ is functorial as follows: for every commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{f'} & N \\
\downarrow{m} & & \downarrow{n} \\
A & \xrightarrow{f} & B
\end{array}$$

in $C$ with $m, n \in C$, there is a unique morphism $f''$ making the following diagram commute (see Exercise 5.G for a specification of the functoriality claim).

A subobject $m$ is $c$-closed if $j_m$ is an isomorphism, and it is $c$-dense if $c(m)$ is an isomorphism. More generally, a morphism $f : A \to B$ is $c$-dense if $f(1_A) : f(A) \to B$ is $c$-dense. The
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Closure operator $c$ is idempotent if $c(m)$ is $c$-closed (so that $c(c(m)) \cong c(m)$ for all $m \in \mathcal{M}$), and $c$ is weakly hereditary if $j_m$ is $c$-dense for all $m \in \mathcal{M}$.

5.4.1 Proposition. The following conditions on a closure operator $c$ are equivalent:

(i) $c$ is idempotent and weakly hereditary;

(ii) $c$ is idempotent, and the class $\mathcal{M}^c$ of $c$-closed subobjects in $\mathcal{C}$ is closed under composition;

(iii) the class $\mathcal{M}^c$ is a right factorization class for morphisms in $\mathcal{C}$.

If these conditions hold, the left companion of $\mathcal{M}^c$ is the class $\mathcal{E}^c$ of all $c$-dense morphisms in $\mathcal{C}$. The $(\mathcal{E}^c, \mathcal{M}^c)$-system factors a morphism $f : A \rightarrow B$ through $c_B(f(A))$.

Proof. The statement is easily proved by using functoriality as described above.

It is also a straightforward exercise to show that for any closure operator $c$ the class $\mathcal{M}^c$ is closed under limits, in particular stable under pullbacks and under multiple pullbacks.

These latter properties are characteristic in the following sense:

5.4.2 Theorem. Let $\mathcal{M}$ be a right factorization class for sinks in $\mathcal{C}$, and consider a subclass $\mathcal{K} \subseteq \mathcal{M}$. Then $\mathcal{K} = \mathcal{M}^c$ for an idempotent closure operator $c$ if and only if $\mathcal{K}$ is stable under pullbacks and multiple pullbacks. In fact, the closure operator $c$ is uniquely determined by $\mathcal{K}$.

Moreover, $c$ is weakly hereditary if and only if $\mathcal{K}$ is closed under composition.

Proof. See Exercise 5.H.

A closure operator is called hereditary if for all $m : M \rightarrow A$, $k : A \rightarrow B$ in $\mathcal{M}$, the lower rectangle in

$$
\begin{array}{ccc}
M & \xrightarrow{1_M} & M \\
j_m & \downarrow & j_{k_m} \\
c_A(M) & \xrightarrow{c_A(m)} & c_B(M) \\
c_A(m) & \downarrow & c_B(k_m) \\
A & \xrightarrow{k} & B
\end{array}
$$

is a pullback diagram; that is, if $c_A(m) \cong k^{-1}(c_B(k \cdot m))$. Exploitation of this property with $m = j_n$ and $k = c_B(n)$ for any $N \rightarrow B$ in $\mathcal{M}$ yields

$$c_B(n)(j_n) \cong c_B(n)^{-1}(c_B(n)) \cong j_n,$$

so that $j_n$ is $c$-dense. Hence, heredity implies weak heredity. The following result describes the extent to which heredity is stronger than weak heredity.
5.4.3 Proposition. A closure operator $c$ is hereditary if and only if $c$ is weakly hereditary and satisfies the following cancellation condition: for all $m, k \in M$, if $k \cdot m$ is $c$-dense, then $m$ is also $c$-dense.

Proof. The cancellation condition is certainly necessary for heredity of $c$ since when $c_B(k \cdot m)$ is an isomorphism, any pullback of it is also an isomorphism. Conversely, let the weakly hereditary closure operator $c$ satisfy the cancellation condition. We want to see that the canonical morphism $t : c_A(M) \to k^{-1}(c_B(M))$ induced by the commutative lower rectangle of (5.4.i) is an isomorphism. But weak heredity makes

$$j_{k \cdot m} = k^{-1}(c_B(k \cdot m)) \cdot t \cdot j_m$$

c-dense, so that $s := t \cdot j_m : M \to k^{-1}(c_B(M))$ is $c$-dense by hypothesis. Functoriality of $c$ then yields the inverse of $t$:

$$M \xrightarrow{1_M} M$$

$$\downarrow j_s \quad \downarrow j_m$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$\cong \quad \downarrow c_A(M)$$

$$\cong \downarrow c_A(m)$$

$$k^{-1}(c_B(M)) \xrightarrow{c_A(m)} A.$$

5.4.4 Examples.

(1) We may think of an idempotent closure operator $c$ as a family $(c_A)_{A \in \text{ob} C}$ of closure operations $c_A$ on sub $A$ (in the sense of [1.6]) which collectively must satisfy the continuity condition. (The notion of $c$-closedness defined here then coincides with the one defined in [1.6]).

(2) The down- and up-closure for subsets of ordered sets define idempotent and hereditary closure operators of $\text{Ord}$ (with its (Epi, RegMono)-factorization system, see [1.7] where RegMono refers to RegMono $C$, the class of all regular monomorphisms in $C$). Likewise, Kuratowski closure defines an idempotent and hereditary closure operator of $\text{Top}$ (with its (Epi, RegMono)-factorization system, see Exercise [1.F]).

(3) In terms of Theorem 5.4.2, the Kuratowski closure operator of $\text{Top}$ corresponds to the class $\mathcal{K}$ of closed subspace injections in $\text{Top}$. The class $\mathcal{O}$ of open subspace injections in $\text{Top}$ still induces a closure operator $\theta$ via

$$\theta_X(M) = \cap \{O \subseteq X \mid M \subseteq O \text{ and } O \text{ open}\}$$

$$= \{x \in X \mid \forall V \text{ neighborhood of } x : M \cap V \neq \emptyset\}$$

(with $\nabla$ the Kuratowski closure of $V$ in $X$). But failure of $\mathcal{O}$ to be closed under multiple pullbacks (that is, intersections) makes $\theta$ fail to be idempotent, and $\theta$ is not weakly hereditary either.
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5.5 Generators and cogenerators. A class \( \mathcal{G} \) of objects in a category \( \mathcal{C} \) is generating if for every object \( A \) in \( \mathcal{C} \) the family \( \mathcal{C}(\mathcal{G}, A) \) of all morphisms with codomain \( A \) and domain in \( \mathcal{G} \) forms an epi-sink. Hence, for all \( f, g : A \to B \) in \( \mathcal{C} \), one has \( f = g \) whenever \( f \cdot x = g \cdot x \) for all \( x : G \to A, G \in \mathcal{G} \). Equivalently, \( \mathcal{G} \) is generating if the generalized hom-functor

\[
\mathcal{C}(\mathcal{G}, -) : \mathcal{C} \to \text{Set}^\mathcal{G}, \quad A \mapsto (\mathcal{C}(G, A))_{G \in \mathcal{G}}
\]

is faithful (where \( \mathcal{C} \) is assumed to be locally small). A class \( \mathcal{G} \) is strongly generating in \( \mathcal{C} \) if \( \mathcal{C}(\mathcal{G}, A) \) is an extremal epi-sink for all objects \( A \), so that one has the additional property that for a monomorphism \( m : B \to A \), one can have \( x = m \cdot h_x \) for all \( x \) in \( \mathcal{C}(\mathcal{G}, A) \) only if \( m \) is an isomorphism (the dual notion is that of an extremal mono-source). Equivalently, \( \mathcal{G} \) is strongly generating if \( \mathcal{C}(\mathcal{G}, -) \) is faithful and reflects isomorphisms. We call a small generating class a generator of \( \mathcal{C} \), and a single object \( G \) in \( \mathcal{C} \) a generator of \( \mathcal{C} \) if \( \{G\} \) is one. Similarly, one says that a class is a strong generator if it is a small strongly generating class, and a single object \( G \) can similarly be called a strong generator. Note that one commonly uses separator as an alternative name for generator. The terms cogenerating, cogenerator, or coseparator are used in the dual situation.

When \( \mathcal{G} \) is small, the functor \( \mathcal{C}(\mathcal{G}, -) \) has a left adjoint \( F \) if and only if all coproducts

\[
FX = \coprod_{G \in \mathcal{G}} X_G \cdot G
\]

(with \( X = (X_G)_{G \in \mathcal{G}} \) an object in \( \text{Set}^\mathcal{G} \)) exist; here \( X_G \cdot G \) denotes the coproduct of \( X_G \)-many copies of \( G \) in \( \mathcal{C} \). The counits

\[
\varepsilon_A : \coprod_{G \in \mathcal{G}} \mathcal{C}(G, A) \cdot G \to A
\]

are the canonical morphisms with \( \varepsilon_A \cdot i_x = x \) for all \( x : G \to A, G \in \mathcal{G} \), with \( i_x \) denoting a coproduct injection.

5.5.1 Proposition. The following conditions on a set \( \mathcal{G} \) of objects in a locally small category \( \mathcal{C} \) with coproducts are equivalent:

- (i) \( \mathcal{G} \) is generating;
- (ii) for all objects \( A \), the canonical morphisms \( \varepsilon_A \) are epimorphisms;
- (iii) for every object \( A \), there is some epimorphism

\[
\coprod_{i \in I} G_i \to A
\]

with a small family \( (G_i)_{i \in I} \) of objects in \( \mathcal{G} \).

The same equivalence holds if one specializes to a strongly generating set \( \mathcal{G} \) in (i), and extremal epimorphisms in (ii) and (iii).

Proof. The equivalence follows from Exercise 5.1. \( \square \)
A singleton set (in fact, every non-empty set) is a single-object strong generator of $\text{Set}$. The terminal object is also a generator in $\text{Ord}$ or $\text{Top}$, but it is not strong. The free algebra over a singleton set is a strong generator in every Eilenberg–Moore category over $\text{Set}$ (see 3.2). Hence, the additive group $\mathbb{Z}$ is a generator of both $\text{Grp}$ and $\text{AbGrp}$.

A two-element set is a single-object strong cogenerator of $\text{Set}$. Provided with the indiscrete structure, it is a cogenerator in both $\text{Ord}$ and $\text{Top}$, but it is not strong. The “rational circle” $\mathbb{Q}/\mathbb{Z}$ is a strong cogenerator in $\text{AbGrp}$ but $\text{Grp}$ has no cogenerator at all.

5.6 $U$-initial morphisms and sources. For a functor $U : A \to X$, a source $(g_i : A \to B_i)_{i \in I}$ of $A$-morphisms is $U$-initial if, for every source $(h_i : C \to B_i)_{i \in I}$ in $A$ and every $X$-morphism $s : UC \to UA$ with $Ug_i \cdot s = Uh_i$, there is exactly one morphism $t : C \to A$ in $A$ with $Ut = s$ and $g_i \cdot t = h_i$ for all $i \in I$:

\[
\begin{array}{ccc}
A & \xrightarrow{Ug_i} & UB_i \\
\downarrow{t} & & \downarrow{Uh_i} \\
C & \xrightarrow{s} & UC
\end{array}
\]

Of course, uniqueness of $t$ as well as $g_i \cdot t = h_i (i \in I)$ follow from $Ut = s$ when $U$ is faithful. Hence, for $U$ faithful, $U$-initiality of $(g_i)_{i \in I}$ simply means that any $X$-morphism $s : UC \to UA$ can be lifted to an $A$-morphism $t : C \to A$ along $U$ whenever all $Ug_i \cdot s : UC \to UB_i$ can be lifted to $A$-morphisms $h_i : C \to B_i$ along $U$.

If the given source consists of a single morphism $f : A \to B$ (hence, if $|I| = 1$), it is more customary to say that $f$ is $U$-cartesian instead of $U$-initial, and we shall do so especially when $U$ is not necessarily faithful.

In the case where $I = \emptyset$, the source $(g_i)_{i \in I}$ is given by an object $A$, and we say that $A$ is $U$-indiscrete when $A$ is $U$-initial as an empty source. The dual notions for sinks are those of $U$-final sink, $U$-cocartesian morphism, and $U$-discrete object, with the universal property depicted by

\[
\begin{array}{ccc}
UA_i & \xrightarrow{Uf_i} & UB \\
\downarrow{uk_i} & & \downarrow{t} \\
UC & \xrightarrow{s} & C
\end{array}
\]

(We note a certain contradiction in the common terminology here, since $U$-final sinks with domain $(A_i)_{i \in I}$ are characterized as initial objects in a certain category; but the first two examples below give some justification.)

5.6.1 Examples.

1. For the forgetful functor $U : \text{Ord} \to \text{Set}$, a source $(g_i : A \to B_i)_{i \in I}$ is $U$-initial precisely when

\[x \leq y \iff \forall i \in I (g_i(x) \leq g_i(y))\]
for all \( x, y \in A \). A sink \((f_i : A_i \to B_i)_{i \in I}\) is \(U\)-final precisely when, for all \( z \neq w \) in \( A \),
\[
  z \leq w \iff \exists i_0, \ldots, i_n \in I \exists x_0 \leq y_0 \text{ in } A_{i_0}, \ldots, x_n \leq y_n \text{ in } A_{i_n} : \\
  z = f_{i_0}(x_0), f_{i_0}(y_0) = f_{i_1}(x_1), \ldots, f_{i_{n-1}}(y_{n-1}) = f_{i_n}(x_n), f_{i_n}(y_n) = w .
\]
Briefly, \( B \) carries the least order making all \( f_i \) monotone.

(2) For the forgetful functor \( U : \text{Top} \to \text{Set} \), a source \((g_i : A \to B_i)_{i \in I}\) is \(U\)-initial precisely when \( \{g_i^{-1}(V) \mid i \in I, V \subseteq B_i \text{ open}\} \) is a generating system of open sets for \( A \). A sink \((f_i : A_i \to B)_{i \in I}\) is \(U\)-final precisely if \( V \subseteq B \) is open whenever all \( f_i^{-1}(V) \subseteq A_i \) are open.

(3) Let \( \mathbb{T} \) be a monad on a category \( X \). Then every mono-source in \( X^\mathbb{T} \) is initial with respect to the forgetful functor \( G^\mathbb{T} : X^\mathbb{T} \to X \).

(4) For a category \( C \), the functor category \( C^2 \) (with \( 2 = \{\bot, \top\} \) considered as a category) can be thought of as having the morphisms of \( C \) as its objects, and a morphism \((u, v) : f \to g \) in \( C^2 \) is given by a commutative square
\[
\begin{array}{ccc}
  f & \xrightarrow{u} & g \\
  \downarrow{v} & & \downarrow{v} \\
\end{array}
\]
(5.6.i)
in \( C \). The evaluation functor at \( \top \in \text{ob} 2 \) now appears as the codomain functor
\[
\text{cod} : C^2 \to C , \quad f \mapsto \text{cod } f , \quad (u, v) \mapsto v .
\]
The morphism \((u, v) : f \to g \) is cod-cartesian if and only if (5.6.i) is a cartesian square in \( C \), that is, a pullback diagram in \( C \).

For \( U : A \to X \), we denote the classes of \( U\)-initial and \( U\)-final morphisms in \( A \) by
\[
\text{Ini} U \quad \text{and} \quad \text{Fin} U ,
\]
respectively, and list some easily established properties for them.

5.6.2 Proposition.

(1) \( \text{Ini} U \) contains all isomorphisms, is closed under composition, and is weakly left-cancelable; it is even left-cancelable when \( U \) is faithful.

(2) \( \text{Ini} U \) is stable under pullbacks and multiple pullbacks when \( U \) preserves them, and is closed under those limits in \( A \) that are preserved by \( U \).

Proof. The statements follow by routine verifications. \( \square \)

We leave it to the reader to formulate the corresponding generalized statements for sources, as well as their dualizations for morphisms and sinks.
5.7 Fibrations and cofibrations. A functor $U : A \to X$ is a *fibration* (more precisely, a *cloven fibration*) when, for all $f : X \to UB$ in $X$ with $B \in \text{ob} A$, there is a (tacitly chosen) $U$-cartesian lifting, i.e., a morphism $g \in \text{Ini} U$ with $Ug = f$:

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow U \\
X & \rightarrow & UB
\end{array}
$$

It is easy to show that $U$ is a fibration if and only if the functors $U_B : A/B \to X/UB$, $(A,g) \mapsto (UA,Ug)$, have right adjoints $\Gamma_B$ such that the counits are identity morphisms. Hence, $U_B \Gamma_B = 1_{X/UB}$ and $\Gamma_B$ is a full embedding for all $B \in \text{ob} A$.

A functor $U : A \to X$ is a *cofibration* (also called an *opfibration*) when $U^{\text{op}} : A^{\text{op}} \to X^{\text{op}}$ is a fibration; explicitly, when for all $f : UA \to Y$ in $X$ with $A \in \text{ob} A$, there is a $U$-cocartesian lifting (again, tacitly chosen) $g : A \to B$ in Fin$U$:

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow U \\
UA & \rightarrow & Y
\end{array}
$$

For $X \in \text{ob} X$, let $U^{-1}X$ denote the fibre of $U$ over $X$: its objects are the $A$-objects $A$ with $UA = X$, and a morphism $t : A \to A'$ in $U^{-1}X$ is an $A$-morphism $t$ with $Ut = 1_X$ (and composition as in $A$). When $U$ is a fibration, any $f : X \to Y$ in $X$ gives rise to a functor $f^* : U^{-1}Y \to U^{-1}X$ which assigns to $B \in \text{ob}(U^{-1}Y)$ the domain $A$ of the chosen $U$-cartesian lifting of $f : X \to UB$. Dually, when $U$ is a cofibration, there is a functor $f_* : U^{-1}X \to U^{-1}Y$ which assigns to $A \in \text{ob}(U^{-1}X)$ the codomain $B$ of the chosen $U$-cocartesian lifting of $f : UA \to Y$. One easily checks that there is an adjunction

$$
\begin{array}{ccc}
U^{-1}Y & \rightleftarrows & U^{-1}X \\
\downarrow f^* & & \downarrow f_*
\end{array}
$$

when $U$ is a fibration and a cofibration.

This adjunction describes best the “transport of structure along $f$”. For example, for the forgetful functor $U : \text{Top} \to \text{Set}$ and a set $X$, the fibre $U^{-1}X$ is the lattice of topologies on $X$, and for a map $f : X \to Y$ of sets and a topology $\tau \in U^{-1}X$, the topology $\sigma = f_*(\tau)$ is described by $(V \in \sigma \iff f^{-1}(V) \in \tau)$. Likewise, for $\sigma \in U^{-1}Y$, the topology $\tau = f^*(\sigma)$ is given by $(V \in \tau \iff \exists W \in \sigma : V = f^{-1}(W))$. This way, one sees that $U$ is a fibration and a cofibration.

The forgetful functor $\text{Ord} \to \text{Set}$ is another example of a fibration and a cofibration. The (non-faithful) object-functor $\text{ob} : \text{Cat} \to \text{Set}$ is a fibration, but not a cofibration (see Exercise 5.R).
A useful generalization of the notions of fibration and cofibration is that of an $\mathcal{M}_0$-fibration and $\mathcal{E}_0$-fibration, for classes of morphisms $\mathcal{M}_0$ and $\mathcal{E}_0$ in $X$. For an $\mathcal{M}_0$-fibration, one requires the existence of a $U$-cartesian lifting of $f : X \to UB$ in $X$ only when $f \in \mathcal{M}_0$; dually, for an $\mathcal{E}_0$-fibration, $U$-cocartesian liftings are only required for morphisms in $\mathcal{E}_0$.

For example, the forgetful functor $\text{Ord}_{\text{sep}} \to \text{Set}$ of the category of separated ordered sets is not a fibration, but it is a mono-fibration (that is, a Mono $\text{Set}$-fibration). Similarly, the forgetful functor $\text{Haus} \to \text{Set}$ from the category $\text{Haus}$ of Hausdorff topological spaces is a mono-fibration. (A topological space $X$ is Hausdorff if, for all $x, y \in X$ with $x \neq y$, there exist open subsets $A, B \subseteq X$ with $x \in A$, $y \in B$ and $A \cap B = \emptyset$.) The following result offers a general explanation of these examples.

5.7.1 Proposition. For a functor $U : A \to X$ and a factorization system $(\mathcal{E}_0, \mathcal{M}_0)$ in $X$, let

$$\mathcal{E} = U^{-1}\mathcal{E}_0 = \{e \in \text{mor} A \mid Ue \in \mathcal{E}_0\}, \quad \mathcal{M} = U^{-1}\mathcal{M}_0 \cap \text{Ini} \ U.$$ 

(1) If $U$ is an $\mathcal{M}_0$-fibration, then $(\mathcal{E}, \mathcal{M})$ is a factorization system in $A$.

(2) If $U$ is a fibration and $B$ is an $\mathcal{E}$-reflective subcategory of $A$, then $U|_B$ is an $\mathcal{M}_0$-fibration.

Proof. (1) In order to $(\mathcal{E}, \mathcal{M})$-factorize $f : A \to B$ in $A$, let $m_0 \cdot e_0 = Uf$ be an $(\mathcal{E}_0, \mathcal{M}_0)$-factorization in $X$, and consider a $U$-cartesian lifting $m : C \to B$ of $m_0 : Z \to UB$. Then $e : A \to C$ is the only morphism with $Ue = e_0$ and $m \cdot e = f$. The fact that $\mathcal{E} \perp \mathcal{M}$ follows from routine diagram chasing. (2) follows from (1) and Proposition 5.1.3. 

5.7.2 Corollary. If $U : A \to X$ is a fibration, then $U^{-1}(\text{Iso} X)$ is a left factorization class in $A$.

Proof. Apply Proposition 5.7.1 with $\mathcal{E}_0 = \text{Iso} X$. 

5.8 Topological functors. A functor $U : A \to X$ is topological if every source $(f_i : X \to UB_i)_{i \in I}$ with a family $(B_i)_{i \in I}$ of $A$-objects admits a $U$-initial lifting (once more, assumed to be chosen), that is, an $U$-initial source $(g_i : A \to B_i)_{i \in I}$ with $UA = X$ and $Ug_i = f_i$ for all $i \in I$:

$$\begin{array}{ccc}
A & \xrightarrow{g_i} & B_i \\
\downarrow & & \downarrow \\
X & \xrightarrow{f_i} & UB_i.
\end{array}$$

Hence, in generalization of the corresponding statement for fibrations, $U$ is topological if for every discrete category $I$ and every object $B = (B_i)_{i \in I} \in \text{ob} A^I$, the induced functor

$$U_B : (\Delta_X \downarrow B) \to (\Delta_X \downarrow UB)$$

has a right adjoint $\Gamma_B$ such that the counits are identity morphisms; here $\Delta_A : A \to A^I$ is as in 2.8.
Obviously, every topological functor is a fibration ($|I| = 1$) and has a full and faithful right adjoint ($I = \emptyset$; in this case, for the only object $B$ of $A^\emptyset$, one has $(\Delta_A \downarrow B) \cong A$ and $(\Delta_\times \downarrow UB) \cong X$).

The forgetful functor $U : \text{Ord} \to \text{Set}$ is topological. For a source $(g_i : X \to B_i)_{i \in I}$ with ordered sets $B_i$, make the set into an ordered set $A$ by providing it with the appropriate order described in Example 5.6.1(1). Similarly, $U : \text{Top} \to \text{Set}$ is topological: provide $X$ with the topology generated by the sets $g_i^{-1}(V)$ ($i \in I$, $V \subseteq B_i$ open), see Example 5.6.1(2).

Topological functors possess extremely good lifting properties. We already saw that, as fibrations, they lift factorization systems for morphisms (see Proposition 5.7.1(1)). In fact, there is a common generalization for both lemmata, see [Böger and Tholen, 1978].

Next, we show how to “lift” limits. For that, one first proves a lemma similar to 5.3.1. (In fact, there is a common generalization for both lemmata, see [Böger and Tholen, 1978].)

5.8.1 Proposition. For a functor $U : A \to X$ and a factorization system $(\mathcal{E}, \mathcal{M})$ for sources in $X$, let

$$\mathcal{E} = U^{-1}\mathcal{E}_0, \quad \mathcal{M} = \{(f_i)_{i \in I} \mid (Uf_i)_{i \in I} \in \mathcal{M}_0\}.$$ 

(1) If $U$ is $\mathcal{M}_0$-topological, then $(\mathcal{E}, \mathcal{M})$ is a factorization system for sources in $A$.

(2) If $U$ is topological and $B$ is an $\mathcal{E}$-reflective subcategory of $A$, then $U|_B$ is $\mathcal{M}_0$-topological.

Proof. The proof of Proposition 5.7.1 can be adapted to the present situation. \hfill \Box

Next, we show how to “lift” limits. For that, one first proves a lemma similar to 5.3.1. (In fact, there is a common generalization for both lemmata, see [Böger and Tholen, 1978].)

5.8.2 Lemma. A topological functor $U : A \to X$ is faithful.

Proof. For $x, y : A \to D$ in $A$, assume $Ux = Uy = h$, and consider the constant source $(h)_{i \in I}$ with $I := \text{mor}U^{-1}(UA)$. Let $(e_i : C \to D)_{i \in I}$ be a $U$-initial lifting of $(h)_{i \in I}$, and consider

$$J := \{w \in I \mid \forall i \in I : e_i \cdot w \in \{x, y\}\}.$$ 

With a retraction $\sigma : I \to J$, consider for $i \in I$

$$u_i := \begin{cases} x & \text{if } e_i \cdot \sigma(i) = y \\ y & \text{if } e_i \cdot \sigma(i) = x, \end{cases}$$

and derive $x = y$ as in Lemma 5.3.1. \hfill \Box

5.8.3 Proposition. For a faithful functor $U : A \to X$, the $U$-initial lifting of a limit cone $\lambda : \Delta L \to UD$ in $X$ (with $D : J \to A$) yields a limit cone $\alpha : \Delta A \to D$ in $A$.

Proof. Faithfulness of $U$ makes sure that the $U$-initial lifting $(\alpha_i : A \to D_i)_{i \in \text{ob}J}$ of the source $(\lambda_i : L \to UD_i)_{i \in \text{ob}J}$ does give a cone in $A$. Its limit property follows routinely. \hfill \Box
5.8.4 Corollary. For a topological functor $U : A \to X$, if $X$ is $J$-complete, then so is $A$, and $U$ preserves $J$-limits.

Proof. This follows immediately from the Proposition.

We note that a topological functor preserves in fact all limits since it has a left adjoint (see Theorem 5.9.1 below).

The Proposition and its Corollary fully explain how to construct limits in $\mathbf{Top}$. Indeed, one just needs to provide the limit of the underlying sets with the $U$-initial structure with respect to the limit projections.

The previous assertions hold analogously for colimits since topologicity of a functor is a self-dual concept, as we show next.

5.9 Selfdual characterization of topological functors. A functor $U : A \to X$ is transportable if for every isomorphism $f : X \to UB$ in $X$ with $B \in \text{ob} A$, there is a (chosen) isomorphism $g : A \to B$ in $A$ with $Ug = f$. Since $U$-cartesian liftings of isomorphisms are isomorphisms, transportability of $U$ means precisely that $U$ is an $\text{Iso}_X$-fibration.

When $U$ is faithful, all fibres $U^{-1}X$ ($X \in \text{ob} X$) are ordered classes. We call them large-complete if the infimum (or supremum) of any subclass exists.

5.9.1 Theorem. The following conditions are equivalent for a functor $U : A \to X$:

(i) $U$ is topological;

(ii) every sink $(f_i : UA_i \to Y)_{i \in I}$ admits a $U$-final lifting $(g_i : A_i \to B)_{i \in I}$;

(iii) $U$ is a faithful fibration and a cofibration with large-complete fibres;

(iv) $U$ is faithful and transportable with a fully faithful left adjoint, and $U^{-1}(\text{Iso}_X)$ is a left factorization class for sources in $A$;

(v) $U$ is faithful and transportable with a fully faithful right adjoint, and $U^{-1}(\text{Iso}_X)$ is a right factorization class for sinks in $A$.

Proof. (i) $\implies$ (iii): An infimum of a family $(B_i)_{i \in I}$ in $U^{-1}Y$ is obtained from a $U$-initial lifting of the source $(1_Y : Y \to UB_i)_{i \in I}$. Hence, the fibres of $U$ are large-complete. It remains to be shown that $U$ is a cofibration. For $f : UA \to Y$ in $X$ with $A \in \text{ob} A$, one considers all morphisms $g_i : A \to B_i$ in $A$ with $Ug_i = f$ ($i \in I$), and then a $U$-initial lifting $(e_i : B \to B_i)_{i \in I}$ of the source $(1_Y : Y \to UB_i)_{i \in I}$. There is then a morphism $g : A \to B$ with $Ug = f$, and we should check that it is $U$-final. Hence, let $k : A \to C$ in $A$ and $s : UB \to UC$ in $X$ with $s \cdot Ug = Uk$. With a $U$-initial lifting $t : D \to C$ of $s$, one obtains $g' : A \to D$ with $Ug' = f$, so that $g' = g_i$ for some $i \in I$. Now, $t \cdot e_i : B \to C$ satisfies $U(t \cdot e_i) = s$, as desired.
(iii) \implies (i): Given a source \((f_i : X \to UB_i)_{i \in I}\) in \(X\) with \(B_i \in \text{ob} A\) for every \(i \in I\), let \(g_i : A_i \to B_i\) be a \(U\)-initial lifting of \(f_i : X \to UB_i\), and let \((e_i : A \to A_i)_{i \in I}\) represent an infimum in \(U^{-1}X\). We need to show that \((g_i \cdot e_i : A \to B_i)_{i \in I}\) is \(U\)-initial. Hence, we consider \(s : UC \to X\) in \(X\) and \(k_i : C \to B_i\) in \(A\) with \(f_i \cdot s = Uk_i\) for all \(i \in I\). Now, \(U\)-initiality of every \(g_i\) gives \(t_i : C \to A_i\) in \(A\) with \(Ut_i = s\), and every \(t_i\) factors through the \(U\)-final lifting \(t : C \to D\) of \(s : UC \to X\). The infimum property of \(A\) in \(U^{-1}X\) then yields a morphism \(j : D \to A\) in \(U^{-1}X\), and we have \(U(j \cdot t) = s\), as desired.

Since \((iii)^{op} = (iii)\) and \((i)^{op} = (ii)\), we have established the equivalence of \((i), (ii), (iii)\). In particular, a topological functor has both a fully faithful left adjoint and a fully faithful right adjoint, and it is faithful and transportable. Furthermore, for a source \((f_i : A \to B_i)_{i \in I}\) in \(A\), one obtains a \((U^{-1}(Iso), \{U\text{-initial sources}\})\)-factorization by \(U\)-initially lifting the source \((Uf_i : UA \to UB_i)_{i \in I}\). The unique diagonalization property follows as in the morphism case. Hence, \((i) \implies (iv)\) is shown.

(iv) \implies (i): Let \(D \dashv U\) with unit \(\eta\) an isomorphism. A source \((f_i : X \to UB_i)_{i \in I}\) in \(X\) gives rise to a source \((g_i : DX \to B_i)_{i \in I}\), for which there is then a morphism \(e : DX \to C\) in \(U^{-1}(IsoX)\) and a \(U\)-initial source \((m_i : C \to B_i)_{i \in I}\) with \(m_i \cdot e = g_i\) for all \(i \in I\). The \(X\)-isomorphism \(U e \cdot \eta_X : X \to UC\) may be lifted to an \(A\)-isomorphism \(j : A \to C\) with \(U j = U e \cdot \eta_X\). Hence, \((m_i \cdot j)_{i \in I}\) is the desired \(U\)-initial lifting of \((f_i)_{i \in I}\). Since \((iv)^{op} = (v)\), this completes the proof. \(\square\)

The factorization needed in \((v)\) may be constructed with Theorem 5.3.2. For this, we say that \(X\) has small connected limits if every diagram \(D : J \to X\) with \(J\) small and connected (see Exercise 2.Q) has a chosen limit in \(X\).

\(\copyright\) 5.9.2 Corollary. Let \(X\) have small connected limits, and let \(U : A \to X\) have small fibres. The functor \(U\) is topological if and only if the following conditions hold:

1. \(U\) is faithful and transportable;
2. \(A\) has small connected limits, and \(U\) preserves them;
3. \(U\) has a fully faithful right adjoint.

Proof. For the necessity of the conditions, see Proposition 5.8.3 and Theorem 5.9.1. For their sufficiency, after 5.9.1(v) we must only show that \(\mathcal{M} = U^{-1}(IsoX)\) satisfies the conditions of Theorem 5.3.2. Certainly, \(\mathcal{M}\) satisfies \(\text{Iso}A \subseteq \mathcal{M} \subseteq \text{Mono} A\), is closed under composition, and also stable under pullbacks since \(U\) preserves them. The only delicate point is the existence of (not necessarily small) intersections of morphisms in \(\mathcal{M}\). Hence, consider \(m_i : A_i \to B\) in \(\mathcal{M}\) \((i \in I)\). Transportability gives isomorphisms \(f_i : A_i \to B_i\) with \(Uf_i = Um_i\) \((i \in I)\), and

\[\{B_i \in \text{ob} X \mid i \in I\} \subseteq U^{-1}(UB)\]

is just a set. Furthermore, if \(B_i = B_j\), then \((A_i, m_i) \cong (A_j, m_j)\) in \(A/B\) for all \(i, j \in I\). Consequently, in order to form the multiple pullback of \((m_i)_{i \in I}\) in \(A\), it suffices to form the
multiple pullback of a small subfamily, which exists and is preserved by $U$, so that it lies in $\mathcal{M}$ again. □

5.10 Epireflective subcategories. In Proposition 5.10.1, we noted that a replete reflective subcategory $\mathcal{B}$ of a category $\mathcal{C}$ with a factorization system $(\mathcal{E}, \mathcal{M})$ for morphisms has its reflection morphisms in $\mathcal{E}$ if and only if $\mathcal{B}$ is closed under $\mathcal{M}$-morphisms in $\mathcal{C}$, that is, if $m : A \to B$ in $\mathcal{M}$ with $B \in \mathcal{B}$ implies $A \in \mathcal{B}$. More generally, for a collection $\mathcal{M}$ of sources in $\mathcal{C}$, one says that $\mathcal{B}$ is closed under $\mathcal{M}$-sources in $\mathcal{C}$ if $(m_i : A \to B_i)_{i \in I}$ in $\mathcal{M}$ with $B_i \in \mathcal{B}$ implies $A \in \mathcal{B}$, and one proves the following result.

5.10.1 Proposition. In a category $\mathcal{C}$ with a factorization system $(\mathcal{E}, \mathcal{M})$ for sources, a full replete subcategory $\mathcal{B}$ of $\mathcal{C}$ is $\mathcal{E}$-reflective if and only if $\mathcal{B}$ is closed under $\mathcal{M}$-sources in $\mathcal{C}$.

Proof. For the “only if” part, one proceeds as in Proposition 5.10.1. For the “if” part, given $C \in \mathcal{B}$, one considers the source $(f_i : C \to A_i)_{i \in I}$ of all morphisms with domain $C$ and codomain in $\mathcal{B}$, and one $(\mathcal{E}, \mathcal{M})$-factors it as $f_i = g_i \cdot e$ with $e : C \to B$ in $\mathcal{E}$. Since $B \in \mathcal{B}$, one considers the source $(f_i : A \to B_i)_{i \in I}$ in $\mathcal{M}$ with $B_i \in \mathcal{B}$ by hypothesis and $\mathcal{E} \subseteq \text{Epi} \mathcal{C}$ (by the dual of Lemma 5.3.1), $e$ is a $\mathcal{B}$-reflection morphism for $C$. □

5.10.2 Corollary. Let $\mathcal{C}$ have products and a factorization system $(\mathcal{E}, \mathcal{M})$ for morphisms, with $\mathcal{E} \subseteq \text{Epi} \mathcal{C}$, and suppose that $\mathcal{C}$ is $\mathcal{E}$-cowellpowered. Then a full replete subcategory $\mathcal{B}$ of $\mathcal{C}$ is $\mathcal{E}$-reflective if and only if $\mathcal{B}$ is closed under products and $\mathcal{M}$-morphisms in $\mathcal{C}$.

Proof. After Proposition 5.10.1, it suffices to guarantee the existence of a factorization system $(\mathcal{E}, \mathcal{M})$ for sources. Given a source $(f_i : A \to B_i)_{i \in I}$ in $\mathcal{C}$, one may $(\mathcal{E}, \mathcal{M})$-factor each $f_i = m_i \cdot e_i$, and then choose a small representative family $(e_j : A \to C_j)_{j \in J}$ (with $J \subseteq I$) among the morphisms $e_i \in \mathcal{E}$ ($i \in I$). By $(\mathcal{E}, \mathcal{M})$-factoring the induced morphism $g : A \to \prod_{j \in J} C_j$ with $p_j \cdot g = e_j$ (for $j \in J$) as $g = m \cdot e$, one obtains

$$f_j = (m_j \cdot p_j \cdot m) \cdot e \quad (j \in J)$$

with $e \in \mathcal{E}$ and $d \perp (m_j \cdot p_j \cdot m)_{j \in J}$ for all $d \in \mathcal{E}$. Since each $e_i$ ($i \in I$) is isomorphic to some $e_j$ ($j \in J$), this factorization extends to the original source. □

In the presence of a topological functor $U : \mathcal{C} \to \mathcal{X}$, we may apply Proposition 5.10.1 in particular to the $(U^{-1}(\text{Iso} \mathcal{X}), \mathcal{M})$-factorization system for sources in $\mathcal{C}$, where now $\mathcal{M}$ consists of all $U$-initial sources, and obtain at once the equivalence $(i) \iff (ii)$ in the following Theorem.

5.10.3 Theorem. Let $U : \mathcal{C} \to \mathcal{X}$ be a topological functor. The following assertions for a full replete subcategory $\mathcal{B}$ of $\mathcal{C}$ are equivalent:

(i) $\mathcal{B}$ is $U^{-1}(\text{Iso} \mathcal{X})$-reflective in $\mathcal{C}$;

(ii) $\mathcal{B}$ is closed under $U$-initial sources in $\mathcal{C}$;
(iii) $U|_B$ is topological, and $U|_B$-initial sources in $B$ are also $U$-initial in $C$.

**Proof.** From the comment preceding the Theorem, we are left to verify (ii) $\iff$ (iii), but this is immediate. \hfill \Box

Let us finally discuss an important sufficient condition for the inclusion functor $B \hookrightarrow C$ to preserve initiality, as in (iii). One says that a full replete subcategory $B$ is $U$-finally dense (or simply finally dense) if every object $C$ in $C$ is the codomain of some $U$-final sink $(f_i : A_i \to C)_{i \in I}$ with all $A_i \in \text{ob}B$; the dual notion is that of a $U$-initially dense (or just initially dense) subcategory. Without the topologicity assumption on $U$, one can still prove:

**5.10.4 Proposition.** Let $U : C \rightarrow X$ be a functor and $B$ a full replete subcategory of $C$ that is finally dense in $C$. Then $U|_B$-initial sources in $B$ are $U$-initial in $C$, and when $U$ is faithful and $U|_B$ is topological, $B$ is $U^{-1}(\text{Iso}X)$-reflective in $C$.

**Proof.** In order to show $U$-initiality of a $U|_B$-initial source $(f_i : A \to B)_{i \in I}$ in $B$, consider $(g_i : C \to B_i)_{i \in I}$ in $C$ and $s : UC \to UA$ in $X$ with $Uf_i \cdot s = Ug_i$ for all $i \in I$, and let $(h_j : A_j \to C)_{j \in J}$ be $U$-final with all $A_j \in \text{ob}B$. $U|_B$-initiality then gives, for every $j \in J$, a unique $t_j : C \to A_j$ with $Ut_j = s \cdot Uh_j$ and $f_i \cdot t_j = g_i \cdot h_j$ for all $i, j \in I$, and $U$-finality yields a unique $t : C \to A$ in $B$ with $Ut = s$ and $t \cdot h_j = t_j$ for all $j \in J$. By $U|_B$-initiality again, $t$ also satisfies $f_i \cdot t = g_i$ for all $i \in I$, and one easily sees that $t$ is the only morphism “over $s$” satisfying these equations:

![Diagram](https://example.com/diagram.png)

For the reflexivity assertion, one proceeds as in the proof of Proposition 5.10.1 and, given $C \in \text{ob}C$, one considers the source $(f_i : C \to A_i)_{i \in I}$ of all morphisms with domain $C$ and codomain in $B$. Let $(g_i : B \to A_i)_{i \in I}$ be a $U|_B$-initial lifting of $(Uf_i : UC \to UA_i)_{i \in I}$. Since $(g_i)_{i \in I}$ must be even $U$-initial, there is $e : C \to B$ in $C$ with $Ue = 1_{UC}$ which is easily seen to the $B$-reflection morphism for $C$. \hfill \Box

**5.10.5 Example.** There is a full coreflective embedding $E : \text{Ord} \to \text{Top}$ which provides an ordered set $(X, \leq)$ with the topology of open sets generated by the down-sets $\downarrow x$, for $x \in X$. Its right adjoint $S$ provides a topological space with its underlying order (see 1.9). A topological space $X$ is in the image of $E$ precisely when it is an Alexandroff space, that is, when arbitrary intersections of open sets in $X$ are open. By the dual of Theorem 5.10.3, $\text{Ord}$ is closed under $U$-final sinks in $\text{Top}$ (with the forgetful functor $U : \text{Top} \to \text{Set}$), and $U|_{\text{Ord}}$-final sinks in $\text{Ord}$ are $U$-final in $\text{Top}$. In fact, $\text{Ord}$ is even initially dense in $\text{Top}$: for a topological space $X$, consider the source of characteristic functions of all open sets of $X$ into the two-chain. But $\text{Ord}$ is not closed under $U$-initial sources, not even under products, as no infinite power in $\text{Top}$ of the two-chain is Alexandroff.
5.11 The Taut Lift Theorem. We consider a commutative diagram of functors

\[
\begin{array}{ccc}
A & \xrightarrow{G} & B \\
U & \downarrow & V \\
X & \xrightarrow{J} & Y
\end{array}
\]

such that \( J \) has a left adjoint \( H \xrightarrow{\gamma} J \). Our goal is to find a left adjoint \( F \xrightarrow{\eta} G \) when \( U \) has good lifting properties.

5.11.1 Theorem (Taut Lift Theorem). Let \( U \) be a topological functor. Then in the previous displayed diagram, \( G \) has a left adjoint \( F \) with \( UF = HV \) if and only if \( G \) maps \( U \)-initial sources to \( V \)-initial sources.

Proof. For the necessity of the stated condition, see Exercise 5.1. For its sufficiency, we consider an object \( B \) in \( B \) and the source of all morphisms \( t: B \to GA_t \) with \( A_t \in A \). The source of all

\[
HV_B \xrightarrow{HVt} HVGA_t = HJU A_t \xrightarrow{\delta_{UA_t}} UA_t
\]

has a \( U \)-initial lifting

\[
FB \xrightarrow{f_t} A_t
\]

so that \( Uf_t = \delta_{UA_t} \cdot HVt \). As \( JHVB = JUF B \), the codomain of \( \gamma_{VB} \) is the domain of \( JUf_t \), and

\[
JUf_t \cdot \gamma_{VB} = J\delta_{UA_t} \cdot JHVt \cdot \gamma_{VB} = J\delta_{UA_t} \cdot \gamma_{JA_t} \cdot Vt = Vt
\]

By hypothesis \( G \) transforms the source into a \( V \)-initial source, so the diagram

\[
\begin{array}{ccc}
JUF B & \xrightarrow{\gamma_{VB}=\gamma_{VB}} & JU A_t = VGA_t \\
\downarrow & & \downarrow Vt \\
V B & & V A_t
\end{array}
\]

produces a morphism \( \eta_B : B \to GFB \) with \( V\eta_B = \gamma_{VB} \) and \( Gf_t \cdot \eta_B = t \) for all \( t \). By adjointness of \( J \) and faithfulness of \( U \), the factorization \( Vt = JUf_t \cdot \gamma_{VB} \) determines \( f_t \) uniquely.

5.11.2 Example. Consider the diagram of forgetful functors

\[
\begin{array}{ccc}
\text{TopGrp} & \longrightarrow & \text{Top} \\
\downarrow & & \downarrow \\
\text{Grp} & \longrightarrow & \text{Set}
\end{array}
\]
where $\text{TopGrp}$ is the category of topological groups (that is, groups $G$ with a topology that makes the group operations $G \times G \to G$, $(x, y) \mapsto x \cdot y$, and $G \to G$, $x \mapsto x^{-1}$ continuous), and of continuous group homomorphisms. It is easy to see that $\text{TopGrp} \to \text{Grp}$ is topological with initial structures formed as for $\text{Top} \to \text{Set}$. Hence, the free-group functor $F : \text{Set} \to \text{Grp}$ may be lifted to a left adjoint functor $\text{Top} \to \text{TopGrp}$. In lieu of groups, the example generalizes to any algebraic structures defined by a set of operations and equations between them, and that admit free functors (see Example 2.12.2).

**Exercises**

5.A More cancellation rules. Let $\mathcal{M}$ be a pullback-stable class of morphisms in $\mathcal{C}$. Then $\mathcal{M}$ satisfies the cancellation property: if $g \cdot f \in \mathcal{M}$ with $g$ monic implies $f \in \mathcal{M}$. If $\mathcal{M}$ belongs to a factorization system $(\mathcal{E}, \mathcal{M})$, then $g \cdot f \in \mathcal{M}$ with a split epimorphism $f$ implies $g \in \mathcal{M}$. The family $(f_i : A \to B_i)_{i \in I}$ is a mono-source if $(g_{ij} \cdot f_i)_{i \in I, j \in J}$ is a mono-source for sources $(g_{ij} : B_i \to C_{ij})_{j \in J, i \in I}$.

5.B More on proper systems. For a factorization system $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{C}$ with equalizers, one has $\mathcal{E} \subseteq \text{Epi} \mathcal{C}$ if and only if $\text{RegMono} \mathcal{C} \subseteq \mathcal{M}$.

5.C Extremal, strong and regular epimorphisms. The class of strong epimorphisms in $\mathcal{C}$ is

$$\text{StrongEpi} \mathcal{C} := \text{Epi} \mathcal{C} \cap \left(\downarrow (\text{Mono} \mathcal{C})\right).$$

One has

$$\text{RegEpi} \mathcal{C} \subseteq \text{StrongEpi} \mathcal{C} \subseteq \text{ExtEpi} \mathcal{C},$$

where $\text{RegEpi} \mathcal{C}$ and $\text{ExtEpi} \mathcal{C}$ are the classes of regular and extremal epimorphisms, respectively (see 5.1 for the latter). If $\mathcal{C}$ has pullbacks, then $\text{StrongEpi} \mathcal{C} = \text{ExtEpi} \mathcal{C}$. Of course, the dual statements hold for strong monomorphisms. Furthermore, each of the following statements implies the next, and all are equivalent when $\mathcal{C}$ has kernel pairs and coequalizers of kernel pairs:

(i) $\text{Mono} \mathcal{C} \cdot \text{RegEpi} \mathcal{C} = \text{mor} \mathcal{C}$;

(ii) $(\text{RegEpi} \mathcal{C}, \text{Mono} \mathcal{C})$ is a factorization system in $\mathcal{C}$;

(iii) $\text{RegEpi} \mathcal{C} = \text{ExtEpi} \mathcal{C}$;

(iv) $\text{RegEpi} \mathcal{C}$ is closed under composition.

5.D Regular monomorphisms in a topological category. If $U : \mathcal{A} \to \mathcal{X}$ is topological (or just a faithful functor with both a left and a right adjoint), an $\mathcal{A}$-morphism $f$ is a regular monomorphism if and only if it is $U$-initial and $Uf$ is a regular monomorphism. The same statement holds if “regular” is replaced by “extremal”.
5. **Factoriality of factorizations.** For every factorization system \((\mathcal{E}, \mathcal{M})\), there is a functor \(F : \mathcal{C}^2 \to \mathcal{C}\) which assigns to a morphism \(f\) in \(\mathcal{C}\) the object \(\text{dom } m = \text{cod } e\) for a chosen \((\mathcal{E}, \mathcal{M})\)-factorization \(f = m \cdot e\).

5.F **Wellpowered and cowellpowered categories over \(\text{Set}\).** The category \(\text{Set}\) and every topological category over \(\text{Set}\) is wellpowered and cowellpowered. Every monadic category over \(\text{Set}\) is wellpowered and \(\mathcal{E}\)-cowellpowered with \(\mathcal{E}\) the class of regular epimorphisms. However, a monadic category over \(\text{Set}\) is not necessarily cowellpowered: \(\text{Frm}\) is not cowellpowered (see [Johnstone, 1982]).

5.G **Closure operators as functors.** A class \(\mathcal{M}\) of morphisms in \(\mathcal{C}\) can be considered as a full subcategory of \(\mathcal{C}^2\) (see 5.6.1(4)). Show that an \(\mathcal{M}\)-closure operator \(c\) defines a functor \(c : \mathcal{M} \to \mathcal{M}\), together with a natural transformation \(j : 1_{\mathcal{M}} \to c\). If there is a monofibration \(U : \mathcal{C} \to \text{Set}\) such that \(\mathcal{M} = U^{-1}(\text{Mono} \cap \text{Ini} U)\) (see 5.9) and \(\mathcal{C}\) is \(\mathcal{M}\)-wellpowered, then \(c\) defines a functor
\[
c : \mathcal{C} \to \text{Cls} , \quad X \mapsto (UX, c_X) .
\]

5.H **Closure operators and closed subobjects.** Let \(\mathcal{M}\) be a right factorization class for sinks in \(\mathcal{C}\), and let \(\mathcal{K} \subseteq \mathcal{M}\) be stable under pullbacks. Then
\[
c_A(m) = \bigwedge\{k \in \text{sub } A \mid k \in \mathcal{K}, k \leq m\}
\]
defines an idempotent \(\mathcal{M}\)-closure operator of \(\mathcal{C}\), and one has \(\mathcal{M}^c = \mathcal{K}\) if and only if \(\mathcal{K}\) is stable under multiple pullbacks. Furthermore, as an idempotent closure operator, \(c\) is uniquely determined by the condition \(\mathcal{M} \subseteq \mathcal{K}\), and \(c\) is weakly hereditary if and only if \(\mathcal{K}\) is closed under composition.

5.I **Adjunctions, epimorphisms and generating classes.** For an adjunction \(F \dashv G : \mathcal{A} \to \mathcal{X}\), one has that if \(G\) is faithful and \(\mathcal{H}\) is generating in \(\mathcal{X}\), then
\[
F\mathcal{H} = \{FH \mid H \in \mathcal{H}\}
\]
is generating in \(\mathcal{A}\). Furthermore, the following conditions are equivalent:

(i) \(G\) is faithful;

(ii) \(G\) reflects epimorphisms;

(iii) the counits \(\varepsilon_A\) are epimorphisms;

(iv) the class \(\{FX \mid X \in \mathcal{X}\}\) is generating in \(\mathcal{A}\).

The same equivalence holds if one specializes to a faithful functor that reflects isomorphisms in (i), extremal epimorphisms in (ii) and (iii), and a strongly generating class in (iv).
5. J **Special Adjoint Functor Theorem.** Suppose that $\mathcal{A}$ is locally small, small-complete, has a cogenerator $\mathcal{G}$, and is wellpowered, so that for every $A \in \text{ob} \mathcal{A}$ there is a chosen set $\mathcal{J}_A \subseteq \text{ob} \mathcal{A}$ such that

if $m : B \to A$ is a monomorphism, then there is $J \in \mathcal{J}_A$ with $J \cong B$.

Then a functor $G : \mathcal{A} \to \mathcal{X}$ into a locally small category $\mathcal{X}$ is right adjoint if and only if it preserves small limits.

**Hint.** Use Proposition 5.5.1 and the sets $\mathcal{J}_A$ to construct a $G$-solution set for every $X \in \text{ob} \mathcal{X}$.

5. K **Special Adjoint Functor Theorem for a class of monomorphisms.** For some class $\mathcal{M}$ of monomorphisms closed under composition with isomorphisms, let the locally small category $\mathcal{A}$ have pullbacks of morphisms in $\mathcal{M}$ (along arbitrary morphisms) and intersections of arbitrarily large families of morphisms in $\mathcal{M}$, and suppose that both belong to $\mathcal{M}$ again. Furthermore, let $\mathcal{A}$ have an $\mathcal{M}$-cogenerator, that is, a set $\mathcal{G}$ of objects in $\mathcal{A}$ such that the product $P_A = \prod_{G \in \mathcal{G}} G^{A(G,A)}$ exists for all $A \in \text{ob} \mathcal{A}$, and the canonical morphism $P_A \to A$ lies in $\mathcal{M}$. Then a functor $G : \mathcal{A} \to \mathcal{X}$ has a left adjoint if and only if $G$ preserves all limits whose existence is guaranteed by the hypotheses.

5. L **Dense generators.** A class $\mathcal{G}$ of objects in a locally small category $\mathcal{C}$ is densely generating in $\mathcal{C}$ if $\mathcal{C}(\mathcal{G},A)$ is a strict epi-sink for all objects $A$, that is: whenever a sink $(h_x : G_x \to B)_{x \in \mathcal{C}(\mathcal{G},A)}$ has the property

$$x \cdot a = y \cdot b \implies h_x \cdot a = h_y \cdot b$$

for all $a : D \to G_x$, $b : D \to G_y$, $x, y \in \mathcal{C}(\mathcal{G},A)$, then $h_x = f \cdot x$ for all $x$, with a uniquely determined morphism $f : A \to B$. Show that $\mathcal{G}$ is densely generating if and only if the functor

$$y_\mathcal{G} : \mathcal{C} \to \text{Set}^{\mathcal{G}_{\text{op}}} \quad A \mapsto \left( \mathcal{C}(-,A) : \mathcal{G}_{\text{op}} \to \text{Set} \right)$$

(which has the Yoneda embedding as a factor) is full and faithful; here, and differently from 5.5, we consider $\mathcal{G}$ as a full subcategory of $\mathcal{C}$. A densely generating set is considered $\mathcal{G}$ as a full subcategory of $\mathcal{C}$. A densely generating set is strongly generating, but not vice versa: a one-point space is a strong generator of $\text{CompHaus}$, the category of compact Hausdorff spaces and continuous maps, but it is not dense (see Gabriel and Ulmer 1971).

5. M **$\mathcal{M}$-injective objects.** An object $C$ in a category $\mathcal{C}$ is $\mathcal{M}$-injective for a class $\mathcal{M}$ of morphisms in $\mathcal{C}$ if for any $m : A \to B$ with $m \in \mathcal{M}$ and $f : A \to C$, there exists a map $g : B \to C$ extending $f$, that is, such that the following diagram commutes:

$$\begin{array}{ccc}
A & \xrightarrow{m} & B \\
\downarrow f & & \downarrow g \\
C & \downarrow & \\
\end{array}$$
5. FACTORIZATIONS, FIBRATIONS AND TOPOLOGICAL FUNCTORS

5.N \( U \)-cocartesian liftings of regular epimorphisms. Let \( A \) have kernel pairs, and suppose that \( U : A \to X \) preserves them. Then a \( U \)-cocartesian lifting of a regular epimorphism \( f : UA \to Y \) in \( X \) is a regular epimorphism in \( A \). Morphisms in \( \text{Fin} U \cap U^{-1}(\text{RegEpi} X) \) are also called quotient morphisms of \( A \) (with respect to \( U \)).

5.O Fibrations and equivalences. A fibration \( U : A \to X \) which reflects isomorphisms is an equivalence, provided that \( A \) has a terminal object and \( U \) preserves it. The provision is essential: for any category \( C \) and \( T \in \text{ob} C \), the domain functor \( \text{dom} = \text{dom}_T : C/T \to C \) is a fibration and reflects isomorphisms, but it is not an equivalence, unless \( T \) is terminal in \( C \).

5.P The codomain functor as a fibration. The functor \( \text{cod} : C^2 \to C \) of Example 5.6.1(4) is a fibration if and only if \( C \) has pullbacks. It is topological if and only if \( C \) has multiple pullbacks (of any size).

5.Q The domain functor is not necessarily topological. Let \( C \) be locally small and small-complete. For any \( T \in \text{ob} C \), the domain functor \( \text{dom} : C/T \to C \) satisfies all conditions of Corollary 5.9.2 except that the right adjoint of \( \text{dom} \) generally fails to be fully faithful. Thus, in general, \( \text{dom} \) fails to be topological.

5.R The set-of-objects functor as a fibration. The set-of-objects functor \( \text{ob} : \text{Cat} \to \text{Set} \) is a fibration, but not a cofibration. Hence, it is not topological.

5.S Topological restrictions of topological functors. Let \( U : C \to X \) be topological and \( A \) a full replete subcategory of \( C \), and set \( \mathcal{J} := U^{-1}(\text{Iso} X) \). The following conditions are equivalent:

(i) \( U|_A \) is topological;

(ii) there is a full \( \mathcal{J} \)-reflective subcategory \( B \) of \( C \) which contains \( A \) as a \( \mathcal{J} \)-coreflective subcategory;

(iii) there is a full \( \mathcal{J} \)-coreflective subcategory \( B \) of \( C \) which contains \( A \) as a \( \mathcal{J} \)-reflective subcategory;

(iv) there is a functor \( R : C \to A \) with \((U|_A)R = U \) and \( R|_A = 1_A \).

\text{Hint.} \ For \ (i) \implies (ii), \ take \ for \ \text{ob} B \ all \ C \text{-objects} \ C \ for \ which \ there \ exist \ a \ U \text{-initial source} \ with \ domain \ C \ and \ all \ codomains \ in \ A.

5.T Preservation of initiality. For a commutative diagram of functors

\[
A \overset{G}{\longrightarrow} B \\
\downarrow{U} \quad \downarrow{V} \\
X \overset{J}{\longrightarrow} Y
\]
such that there are adjunctions \( \eta : F \cong G \) and \( \delta : H \cong J \), with the canonical transformation\( HV \to UF \) an isomorphism, show that \( G \) maps \( U \)-initial sources to \( V \)-initial sources. (Note that no topologicity assumption for \( U \) or \( V \) is required.)

**5.U Generalized Taut Lift Theorem.** In the commutative diagram of functors

\[
\begin{array}{ccc}
A & \xrightarrow{G} & B \\
U \downarrow & & \downarrow V \\
X & \xrightarrow{F} & Y
\end{array}
\]

let \( A \) have an \((\mathcal{E}, \mathcal{M})\)-factorization system for sources such that \( G \) maps sources in \( \mathcal{M} \) to \( V \)-initial sources. If \( U \) and \( J \) have left adjoints, then \( G \) has also a left adjoint.

**5.V Grothendieck construction versus faithful fibrations and topological functors.**

1. For a faithful cloven fibration \( U : A \to X \) one obtains (in the notation of \([5.7]\)) a pseudo-functor

\[
U^{-1} : X^{\text{op}} \to \text{ORD}
\]

into the ordered metacategory of ordered classes; it assigns to \( f : X \to Y \) in \( X \) the monotone function \( f^* : U^{-1}Y \to U^{-1}X \). If \( U \) is topological, \( \text{ORD} \) may be replaced by \( \text{INF} \), where objects have all infima and whose morphisms preserve them.

2. For a pseudo-functor \( T : X^{\text{op}} \to \text{ORD} \), let the objects of the category \( X_T \) be pairs \((X, \tau)\) with \( X \in \text{ob} X \), \( \tau \in TX \); a morphism \( f : (X, \tau) \to (Y, \sigma) \) is an \( X \)-morphism \( f : X \to Y \) with \( \tau \leq Tf(\sigma) \). The forgetful functor

\[
U_T : X_T \to X , \quad (X, \tau) \to X
\]

is a faithful fibration. When \( T \) takes values in \([\text{INF}]\) the functor \( U_T \) is topological.

3. For a faithful fibration \( U : A \to X \) and \( T = U^{-1} \), there is an isomorphism \( G : A \to X_T \) with \( U_T G = U \).

4. For a pseudo-functor \( T : X^{\text{op}} \to \text{ORD} \), there is a pseudo-natural isomorphism \( \gamma : T \to (U_T)^{-1} \).

**5.W Equivalence of topological functors and \( \text{INF} \)-valued pseudo-functors.**

1. Let the objects of the metacategory \( \text{FFIB} \) be faithful fibrations, and let morphisms \((F, G) : U \to V \) be commutative diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{G} & B \\
U \downarrow & & \downarrow V \\
X & \xrightarrow{F} & Y
\end{array}
\]
in \text{CAT}, such that \( G \) transforms \( U \)-initial morphisms into \( V \)-initial morphisms. The objects of the metacategory \( \text{CAT} \triangleright \text{ORD} \) are contravariant pseudo-functors with values in \( \text{ORD} \); a morphism

\[
(F, \gamma) : T \rightarrow S
\]

is given by a diagram

\[
\begin{array}{ccc}
X^{\text{op}} & \xrightarrow{F^{\text{op}}} & Y^{\text{op}} \\
\downarrow \gamma \downarrow & & \uparrow \gamma \uparrow \\
T & \xrightarrow{} & S \\
\text{ORD} & \text{ORD} & \text{ORD}
\end{array}
\]

with a functor \( F : X \rightarrow Y \) and a pseudo-natural transformation \( \gamma : T \rightarrow SF^{\text{op}} \), and its composite with \( (H, \delta) : S \rightarrow R \) is given by

\[
(H, \delta)(F, \gamma) = (HF, \delta F^{\text{op}} \cdot \gamma).
\]

Show that \( \text{FFIB} \) and \( \text{CAT} \triangleright \text{ORD} \) are equivalent metacategories.

(2) Let \( \text{TOPFUN} \) be the (non-full) subcategory of \( \text{FFIB} \) formed by those \( (F, G) : U \rightarrow V \) with \( U, V \) topological and \( G \) transforming \( U \)-initial sources into \( V \)-initial sources. Prove that \( \text{TOPFUN} \) is equivalent to \( \text{CAT} \triangleright \text{INF} \).

(3) Formulate (1) and (2) for cofibrations instead of fibrations.
Notes on Chapter II

Most of the topics presented in this chapter may be found in more elaborate form in the standard books on category theory: [Mac Lane, 1971], [Adámek, Herrlich, and Strecker, 1990], [Borceux, 1994a,b,c]. Each of these books contains remarks or brief sections on the foundations of category theory and its connections with logic or set theory, a topic that has been addressed in many articles, including [Lawvere, 1966], [Mac Lane, 1969], [Feferman, 1969, 1977], [Bénabou, 1985], as well as in books on topos theory, such as [Mac Lane and Moerdijk, 1994]. Beginners of category theory will enjoy the books [Lawvere and Rosebrugh, 2003] and [Awodey, 2006] while advanced readers are referred to [Kelly, 1982] as the standard text on enriched category theory, and to [Johnstone, 2002a,b] as a rich resource for a broad range of categorical topics. Readers looking for further reading on order and quantale theory as it pertains to Section 1 of this chapter are referred to [Johnstone, 1982], [Wood, 2004], [Rosenthal, 1990].

We highlight some particular aspects that distinguish this chapter from the literature mentioned thus far and give some additional references. The chapter covers some aspects of monad theory in greater detail than the standard texts on category theory, as it includes Duskin’s Monadicity Criterion (see Theorem 3.5.1, originally established in [Duskin, 1969]) and a discussion of Beck’s distributive laws [Beck, 1969] (see also [Manes and Mulry, 2007]), as well as a treatment of Kock-Zöberlein monads (see [Kock, 1995] and references therein), albeit in the simplified context of ordered categories. For further reading on monads and their connection with algebraic theories, we refer the reader to [Manes, 1976], [Barr and Wells, 1985], [MacDonald and Sobral, 2004] and [Adámek, Rosický, and Vitale, 2011]. In a higher-order context, they are treated in [Street, 1974] and [Lack and Street, 2002]. In this chapter, however, we touch upon higher-categorical structures only to a minimal level, with 2-cells most often given simply by order, the notion of quantaloid being an important example as treated in Stubbe’s articles [2005, 2006, 2007].

Another non-standard emphasis of this chapter concerns factorization systems and topological functors which, unlike in [Adámek, Herrlich, and Strecker, 1990], are presented in concert with fibrations (see [Grothendieck, Verdier, and Deligne, 1972], [Bénabou, 1985], [Streicher, 1998–2012]), as highlighted by Theorem 5.9.1. A predecessor of its Corollary 5.9.2 first appeared in Hoffmann’s thesis 1972. Wyler’s Taut Lift Theorem 5.11.1 was first proved in [Wyler, 1971] and presented in a general categorical context in [Tholen, 1978]. The key existence theorem on factorization systems (Theorem 5.3.2) appeared in generalized form in [Tholen, 1979], see also [Tholen, 1983]. For a categorical treatment of closure operators, the reader is referred to [Dikranjan and Tholen, 1995].
III Lax Algebras

Dirk Hofmann, Gavin J. Seal, Walter Tholen

For a quantale $\mathcal{V}$ and a monad $\mathbb{T}$ on $\text{Set}$, laxly extended to the category $\mathcal{V}\text{-Rel}$ of sets and $\mathcal{V}$-valued relations, this chapter introduces the key category of interest to this book, the category $(\mathcal{T}, \mathcal{V})\text{-Cat}$ whose objects, depending on context, may be called $(\mathcal{T}, \mathcal{V})$-algebras, $(\mathcal{T}, \mathcal{V})$-spaces, or $(\mathcal{T}, \mathcal{V})$-categories. After a first introduction to the $\mathcal{V}$-relational setting and the required lax monad extension, the guiding examples (ordered sets, metric spaces, topological spaces, approach spaces) are presented in full detail, followed by the basic properties of the category $(\mathcal{T}, \mathcal{V})\text{-Cat}$, like its topologicity over $\text{Set}$ and its embeddability into the quasitopos of $(\mathbb{T}, \mathcal{V})$-graphs. The seemingly “technical” lax extendability of $\mathbb{T}$ to the “syntactical” category $\mathcal{V}\text{-Rel}$ permits us to consider $\mathbb{T}$ as a monad on $\mathcal{V}\text{-Cat}$ and even $(\mathcal{T}, \mathcal{V})\text{-Cat}$, the Eilenberg–Moore algebras of which lead to the consideration of objects that combine relevant ordered, topological and metric structures in a very natural way.

1 Basic Concepts

1.1 $\mathcal{V}$-relations. Recall from [II.1.2] that a relation $r$ from a set $X$ to a set $Y$ associates to every pair $(x, y) \in X \times Y$ a truth value in $2 = \{\text{false}, \text{true}\}$ which tells us whether $x$ and $y$ are $r$-related or not. In order to model situations where quantitative information is available, $r$ can be allowed to take values in any quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ rather than just in $2 = (2, \land, \top)$. (The quantale $\mathcal{V}$ is associative and unital, as defined in [II.1.10].) A $\mathcal{V}$-relation $r : X \rightarrow Y$ from $X$ to $Y$ is therefore presented by a map $r : X \times Y \rightarrow \mathcal{V}$. As for ordinary relations, a $\mathcal{V}$-relation $r : X \rightarrow Y$ can be composed with another $\mathcal{V}$-relation $s : Y \rightarrow Z$ via “matrix multiplication”

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

(for all $x \in X$, $z \in Z$) to yield a $\mathcal{V}$-relation $s \cdot r : X \rightarrow Z$. This composition is associative (see Exercise [II.1.10]), and the $\mathcal{V}$-relation $1_X : X \rightarrow X$ that sends every diagonal element $(x, x)$ to $k$, and all other elements to the bottom element $\bot$ of $\mathcal{V}$ serves as the identity morphism on $X$. Thus, sets and $\mathcal{V}$-relations form a category, denoted by $\mathcal{V}\text{-Rel}$. 

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The set $\mathcal{V}$-Rel$(X, Y)$ of all $\mathcal{V}$-relations from $X$ to $Y$ inherits the pointwise order induced by $\mathcal{V}$: given $r : X \rightarrow Y$ and $r' : X \rightarrow Y$, we have

$$r \leq r' \iff \forall (x, y) \in X \times Y \ (r(x, y) \leq r'(x, y)).$$

Since the order on $\mathcal{V}$ is complete, so is the pointwise order on $\mathcal{V}$-Rel$(X, Y)$, and since the tensor in $\mathcal{V}$ distributes over suprema, $\mathcal{V}$-relational composition preserves suprema in each variable:

$$s \cdot \bigvee_{i \in I} r_i = \bigvee_{i \in I} (s \cdot r_i) \quad \text{and} \quad \bigvee_{i \in I} r_i \cdot t = \bigvee_{i \in I} (r_i \cdot t)$$

for $\mathcal{V}$-relations $r_i : X \rightarrow Y$ ($i \in I$), $s : Y \rightarrow Z$, and $t : W \rightarrow X$. Thus, $\mathcal{V}$-Rel is not just an ordered category, but even a quantaloid (see II.4.5 and II.4.8).

The canonical isomorphism $X \times Y \cong Y \times X$ induces a bijection between $\mathcal{V}$-Rel$(X, Y)$ and $\mathcal{V}$-Rel$(Y, X)$, so that for every $\mathcal{V}$-relation $r : X \rightarrow Y$ one has the opposite (or dual) $\mathcal{V}$-relation $r^\circ : Y \rightarrow X$ defined by

$$r^\circ(x, y) = r(y, x)$$

for all $x \in X$, $y \in Y$. This operation preserves the order on $\mathcal{V}$-Rel$(X, Y)$:

$$r \leq r' \implies r^\circ \leq (r')^\circ,$$

and one has $1_X^\circ = 1_X$ as well as $r^{\circ\circ} = r$. Let us also note that the equality

$$(s \cdot r)^\circ = r^\circ \cdot s^\circ$$

holds whenever $\mathcal{V}$ is commutative.

1.1.1 Examples.

(1) As mentioned above, a 2-relation is just an ordinary relation. For relations $r : X \rightarrow Y$ and $s : Y \rightarrow Z$, the previous “matrix multiplication” formula specializes to the usual relational composition:

$$x \ (s \cdot r) \ z \iff \exists y \in Y \ (x \ r \ y \ \& \ y \ s \ z).$$

Therefore, the category of 2-relations is just the category of relations:

$$\text{2-Rel} \cong \text{Rel}.$$

(2) For $\mathcal{V} = \mathcal{P}_+$ (see II.1.10), a $\mathcal{P}_+$-relation is a “distance function” $r : X \times Y \rightarrow \mathcal{P}_+$, and composition with $s : Y \times Z \rightarrow \mathcal{P}_+$ yields

$$s \cdot r(x, z) = \inf \{r(x, y) + s(y, z) \mid y \in Y\}$$

for all $x \in X$ and $z \in Z$. Therefore, $\mathcal{P}_+$-Rel can be seen as the category of sets and metric relations.
1. BASIC CONCEPTS

(3) For \(2^2 = \{\bot, u, v, \top\}\) the diamond lattice of Exercise II.1.H a \(2^2\)-relation is a “choice relation” that chooses between the truth values \(u\) and \(v\), taking value \(\bot\) if none is selected, and \(\top\) if both are. Each \(2^2\)-relation \(r\) can therefore be considered as a pair of relations \((r_u, r_v)\), and \(\mathcal{V}\)-Rel as the category of sets and birelations:

\[
\begin{align*}
X \times Y & \xrightarrow{r} 2^2 \cong 2 \times 2 \\
X \times Y & \xrightarrow{r_u} 2, \quad X \times Y \xrightarrow{r_v} 2.
\end{align*}
\]

1.2 Maps in \(\mathcal{V}\)-Rel. There is a functor from \(\text{Set}\) to \(\mathcal{V}\)-Rel that interprets the graph of a \(\text{Set}\)-map \(f : X \to Y\) as the \(\mathcal{V}\)-relation \(f : X \dashv Y\) given by

\[
f_o(x, y) = \begin{cases} 
k & \text{if } f(x) = y, \\
\bot & \text{otherwise.}
\end{cases}
\]

The functor \((-)_o : \text{Set} \to \mathcal{V}\)-Rel

is faithful if and only if \(\bot < k\) in \(\mathcal{V}\). Therefore, from now on, the quantale \(\mathcal{V}\) is assumed to be non-trivial, so that \(\mathcal{V}\) is not reduced to a singleton (see Exercise 1.A). To keep notations simple, we usually write \(f : X \to Y\) instead of \(f_o : X \dashv Y\) to designate a \(\mathcal{V}\)-relation induced by a map.

The formula for \(\mathcal{V}\)-relational composition becomes considerably easier if one of the \(\mathcal{V}\)-relations comes from a \(\text{Set}\)-map:

\[
\begin{align*}
s \cdot f(x, z) &= s(f(x), z), \\
g \cdot r(x, z) &= \bigvee_{y \in g^{-1}(z)} r(x, y), \\
h^o \cdot s(y, w) &= s(y, h(w)), \\
t \cdot f^o(y, z) &= \bigvee_{x \in f^{-1}(y)} t(x, z),
\end{align*}
\]

for all maps \(f : X \to Y\), \(g : Y \to Z\), \(h : W \to Z\), \(\mathcal{V}\)-relations \(r : X \dashv Y\), \(s : Y \dashv Z\), \(t : X \dashv Z\), and elements \(w \in W\), \(x \in X\), \(y \in Y\), \(z \in Z\). In particular, one should keep in mind that the pointwise expression of a relation of the form \(h^o \cdot s \cdot f\) is

\[
h^o \cdot s \cdot f(x, w) = s(f(x), h(w)),
\]

as this formula will be used systematically from now on. We note that, without any commutativity assumption on \(\mathcal{V}\), composition of \(\mathcal{V}\)-relations with \(\text{Set}\)-maps is also compatible with the involution \((-)^o\):

\[
(s \cdot f)^o = f^o \cdot s^o \quad \text{and} \quad (g \cdot r)^o = r^o \cdot g^o.
\]

The formulas for “relation-with-map composition” imply at once that every \(\text{Set}\)-map \(f : X \to Y\) satisfies the inequalities

\[
1_X \leq f^o \cdot f_o \quad \text{and} \quad f_o \cdot f^o \leq 1_Y.
\]
in $\mathcal{V}$-Rel, so that $f_\circ$ is a map in the sense of \ref{II.4.7}, thus providing further justification for the identification $f_\circ = f$. Given Set-maps $f : X \to Y$ and $g : Y \to Z$, we therefore obtain the adjunction rules:

$$g \cdot r \leq t \iff r \leq g^\circ \cdot t \quad \text{and} \quad t \cdot f^\circ \leq s \iff t \leq s \cdot f \quad (1.2.i)$$

for $\mathcal{V}$-relations $r : X \to Y$, $s : Y \to Z$, and $t : X \to Z$ (see Proposition II.4.7.1).

For $\mathcal{V} = 2$, every map (that is, every left-adjoint morphism) in $\mathcal{V}$-Rel = Rel is given by a Set-map: whenever $r \dashv s : Y \to X$, one has $r = f_\circ$, $s = f^\circ$ for a uniquely determined map $f : X \to Y$ (see \ref{II.4.7}). We briefly discuss to which extent this fact holds more generally and call the quantale $\mathcal{V}$ \textit{integral} if $k = \top$, that is, if the top element is the neutral element of the tensor. In an integral quantale $\mathcal{V}$, one has $u \otimes v \leq u \lor v$ for all $u, v \in \mathcal{V}$ (since $u \otimes v \leq u \otimes \top = u \otimes k = u$ and likewise $u \otimes v \leq v$). We say that $\mathcal{V}$ is \textit{lean} if

$$(u \lor v = \top \text{ and } u \otimes v = \bot) \implies (u = \top \text{ or } v = \top)$$

for all $u, v \in \mathcal{V}$. In an integral and lean quantale, $\top$ and $\bot$ are the only \textit{complemented} elements (that is, elements $u$ for which there is $v$ with $u \lor v = \top$, $u \land v = \bot$). The quantales $2$, $\mathcal{P}_+$, $\mathcal{P}_\text{max}$ are integral and lean, $3$ and $\mathcal{P}_\times$ are lean but not integral, and $2^2$ is integral but not lean.

\textbf{1.2.1 Proposition.} \textit{For an integral quantale $\mathcal{V}$, all left-adjoint $\mathcal{V}$-relations are Set-maps if and only if $\mathcal{V}$ is lean.}

\textit{Proof.} Let $\mathcal{V}$ be integral and lean, and assume $r \dashv s : Y \to X$ in $\mathcal{V}$-Rel. If $X = \emptyset$, then $r : \emptyset \to Y$ is the inclusion map. Hence, one can consider $x \in X$. Since

$$\bot < k = (s \cdot r)(x, x) = \bigvee_{y \in Y} r(x, y) \otimes s(y, x),$$

there is some $y \in Y$ with $u := r(x, y) \otimes s(y, x) > \bot$, and we can write $k = \top = u \lor v$, where $v := \bigvee_{y \neq y} r(x, y') \otimes s(y', x)$. From $r \cdot s(y, y') \leq \bot$ one obtains $s(y, x) \otimes r(x, y') = \bot$ for all $y' \neq y$, and therefore

$$u \otimes v = \bigvee_{y \neq y} r(x, y) \otimes (s(y, x) \otimes r(x, y')) \otimes s(y', x) = \bot.$$ 

Consequently, $u = r(x, y) \otimes s(y, x) = \top$ (since $v = \top = k$ would force $u = u \otimes v = \bot$), so that $r(x, y) = \top = s(y, x)$ because $\top = r(x, y) \otimes s(y, x) \leq r(x, y) \land s(y, x)$. Since $u = \top$ and $v = \bot$, we have shown that for every $x \in X$ there is precisely one $y := f(x)$ in $Y$ with $r(x, y) \otimes s(y, x) > \bot$, and then $r(x, y) = s(y, x) = k$. Consequently, $f \leq r$ and $f^\circ \leq s$, which actually forces $r = f$:

$$r = r \cdot 1_X \leq r \cdot f^\circ \cdot f \leq r \cdot s \cdot f \leq 1_Y \cdot f = f.$$ 

Conversely, for maps in $\mathcal{V}$-Rel to be Set-maps we must show that $\mathcal{V}$ is necessarily lean. If $u \lor v = \top$ and $u \otimes v = \bot$, with $X := \{u, v\}$ we may define $r : 1 = \{\ast\} \to X$ by $r(\ast, x) = x$ otherwise.
and claim $r \vdash r^\circ$. Indeed, since $k = \top$ and $(r \cdot r^\circ)(u, v) = u \otimes v = \bot$, one has $r \cdot r^\circ \leq 1_Y$; for $1_1 \leq r^\circ \cdot r$ we first observe
\[ u = u \otimes k = u \otimes (u \lor v) = (u \otimes u) \lor (u \otimes v) = u \otimes u \]
and likewise $v = v \otimes v$ which implies
\[ (r^\circ \cdot r)(\star, \star) = (u \otimes u) \lor (v \otimes v) = u \lor v = k . \]
By hypothesis, $r$ is then given by the maps $\star \mapsto u$ or $\star \mapsto v$, which means $u = k$ or $v = k$.

The adjunction $f \dashv f^\circ$ detects injectivity and surjectivity of the $\mathsf{Set}$-map $f$ by the equivalent conditions $f^\circ \cdot f = 1_X$ and $f \cdot f^\circ = 1_Y$, respectively. In fact, one can easily prove a more general fact:

1.2.2 Proposition.

1. The $\mathsf{Set}$-maps $f_i : X \to Y_i$ ($i \in I \neq \emptyset$) form a mono-source if and only if $\bigwedge_{i \in I} f_i^\circ \cdot f_i = 1_X$.

2. The $\mathsf{Set}$-maps $g_i : X_i \to Y$ ($i \in I$) form an epi-sink if and only if $\bigvee_{i \in I} g_i \cdot g_i^\circ = 1_Y$.

Proof. The statements follow from
\[ (\bigwedge_{i \in I} f_i^\circ \cdot f_i)(x, x') = \bigwedge_{i \in I} 1_Y(f_i(x), f_i(x')) = k \iff \forall i \in I (f_i(x) = f_i(x')) \]
for all $x, x' \in X$, and
\[ (\bigvee_{i \in I} g_i \cdot g_i^\circ)(x, x') = \bigvee_{i \in I} \bigvee_{x \in g_i^{-1}y} k = k \iff \bigcup_{i \in I} g_i^{-1}y \neq \emptyset \]
for all $y \in Y$. \qed

1.2.3 Remarks.

1. The empty source with domain $X$ is a mono-source in $\mathsf{Set}$ if and only if $|X| \leq 1$. Hence, the assertion of Proposition 1.2.2(1) remains valid also in the case $I = \emptyset$ precisely when either $X = \emptyset$, or when $|X| = 1$ and $\top = k$. Consequently, for an integral quantale $\mathcal{V}$, the restriction $I \neq \emptyset$ of Proposition 1.2.2(1) may be dropped.

2. The functor $(-)_o : \mathsf{Set} \to \mathcal{V}-\mathsf{Rel}$ has a right adjoint $\mathcal{V}-\mathsf{Rel} \to \mathsf{Set}$ that sends a set $X$ to the set $\mathcal{V}^X$ of all maps $\phi : X \to \mathcal{V}$, and a $\mathcal{V}$-relation $r : X \to Y$ to the map $r^\mathcal{V} : \mathcal{V}^X \to \mathcal{V}^Y$ defined by
\[ r^\mathcal{V}(\phi)(y) = \bigvee_{x \in X} \phi(x) \otimes r(x, y) \]
for all $\phi \in \mathcal{V}^X$ and $y \in Y$. The monad $\mathbb{P}_\mathcal{V}$ induced by this adjunction is the $\mathcal{V}$-powerset monad—whose Kleisli category is $\mathcal{V}-\mathsf{Rel}$ (see Exercise 1.D).
1.3 \(\mathcal{V}\)-categories, \(\mathcal{V}\)-functors and \(\mathcal{V}\)-modules. We introduced \(\mathcal{V}\)-categories and \(\mathcal{V}\)-functors for a monoidal category \(\mathcal{V}\) in II.4.10. Here we recall the definition in the highly simplified case where \(\mathcal{V} = (\mathcal{V}, \otimes, k)\) is a quantale.

A \(\mathcal{V}\)-relation \(a : X \to X\) is transitive if \(a \cdot a \leq a\) and reflexive if \(1_X \leq a\). A \(\mathcal{V}\)-category \((X, a)\) is a set \(X\) with a transitive and reflexive \(\mathcal{V}\)-relation \(a\). A \(\mathcal{V}\)-functor \(f : (X, a) \to (Y, b)\) of \(\mathcal{V}\)-categories is given by a map \(f : X \to Y\) with \(f \cdot a \leq b \cdot f\) or, equivalently, \(a \leq f^\circ \cdot b \cdot f\). Hence, in pointwise notation, the characteristic conditions for a \(\mathcal{V}\)-category read as

\[ a(x, y) \otimes a(y, z) \leq a(x, z) \quad \text{and} \quad k \leq a(x, x), \]

and for a \(\mathcal{V}\)-functor as

\[ a(x, y) \leq b(f(x), f(y)) \]

for all \(x, y, z \in X\). Since identity maps and composites of \(\mathcal{V}\)-functors are \(\mathcal{V}\)-functors, \(\mathcal{V}\)-categories and \(\mathcal{V}\)-functors form a category

\[ \mathcal{V}\text{-Cat}. \]

1.3.1 Examples.

(1) For \(\mathcal{V} = 2 = \{\text{true}, \text{false}\}\), writing \(x \leq y\) for \(a(x, y) = \text{true}\), the transitivity and reflexivity conditions read as expected:

\[ (x \leq y \& y \leq z \Longrightarrow x \leq z) \quad \text{and} \quad x \leq x \]

for all \(x, y, z \in X\). Thus, a 2-category \((X, \leq)\) is just an ordered set. (Recall from II.1.3 that we do not require an order to be antisymmetric.) A 2-functor \(f : (X, \leq) \to (Y, \leq)\) is a map \(f : X \to Y\) with

\[ x \leq y \Longrightarrow f(x) \leq f(y) \]

(for all \(x, y \in X\), so the category of 2-categories is just the category of ordered sets:

\[ 2\text{-Cat} = \text{Ord}. \]

(2) For \(\mathcal{V} = \mathbb{P}_+\) (see Example II.10.1(3)), a transitive and reflexive \(\mathbb{P}_+\)-relation is equivalently described as a metric on \(X\), that is, a map \(a : X \times X \to \mathbb{P}_+\) such that

\[ a(x, y) + a(y, z) \geq a(x, z) \quad \text{and} \quad 0 = a(x, x), \]

for all \(x, y, z \in X\), and we say that \(X\) is a metric space. Whenever we require any of the other traditionally assumed conditions, namely symmetry \((a(x, y) = a(y, x))\), separation \((a(x, y) = 0 = a(y, x) \Longrightarrow x = y)\), and finiteness \((a(x, y) < \infty)\), we will say so explicitly, thus calling \(X\) a symmetric, separated or finitary metric
1. BASIC CONCEPTS

A $\mathcal{P}_+$-functor $f : (X, a) \to (X, b)$ is a map $f : X \to Y$ that is non-expansive:

$$a(x, y) \geq b(f(x), f(y))$$

(for all $x, y \in X$). Hence, the category of $\mathcal{P}_+$-categories is equivalently described as the category $\text{Met}$ of metric spaces in the generalized sense as specified above:

$$\mathcal{P}_+\text{-Cat} = \text{Met}.$$  

It contains the full subcategories $\text{Met}_{\text{sym}}$ and $\text{Met}_{\text{sep}}$ of symmetric and separated metric spaces, respectively.

We sketch some general procedures for creating $\mathcal{V}$-categories, starting with $\mathcal{V}$ itself: the $\mathcal{V}$-valued binary operation $\otimes$ with $v \otimes t \leq w \iff t \leq v \multimap w$ defines a $\mathcal{V}$-relation, and one obtains:

1.3.2 Proposition. The quantale $\mathcal{V}$ endowed with the $\mathcal{V}$-relation $\multimap$ is a $\mathcal{V}$-category.

Proof. Since $v \otimes k \leq v$, one has $k \leq v \multimap v$, and from $v \multimap w \leq v \multimap w$, one obtains $v \otimes (v \multimap w) \leq w$. Consequently,

$$v \otimes (v \multimap w) \otimes (w \multimap z) \leq w \otimes (w \multimap z) \leq z,$$

which yields transitivity: $(v \multimap w) \otimes (w \multimap z) \leq v \multimap z$. □

Note that the more general situation of this Proposition where $\mathcal{V}$ is not just a quantale but a monoidal category was sketched in Exercise II.4.1.

For $\mathcal{V} = 2$, $\multimap$ returns the order of 2 (false < true), while for $\mathcal{V} = \mathcal{P}_+$, $\multimap$ is the truncated difference: $v \multimap w = w - v$ if $v \leq w < \infty$, $v \multimap w = 0$ if $w \leq v$, and $v \multimap \infty = \infty$ if $v < \infty$ (see II.1.10).

Let now $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{V}$-categories. One defines a $\mathcal{V}$-relation $[-,-]$ on $\mathcal{V}\text{-Cat}(X, Y) = \{f : X \to Y \mid f$ is a $\mathcal{V}$-functor\} by

$$[f, g] = \Lambda_{x \in X} b(f(x), g(x)),$$

and a $\mathcal{V}$-relation $a \otimes b$ on $X \times Y$ by

$$(a \otimes b)((x, y), (x', y')) = a(x, x') \otimes b(y, y')$$

for all $x, x' \in X$, $y, y' \in Y$.

1.3.3 Proposition. Let $X = (X, a)$, $Y = (Y, b)$ be $\mathcal{V}$-categories.

1. $[X, Y] = (\mathcal{V}\text{-Cat}(X, Y), [-,-])$ is a $\mathcal{V}$-category.
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(2) If $\mathcal{V}$ is commutative, then $X \otimes Y := (X \times Y, a \otimes b)$ is a $\mathcal{V}$-category.

Proof. Straightforward verifications. \hfill \Box

1.3.4 Example. For $\mathcal{V} = P_+$, $[f, g] = \sup_{x \in X} b(f(x), g(x))$ is the usual "sup-metric" on the function space $[X, Y]$, and $(a \otimes b)((x, y), (x', y')) = a(x, x') + b(y, y')$ endows $X \times Y$ with the usual "+-metric".

1.3.5 Remark. The structure on $[X, Y]$ may be rewritten as

$$[f, g] = \bigwedge_{x,x' \in X} a(x, x') \multimap b(f(x), g(x')) .$$

(1.3.i) Indeed, since $g : X \to Y$ is a $\mathcal{V}$-functor, for all $x, x' \in X$, one has

$$b(f(x), g(x)) \otimes a(x, x') \leq b(f(x), g(x)) \otimes b(g(x), g(x')) \leq b(f(x), g(x'))$$

and then $b(f(x), g(x)) \leq a(x, x') \multimap b(f(x), g(x'))$, which proves "$$" of \cite{1.3.i}. "\geq" follows from

$$a(x, x) \multimap b(f(x), g(x)) \leq k \multimap b(f(x), g(x)) \leq b(f(x), g(x))$$

for all $x \in X$.

1.3.6 Theorem. For a commutative quantale $\mathcal{V}$, $\mathcal{V}$-Cat is a symmetric monoidal closed category.

Proof. We prove that

$$Z \otimes X \xrightarrow{\phi} [X, Y] \xrightarrow{\phi} Y$$

with $\phi(z)(x) = \tilde{\phi}(z, x)$ for all $z \in Z, x \in X$ establishes a bijective correspondence of $\mathcal{V}$-functors $\phi$ and $\tilde{\phi}$ when $X = (X, a)$, $Y = (Y, b)$, $Z = (Z, c)$ are $\mathcal{V}$-categories. In fact, by Remark \cite{1.3.5} $\mathcal{V}$-functoriality of $\phi$ means equivalently

$$c(z, z') \leq a(x, x') \multimap b(\phi(z)(x), \phi(z)(x'))$$

for all $z, z' \in Z, x, x' \in X$, which may be equivalently rewritten as

$$(c \otimes a)((z, x), (z', x')) = c(z, z') \otimes a(x, x') \leq b(\tilde{\phi}(z, x), \tilde{\phi}(z, x')) ,$$

and this last inequality describes the $\mathcal{V}$-functoriality of $\tilde{\phi}$. We note that this inequality also entails $\mathcal{V}$-functoriality of $\phi(z) : X \to Y$ for all $z \in Z$, since

$$a(x, x') = k \otimes a(x, x') \leq c(z, z) \otimes a(x, x') \leq b(\phi(z)(x), \phi(z)(x')) .$$

The correspondence is obviously bijective and natural in $Z$, which shows monoidal closure, and symmetry $(X \otimes Y \cong Y \otimes X)$ holds trivially. \hfill \Box
1.3.7 Examples. For $\mathcal{V} = 2$, Theorem 1.3.6 confirms that $\text{Ord}$ is cartesian closed and, for $\mathcal{V} = P_+$, that $\text{Met}$ is monoidal closed. But note that the tensor product $X \otimes Y$ in $\text{Met}$ must not be confused with the product $X \times Y$: while the structure of $X \otimes Y$ is given by the “+-metric”, $X \times Y$ carries the “max-metric”; see Exercise I.G.

The notion of module for ordered sets (see II.1.4) extends naturally from the case $\mathcal{V} = 2$ to the arbitrary case: for $\mathcal{V}$-categories $(X,a)$, $(Y,b)$ one calls a $\mathcal{V}$-relation $r: X \to Y$ a $\mathcal{V}$-module (also $\mathcal{V}$-bimodule, $\mathcal{V}$-profunctor or $\mathcal{V}$-distributor) if

$$r \cdot a \leq r \quad \text{and} \quad b \cdot r \leq r.$$ 

Since the reversed inequalities always hold, these are in fact equalities:

$$r \cdot a = r \quad \text{and} \quad b \cdot r = r.$$ 

We write

$$r : (X,a) \leadsto (Y,b)$$

if the $\mathcal{V}$-relation $r$ is a $\mathcal{V}$-module. The module inequalities are stable under $\mathcal{V}$-relational composition, and

$$a : (X,a) \leadsto (X,a)$$

serves as an identity morphism in the category $\mathcal{V}$-$\text{Mod}$ whose objects are $\mathcal{V}$-categories and morphisms are $\mathcal{V}$-modules. This category is ordered, with the order inherited from $\mathcal{V}$-$\text{Rel}$; in fact, $\mathcal{V}$-$\text{Mod}$ is a quantaloid, with suprema in its hom-sets formed as in $\mathcal{V}$-$\text{Rel}$.

There is now a structured version of the functors

$$\text{Set} \xrightarrow{(-)_{\circ}} \mathcal{V}$-$\text{Rel} \xleftarrow{(-)^{\circ}} \text{Set}^{\text{op}},$$

as follows. For a $\mathcal{V}$-functor $f : (X,a) \to (Y,b)$ one defines $\mathcal{V}$-modules

$$f_* : (X,a) \leadsto (Y,b) \quad \text{and} \quad f^* : (Y,b) \leadsto (X,a)$$

by

$$f_* := b \cdot f \quad \text{and} \quad f^* := f^\circ \cdot b,$$

that is:

$$f_*(x,y) = b(f(x),y) \quad \text{and} \quad f^*(y,x) = b(y,f(x))$$

for all $x \in X$, $y \in Y$. One easily verifies the $\mathcal{V}$-module conditions and functoriality:

$$\mathcal{V}$-$\text{Cat} \xrightarrow{(-)_{\circ}} \mathcal{V}$-$\text{Mod} \xleftarrow{(-)^{\circ}} (\mathcal{V}$-$\text{Cat})^{\text{op}}.$$
In particular,
\[ 1_{(X,a)} = a = (1_X)^* = (1_X)_* , \]
so that we can simply write \( 1^*_X \) for the identity \( \mathcal{V} \)-module on \((X,a)\). Moreover, there is also a structured version of the adjunction \( f_* \dashv f^* \):

**1.3.8 Proposition.** For a \( \mathcal{V} \)-functor \( f : (X,a) \to (Y,b) \), one has \( f_* \dashv f^* \) in \( \mathcal{V} \)-Mod.

**Proof.** One writes
\[ f_* \cdot f^* = b \cdot f \cdot f^* \cdot b \leq b \cdot 1_Y \cdot b \leq b = 1^*_Y , \]
and
\[ 1^*_X = a \leq a \cdot 1_X \cdot a \leq a \cdot f^* \cdot f \cdot a \leq f^* \cdot b \cdot b \cdot f = f^* \cdot f_* , \]
which proves the claim. \( \square \)

Let us finally observe that if \( \mathcal{V} \) is commutative, then for every \( \mathcal{V} \)-category \( X = (X,a) \), the pair
\[ X^{op} := (X,a^{op}) \]
is also a \( \mathcal{V} \)-category, called the *dual* of \( X \). For every \( \mathcal{V} \)-functor \( f : (X,a) \to (Y,b) \) one has a \( \mathcal{V} \)-functor \( f^{op} : X^{op} \to Y^{op} \) given by \( f \), so there is a functor
\[ (-)^{op} : \mathcal{V} \text{-Cat} \to \mathcal{V} \text{-Cat} . \]
Furthermore,
\[ (f^{op})_* = (f^*)^o \quad \text{and} \quad (f^{op})^* = (f_*)^o . \]

**1.4 Lax extensions of functors.** Subsection 1.2 shows that the category \( \mathcal{V} \text{-Rel} \) of \( \mathcal{V} \)-relations can be seen as an extension of \( \text{Set} \). For a given monad \( \mathbb{T} = (T,m,e) \) on \( \text{Set} \), we now consider extensions of \( \mathbb{T} \) to \( \mathcal{V} \text{-Rel} \). For this, we first concentrate on the underlying \( \text{Set} \)-functor \( T \); the natural transformations \( e \) and \( m \) will be considered afterwards, in 1.5.

**1.4.1 Definition.** For a quantale \( \mathcal{V} \) and a functor \( T : \text{Set} \to \text{Set} \), a *lax extension* \( \hat{T} : \mathcal{V} \text{-Rel} \to \mathcal{V} \text{-Rel} \) of \( T \) to \( \mathcal{V} \text{-Rel} \) is given by functions
\[ \hat{T}_{X,Y} : \mathcal{V} \text{-Rel}(X,Y) \to \mathcal{V} \text{-Rel}(TX,TY) \]
for all sets \( X,Y \) (with \( \hat{T}_{X,Y} \) simply written as \( \hat{T} \)), such that

1. \( r \leq r' \implies \hat{T}r \leq \hat{T}r' , \)
2. \( \hat{T}s \cdot \hat{T}r \leq \hat{T}(s \cdot r) , \)
3. \( Tf \leq \hat{T}f \) and \( (Tf)^o \leq \hat{T}(f^o) , \)

for all sets \( X,Y,Z \), \( \mathcal{V} \)-relations \( r,r' : X \leftrightarrow Y \), \( s : Y \leftrightarrow Z \), and maps \( f : X \to Y \). By setting \( \hat{T}X = TX \) for all sets \( X \), and observing that condition (3) yields \( 1_{TX} \leq \hat{T}1_X \), one can define a lax extension of a \( \text{Set} \)-functor \( T \) equivalently as a lax functor \( \hat{T} : \mathcal{V} \text{-Rel} \to \mathcal{V} \text{-Rel} \) (see II.4.6) that agrees with \( T \) on objects of \( \mathcal{V} \text{-Rel} \) and satisfies the extension condition (3).
1. BASIC CONCEPTS

1.4.2 Examples.

(1) The identity functor on Set has a lax extension given by the identity functor on $\mathcal{V}$-$\text{Rel}$.

(2) For $\mathcal{V} = 2$, the covariant powerset functor $P : \text{Set} \to \text{Set}$ has lax extensions $\hat{P}, \tilde{P} : \text{Rel} \to \text{Rel}$ given by

\[
A (\hat{P} r) B \iff A \subseteq \rho(B) \iff \forall x \in A \exists y \in B (x r y)
\]

\[
A (\tilde{P} r) B \iff B \subseteq \rho(A) \iff \forall y \in B \exists x \in A (x r y)
\]

for every relation $r : X \to Y$, and all $A \subseteq X, B \subseteq Y$.

(3) Every functor $T$ on Set admits a largest lax extension to $\mathcal{V}$-$\text{Rel}$ given by

\[
T \uparrow r : TX \times TY \to V , \quad (x, y) \mapsto \top
\]

for all $\mathcal{V}$-relations $r : X \to Y$.

Although a lax extension $\hat{T}$ preserves composition of $\mathcal{V}$-relations only up to inequality, it operates more strictly on composites of $\mathcal{V}$-relations with Set-maps, as the Corollary to the following Proposition shows.

1.4.3 Proposition. Given functions $\hat{T}_{X,Y} : \mathcal{V}$-$\text{Rel}(X,Y) \to \mathcal{V}$-$\text{Rel}(TX,TY)$ that satisfy the conditions (1) and (2) of 1.4.1, the following are equivalent:

(i) $Tf \leq \hat{T} f$ and $(Tf)^{\circ} \leq \hat{T} (f^{\circ})$ for all $f : X \to Y$ (this is condition 1.4.1(3));

(ii) $Tf \leq \tilde{T} f$ and $\tilde{T} (s \cdot f) = \tilde{T} s \cdot Tf$ for all $f : X \to Y$ and $s : Y \to Z$;

(iii) $(Tf)^{\circ} \leq \hat{T} (f^{\circ})$ and $\hat{T} (f^{\circ} \cdot r) = \hat{T} f^{\circ} \cdot \hat{T} r$ for all $f : X \to Y$ and $r : X \to Z$.

The next condition is a consequence of any of the previous ones, and is equivalent to each of them if $\hat{T}$ also satisfies $1_{TX} \leq \hat{T} 1_X$:

(iv) $\hat{T} (s \cdot f) = \hat{T} s \cdot Tf$ and $\hat{T} (f^{\circ} \cdot r) = (Tf)^{\circ} \cdot \hat{T} r$ for all $f : X \to Y$ and $r : X \to Z$, $s : Y \to Z$.

Proof. For (i) $\implies$ (ii), we observe

\[
\hat{T} s \cdot Tf \leq \hat{T} s \cdot \hat{T} f \leq \hat{T} (s \cdot f) \leq \hat{T} (s \cdot f) \cdot (Tf)^{\circ} \cdot Tf \\
\leq \hat{T} (s \cdot f) \cdot \hat{T} (f^{\circ}) \cdot Tf \leq \hat{T} (s \cdot f \cdot f^{\circ}) \cdot Tf \leq \hat{T} s \cdot Tf
\]

so these inequalities are all equalities, and (i) $\implies$ (iii) is shown in the same way. For (ii) $\implies$ (iv), we see that $1_{TX} \leq \hat{T} 1_X$ follows immediately from $Tf \leq \hat{T} f$; moreover, one observes

\[
Tf \cdot \hat{T} (f^{\circ} \cdot r) \leq \hat{T} f \cdot \hat{T} (f^{\circ} \cdot r) \leq \hat{T} (f \cdot f^{\circ}) \cdot r \leq \hat{T} r
\]
so \( \hat{T}(f^o \cdot r) \leq (Tf)^o \cdot \hat{T}r \) follows by the first adjunction rule in \([1.2.3]\); for the other inequality, we apply \( \hat{T} \) to \( 1_X \leq f^o \cdot f \) to obtain \( 1_X \leq \hat{T}(f^o) \cdot Tf \) thanks to the hypothesis, and \((Tf)^o \leq \hat{T}(f^o)\) by the second adjunction rule, so that

\[
(Tf)^o \cdot \hat{T}r \leq \hat{T}(f^o) \cdot \hat{T}r \leq \hat{T}(f^o \cdot r).
\]

The implication (iii) \( \implies \) (iv) is proved similarly. Finally, for (iv) \( \implies \) (i), we observe that \( 1_X \leq f^o \cdot f \) yields

\[
1_{TX} \leq \hat{T}(1_X) \leq \hat{T}(f^o \cdot f) = \hat{T}(f^o) \cdot Tf,
\]

which implies \((Tf)^o \leq \hat{T}(f^o)\), and \( Tf \leq \hat{T}f \) follows in a similar way.

\[\square\]

1.4.4 Corollary. For a lax extension \( \hat{T} : \mathcal{V}\text{-Rel} \to \mathcal{V}\text{-Rel} \) of a \textit{Set}-functor \( T \) one has

\[
\hat{T}(s \cdot f) = \hat{T}s \cdot \hat{T}f = \hat{T}s \cdot Tf, \quad \hat{T}(f^o \cdot r) = \hat{T}(f^o) \cdot \hat{T}r = (Tf)^o \cdot \hat{T}r
\]

for all maps \( f : X \to Y \) and \( \mathcal{V}\text{-relations} \ r : X \leftrightarrow Z, \ s : Y \leftrightarrow Z \).

\[\text{Proof.}\] This follows immediately from the Proposition since a lax extension is a lax functor, and \( \hat{T}s \cdot \hat{T}f \leq \hat{T}(s \cdot f) = \hat{T}s \cdot Tf \leq \hat{T}s \cdot \hat{T}f \); the other equalities are obtained in the same way.

\[\square\]

A lax extension \( \hat{T} \) of \( T \) is \textit{flat} if

\[
\hat{T}1_X = T1_X = 1_{TX},
\]

that is, if both diagrams

\[
\begin{array}{ccc}
\mathcal{V}\text{-Rel} & \xrightarrow{T} & \mathcal{V}\text{-Rel} \\
\scriptstyle{(-)_o} \downarrow & \quad & \quad \downarrow \scriptstyle{(-)_o} \\
\text{Set} & \xrightarrow{T} & \text{Set}
\end{array}
\quad
\begin{array}{ccc}
\mathcal{V}\text{-Rel} & \xrightarrow{T} & \mathcal{V}\text{-Rel} \\
\scriptstyle{(-)^o} \downarrow & \quad & \quad \downarrow \scriptstyle{(-)^o} \\
\text{Set}^{\text{op}} & \xrightarrow{T^{\text{op}}} & \text{Set}^{\text{op}}
\end{array}
\]

commute. Indeed, if \( \hat{T} \) is flat, by the previous Proposition one obtains

\[
\hat{T}f = \hat{T}1_Y \cdot Tf = Tf \quad \text{and} \quad \hat{T}(f^o) = (Tf)^o \cdot \hat{T}1_X = (Tf)^o
\]

for all \( f : X \to Y \) in \text{Set}. Note that, of all the Examples \([1.4.2]\), only the given lax extension of the identity functor is flat.

1.5 Lax extensions of monads. Let us now turn our attention to the natural transformations \( e \) and \( m \) that we wish to extend from \text{Set} to \( \mathcal{V}\text{-Rel} \) together with the functor \( T \).

1.5.1 Definition. A triple \( \hat{T} = (\hat{T}, m, e) \) is a \textit{lax extension of the monad} \( \hat{T} = (T, m, e) \) if \( \hat{T} \) is a lax extension of \( T \) which makes both \( m : \hat{T}\hat{T} \to \hat{T} \) and \( e : 1_{\mathcal{V}\text{-Rel}} \to \hat{T} \) oplax (see \([II.4.6]\)), that is:

\[(4) \; m_Y \cdot \hat{T}\hat{T}r \leq \hat{T}r \cdot m_X,\]
1. BASIC CONCEPTS

(5) \( e_Y \cdot r \leq \hat{T}_r \cdot e_X, \)

for all \( \mathcal{V} \)-relations \( r : X \to Y. \)

By using both adjunction rules (1.2.i) of 1.2 for the maps \( m_X \) and \( e_X, \) we obtain the following equivalent formulations of (4) and (5):

\[
\begin{align*}
& (4^o) \quad \hat{T}_r \cdot m_X \leq m_Y \cdot \hat{T}_r, \\
& (5^o) \quad r \cdot e_X \leq e_Y \cdot \hat{T}_r.
\end{align*}
\]

Similarly, these conditions are equivalent to:

\[
\begin{align*}
& (4') \quad \hat{T}_r \leq m_Y \cdot \hat{T}_r \cdot m_X, \\
& (5') \quad r \leq e_Y \cdot \hat{T}_r \cdot e_X.
\end{align*}
\]

These inequalities then yield the following pointwise expressions:

\[
\begin{align*}
& (4^*) \quad \hat{T}_r(X, \mathcal{Y}) \leq \hat{T}_r(m_X(X), m_Y(\mathcal{Y})), \\
& (5^*) \quad r(x, y) \leq \hat{T}_r(e_X(x), e_Y(y)),
\end{align*}
\]

for all \( x \in X, y \in Y, X \in TTX, \mathcal{Y} \in TTY, \) and \( \mathcal{V} \)-relations \( r : X \to Y. \)

One says that a lax extension \( \hat{T} = (\hat{T}, m, e) \) of the monad \( T \) is flat if the lax extension \( \hat{T} \) of the functor \( T \) is flat.

The construction of lax extensions in a general setting can be rather technical, and is postponed until Chapter IV. In this Chapter, however, we will describe especially important extensions in 1.10 and 2.4. Here, we restrict ourselves to reconsidering the easy examples of 1.4.2. These demonstrate in particular that a monad on \( \text{Set} \) may generally admit more than one lax extension to \( \mathcal{V} \)-Rel.

1.5.2 Examples.

(1) The identity monad \( I \) on \( \text{Set} \) can be extended to the identity monad \( I \) on \( \mathcal{V} \)-Rel, and unless otherwise stated, this is the flat lax extension that will be used from now on for this monad.

(2) The lax extensions \( \check{P}, \hat{P} \) of 1.4.2 provide non-flat lax extensions \( \check{P}, \hat{P} \) of the powerset monad \( P \) (see Example II.3.1.1(3)) to \( \text{Rel}. \)

(3) Every monad \( \mathbb{T} \) on \( \text{Set} \) admits a largest lax extension \( \mathbb{T}^\uparrow \) to \( \mathcal{V} \)-Rel. It fails to be flat.
1.6 \((\mathbb{T}, \mathcal{V})\)-categories and \((\mathbb{T}, \mathcal{V})\)-functors. Let \(\mathcal{V}\) be a quantale, and \(\hat{\mathbb{T}} = (\hat{T}, m, e)\) a lax extension to \(\mathcal{V}\)-Rel of a monad \(\mathbb{T} = (T, m, e)\) on Set. A \((\mathbb{T}, \mathcal{V})\)-relation \(a : TX \leftrightarrow X\) is transitive if it satisfies
\[ a \cdot \hat{T}a \cdot m^o_X \leq a \quad \text{or equivalently} \quad a \cdot \hat{T}a \leq a \cdot m_X \]
by adjunction (see 1.2). In pointwise notation, this transitivity condition becomes
\[ \hat{T}a(x, y) \otimes a(y, z) \leq a(m_X(x), z) \]
for all \(x \in TTX\), \(y \in TX\), and \(z \in X\). A \((\mathbb{T}, \mathcal{V})\)-relation \(a : TX \leftrightarrow X\) is reflexive if it satisfies
\[ e^o_X \leq a \quad \text{or equivalently} \quad 1_X \leq a \cdot e_X . \]
In pointwise notation, \(a : TX \leftrightarrow X\) is reflexive if and only if
\[ k \leq a(e_X(x), x) \]
holds for all \(x \in X\).

1.6.1 Definition. A \((\mathbb{T}, \mathcal{V})\)-category, depending on context also referred to as a lax algebra, a \((\mathbb{T}, \mathcal{V})\)-algebra or a \((\mathbb{T}, \mathcal{V})\)-space, is a pair \((X, a)\) consisting of a set \(X\) and a transitive and reflexive \((\mathbb{T}, \mathcal{V})\)-relation \(a : TX \leftrightarrow X\); that is, it is a set \(X\) with a \(\mathcal{V}\)-relation \(a : TX \leftrightarrow X\) satisfying the two laws for an Eilenberg–Moore algebra laxly:

\[
\begin{array}{ccc}
TTX & \xrightarrow{Ta} & TX \\
m_X & \geq & a \\
TX & \xrightarrow{a} & X
\end{array}
\]
\[
\begin{array}{ccc}
X & \xrightarrow{e_X} & TX \\
1_X & \leq & a \\
X & \xrightarrow{a} & X
\end{array}
\]

Note that the notion depends in fact not just on \(\mathbb{T}\) but also on \(\hat{\mathbb{T}}\), hence, whenever needed, we will refer to a \((\mathbb{T}, \mathcal{V})\)-category more precisely as a \((\mathbb{T}, \mathcal{V}, \hat{\mathbb{T}})\)-category.

We already considered an important special type of \((\mathbb{T}, \mathcal{V})\)-categories in 1.3. When \(\mathbb{T}\) is the identity monad \(\mathbb{I}\) identically extended to \(\mathcal{V}\)-Rel, an \((\mathbb{I}, \mathcal{V})\)-category is simply a \(\mathcal{V}\)-category. Hence, in what follows we consider easy examples with other choices of \(\mathbb{T}\). Further examples will follow in Section 2.

1.6.2 Examples.

1. With \(\mathcal{V} = 2\) and \(T = P\) laxly extended by \(\check{P}\) (Example 1.5.2(2)), a transitive and reflexive relation \(a : PX \leftrightarrow X\) must satisfy the conditions
\[
(A \subseteq a^o(B) \& B a z \implies (\bigcup A) a z) \quad \text{and} \quad \{x\} a x
\]
for all \( x \in X, B \subseteq X, A \subseteq PX \). Since \( \{x\} a y \) may be re-written as \( \{x\} \subseteq a^\circ(\{y\}) \), by
\[
x \leq y \iff \{x\} a y
\]
one defines an order on \( X \). We claim that this order completely determines \( a \), since
\[
A a y \iff \forall x \in A (\{x\} a y) \iff A \subseteq \downarrow y . \tag{1.6.i}
\]
Indeed, when \( A a y \) and \( x \in A \) one has \( \{x\} a y \) by transitivity; when \( \{x\} a y \) for all \( x \in A \) one uses \( \{x\} \subseteq a^\circ(\{y\}) \) to obtain \( A a y \) by transitivity.

Conversely, starting with an order \( \leq \) on \( X \), (1.6.i) defines a \( (P, 2, \check{\mathcal{P}}) \)-category structure \( a \) on \( X \) which reproduces the original order.

(2) Trading \( \check{\mathcal{P}} \) for \( \hat{\mathcal{P}} \) (Example 1.5.2(2)), for a transitive and reflexive relation \( a : PX \to X \)
we may define a closure operation \( c \) on \( PX \) by
\[
x \in c(A) \iff A a x .
\]
(For idempotency of \( c \), consider \( A = \{c(\{x\}) \mid x \in A\} \), where \( A \subseteq X \).) Conversely, given \( c \), this definition yields a \( (P, 2, \check{\mathcal{P}}) \)-category structure \( a \) on \( X \).

(3) For arbitrary \( \mathcal{V} \) and the largest lax extension \( \mathcal{T}^\top \) of a monad \( \mathcal{T} \) (Example 1.5.2(3)),
the only \( (\mathcal{T}, \mathcal{V}, \mathcal{T}^\top) \)-category structure \( t \) on a set \( X \) is given by \( t(x, y) = \mathcal{T} \) for all \( x \in TX, y \in X \). Indeed, for any \( (\mathcal{T}, \mathcal{V}, \mathcal{T}^\top) \)-category structure \( a \) on \( X \), one has
\[
a = a \cdot e_T^o \cdot m_X^o \leq e_X^o \cdot T^\top a \cdot m_X^o \leq a \cdot T^\top a \cdot m_X^o \leq a ,
\]
so
\[
a(x, y) = e_X^o \cdot T^\top a \cdot m_X^o (x, y) = \bigvee_{x \in m_X^{-1}(x)} T^\top a(x, e_X(y)) = \mathcal{T}
\]
since \( e_{TX}(x) \in m_X^{-1}(x) \neq \emptyset \).

1.6.3 Definition. A map \( f : X \to Y \) between \( (\mathcal{T}, \mathcal{V}) \)-categories \((X, a)\) and \((Y, b)\) is a \( (\mathcal{T}, \mathcal{V}) \)-functor if it satisfies
\[
f \cdot a \leq b \cdot Tf .
\]
Diagrammatically this means that \( f \) is a lax homomorphism of lax algebras:
\[
\begin{array}{ccc}
TX & \xrightarrow{Tf} & TY \\
\scriptstyle a \downarrow & \leq & \downarrow \scriptstyle b \\
X & \xrightarrow{f} & Y 
\end{array}
\]
We can transcribe this condition equivalently as \( a \leq f^\circ b \cdot T f \) which, in pointwise notation, reads as
\[
a(\chi, x) \leq b(T f(\chi), f(x))
\]
for all \( \chi \in TX \) and \( x \in X \).

The identity map \( 1_X : (X, a) \to (X, a) \) is a \((\mathbb{T}, \mathcal{V})\)-functor, and so is the composite of \((\mathbb{T}, \mathcal{V})\)-functors. Hence, \((\mathbb{T}, \mathcal{V})\)-categories and \((\mathbb{T}, \mathcal{V})\)-functors form a category, denoted by \((\mathbb{T}, \mathcal{V})\)-Cat.

Of course, this category depends on the lax extension \( \hat{T} \) of \( T \), but we will always assume that such an extension is given beforehand and will therefore write more precisely \((\mathbb{T}, \mathcal{V}, \hat{T})\)-Cat in lieu of \((\mathbb{T}, \mathcal{V})\)-Cat only if there is a danger of ambiguity. When \( \mathbb{T} = \mathbb{I} \) is identically extended to \( \mathcal{V}\)-Rel, an \((\mathbb{I}, \mathcal{V})\)-functor is simply a \( \mathcal{V}\)-functor. Hence,
\[
(\mathbb{I}, \mathcal{V})\text{-Cat} = \mathcal{V}\text{-Cat}.
\]

### 1.6.4 Examples.

1. For a map between \((P, 2, \hat{P})\)-categories to be a \((P, 2)\)-functor means equivalently that the map must be monotone with respect to the induced orders (1.6.2(1)). As a consequence, one obtains an isomorphism
\[
(P, 2, \hat{P})\text{-Cat} \cong \text{Ord}
\]
which leaves underlying sets invariant.

2. Similarly, with 1.6.2(2) one has an isomorphism
\[
(P, 2, \hat{P})\text{-Cat} \cong \text{Cls}.
\]

3. Because of 1.6.2(3) there is an isomorphism
\[
(\mathbb{T}, \mathcal{V}, \mathbb{T}^\top)\text{-Cat} \cong \text{Set}
\]
for every monad \( \mathbb{T} \) on \( \text{Set} \) and every quantale \( \mathcal{V} \).

A \( \mathbb{T}\)-algebra \((X, a)\) (where \( a \) is a map satisfying \( a \cdot e_X = 1_X \) and \( a \cdot Ta = a \cdot m_X \)) is generally not a \((\mathbb{T}, \mathcal{V})\)-algebra \((X, a)\) (which requires \( a \cdot \hat{T}a \leq a \cdot m_X \)), as one may see already in the case \( \mathbb{T} = P \) with its extensions \( \hat{P} \) or \( \hat{P}^\circ \). There is, of course, no problem when \( \hat{T} \) is flat:

### 1.6.5 Proposition. If \( \hat{T} \) is a flat lax extension of \( T \) to \( \mathcal{V}\text{-Rel} \), then a \( \mathbb{T}\)-algebra \((X, a : TX \to X)\) is also a \((\mathbb{T}, \mathcal{V})\)-category. In this case, morphisms of \( \mathbb{T}\)-algebras yield \((\mathbb{T}, \mathcal{V})\)-functors between the corresponding \((\mathbb{T}, \mathcal{V})\)-categories, and there is a full embedding
\[
\text{Set}^\mathbb{T} \hookrightarrow (\mathbb{T}, \mathcal{V})\text{-Cat}.
\]
Proof. The only non-obvious fact in the statement of the Proposition is that the embedding of Set\textsuperscript{T} in (\mathbb{T}, \mathcal{V})-Cat is full. To see this, consider \mathbb{T}-algebras (X, a) and (Y, b) with a (\mathbb{T}, \mathcal{V})-functor f : (X, a) \rightarrow (Y, b), that is, a map f : X \rightarrow Y satisfying
\[ f \cdot a \leq b \cdot Tf \]
in \mathcal{V}-Rel. As f \cdot a and b \cdot Tf are really Set-maps, the inequality means that the graph of the first is contained in the second, but an inclusion of graphs of Set-maps with same domain is an equality.

1.7 Kleisli convolution. The relations representing (\mathbb{T}, \mathcal{V})-category structures are of the form \( a : TX \rightarrow X \). More generally, a (\mathbb{T}, \mathcal{V})-relation is a \mathcal{V}-relation \( r : TX \rightarrow Y \), also denoted by \( r : X \rightarrow Y \). In order to compose such relations, we introduce the Kleisli convolution of (\mathbb{T}, \mathcal{V})-relations as a variation of the Kleisli composition presented in [II.3.6]. Let us emphasize that associativity of this operation turns out to depend on the monad lax extension, so that sets with \mathcal{V}-relations \( r : TX \rightarrow Y \) only form a category in particular cases. In [1.8] we will provide a context in which the Kleisli convolution allows for identity morphisms, and in [1.9] we will introduce a category that has the Kleisli convolution as its composition.

1.7.1 Definition. Given a lax extension \( \hat{T} = (\hat{T}, m, e) \) of a Set-monad \( T = (T, m, e) \), the Kleisli convolution \( s \circ r : X \rightarrow Z \) of (\mathbb{T}, \mathcal{V})-relations \( r : X \rightarrow Y \) and \( s : Y \rightarrow Z \) is the (\mathbb{T}, \mathcal{V})-relation defined by
\[ s \circ r := s \cdot \hat{T}r \cdot m^0_X , \]
an operation that may be depicted as
\[ ( TX \xrightarrow{r} Y , TY \xrightarrow{s} Z ) \quad \mapsto \quad ( TX \xrightarrow{m^0_X} TTX \xrightarrow{\hat{T}r} TY \xrightarrow{s} Z ) . \]
When \( \mathbb{T} = \mathbb{I} \), then
\[ s \circ r = s \cdot r \]
is just the relational composition of \mathcal{V}-relations.
The set of all (\mathbb{T}, \mathcal{V})-relations from X to Y inherits the order of \mathcal{V}-Rel(TX,Y):
\[ r \leq r' \iff \forall (x, y) \in TX \times Y \ (r(x, y) \leq r'(x, y)) , \]
and the Kleisli convolution preserves this order in each variable:
\[ r \leq r', s \leq s' \quad \Rightarrow \quad r \circ s \leq r' \circ s' , \]
for all \( r, r' : X \rightarrow Y \) and \( s, s' : Y \rightarrow Z \). The (\mathbb{T}, \mathcal{V})-relation \( e^0_X : X \rightarrow X \) is a lax identity for this composition: one has
\[ e^0_Y \circ r = e^0_Y \cdot \hat{T}r \cdot m^0_X \geq r \cdot e^0_{TX} \cdot m^0_X = r , \]
CHAPTER III. LAX ALGEBRAS

with equality holding if \( e^o = (e^o_X)_X : \hat{T} \to 1 \) is a natural transformation, and

\[
r \circ e^o_X = r \cdot \hat{T}(e^o_X) \cdot m^o_X \geq r \cdot (Te_X)^o \cdot m^o_X = r ,
\]

with equality holding if \( \hat{T} \) is flat. In particular, \( e^o_X \circ e^o_X \geq e^o_X \), but generally this inequality is strict when \( \hat{T} \) fails to be flat. It turns out that \( e^o_X \) can be modified to become idempotent without loss of its lax identity properties. To this end, we first prove the following result.

1.7.2 Lemma. For a lax extension \( \hat{T} = (\hat{T},m,e) \) to \( V\text{-Rel} \) of a monad \( T = (T,m,e) \) on \( \text{Set} \) one has:

\[
\hat{T}1_X = \hat{T}(e^o_X) \cdot m^o_X .
\]

Proof. On one hand, we can exploit \( 1_{TX} = 1_X \circ TX = (m_X \cdot Te_X)^o \cdot m_X \cdot m^o_X \) to obtain

\[
\begin{align*}
\hat{T}1_X &= \hat{T}1_X \cdot 1^o_{TX} \\
&= \hat{T}1_X \cdot (Te_X)^o \cdot m^o_X \\
&\leq \hat{T}(1_X) \cdot \hat{T}(e^o_X) \cdot m^o_X \\
&\leq \hat{T}(e^o_X) \cdot m^o_X
\end{align*}
\]

On the other hand,

\[
\begin{align*}
\hat{T}(e^o_X) \cdot m^o_X &\leq \hat{T}(e^o_X \cdot \hat{T}1_X) \cdot m^o_X \\
&= (Te_X)^o \cdot \hat{T}1_X \cdot m^o_X \\
&\leq (Te_X)^o \cdot m_X \cdot \hat{T}1_X \\
&= \hat{T}1_X
\end{align*}
\]

which concludes the proof. \( \square \)

We set

\[
1^\sharp_X := e^o_X \circ e^o_X ,
\]

hence

\[
1^\sharp_X = e^o_X \cdot \hat{T}1_X
\]

by the Lemma, and we can prove:

1.7.3 Proposition. If \( \hat{T} = (\hat{T},m,e) \) is a lax extension of the monad \( \mathbb{T} = (T,m,e) \) to \( V\text{-Rel} \), then

\[
r \circ e^o_X = r \cdot \hat{T}1_X = r \circ 1^\sharp_X \quad \text{and} \quad e^o_Y \circ r = 1^\sharp_Y \circ r
\]

for all \( (\mathbb{T},V) \)-relations \( r : X \leftrightarrow Y \). In particular, \( 1^\sharp_X \circ 1^\sharp_X = 1^\sharp_X \), so that \( (X,1^\sharp_X) \) is a \( (\mathbb{T},V) \)-algebra.
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Proof. We first observe that
\[ r \circ 1^\sharp_X = r \cdot \hat{T}(e^\circ_X \cdot \hat{T}1_X) \cdot m^\circ_X \]
\[ = r \cdot (Te_X)^\circ \cdot \hat{T}1_X \cdot m^\circ_X \quad \text{(Corollary 1.4.4)} \]
\[ \leq r \cdot (Te_X)^\circ \cdot m^\circ_X \cdot \hat{T}1_X \quad \text{(m oplax)} \]
\[ = r \cdot \hat{T}1_X \quad \text{(1_X = (Te_X)^\circ \cdot m^\circ_X)} \]
\[ = r \cdot \hat{T}(e^\circ_X) \cdot m^\circ_X \quad \text{(Lemma 1.7.2)} \]
\[ = r \circ e^\circ_X . \]

This inequality suffices to prove the first set of equalities, since \( e^\circ_X \leq 1^\sharp_X \) implies \( r \circ e^\circ_X \leq r \circ 1^\sharp_X \).

The other equality follows directly from Corollary [1.4.4] as
\[ 1^\sharp_Y \circ r = e^\circ_Y \cdot \hat{T}(1^\sharp_Y) \cdot \hat{T}r \cdot m^\circ_X = e^\circ_Y \cdot \hat{T}r \cdot m^\circ_X = e^\circ_Y \circ r . \]

Finally,
\[ 1^\sharp_X \circ 1^\sharp_X = 1^\sharp_X \cdot \hat{T}1_X = e^\circ_X \cdot \hat{T}1_X \cdot \hat{T}1_X = e^\circ_X \cdot \hat{T}1_X = 1^\sharp_X . \]

We call \((X, 1^\sharp_X)\) the discrete \((\mathbb{T}, \mathcal{V})\)-category over \(X\), see [3.2] below.

1.8 Unitary \((\mathbb{T}, \mathcal{V})\)-relations. Our candidates for the identities of the Kleisli convolution are the \((\mathbb{T}, \mathcal{V})\)-relations \(1^\sharp_X\), but the array of \((\mathbb{T}, \mathcal{V})\)-relations from \(X\) to \(Y\) that are left invariant by composition with these identities must still be determined.

1.8.1 Definition. A \((\mathbb{T}, \mathcal{V})\)-relation \(r : X \rightarrow Y\) is right unitary if it satisfies
\[ r \circ e^\circ_X \leq r , \]
while it is left unitary if
\[ e^\circ_Y \circ r \leq r \]
holds. In terms of the relational composition, these conditions amount to
\[ r \cdot \hat{T}1_X \leq r \quad \text{and} \quad e^\circ_Y \cdot \hat{T}r \cdot m^\circ_X \leq r , \]
respectively. The \((\mathbb{T}, \mathcal{V})\)-relation \(r\) is unitary if it is both left and right unitary.

The \((\mathbb{T}, \mathcal{V})\)-relation \(e^\circ_X\) itself is not unitary in general, but Proposition [1.7.3] shows that we can replace it in the previous definitions by \(1^\sharp_X\). It also follows from the discussion preceding Lemma [1.7.2] that the inequalities appearing in the left and right unitary conditions are in fact equalities. Hence, a \((\mathbb{T}, \mathcal{V})\)-relation \(r\) is right unitary, respectively left unitary, if
\[ r \circ 1^\sharp_X = r , \quad \text{respectively} \quad 1^\sharp_Y \circ r = r . \]
Let us examine \((T, \mathcal{V})\)-categories and \((T, \mathcal{V})\)-functors in the light of Kleisli convolution and unitary \((T, \mathcal{V})\)-relations. By definition, a \((T, \mathcal{V})\)-category structure is a relation \(a : TX \rightarrow X\) such that
\[
a \circ a \leq a \quad \text{and} \quad e_X^\circ \leq a .
\]
These conditions imply that such a \((T, \mathcal{V})\)-relation \(a : X \rightarrow X\) is always unitary:
\[
a \circ 1_X^\sharp = a = 1_X^\sharp \circ a ,
\]
since \(a \circ e_X^\circ \leq a \circ a \leq a\) and \(e_X^\circ \circ a \leq a \circ a \leq a\). As a consequence, \(a : TX \rightarrow X\) is a \((T, \mathcal{V})\)-category structure if and only if
\[
a \circ a = a \quad \text{and} \quad 1_X^\sharp \leq a . \quad (1.8.i)
\]
Indeed, the first condition follows from transitivity: \(a \leq a \circ e_X \leq a \circ a \leq a\), and the second condition follows from reflexivity: \(1_X^\sharp = e_X^\circ \circ e_X \leq a \circ e_X^\circ \leq a\) (the converse resulting from \(e_X^\circ \leq 1_X^\sharp\)). Hence, a \((T, \mathcal{V})\)-algebra structure \(a\) can also be seen as a monoid in the ordered set of unitary \((T, \mathcal{V})\)-relations from \(X\) to \(X\), considered as a category that is provided with the \(\circ\)-operation. But we recall that associativity of \(\circ\) is guaranteed only under additional hypotheses (see 1.9 below), a property that is needed to consider \(\circ\) as a monoidal structure.

By definition, a \((T, \mathcal{V})\)-functor \(f : (X, a) \rightarrow (Y, b)\) is a map \(f : X \rightarrow Y\) satisfying
\[
f \cdot a \leq b \cdot Tf .
\]
Since a \((T, \mathcal{V})\)-category structure \(b\) is right unitary, it satisfies \(b \cdot \hat{T}1_Y = b \circ e_Y \circ = b\) by Proposition 1.7.3; one then obtains by Proposition 1.4.3 the equalities \(b \cdot \hat{T}f = b \cdot \hat{T}1_Y \cdot Tf = b \cdot Tf\). Hence, the \((T, \mathcal{V})\)-functor condition can equivalently be expressed by using the lax extension of \(T\):
\[
f \cdot a \leq b \cdot \hat{T}f .
\]
Setting
\[
f^\sharp := f \circ \cdot 1_Y^\sharp : Y \rightarrow X ,
\]
one can also express \((T, \mathcal{V})\)-functoriality of a map \(f : X \rightarrow Y\) via Kleisli convolution as
\[
a \circ f^\sharp \leq f^\sharp \circ b ,
\]
see Exercise 1.M.

1.9 Associativity of unitary \((T, \mathcal{V})\)-relations. With respect to the Kleisli convolution, the unitary \((T, \mathcal{V})\)-relation \(1_X^\sharp\) serves as an identity for all unitary \((T, \mathcal{V})\)-relations composable with \(1_X^\sharp\). In general however, unitary \((T, \mathcal{V})\)-relations do not compose associatively, even when \(T\) is the identity monad (see Proposition 1.9.7 below).
1.9.1 Definition. A lax extension $\hat{T}$ to $\mathcal{V}$-$\text{Rel}$ of a monad $\mathcal{T} = (T, m, e)$ on $\text{Set}$ is associative whenever the Kleisli convolution of unitary $(\hat{T}, \mathcal{V})$-relations is associative. Explicitly, a lax extension $\hat{T}$ is associative whenever

$$t \circ (s \circ r) = (t \circ s) \circ r,$$

or equivalently,

$$t \cdot \hat{T}(s \cdot \hat{T}r \cdot m_X) \cdot m_X = t \cdot \hat{T}s \cdot m_Y \cdot \hat{T}r \cdot m_X \quad (1.9.i)$$

for all unitary $(\mathcal{T}, \mathcal{V})$-relations $r : X \leftrightarrow Y$, $s : Y \leftrightarrow Z$, and $t : Z \leftrightarrow W$.

For $\hat{T}$ associative, unitary $(\mathcal{T}, \mathcal{V})$-relations are closed under Kleisli convolution:

$$(s \circ r) \circ 1^X_X = s \circ (r \circ 1^Y_Y) = s \circ r \quad \text{and} \quad 1^X_X \circ (s \circ r) = (1^X_X \circ s) \circ r = s \circ r.$$

Hence, in the presence of an associative lax extension $\hat{T}$, we can form the category

$$(\mathcal{T}, \mathcal{V})-$\text{UREl}$$

whose objects are sets, and whose morphisms are unitary $(\mathcal{T}, \mathcal{V})$-relations that compose via Kleisli convolution. We note that, like $(\mathcal{T}, \mathcal{V})$-$\text{Cat}$, also $(\mathcal{T}, \mathcal{V})$-$\text{UREl}$ depends on the lax extension $\hat{T}$; we write $(\mathcal{T}, \mathcal{V}, \hat{T})$-$\text{UREl}$ whenever this dependency needs to be emphasized. When the hom-sets $(\mathcal{T}, \mathcal{V})$-$\text{UREl}(X, Y)$ are equipped with the pointwise order induced by $\mathcal{V}$,

$$(\mathcal{T}, \mathcal{V})$-$\text{UREl}$$

becomes an ordered category.

Condition (1.9.i) appears daunting to verify directly, so we postpone examples of associative lax extensions until after Proposition 1.9.4 which presents more practical conditions. To this end, we introduce the unitary $(\mathcal{T}, \mathcal{V})$-relation

$$r^\sharp := e^\mathcal{V}_Y \cdot \hat{T}r : TX \leftrightarrow Y$$

associated to a $\mathcal{V}$-relation $r : X \leftrightarrow Y$ (see Exercise 1.N). Notice that $(1_X)^\sharp = 1^X_X$ as defined in 1.7.

1.9.2 Lemma. Let $\hat{T}$ be a lax extension to $\mathcal{V}$-$\text{Rel}$ of a monad $\mathcal{T} = (T, m, e)$ on $\text{Set}$. Then

$$\hat{T}(s^\sharp \cdot \hat{T}r) \cdot m_X^\mathcal{V} = \hat{T}(s \cdot r)$$

for all $\mathcal{V}$-relations $r : X \leftrightarrow Y$ and $s : Y \leftrightarrow Z$. In particular,

$$\hat{T}(s^\sharp) \cdot m_Y^\mathcal{V} = \hat{T}s$$

for all $\mathcal{V}$-relations $s : Y \leftrightarrow Z$. 
Proof. The first stated equality follows from

\[
\hat{T}(s \cdot r) = \hat{T}(s \cdot r) \cdot \hat{T}(e_X^0) \cdot m_X^0 \quad \text{(Lemma 1.7.2)}
\]
\[
\leq \hat{T}(s \cdot r \cdot e_X^0) \cdot m_X^0 \quad \text{(\(\hat{T}\) lax functor)}
\]
\[
\leq \hat{T}(e_X^0 \cdot \hat{T} s \cdot \hat{T} r) \cdot m_X^0 = \hat{T}(s_X \cdot \hat{T} r) \cdot m_X^0 \quad \text{(\(e^0\) lax natural)}
\]
\[
\leq \hat{T}(e_X^0 \cdot \hat{T}(s \cdot r)) \cdot m_X^0 \quad \text{(\(\hat{T}\) lax functor)}
\]
\[
= (T \eta_Z)^o \cdot \hat{T}(s \cdot r) \cdot m_X^0 \quad \text{(Corollary 1.4.4)}
\]
\[
\leq (T \eta_Z)^o \cdot \hat{T}(s \cdot r) \quad \text{(\(m^o\) lax natural)}
\]
\[
= \hat{T}(s \cdot r) .
\]

The particular case is obtained by setting \(r = 1_Y\).

\[\square\]

1.9.3 Lemma. Let \(\hat{\mathbb{T}}\) be a lax extension to \(\mathbb{V}\)-\text{Rel} of a monad \(\mathbb{T} = (T, m, e)\) on \text{Set}. Then

\[
m_X^0 \cdot \hat{T} 1_X = \hat{T} 1_{TX} \cdot m_X^0 \cdot \hat{T} 1_X = \hat{T} \hat{T} 1_X \cdot m_X^0 \cdot \hat{T} 1_X .
\]

Proof. Since \(1_{TX} \leq \hat{T} 1_{TX} \leq \hat{T} \hat{T} 1_X\), we have

\[
m_X^0 \cdot \hat{T} 1_X \leq \hat{T} 1_{TX} \cdot m_X^0 \cdot \hat{T} 1_X \leq \hat{T} \hat{T} 1_X \cdot m_X^0 \cdot \hat{T} 1_X \leq m_X^0 \cdot \hat{T} 1_X .
\]

by lax naturality of \(\hat{\mathbb{T}}\).

\[\square\]

1.9.4 Proposition. Let \(\hat{\mathbb{T}}\) be a lax extension to \(\mathbb{V}\)-\text{Rel} of a monad \(\mathbb{T} = (T, m, e)\) on \text{Set}. The following are equivalent:

\begin{enumerate}
\item \(\hat{\mathbb{T}}\) is associative;
\item \(\hat{T} : \mathbb{V}\)-\text{Rel} \to \mathbb{V}\)-\text{Rel} preserves composition and \(m^o : \hat{T} \to \hat{T} \hat{T}\) is natural;
\item \(t \circ (s \circ r) = (t \circ s) \circ r\) for all \(\mathbb{V}\)-relations \(t : T Z \to W\), \(s : T Y \to Z\) and right unitary \(\mathbb{V}\)-relations \(r : T X \to Y\).
\end{enumerate}

Proof. For (i) \(\implies\) (ii), consider \(\mathbb{V}\)-relations \(r : X \to Y\) and \(s : Y \to Z\). We first prove that an associative lax extension preserves composition. Since all of \(r_X^e, s_X^e,\) and \(T 1_Z\) are unitary (Exercise 1.1.1.1), one has

\[
\hat{T} 1_Z \circ (s_X^e \circ r_X^e) = (\hat{T} 1_Z \circ s_X^e) \circ r_X^e .
\]

This identity is equivalent to \(\hat{T}(s \cdot r) = \hat{T} s \cdot \hat{T} r\): indeed,

\[
\hat{T} 1_Z \circ (s_X^e \circ r_X^e) = \hat{T}(s_X^e \cdot \hat{T}(r_X^e) \cdot m_X^0) \cdot m_X^0 = \hat{T}(s_X^e \cdot \hat{T} r) \cdot m_X^0 = \hat{T}(s \cdot r)
\]

by using Lemma 1.9.2 twice, and

\[
(\hat{T} 1_Z \circ s_X^e) \circ r_X^e = \hat{T}(s_X^e) \cdot m_Y^e \cdot \hat{T}(r_X^e) \cdot m_X^0 = \hat{T} s \cdot \hat{T} r
\]
by Lemma 1.9.2 again. To see that \( m^\circ \) is natural, we compute
\[
\hat{T}_1 \circ (\hat{T}_1 \circ r_z) = \hat{T}_1 \cdot \hat{T}(\hat{T}_1 \cdot \hat{T}(r_z) \cdot m^\circ_X) \cdot m^\circ_X = \hat{T}_1 \cdot \hat{T} \cdot m^\circ_X = \hat{T} \cdot m^\circ_X
\]
and
\[
(\hat{T}_1 \circ \hat{T}_1) \circ r_z = \hat{T}_1 \cdot \hat{T} \cdot m^\circ_X \cdot m^\circ_X = \hat{T} \cdot m^\circ_X
\]
using Lemmata 1.9.2 and 1.9.3. Since Kleisli convolution is associative on unitary relations, we obtain \( \hat{T} \cdot m^\circ_X = m^\circ_Y \cdot \hat{T}r \).

For (ii) \( \implies \) (iii), we use right unitariness of \( r \) to write
\[
t \circ (s \circ r) = t \cdot \hat{T}(s \cdot \hat{T}(r \cdot \hat{T}_1 \cdot m^\circ_X) \cdot m^\circ_X
\]
\[
= t \cdot \hat{T}(s \cdot \hat{T} \cdot m^\circ_X \cdot \hat{T}_1 \cdot m^\circ_X
\]
\[
= t \cdot \hat{T} \cdot m^\circ_X \cdot \hat{T}_1 \cdot m^\circ_X
\]
\[
= t \cdot \hat{T} \cdot m^\circ_X \cdot m^\circ_X \cdot \hat{T}_1
\]
\[
= t \cdot \hat{T} \cdot m^\circ_X \cdot \hat{T}_1
\]
\[
= t \cdot \hat{T} \cdot m^\circ_X \cdot \hat{T}(r \cdot \hat{T}_1 \cdot m^\circ_X) = (t \circ s) \circ r .
\]

(iii) \( \implies \) (i) is immediate by definition of an associative lax extension. \( \square \)

1.9.5 Remark. In the case where the lax extension \( \hat{T} \) to \( V\text{-Rel} \) satisfies \( \hat{T}(r^\circ) = (\hat{T}r)^\circ \) for all \( V \)-relations \( r \), the naturality condition for \( m^\circ \) in Proposition 1.9.4(ii) can equivalently be expressed as naturality of \( m : \hat{T} \hat{T} \to \hat{T} \) (see Exercise 1.J).

1.9.6 Examples.

(1) The identity extension \( \hat{I} \) to \( V\text{-Rel} \) of the identity monad is associative. Indeed, the Kleisli convolution of \( V \)-relations is their usual composition, which is associative (Exercise 1.C).

(2) The lax extensions \( \hat{P}, \hat{\hat{P}} \) of the powerset monad \( \hat{P} \) (Example 1.5.2(2)) to \( \text{Rel} \) are both associative. Let us verify (ii) of Proposition 1.9.4 for \( \hat{P} \) (the verifications for \( \hat{\hat{P}} \) are similar). Since \( A \) (\( \hat{P}r \)) \( B \) is defined as \( A \subseteq r^\circ(B) \), the equivalence
\[
A \subseteq (s \cdot r)^\circ(C) \iff \exists B \subseteq Y (A \subseteq r^\circ(B) \& B \subseteq s^\circ(C))
\]
(for all \( A \subseteq X, C \subseteq Y, \) and relations \( r : X \to Y, s : Y \to X \)) shows that the lax extension \( \hat{P} \) preserves relational composition. As the monad multiplication \( \cup \) yields a lax natural transformation \( \cup^\circ \) in \( \text{Rel} \), one only needs to verify that \( \cup^\circ : \hat{P} \hat{P}r \subseteq \hat{P} \hat{P}r \cdot \cup^\circ_X \); suppose that for \( A \subseteq X \) and \( B \subseteq PY \), one has \( A \subseteq r^\circ(\cup B) \); then the subset \( A = \{ \{x\} \mid x \in A \} \subseteq PX \) is such that \( \cup A = A \) and for all \( A' \in A \), there is \( B \in B \) such that \( A' \subseteq r^\circ(B) \).
(3) The largest lax extension $\top^\top$ to $\mathcal{V}$-$\text{Rel}$ of a monad $\top$ on $\text{Set}$ is associative.

Let us exhibit now a non-associative lax extension. To this end, we consider the five-element frame $C$ depicted by

```
\[
\begin{array}{c}
\top \\
\downarrow \\
w \\
\downarrow \\
u \\
\downarrow \\
v \\
\downarrow \\
\bot \\
\end{array}
\]
```

(considered as a quantale with $\otimes = \land$). The identity monad $\mathbb{1} = (I = 1_{\text{Set}}, 1, 1)$ on $\text{Set}$ may be extended non-identically to $C$-$\text{Rel}$ by

$$\hat{I}r(x, y) = \begin{cases} 
\top & \text{if } w \leq r(x, y) \\
r(x, y) & \text{otherwise}, 
\end{cases}$$

for all $x \in X$, $y \in Y$, and $C$-relations $r : X \rightarrow Y$.

**1.9.7 Proposition.** $C$ is a commutative, integral and lean quantale, and $\hat{\mathbb{1}} = (\hat{I}, m = 1, e = 1)$ is a flat lax extension of the identity monad $\mathbb{1}$ to $C$-$\text{Rel}$ which makes $m^\circ : \hat{I} \rightarrow \hat{I}\hat{I}$ a natural transformation. But $\hat{I}$ fails to preserve Kleisli convolution, so $\hat{\mathbb{1}}$ is not associative.

**Proof.** The claims about $C$ are immediate. One also easily sees that

$$r \leq \hat{I}r = \hat{I}\hat{I}r \quad \text{and} \quad \hat{I}f = f, \quad \hat{I}(f^\circ) = f^\circ$$

for all $r : X \rightarrow Y$ in $C$-$\text{Rel}$, and $f : X \rightarrow Y$ in $\text{Set}$. Hence, all claims but the last will follow from

$$\hat{I}s \cdot \hat{I}r \leq \hat{I}(s \cdot r)$$

(with $s : Y \rightarrow Z$). To see this, suppose first that for $x \in X$ and $z \in Z$, we have $\hat{I}s \cdot \hat{I}r(x, z) = \top$. Then necessarily $\hat{I}r(x, y) = \top = \hat{I}s(y, z)$ for some $y \in Y$, that is, $w \leq r(x, y)$ and $w \leq s(y, z)$. In this case, $w \leq (s \cdot r)(x, z)$ and $\hat{I}(s \cdot r)(x, z) = \top$ follows. Suppose now that

$$(\hat{I}s \cdot \hat{I}r)(x, z) = \bigvee_{y \in Y} \hat{I}r(x, y) \land \hat{I}s(y, z) \leq w < \top,$$

and consider any $y \in Y$. We may assume $\hat{I}r(x, y) < w$ (since $\hat{I}r(x, y)$ and $\hat{I}s(y, z)$ cannot both have value $w$), so that $\hat{I}r(x, y) = r(x, y)$. But then, regardless of whether $\hat{I}s(y, z) = \top$ (in which case $w \leq s(y, z)$) or $\hat{I}s(y, z) < \top$ (in which case $\hat{I}s(y, z) = s(y, z)$), we obtain

$$\hat{I}r(x, y) \land \hat{I}s(y, z) = r(x, y) \land s(y, z).$$

Consequently,

$$(\hat{I}s \cdot \hat{I}r)(x, z) = \bigvee_{y \in Y} \hat{I}r(x, y) \land \hat{I}s(y, z) = (s \cdot r)(x, z) < \top,$$
and \((\hat{I}s \cdot \hat{I}r)(x, z) \leq \hat{I}(s \cdot r)(x, z)\) follows.

For the last claim, consider \(X = \{0, 1, 2, 3\}\) and \(r : X \to X\) with
\[
r(0, 1) = \top, \quad r(1, 3) = u, \quad r(0, 2) = v, \quad r(2, 3) = \top,
\]
while \(r(x, y) = \bot\) for all other pairs \((x, y)\). Then \(\hat{I}r = r\), and therefore,
\[
(\hat{I}r \cdot \hat{I}r)(0, 3) = (r \cdot r)(0, 3) = \bigvee_{y \in X} r(0, y) \land r(y, 3) = u \lor v = w,
\]
which, however, implies \((\hat{I}r \cdot \hat{I}r)(0, 3) < \top = \hat{I}(r \cdot r)(0, 3)\).

### 1.10 The Barr extension

Finding a lax extension of a functor \(T : \text{Set} \to \text{Set}\) to \(\mathcal{V}\)-Rel requires some effort in general, but the problem is much simpler for \(\mathcal{V} = 2\). Recall that \(2\)-Rel \(\cong \text{Rel}\). Given a relation \(r : X \times Y \to 2\), we will denote its representation as a subset of \(X \times Y\) by \(R\) (compare with [II.1.2]). With \(\pi_1 : R \to X\) and \(\pi_2 : R \to Y\) the respective projections, \(r\) is represented as a span, that is, as a diagram of the form

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_1} & R \xleftarrow{\pi_2} Y \\
\end{array}
\]

and we have
\[
r = \pi_2 \cdot \pi_1^\circ
\]
in \(\text{Rel}\).

### 1.10.1 Definition

The **Barr extension** of a functor \(T : \text{Set} \to \text{Set}\) to \(\text{Rel}\) is defined by
\[
\text{Tr} := T \pi_2 \cdot (T \pi_1)^\circ.
\]

Elementwise, for \(\chi \in TX\) and \(y \in TY\) the Barr extension is given by
\[
\chi \text{ Tr } y \iff \exists w \in TR (T \pi_1(w) = \chi \land T \pi_2(w) = y).
\]

### 1.10.2 Remarks

1. The Barr extension \(\text{Tr}\) preserves the order on hom-sets. Indeed, if \(s \leq r\), then we may assume \(S \subseteq R\) with \(S\) the domain of the span representing \(s\); hence, with \(i : S \leftrightarrow R\) denoting the inclusion map,
\[
\text{Ts} = T \pi_2 \cdot Ti \cdot (Ti)^\circ \cdot (T \pi_1)^\circ \leq T \pi_2 \cdot (T \pi_1)^\circ = \text{Tr}.
\]
(2) One easily verifies that
\[ T(r^\circ) = (Tr)^\circ \quad \text{and} \quad Tf = Tf \]
for all relations \( r : X \to Y \) and Set-maps \( f : X \to Y \). Moreover, given Set-maps \( f : A \to X \) and \( g : Y \to B \), one has by definition
\[ T(g \cdot r) = Tg \cdot Tr \quad \text{and} \quad T(r \cdot f^\circ) = Tr \cdot (Tf)^\circ. \]

(3) In the definition of \( Tr \) the pair \((\pi_1, \pi_2)\) can be replaced by any other mono-source representing \( r \), or even by any other source \((p, q)\) with \( r = q \cdot p^\circ \) if \( T \) sends surjections to surjections. (Recall that, in the presence of the Axiom of Choice, every Set-functor preserves surjections, since every epimorphism in Set splits if and only if the Axiom of Choice holds, see Exercise II.2.C.)

Given any other factorization \( r = q \cdot p^\circ \) via maps \( p : P \to X \) and \( q : P \to Y \), the equation \( r = q \cdot p^\circ \) says precisely that the canonical map \( P \to X \times Y \) has image \( R \) and therefore defines a surjection \( l : P \to R \). Moreover, this map \( l \) is a bijection if \((p, q)\) forms a mono-source: \( l \) is monic by a trivial cancellation rule for mono-sources (see II.5.3 in the dual situation, and Exercise II.5.A).

In general, for a factorization \( r = q \cdot p^\circ \), one has
\[ Tq \cdot (Tp)^\circ = T(\pi_2 \cdot l) \cdot (T(\pi_1 \cdot l))^\circ = T\pi_2 \cdot Tl \cdot (Tl)^\circ \cdot (T\pi_1)^\circ \leq T\pi_2 \cdot (T\pi_1)^\circ \]
with equality holding if \( Tl \cdot (Tl)^\circ = 1_{TX} \), that is, if \( Tl \) is surjective (Proposition II.2.2).

1.10.3 Examples.

(1) The Barr extension \( T\) of the identity functor \( 1 \) on Set is simply the identity functor \( 1 \) on Rel.

(2) For the filter functor \( F : \text{Set} \to \text{Set} \), the Barr extension \( F \) is obtained as follows. First, notice that for filters \( a \in FX, b \in FY \), and a relation \( r : X \to Y \),
\[ a \quad (Fr)\ b \quad \iff \quad \exists c \in FR \quad (\pi_1[c] = a \land \pi_2[c] = b) \quad . \]

If such a filter \( c \) exists, then for all \( A \in a \), one has: \( C := \pi_1^{-1}(A) \in c \), and the set \( r(A) := \{ y \in Y \mid \exists x \in A \ (x \ r \ y) \} \)

must be in \( b \), as it contains \( \pi_2(C) \) and \( \pi_2(C) \in \pi_2[c] = b \). Similarly, one observes that \( r^\circ(B) \in a \) for all \( B \in b \). Conversely, if \( r(A) \in b \) and \( r^\circ(B) \in a \) for all \( A \in a \) and \( B \in b \), the sets \( C_{A,B} = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \) (with \( A \) running through \( a \), and \( B \) through \( b \)) form a filter base for \( c \in FR \) such that \( \pi_1[c] = a \) and \( \pi_2[c] = b \).
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Therefore, the Barr extension of the filter functor is given by

\[ a (\mathcal{F} r) b \iff r[a] \subseteq b \& r^o[b] \subseteq a \]

for all \( a \in FX \) and \( b \in FY \), and relations \( r : X \rightarrow Y \), where \( r[a] \) is the filter generated by the filter base \( \{ r(A) \mid A \in a \} \) (this notation coincides with the image-filter notation for maps of \( \text{II.1.12} \)).

(3) In the previous example, if both \( a \) and \( b \) are ultrafilters, then

\[ r[a] \subseteq b \iff r^o[b] \subseteq a \]

Indeed, for an ultrafilter \( b \in \beta Y \) and \( A' \subseteq X \), one has

\[ A' \in b \iff \forall B \in b \ (A' \cap B \neq \emptyset) . \]

Hence, \( r[a] \subseteq b \) means that for all \( A \in a \) and \( B \in b \), one has \( r(A) \cap B \neq \emptyset \), that is, \( A \cap r^o(B) \neq \emptyset \), and one obtains \( r^o[b] \subseteq a \) (the other implication also follows). The Barr extension of the ultrafilter functor \( \beta \) is therefore described by

\[ a (\mathcal{F} \beta) b \iff \forall B \in b \ (A \cap B \neq \emptyset) . \]

for all \( a \in \beta X \), \( b \in \beta Y \) and relations \( r : X \rightarrow Y \), or equivalently by

\[ a (\mathcal{F} \beta) b \iff \forall A \in a \forall B \in b \ \exists x \in A, y \in B \ (x \ r \ y) . \]

(4) The equivalent descriptions of the Barr extension of the ultrafilter monad lead to distinct extensions when we consider the filter monad instead. Similarly to the extensions \( \hat{P}, \hat{P} \) of the powerset functor \( \text{(1.4.2)} \), one obtains two lax extensions of the filter functor, neither of which is the Barr extension of \( F \). First, by setting

\[ a (\hat{F} r) b \iff a \supseteq r^o[b] \]

\[ \iff \forall B \in b \ \exists A \in a \ \forall x \in A \ \exists y \in B \ (x \ r \ y) . \]

for all relations \( r : X \rightarrow Y \), and filters \( a \in FX \), \( b \in FY \), one obtains a non-flat lax extension whose lax algebras, similarly to the Barr extension of \( \beta \), provide a convergence description of the category of topological spaces (see Theorem \( \text{2.2.5} \) and Corollary \( \text{IV.1.5.4} \)). By contrast, the lax algebras with respect to the lax extension given by

\[ a (\hat{F} \beta) b \iff b \supseteq r[a] \]

\[ \iff \forall A \in a \ \exists B \in b \ \forall y \in B \ \exists x \in A \ (x \ r \ y) \]

and their lax homomorphisms form a category isomorphic to the category of closure spaces (Exercise \( \text{1.O} \)).

We still have to address the question of whether the Barr extension is actually a lax extension of the functor \( T \) or, even better, of the monad \( T \). To this end the functor needs to satisfy an important additional condition that we now proceed to describe. Reassuringly, all the functors presented above satisfy this condition and therefore their Barr extensions are lax extensions (see Examples \( \text{1.12.3} \)).
1.11 The Beck–Chevalley condition. Consider \( f : X \to Z \) and \( g : Y \to Z \), and let

\[
\begin{array}{c}
\begin{array}{ccc}
P & \xrightarrow{p_2} & Y \\
p_1 \downarrow & & \downarrow g \\
X & \xrightarrow{f} & Z \\
\end{array}
\end{array}
\]

be a pullback diagram. The inequality \( p_2 \cdot p_1^\circ \leq g^\circ \cdot f \) holds by commutativity of the diagram, while the pullback property forces equality: \( p_2 \cdot p_1^\circ = g^\circ \cdot f \). More generally, a commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
W & \xrightarrow{h_2} & Y \\
\downarrow h_1 & & \downarrow g \\
X & \xrightarrow{f} & Z \\
\end{array}
\end{array}
\] (1.11.i)

is a Beck–Chevalley square, or simply a BC-square, if the maps involved satisfy \( h_2 \cdot h_1^\circ = g^\circ \cdot f \), or equivalently, if \( h_1 \cdot h_2^\circ = f^\circ \cdot g \), that is, if

\[
\begin{array}{c}
\begin{array}{ccc}
W & \xrightarrow{h_2} & Y \\
\downarrow h_1 & & \downarrow g \\
X & \xrightarrow{f} & Z \\
\end{array}
\end{array} \quad \text{or equivalently} \quad \begin{array}{c}
\begin{array}{ccc}
W & \xleftarrow{h_2^\circ} & Y \\
\downarrow h_1 & & \downarrow g \\
X & \xleftarrow{f^\circ} & Z \\
\end{array}
\end{array}
\]

commutes in \( \text{Rel} \). In fact, we may trade here \( \text{Rel} = 2\text{-Rel} \) with any non-trivial quantale \( \mathbb{V} \):

1.11.1 Lemma. The following conditions are equivalent for the commutative square (1.11.i):

(i) \( h_2 \cdot h_1^\circ = g^\circ \cdot f \) in \( \text{Rel} \);

(ii) \( h_2 \cdot h_1^\circ = g^\circ \cdot f \) in \( \mathbb{V}\text{-Rel} \), for every quantale \( \mathbb{V} \);

(iii) \( h_2 \cdot h_1^\circ = g^\circ \cdot f \) in \( \mathbb{V}\text{-Rel} \), for some non-trivial quantale \( \mathbb{V} \);

(iv) (1.11.i) is a weak pullback diagram in \( \text{Set} \), that is, the canonical map \( c : W \to X \times_Y Z \) is surjective.

Proof. The unique quantale homomorphism \( \iota : 2 \to \mathbb{V} \) (with \( \bot \mapsto \bot \), \( \top \mapsto k \)) induces a faithful functor \( \iota : \text{Rel} \to \mathbb{V}\text{-Rel} \) which sends \( r : X \to Y \) to \( \iota r : X \to Y \) (with \( \iota r(x,y) = \iota(r(x,y)) \)); moreover, \( \iota(f_\circ) = f_\circ \) and \( \iota(f^\circ) = f^\circ \) for all maps \( f = f_\circ \). Hence, \( \iota \) facilitates the proof that (i),(ii),(iii) are equivalent. For (i) \( \iff \) (iv) we just note that \( c(w) = (h_1(w), h_2(w)) \) for all \( w \in W \), and

\[
(x \ (g^\circ \cdot f) \ y \iff f(x) = g(y)), \quad (x \ (h_2 \cdot h_1^\circ) \ y \iff \exists w \in W \ (h_1(w) = x \ & \ h_2(w) = y))
\]

for all \( x \in X, \ y \in Y \).
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1.11.2 Definitions.

(1) A \text{Set}-functor \( T \) satisfies the \textit{Beck–Chevalley condition}, or \( BC \) for short, if it sends BC-squares to BC-squares:

\[
h_2 \cdot h_1^* = g^* \cdot f \implies Th_2 \cdot (Th_1)^* = (Tg)^* \cdot Tf
\]

for all maps \( f, g, h_1, h_2 \) with \( h_1 \cdot f = g \cdot h_2 \).

(2) A natural transformation \( \alpha : S \to T \) between \text{Set}-functors \( S \) and \( T \) satisfies the \textit{Beck–Chevalley condition}, or \( BC \) for short, if its naturality diagrams

\[
\begin{array}{ccc}
SX & \xrightarrow{\alpha_X} & TX \\
\downarrow{Sf} & & \downarrow{Tf} \\
SY & \xrightarrow{\alpha_Y} & TY
\end{array}
\]

are BC-squares for all maps \( f : X \to Y \), that is, if \( \alpha_X \cdot (Sf)^* = (Tf)^* \cdot \alpha_Y \), or equivalently, if \( Sf \cdot \alpha_X^* = \alpha_Y^* \cdot Tf \).

Note that by Lemma 1.11.1 it does not matter whether we read the equational conditions appearing in this definition in \text{Rel} or \text{V-Rel} (for a non-trivial \( V \)). Furthermore, we can easily prove the following characterization:

1.11.3 Proposition. The following statements are equivalent for a \text{Set}-functor \( T \):

(i) \( T \) satisfies BC;

(ii) \( T \) preserves weak pullback diagrams;

(iii) \( T \) transforms pullbacks into weak pullbacks and preserves the surjectivity of maps.

Proof. \( (i) \iff (ii) \) follows from Lemma 1.11.1. The first assertion of \( (iii) \) follows trivially from \( (ii) \), and the second from \( (i) \) and Proposition 1.2.2. \( f : X \to Y \) is surjective precisely when \( f \cdot f^* = 1_Y \), that is, if and only if

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{f} & & \downarrow{1_Y} \\
Y & \xrightarrow{1_Y} & Y
\end{array}
\]

is a BC-square. Finally, for \( (iii) \implies (i) \), with the BC-square (1.11.i) and the canonical surjection \( c : W \to X \times_Z Y \), the canonical map

\[
TW \xrightarrow{Tc} T(X \times_Z Y) \xrightarrow{i} TX \times_{TZ} TY
\]

is by hypothesis the composite of two surjectives, hence surjective. \( \square \)
1.11.4 Remarks.

(1) Assuming the Axiom of Choice, so that epimorphisms in $\textsf{Set}$ split and are preserved by functors, we may add
\[
\text{(d) $T$ transforms pullbacks into weak pullbacks.}
\]
to the list of equivalent statements in Proposition 1.11.3.

(2) This list may be extended further by the following statement that is equivalent to (iv):
\[
\text{(e) for all $\textsf{Set}$-maps $f : X \to Z$, $g : Y \to Z$, the monotone map}
\]
\[
(-) \cdot t : \text{Rel}(Q,W) \to \text{Rel}(TP,W)
\]
is fully faithful for all sets $W$; here, $P = X \times_Z Y$, $Q = TX \times_T Z TY$, and $t : TP' \to Q$ is the comparison map.

In fact, for (iv) ⇒ (v), one considers $r, s : Q \to W$ with $s \cdot t \leq r \cdot t$ and concludes $s = s \cdot t \cdot t^o \leq r \cdot t \cdot t^o = r$ since $t$ is surjective. For (v) ⇒ (iv), one observes that $t$, like any map, satisfies $t = t \cdot t^o \cdot t$, in particular $1_Y \cdot t \leq t \cdot t^o \cdot t$. By hypothesis one concludes $1_Y \leq t \cdot t^o$, so that $t$ must be surjective.

We can now return to our primary purpose of introducing BC.

1.11.5 Theorem. For a functor $T : \text{Set} \to \text{Set}$, the following assertions are equivalent:

(i) the functor $T$ satisfies BC;

(ii) the Barr extension $\overline{T}$ is a flat lax extension of $T$ to $\text{Rel}$ and a functor $\overline{T} : \text{Rel} \to \text{Rel}$;

(iii) there is some functor $\hat{T} : \text{Rel} \to \text{Rel}$ which is a lax extension of $T$ to $\text{Rel}$.

Moreover, any functor $\hat{T} : \text{Rel} \to \text{Rel}$ as in (iii) is uniquely determined, that is, $\hat{T} = T$.

Proof. (i) ⇒ (ii): To see that $\overline{T}$ is a flat lax extension, the only issue lies in verifying $\overline{T}s \cdot \overline{T}r = \overline{T}(s \cdot r)$ for relations $r : X \to Y$ and $s : Y \to Z$ with respective factorizations $r = \pi_2 \cdot \pi_1^o$ and $s = \rho_2 \cdot \rho_1^o$. As the pullback $(p_1,p_2)$ of $\xymatrix{ R \ar[r]^{\pi_2} \ar[d]_{\pi_1} & Y \ar[d]^{\rho_1} \ar[l]_{p_1} \ar[l]_{p_2} & S \ar[l]_{\rho_2} \ar[d]_{\pi_2}}$ yields a mono-source that moreover forms a factorization $p_2 \cdot p_1^o$ of the relation $\rho_1 \cdot \pi_2$, 

\[
\xymatrix{ P 
\ar[dr]_{p_2} & 
\ar[dl]^{p_1} \\
R \ar[dr]_{\pi_1} & 
\ar[dl]^{\pi_2} \\
X & \\
& Y \ar[l]_{\rho_1} \ar[u]_{\rho_2} }
\]
one has $\bar{T}(\rho_1^\circ \cdot \pi_2) = Tp_2 \cdot (Tp_1)^\circ$ (see Remark 1.10.2(3)), and $Tp_2 \cdot (Tp_1)^\circ = (T\rho_1)^\circ \cdot T\pi_2$ since $\bar{T}$ satisfies BC. Consequently, with Remark 1.10.2(2) one obtains

$$\bar{T}(s) \cdot \bar{T}(r) = T\rho_2 \cdot (T\rho_1)^\circ \cdot T\pi_2 \cdot (T\pi_1)^\circ = T\rho_2 \cdot \bar{T}(\rho_1^\circ \cdot \pi_2) \cdot (T\pi_1)^\circ = \bar{T}(T\rho_2 \cdot \rho_1^\circ \cdot \pi_2 \cdot \pi_1) = \bar{T}(s \cdot r).$$

(ii) $\implies$ (iii): This is trivial.

(iii) $\implies$ (i): Let $h_2 \cdot h_1^\circ = g^\circ \cdot f$ as in 1.11.2(1). Since functoriality makes $\hat{T}$ also flat one obtains with Corollary 1.4.4

$$Th_2 \cdot (Th_1)^\circ = \hat{T}(h_2 \cdot h_1^\circ) = \hat{T}(g^\circ \cdot f) = (Tg)^\circ \cdot Tf.$$

For the same reasons one has $\hat{T}r = T\pi_2 \cdot (T\pi_1)^\circ = Tr$, for $r = \pi_2 \cdot \pi_1^\circ$. □

1.12 The Barr extension of a monad. Theorem 1.11.5 proves that if $T$ satisfies the Beck–Chevalley condition, then the Barr extension $\overline{T}$ is a lax extension of $T$ to Rel. It does not require much more effort to show that under the same assumption, the Barr extension yields a lax extension of the monad $\mathbb{T} = (T, m, e)$.

Consider first a natural transformation $\alpha : S \to T$ between functors $S, T : \text{Set} \to \text{Set}$ provided with their lax extensions $\bar{S}, \bar{T}$. Then, for a relation $r : X \leftrightarrow Y$ with $r = \pi_2 \cdot \pi_1^\circ$, we have

$$\alpha_Y \cdot \bar{S}r = \alpha_Y \cdot S\pi_2 \cdot (S\pi_1)^\circ = T\pi_2 \cdot \alpha_R \cdot (S\pi_1)^\circ \leq T\pi_2 \cdot (T\pi_1)^\circ \cdot \alpha_X = \bar{T}r \cdot \alpha_X,$$

that is, $\alpha : \bar{S} \to \bar{T}$ is oplax. From the same computation, we observe that $\alpha : \bar{S} \to \bar{T}$ is a natural transformation if all naturality diagrams

$$\begin{align*}
SX \xrightarrow{\alpha_X} TX \\
Sf \bigg| \downarrow \bigg| \downarrow Tf \\
SY \xrightarrow{\alpha_Y} TY
\end{align*}$$

form BC-squares.

Therefore, if $T$ belongs to a monad $\mathbb{T} = (T, m, e)$, then $m$ and $e$ become oplax natural transformations in Rel:

$$m : \overline{TT} \to \overline{T} \quad \text{and} \quad e : \overline{1_{\text{Rel}}} \to \overline{T}.$$

An issue remains with the domain of the multiplication, which should be $\overline{TT}$ rather than $\overline{TT}$. Hence, in order to obtain a lax extension of the monad $\mathbb{T}$ to Rel, we show that the identities $1_{\overline{TT}X}$ are the components of an oplax natural transformation $\overline{TT} \to \overline{TT}$. It follows from Remark 1.10.2(3) and the equality $\overline{T}r = T\pi_2 \cdot (T\pi_1)^\circ$ that

$$\overline{T}Tr = TT\pi_2 \cdot (TT\pi_1)^\circ \leq \overline{T}(\overline{T}r)$$
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for any relation \( r = \pi_2 \cdot \pi_1^\circ \), with equality holding if \( T \) preserves surjections. Thus, the Barr extension \( \hat{T} = (T, m, e) \) is a lax extension of the \( \text{Set} \)-monad \( T = (T, m, e) \) to \( \text{Rel} \) provided that \( T \) satisfies the Beck–Chevalley condition.

\[ \circ \]

**1.12.1 Theorem.** For a monad \( T = (T, m, e) \) on \( \text{Set} \), the following assertions are equivalent.

(i) the functor \( T \) satisfies \( BC \);

(ii) the Barr extension yields a flat lax extension \( \hat{T} = (T, m, e) \) of \( T \) to \( \text{Rel} \).

**Proof.** The implication (i) \( \implies \) (ii) follows from the previous discussion since one knows from Theorem 1.11.5 that if \( T \) satisfies \( BC \) then \( T \) is a flat lax extension of \( T \). The converse implication is also an immediate consequence of the same result, since a monad is a flat lax extension exactly when its underlying functor is one.

\[ \circ \]

**1.12.2 Corollary.** Suppose that \( T = (T, m, e) \) is a monad on \( \text{Set} \) such that \( T \) and \( m \) satisfy \( BC \). Then \( \hat{T} \) is an associative lax extension of \( T \) to \( \text{Rel} \).

**Proof.** The proof of (i) \( \implies \) (ii) in the previous Theorem yields that \( T s \cdot T r = T(s \cdot r) \) if \( T \) satisfies the \( BC \) condition; the discussion preceding Theorem 1.12.1 shows that if \( T \) preserves epimorphisms (as do functors that satisfy \( BC \), see 1.11) and naturality diagrams of \( m \) are \( BC \)-squares, then \( m : T T \to T \) is a natural transformation. The fact that the Barr extension is flat allows us to conclude that the Barr extension is associative by Proposition 1.9.4.

\[ \circ \]

**1.12.3 Examples.**

(1) The identity functor \( 1_{\text{Set}} \) on \( \text{Set} \) obviously satisfies \( BC \); it is also immediate that the Barr extension \( 1_{\text{Rel}} \) is a lax extension.

(2) The filter functor \( F : \text{Set} \to \text{Set} \) satisfies \( BC \). Indeed, suppose that

\[
\begin{array}{ccc}
W & \xrightarrow{h_2} & Y \\
\downarrow{h_1} & & \downarrow{g} \\
X & \xrightarrow{f} & Z
\end{array}
\]

is a \( BC \)-square. Since the square commutes, one immediately obtains the inequality \( Fh_2 \cdot (Fh_1)^\circ \leq (Fg)^\circ \cdot Ff \). For the other direction, we must show that for all filters \( a \in FX \) and \( b \in FY \),

\[
f[a] = g[b] \implies \exists c \in FW \ (h_1[c] = a \land h_2[c] = b).
\]

But the sets \( h_1^{-1}(A) \cap h_2^{-1}(B) \) (for \( A \in a \), and \( B \in b \)) form a base for a filter \( c \) satisfying \( h_1[c] = a \) and \( h_2[c] = b \); indeed, \( g^\circ \cdot f \leq h_2 \cdot h_1^\circ \) means that for any pair
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(x, y) ∈ A × B with f(x) = g(y), there is an element w ∈ W satisfying h₁(w) = x and h₂(w) = y. Thus, the Barr extension \( F \) described in Example 1.10.3(2) is a lax extension of \( F \) to \( \text{Rel} \).

(3) The ultrafilter functor satisfies BC for similar reasons. In this case, to see that \( (βg)° \cdot βf ≤ βh₂ \cdot (βh₁)° \), one obtains a filter \( c \) from ultrafilters \( a \in FX, b \in FY \) as above. By Proposition II.1.13.2, there is an ultrafilter \( x \) on \( W \) with \( c ⊆ x \), so that \( a ⊆ h₁[x] \) and \( b ⊆ h₂[x] \), and maximality of ultrafilters yields the required equalities. Thus, as in the filter case, the Barr extension \( \overline{β} \) of Example 1.10.3(2) is a flat lax extension of \( β \).

We now show that the multiplication \( m \) of the ultrafilter monad \( β \) satisfies BC. For any map \( f : X \to Y \) and all \( x ∈ βX \) and \( y ∈ ββY \) with \( m_Y(y) = βf(x) \), we must find \( X ∈ ββX \) with \( ββf(X) = y \) and \( m_X(X) = x \).

By hypothesis, \( f(A)^β \cap B ≠ \emptyset \), for all \( A ∈ x \) and \( B ∈ y \). One easily verifies \( f(A)^β = βf(A^β) \), and from \( βf(A^β) \cap B ≠ \emptyset \) one obtains \( A^β \cap (βf)^{-1}(B) ≠ \emptyset \). Therefore,

\[ \{A^β \mid A ∈ x\} ∪ \{(βf)^{-1}(B) \mid B ∈ y\} \]

is a filter base, and any ultrafilter \( X \) containing it has the desired property. Consequently, by Corollary 1.12.2 \( \overline{β} \) is associative.

As in 1.9 one can now consider the category \((β, 2)-\text{URel}\) of sets and unitary \((β, 2)-\text{relations}\). (The same statement holds for the filter monad with its Barr extension, but we will not consider this particular instance any further.)

The unit of neither the filter nor the ultrafilter monad satisfies BC (see Exercise 1.Q). In the case of the ultrafilter monad, there is a general reason for this claim, as follows.

1.12.4 Proposition. Any monad \( T = (T, m, e) \) with \( T1 ≅ 1 \) and \( e \) satisfying BC must be isomorphic to the identity monad.

Proof. Since \( e \) satisfies BC, the diagram

\[
\begin{array}{ccc}
TX & \xrightarrow{T1X} & T1 \\
\downarrow{e_X^*} & & \downarrow{e_1^*} \\
X & \xrightarrow{1_X} & 1
\end{array}
\]

commutes in \( \text{Rel} \) for every set \( X \). Expressed elementwise (with \( 1 = \{⋆\} \)) this reads as

\[
∀ x ∈ TX \left( T1X(x) = e_1(⋆) \iff ∃ x ∈ X (e_X(x) = x) \right),
\]

with \( T1X(x) = e_1(⋆) \) holding since \( T1 ≅ 1 \). Hence, \( e_X \) must be surjective, or even bijective if \( T \) is non-trivial (see Exercise II.3.A). But neither of the two trivial monads on \( \text{Set} \) has a unit satisfying BC (Exercise 1.Q). \( \square \)
1.13 A double-categorical presentation of lax extensions. The notion of a lax extension \( \hat{T} : \mathcal{V} \text{-} \text{Rel} \rightarrow \mathcal{V} \text{-} \text{Rel} \) of a functor \( T : \text{Set} \rightarrow \text{Set} \) as given in 1.4.1 allows for a very natural double-categorical interpretation. We briefly describe this presentation here without formally introducing double categories, their functors and natural transformations. To this end, we consider diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
r & \leq & s \\
U & \xrightarrow{g} & V
\end{array}
\]

consisting of \( \text{Set} \)-maps \( f, g \) and \( \mathcal{V} \)-relations \( r, s \) such that

\[ g \cdot r \leq s \cdot f \]

or, equivalently \( r \leq g^\circ s \cdot f \), that is: \( r(x,u) \leq s(f(x),g(u)) \) for all \( x \in X, u \in U \). We call these diagrams cells. The point is that such a cell may be considered as a morphism \( r \rightarrow s \) horizontally as well as a morphism \( f \rightarrow g \) vertically. With map composition used horizontally and \( \mathcal{V} \)-relational composition used vertically one obtains two intertwined category structures whose main interaction is captured by the middle-interchange law:

\[
\begin{array}{ccc}
\alpha & \beta \\
\gamma & \delta
\end{array}
\]

for cells \( \alpha, \beta, \gamma, \delta \) that fit together as indicated above, one has

\[
(\delta \cdot \gamma) \circ (\beta \cdot \alpha) = (\delta \circ \beta) \cdot (\gamma \circ \alpha)
\]

(with vertical composition denoted by \( \circ \)). Cells and their compositions form a double category \( \mathcal{V} \text{-} \text{Rel} \). A double category functor \( \mathcal{T} : \mathcal{V} \text{-} \text{Rel} \rightarrow \mathcal{V} \text{-} \text{Rel} \) returns for every cell \( \alpha \) a cell \( \mathcal{T} \alpha \), preserves horizontal composition and identity morphisms strictly, and vertical composition and identity morphisms laxly. Hence \( \mathcal{T} \) is in fact given by a functor \( T : \text{Set} \rightarrow \text{Set} \) and a lax functor \( \hat{T} : \mathcal{V} \text{-} \text{Rel} \rightarrow \mathcal{V} \text{-} \text{Rel} \) such that

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\mathcal{T}r & \leq & \mathcal{T}s \\
U & \xrightarrow{g} & V \\
\hat{T}r & \leq & \hat{T}s \\
\mathcal{T}X & \xrightarrow{Tf} & \mathcal{T}Y \\
\hat{T}U & \xrightarrow{Tg} & \hat{T}V
\end{array}
\]

that is, \( g \cdot r \leq s \cdot f \) implies \( Tg \cdot \hat{T}r \leq \hat{T}s \cdot Tf \).

1.13.1 Proposition. Double category functors \( \mathcal{T} = (T, \hat{T}) : \mathcal{V} \text{-} \text{Rel} \rightarrow \mathcal{V} \text{-} \text{Rel} \) are precisely the lax extensions \( \hat{T} : \mathcal{V} \text{-} \text{Rel} \rightarrow \mathcal{V} \text{-} \text{Rel} \) of functors \( T : \text{Set} \rightarrow \text{Set} \).
Proof. It suffices to show that the cell preservation condition for $\mathcal{T} = (T, \hat{T})$ is equivalent to the lax extension conditions $Tf \leq \hat{T}f$ and $(Tf)^{\circ} \leq \hat{T}(f^{\circ})$, given that $T$ is a functor and $\hat{T}$ a lax functor. When one exploits the preservation conditions for the cells

\[
\begin{array}{c}
X \xrightarrow{1_X} Y \\
\downarrow f \quad \downarrow f
\end{array} \quad \text{and} \quad \begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow 1_X \quad \downarrow f^{\circ}
\end{array}
\]

one obtains

\[Tf = Tf \cdot T1_X \leq Tf \cdot \hat{T}1_X \leq \hat{T}f \cdot T1_X = \hat{T}f\]

and

\[(Tf)^{\circ} = 1_{TX} \cdot (Tf)^{\circ} \leq \hat{T}1_X \cdot (Tf)^{\circ} \leq \hat{T}(f^{\circ}) \cdot Tf \cdot (Tf)^{\circ} \leq \hat{T}(f^{\circ}) ,\]

respectively. Conversely, by Corollary 1.4.4 a lax extension satisfies the preservation condition: expressing $g \cdot r \leq s \cdot f$ equivalently by $r \leq (g^{\circ}) \cdot s \cdot f$, one obtains $\hat{T}r \leq (Tg)^{\circ} \cdot \hat{T}s \cdot Tf$, or $Tg \cdot \hat{T}r \leq \hat{T}s \cdot Tf$.

One can now proceed to consider an appropriate monad structure on $\mathcal{T}$, by suitable natural transformations $m : \mathcal{T} \mathcal{T} \to \mathcal{T}$ and $e : 1 \to \mathcal{T}$: we require these transformations to be given by horizontal natural transformations $m : \mathcal{T} \mathcal{T} \to \mathcal{T}$ and $e : 1 \to \mathcal{T}$ (in the ordinary sense) that are compatible with the vertical structure, so that there are cells

\[
\begin{array}{c}
TTX \xrightarrow{m_X} TX \\
\downarrow \hat{T}r \quad \downarrow \hat{T}r
\end{array} \quad \text{and} \quad \begin{array}{c}
X \xrightarrow{e_X} TX \\
\downarrow r \quad \downarrow r
\end{array} \quad \text{and} \quad \begin{array}{c}
TTY \xrightarrow{m_Y} TY \\
\downarrow \hat{T}r \quad \downarrow \hat{T}r
\end{array} \quad \text{and} \quad \begin{array}{c}
Y \xrightarrow{e_Y} TY \\
\downarrow r \quad \downarrow r
\end{array}
\]

for every $\mathcal{V}$-relation $r : X \leftrightarrow Y$. But the existence requirement for these cells gives precisely the condition that $m : \hat{T} \mathcal{T} \to \hat{T}$ and $e : 1 \to \hat{T}$ be op-lax (see 1.5.1). We therefore have:

1.13.2 Corollary. Monads of the double category $\mathcal{V}$-Rel are precisely lax extensions to $\mathcal{V}$-Rel of monads on $\mathcal{S}et$.

Exercises

1.A The trivial and integral quantales. Show that a quantale $\mathcal{V}$ is trivial (that is, $|\mathcal{V}| = 1$) if and only if $\bot = k$, where $k$ denotes the neutral element of $\mathcal{V}$. Furthermore, $\mathcal{V}$ is integral (that is, $k = \top$) if and only if the terminal object of $\mathcal{V}$-Cat is a generator.

1.B Lean but not integral. For a monoid $M$, the powerset $PM$ has a quantale structure as in Exercise II.1.M. Then $PM$ is lean, but integral only if $M$ is trivial.
1.C  **Associativity of \( V \)-relational composition.** Verify that
\[
    t \cdot (s \cdot r) = (t \cdot s) \cdot r
\]
holds for all \( V \)-relations \( r : X \to Y \), \( s : Y \to Z \) and \( t : Z \to A \).

1.D  A \( V \)-powerset monad. Given a quantale \( V \), the \( V \)-powerset functor \( P_V \) sends a set \( X \) to its \( V \)-powerset \( V^X \), and a map \( f : X \to Y \) to \( P_V f : V^X \to V^Y \), where
\[
P_V f(\phi)(y) := \bigvee_{x \in f^{-1}(y)} \phi(x),
\]
for all \( \phi \in V^Y \), \( y \in Y \). The multiplication \( \mu : P_V P_V \to P_V \) and unit \( \delta : 1_{\text{Set}} \to P_V \) of the \( V \)-powerset monad \( P_V \) are defined respectively by
\[
    \mu_X(\Phi)(y) := \bigvee_{\phi \in V^X} \Phi(\phi) \otimes \phi(y)
\]
and
\[
    \delta_X(x)(y) := \begin{cases} 
    k & \text{if } x = y \\
    \bot & \text{otherwise},
    \end{cases}
\]
for all \( x, y \in X \), \( \Phi \in V^V \). The extension operation \(( - )^{P_V} \) of the corresponding Kleisli triple (see II.3.7) is given for any \( f : X \to P_V Y \) by
\[
f^{P_V}(\phi)(y) = \bigvee_{x \in X} \phi(x) \otimes f(x)(y)
\]
for all \( \phi \in V^X \), \( y \in Y \). The 2-powerset monad is simply the powerset monad \( \mathbb{P} \), and the Kleisli category returns \( V\text{-Rel} \):
\[
    \text{Set}_{P_V} = V\text{-Rel}.
\]

1.E  **Extensions and liftings in \( V\text{-Rel} \).** For \( r : X \to Y \) and \( s : X \to Z \), show that the extension \( s \bullet r \) in the quantaloid \( V\text{-Rel} \) (see II.4.8) may be described as
\[
    (s \bullet r)(y, z) = \bigwedge_{x \in X} s(x, z) \bullet r(x, y).
\]
Likewise, for \( t : Z \to Y \), the lifting \( r \cdot t \) is described by
\[
    (r \cdot t)(z, x) = \bigwedge_{y \in Y} r(x, y) \cdot t(z, y).
\]

1.F  **Symmetric and separated metric spaces.** The category \( \text{Met}_{\text{sym}} \) is bicoreflective in \( \text{Met} \), and \( \text{Met}_{\text{sep}} \) is strongly epireflective in \( \text{Met} \) (see Example I.3.12). The notion of symmetry and separation can be introduced for \( V \)-categories and the previous statements generalized to \( V\text{-Cat} \), for arbitrary \( V \) in lieu of \( P_+ \).

1.G  **Limits and colimits in \( V\text{-Cat} \).**

1. (1) Products and coproducts of \( (X_i, a_i)_{i \in I} \) in \( V\text{-Cat} \) are formed by endowing the sets \( \Pi_{i \in I} X_i \) and \( \bigsqcup_{i \in I} X_i \) with the structures
\[
p((x_i)_{i \in I}, (y_i)_{i \in I}) = \bigwedge_{i \in I} a_i(x_i, y_i)
\]
and
\[
s((x, i), (y, j)) = \begin{cases} 
    a_i(x, y) & \text{if } i = j \\
    \bot & \text{otherwise},
    \end{cases}
\]
respectively.
1. BASIC CONCEPTS

(2) Show that the forgetful functor $O : \mathcal{V}\text{-Cat} \to \text{Set}$ is topological. Conclude that $\mathcal{V}\text{-Cat}$ is small-complete and small-cocomplete, with all limits and colimits in $\mathcal{V}\text{-Cat}$ preserved by $O$.

(3) For a surjective map of sets $f : X \to Y$ and $z, w \in Y$, a tuple $(x_0, y_0, x_1, y_1, \ldots, x_n, y_n)$ (with $0 \leq n$) is $f$-admissible for $z, w$ if
\[
z = f(x_0), f(y_0) = f(x_1), f(y_1) = f(x_2), \ldots, f(y_{n-1}) = f(x_n), f(y_n) = w.
\]
For $f : (X, a) \to (Y, b)$ in $\mathcal{V}\text{-Cat}$ prove that $f$ is $O$-final if and only if $b(z, w) = \bigvee_{(x_0, y_0, \ldots, x_n, y_n)} a(x_0, y_0) \otimes \ldots \otimes a(x_n, y_n)$ for all $z, w \in Y$.

(4) Describe equalizers and coequalizers in $\mathcal{V}\text{-Cat}$ and apply the description to $\text{Met}$.

1.H Yoneda Functor, Yoneda Lemma, initial density of $\mathcal{V}$ in $\mathcal{V}\text{-Cat}$. Let $\mathcal{V}$ be a commutative quantale, considered as a $\mathcal{V}$-category $(\mathcal{V}, \cdot)$, and for a $\mathcal{V}$-category $X = (X, a)$, set $\widehat{X} = [X^{\text{op}}, \mathcal{V}]$ (see Proposition 1.3.3).

(1) Show that $y : X \to \widehat{X}$ with $y(x) = a(-, x) : X^{\text{op}} \to \mathcal{V}$ defines a $\mathcal{V}$-functor.

(2) Prove $\delta(y(x), \phi) = \phi(x)$ for all $x \in X$, $\phi \in \widehat{X}$, where $\delta$ denotes the $\mathcal{V}$-category structure of $\widehat{X}$.

(3) Conclude that $y$ is $O$-initial with respect to the forgetful functor $O : \mathcal{V}\text{-Cat} \to \text{Set}$ (see Exercise 1.G).

(4) With $\text{ev}_x$ denoting the evaluation $\mathcal{V}$-functor at $x \in X$, show that the source
\[
(X \xrightarrow{y} \widehat{X} \xrightarrow{\text{ev}_x} \mathcal{V})_{x \in X}
\]
is $O$-initial and conclude that $\mathcal{V}$ is $O$-initially dense in $\mathcal{V}\text{-Cat}$.

1.I Lax distributive laws and lax monad extensions. A lax distributive law of the $\mathcal{V}$-powerset monad $\mathcal{P}\mathcal{V}$ (Exercise 1.D) over the monad $\mathcal{T}$ on $\text{Set}$ is a natural transformation $\lambda_X : TP\mathcal{V}X \to P\mathcal{V}\mathcal{T}X$ such that the diagrams

```
TP\mathcal{V}P\mathcal{V} \xrightarrow{\lambda_{P\mathcal{V}}} P\mathcal{V}TP\mathcal{V} \xrightarrow{P\mathcal{V}\lambda} P\mathcal{V}P\mathcal{V}\mathcal{T} \\
\mu_T \downarrow \quad \geq \quad \downarrow \mu_T \\
TP\mathcal{V} \xrightarrow{\lambda} P\mathcal{V}T \\
m_{P\mathcal{V}} \downarrow \quad \geq \quad \downarrow P\mathcal{V}m \\
TTP\mathcal{V} \xrightarrow{T\lambda} TP\mathcal{V}T \xrightarrow{\lambda_T} P\mathcal{V}TT
```

```
TP\mathcal{V} \xrightarrow{\lambda} P\mathcal{V}T \\
TP\mathcal{V} \xrightarrow{T\delta} T\mathcal{V} \xrightarrow{\delta_T} \mathcal{T}P\mathcal{V} \\
e_{P\mathcal{V}} \downarrow \quad \geq \quad \downarrow P\mathcal{V}e \\
P\mathcal{V} \xrightarrow{P\mathcal{V}e}
```
CHAPTER III. LAX ALGEBRAS

commute up to “≥” as indicated. There is a bijective correspondence between lax extensions \( \hat{T} \) of \( \mathbb{T} \) to \( \mathcal{V}\text{-Rel} \) and lax distributive laws \( \lambda \) of \( P_v \) over \( \mathbb{T} \) satisfying

\[
\forall r, r' \in X \to Y \text{ presented as maps } r, r' : X \to P_v Y: \text{ a lax extension } \hat{T} : \mathcal{V}\text{-Rel} \to \mathcal{V}\text{-Rel} \text{ yields a lax distributive law } \lambda = (\hat{T}1_{P_vX} : TP_vX \to P_vTX)_{X \in \text{ob} \text{-Set}},
\]

and a lax distributive law \( \lambda : TP_v \to P_v T \) determines a lax extension \( \hat{T} \) that sends a \( \mathcal{V}\)-relation \( r : X \to P_v Y \) to the \( \mathcal{V}\)-relation \( \lambda_Y \cdot Tr : TX \to P_v TY \).

1.J The dual lax extension. Let \( \mathcal{V} \) be a commutative quantale. If \( \hat{T} \) is a lax extension to \( \mathcal{V}\text{-Rel} \) of a \( \text{Set}\)-functor \( T \), then for any \( \mathcal{V}\)-relation \( r : X \to Y \), its dual

\[
\hat{T}^o r := (\hat{T}(r^o))^o
\]

is also a lax extension to \( \mathcal{V}\text{-Rel} \) of \( T \), and one has \( (\hat{T}^o)^o = \hat{T} \). If \( \hat{T} = (\hat{T}, m, e) \) is a lax extension of a monad \( \mathbb{T} \), then so is \( \hat{T}^o = (\hat{T}^o, m, e) \). In this case, \( \hat{T}^o \) is associative if and only if \( \hat{T} \) preserves composition of \( \mathcal{V}\)-relations and \( m : \hat{T}\hat{T} \to \hat{T} \) is a natural transformation.

For the lax extensions of the powerset and filter monads coming from Examples 1.4.2(2) and 1.10.3(4), one has

\[
\hat{P} = \hat{P}^o \quad \text{and} \quad \hat{F} = \hat{F}^o.
\]

The Barr extension of a functor \( T \) to \( \text{Rel} \) is always self-dual:

\[
\mathcal{T}^o = T.
\]

1.K Checking \((\mathbb{T}, \mathcal{V})\)-functoriality. Prove that the following equivalences hold:

\[
f \cdot a \leq b \cdot Tf \iff a \leq f^o \cdot b \cdot Tf \iff f \cdot a \cdot (Tf)^o \leq b
\]

for all \( \text{Set}\)-maps \( f : X \to Y \), and \( \mathcal{V}\)-relations \( a : TX \to X \), \( b : TY \to Y \).

1.L Associativity of the Kleisli convolution. Consider a lax extension \( \hat{T} \) to \( \mathcal{V}\text{-Rel} \) of the underlying functor of \( \mathbb{T} = (T, m, e) \), and \( (\mathbb{T}, \mathcal{V})\)-relations \( r : X \to Y \), \( s : Y \to Z \), \( t : Z \to W \).

(1) If \( m_Y \cdot \hat{T}r \leq \hat{T}\hat{T}r \cdot m_X^o \), in particular if \( m^o : \hat{T} \to \hat{T}\hat{T} \) is a natural transformation, then \( (t \circ s) \circ r \leq t \circ (s \circ r) \).

(2) If \( \hat{T}r \cdot m_X^o \cdot \hat{T}1_X \leq \hat{T}r \cdot m_X^o \) and \( \hat{T} \) preserves the composition of \( \mathcal{V}\)-relations, then

\[
t \circ (s \circ r) \leq (t \circ s) \circ r.
\]

1.M Kleisli convolution for \((\mathbb{T}, \mathcal{V})\)-functors. For a monad \( \mathbb{T} \) with a lax extension \( \hat{T} \), one defines for a map \( f : X \to Y \) the unitary \( \mathcal{V}\)-relation \( f^\sharp = (e_Y \cdot f)^o \cdot \hat{T}1_Y : TY \to X \). The condition

\[
a \cdot (Tf)^o \leq f^o \cdot b,
\]
for unitary $\mathcal{V}$-relations $a : TX \twoheadrightarrow X$ and $b : TY \twoheadrightarrow Y$, is then equivalent to

$$a \circ f^z \leq f^z \circ b.$$ 

Briefly put, $(\mathcal{T}, \mathcal{V})$-functoriality of a map $f : X \to Y$ between $(\mathcal{T}, \mathcal{V})$-categories $(X, a)$ and $(Y, b)$ can be expressed in terms of the Kleisli convolution.

1.N Unitary $(\mathcal{T}, \mathcal{V})$-relations. Let $(\hat{T}, m, e)$ be a lax extension of a monad $\mathcal{T} = (T, m, e)$ on $\mathsf{Set}$. Show that $\hat{T}r : X \twoheadrightarrow TY$ and $r^z = e^z_Y \cdot \hat{T}r$ are unitary $(\mathcal{T}, \mathcal{V})$-relations for every $\mathcal{V}$-relation $r : X \twoheadrightarrow Y$. Moreover, if $(\mathcal{T}, \mathcal{V})$-relations $r : X \twoheadrightarrow Y$ and $s : Y \twoheadrightarrow Z$ are respectively right and left unitary, then $s \circ r$ is unitary.

1.O Closure spaces via the filter monad. A lax extension of the filter monad $\mathcal{F}$ to $\mathsf{Rel}$ is obtained via

$$a \tilde{\mathcal{F}} r b \iff b \supseteq r[a]$$

for all $a \in FX, b \in FY$, and relations $r : X \twoheadrightarrow Y$ (Example 1.10.3(4)). Verify that for a map $c : PX \to PX$ and relation $r : FX \twoheadrightarrow X$, the relation $r_c : FX \twoheadrightarrow X$ and map $c_r : PX \to PX$ given by

$$a r_c x \iff x \in \bigcap_{A \in a} c(A) \quad \text{and} \quad x \in c_r(A) \iff \hat{A} r x$$

(where $\hat{A}$ denotes the principal filter over $A$) determine an isomorphism between the categories of $(\mathcal{F}, 2)$-categories and of closure spaces:

$$(\mathcal{F}, 2, \hat{\mathcal{F}})\text{-Cat} \cong \mathsf{Cls}.$$ 

1.P Functors on $\mathsf{Set}$ preserving monomorphisms.

(1) The following statements are equivalent for a functor $T : \mathsf{Set} \to \mathsf{Set}$:

(a) $T$ preserves monomorphisms;
(b) $T$ preserves the monomorphism $\emptyset \to T\emptyset$;
(c) $T$ preserves the pullback

$$\begin{array}{ccc}
\emptyset & \to & \emptyset \\
\downarrow & & \downarrow \\
\emptyset & \to & T\emptyset ;
\end{array}$$

(d) $T$ preserves the pullback of (iii) weakly, that is, the canonical map $T\emptyset \to T\emptyset \times_{TT\emptyset} T\emptyset$ is surjective.

(2) Each of the following conditions implies (i)–(iv) of (1).

(a) $T\emptyset = \emptyset$;
(b) $T$ preserves the disjointness of some binary coproduct, that is, $TX \times_{T(X+Y)} TY = \emptyset$ for some $X, Y$;

c) $T$ preserves some binary coproduct;

d) $T$ preserves kernel pairs;

e) $T$ preserves kernel pairs weakly;

(f) $T$ satisfies BC;

(g) $T$ is the functor of a monad $\mathbb{T} = (T, m, e)$.

(3) Show that there is a functor $T : \textbf{Set} \to \textbf{Set}$, with $T\emptyset = \{\star\} + \{\star\}$ and $TX = X + \{\star\}$ for $X \neq \emptyset$, which does not preserve monomorphisms. But there are natural transformations $m : TT \to T$ and $e : 1_{\textbf{Set}} \to T$ with

$$m \cdot mT = m \cdot Tm \quad \text{and} \quad m \cdot eT = 1.$$

1.Q Units of monads and BC.

(1) The units of the two trivial monads on $\textbf{Set}$ (see Exercise II.3.A) do not satisfy BC.

(2) The units of the powerset and the filter monad do not satisfy BC. (Hint. Consider the map $!_X : X \to 1$ for $|X| \geq 2$.)

(3) The list monad (see Example II.3.1.1(2)) is cartesian, that is, its functor $L$ preserves pullbacks and every naturality square of the unit and of the multiplication is a pullback. The functor $L$ also preserves surjectivity of maps. In particular, $L$ and its monad multiplication satisfy BC.
2. Fundamental examples

In this Section, we present some of the motivating examples of lax algebras that will accompany us throughout this book. Further examples appear in Exercises 2.B and 2.D.

2.1 Ordered sets, metric spaces and probabilistic metric spaces. In Example 1.3.1(1), we saw that 2-categories with 2-functors (that is, (I, 2)-categories with (I, 2)-functors, where I is identically extended to Rel, see 1.6) were equivalently described as ordered sets with monotone maps:

\[(\mathcal{I}, 2)\text{-Cat} = 2\text{-Cat} = \text{Ord} \]

Similarly, P+-categories and P+-functors are the metric spaces with non-expansive maps of Example 1.3.1(2):

\[(\mathcal{I}, \mathbb{P}^+)\text{-Cat} = \mathbb{P}^+\text{-Cat} = \text{Met} \]

The quantale isomorphism between P+ and the unit interval with its multiplication:

\(([0, \infty]^{\text{op}}, +, 0) \cong ([0, 1], \cdot, 1)\]

(see Exercise 2.A) allows for a “probabilistic” interpretation of metric spaces. Say that a map \(\phi : [0, \infty] \to [0, 1]\) is a distance distribution if

\[
\phi(v) = \bigvee_{w < v} \phi(w)
\]

for all \(v \in [0, \infty]\); the convolution of two distance distribution maps \(\phi, \psi : [0, \infty] \to [0, 1]\) yields a distance distribution map \(\phi \otimes \psi : [0, \infty] \to [0, 1]\) via

\[
(\phi \otimes \psi)(u) = \bigvee_{v + w \leq u} \phi(v) \cdot \psi(w)
\]

for all \(u \in [0, \infty]\); with \(\kappa : [0, \infty] \to [0, 1]\) defined by \(\kappa(0) = 0\) and \(\kappa(u) = 1\) if \(0 < u\), the set \(D\) of all distribution maps ordered pointwise forms a quantale

\[D = (D, \otimes, \kappa)\]

A probabilistic metric is then a map \(a : X \times X \to D\) such that

\[a(x, x) = 1 \quad \text{and} \quad a(x, y) \otimes a(y, z) \leq a(x, z)\]

for all \(x, y, z \in X\). Here, the value \(p = a(x, y)(u)\) can be loosely interpreted as the “probability” that a given randomized metric \(\tilde{a} : X \times X \to [0, \infty]\) satisfies \(\tilde{a}(x, y) < u\). A set \(X\) with a probabilistic metric \(a : X \times X \to D\) forms a probabilistic metric space \((X, a)\), and a map \(f : X \to Y\) between probabilistic metric spaces \((X, a)\) and \((X, b)\) is probabilistically non-expansive if

\[a(x, y) \leq b(f(x), f(y))\]
for all \( x, y \in X \). Probabilistic metric spaces with probabilistically non-expansive maps are the objects and morphisms of the category \( \text{ProbMet} \) and one observes
\[
(\mathbb{I}, D)\text{-Cat} = D\text{-Cat} = \text{ProbMet}.
\]
In Example 3.5.2(2) below we exhibit full embeddings
\[
\text{Ord} \longrightarrow \text{Met} \longrightarrow \text{ProbMet}
\]
which make explicit how the structures discussed here generalize each other.

2.2 Topological spaces. Our paradigmatic example of a \((\mathbb{T}, \mathbb{V})\)-category comes from [Bar1970], in which topological spaces are presented as so-called relational algebras for the ultrafilter monad; in our context this result reads as
\[
(\mathbb{B}, 2)\text{-Cat} \cong \text{Top}.
\]
Recall from Examples II.3.1.1 that \( \mathbb{B} \) stands for the ultrafilter monad, and the required lax extension \( \mathbb{B} \) of \( \beta : \text{Set} \to \text{Set} \) to \( \text{Rel} \cong 2\text{-Rel} \) is described in Example 1.10.3(3). In the following, we will freely use the notations introduced in II.1.12 and II.1.13. There are many ways to prove the isomorphism mentioned above, depending in particular on the choice of the standard presentation of topological spaces: open or closed sets, interior or closure operations, or neighborhood systems. Here we work with closure operations, while in Chapter IV, with a more developed theory at our disposal, we will present another approach. The idea of the isomorphism between \((\mathbb{B}, 2)\)-categories and topological spaces is that a relation \( r : \beta X \to X \) represents convergence and specifies which ultrafilters converge to which points of \( X \). We can then associate with \( r \) a finitely additive closure operation \( c : PX \to PX \), and conversely, show that every finitely additive closure operation \( c \) determines a convergence relation \( r : \beta X \to X \).

In II.1.6 closure spaces were introduced as pairs \((X, c)\) consisting of a set \( X \) and a closure operation \( c : PX \to PX \) on the power set \( PX \) of \( X \). We observed that such a closure operation is a monotone map (or equivalently, an element of \( \text{Ord}(PX, PX) \)) carrying a monoid structure with respect to the compositional structure, that is,
\[
c \cdot c(A) \subseteq c(A), \quad A \subseteq c(A)
\]
for all \( A \subseteq X \). Furthermore, \( c : PX \to PX \) defines a topology on \( X \) if and only if \( c \) is finitely additive:
\[
c(A \cup B) = c(A) \cup c(B), \quad c(\emptyset) = \emptyset
\]
for all \( A, B \subseteq X \) (see Exercises II.1.F and II.1.G). To any relation \( r : \beta X \to X \), we can associate a finitely additive map \( \text{clos}(r) : PX \to PX \) given by
\[
\text{clos}(r)(A) = \{ y \in X \mid \exists \chi \in \beta X \ (A \in \chi \; \& \; \chi r y) \}.
\]
Conversely, given a map \( c : PX \to PX \), we define a relation \( \text{conv}(c) : \beta X \to X \) by setting
\[
\chi \text{ conv}(c) y \iff \forall A \in PX \ (A \in \chi \implies y \in c(A)) .
\]
Note that we have the identities
\[
\text{conv}(c)(A^\beta) = c(A) \quad \text{and} \quad \text{clos}(r)(A) = r(A^\beta) , \tag{2.2.i}
\]
for all \( A \subseteq X \), where \( A^\beta = \{ \chi \in \beta X \mid A \in \chi \} \), and
\[
r(A^\beta) = \{ x \in X \mid \exists \chi \in A^\beta (x \in r x) \} .
\]

2.2.1 Lemma. If \( \text{Set}(PX, PX) \) is equipped with the pointwise order, then the maps
\[
\text{clos} : \text{Rel}(\beta X, X) \to \text{Set}(PX, PX) \quad \text{and} \quad \text{conv} : \text{Set}(PX, PX) \to \text{Rel}(\beta X, X)
\]
form an adjunction \( \text{clos} \dashv \text{conv} \). The fixpoints of \( \text{clos} \cdot \text{conv} \) are the maps \( c : PX \to PX \) that are finitely additive.

Proof. Monotonicity of \( \text{clos} \) and \( \text{conv} \) follows immediately from the definitions. One also has
\[
1 \leq \text{conv} \cdot \text{clos} \quad \text{and} \quad \text{clos} \cdot \text{conv} \leq 1 .
\]
Indeed, for \( \chi \in \beta X \) and \( y \in X \),
\[
\chi \text{ r } y \implies \forall A \in PX \ (A \in \chi \implies \chi \text{ r } y) \implies \chi \text{ conv}(\text{clos}(r)) y
\]
and \( \chi \text{ conv}(c) y \) implies that \( y \in c(A) \) for all \( A \in \chi \); thus, for any \( A \in PX \), the set \( \text{clos}((\text{conv}(c))(A)) \) is given by
\[
\{ y \in X \mid \exists \chi \in \beta X (A \in \chi \& \forall B \in PX (B \in \chi \implies y \in c(B))) \} \subseteq c(A) . \tag{2.2.ii}
\]
In particular, for \( c : PX \to PX \) one necessarily has \( \text{clos} \cdot \text{conv}(c)(\emptyset) = \emptyset \) (since no ultrafilter \( \chi \in \beta X \) contains \( \emptyset \)), and
\[
\text{clos} \cdot \text{conv}(c)(A \cup B) = \text{clos} \cdot \text{conv}(c)(A) \cup \text{clos} \cdot \text{conv}(c)(B) ,
\]
as \( A \cup B \in \chi \) implies that either \( A \in \chi \) or \( B \in \chi \) (Lemma II.13.1), and the converse holds because \( \chi \) is an up-set. Thus, the fixpoints of \( \text{clos} \cdot \text{conv} \) must be finitely additive.

Consider now a map \( c : PX \to PX \) that is finitely additive (and therefore monotone) and \( A \in PX \) with \( y \in c(A) \). The set
\[
j := \{ J \in PX \mid y \notin c(J) \}
\]
is an ideal on \( X \). By Corollary II.13.5 (with the principal filter \( a = A \)), one obtains the existence of an ultrafilter \( \chi \in \beta X \) with \( \chi \supseteq A \) and \( \chi \cap j = \emptyset \); in other words, \( A \in \chi \) and \( y \in c(B) \) for all \( B \in \chi \), so one can conclude that \( c(A) = \text{clos} \cdot \text{conv}(c)(A) \) by (2.2.ii) above.

\[\square\]
A crucial observation is that both maps clos and conv are homomorphisms of the monoids $(\beta, 2)$-UREl($X, X$) and $\text{SLat}(PX, PX)$ whose operations are given by Kleisli convolution and map composition, respectively:

\[ \circlearrowleft \text{2.2.2 Proposition.} \text{ The maps conv and clos satisfy} \]
\[
\begin{align*}
\text{clos}(s \circ r) &= \text{clos}(s) \cdot \text{clos}(r), & \text{conv}(d \cdot c) &= \text{conv}(d) \circ \text{conv}(c), \\
\text{clos}(\varepsilon_X^c) &= 1_X, & \text{conv}(1_{PX}) &= e_X^c,
\end{align*}
\]

for all $(\beta, 2)$-relations $r, s : X \leftrightarrow X$ and finitely additive maps $c, d : PX \rightarrow PX$.

\[ \circlearrowleft \text{Proof.} \text{ For } \chi \in \beta X \text{ and } x \in X, \text{ if } \chi \text{ conv}(1_{PX}) x, \text{ then } x \in A \text{ for all } A \in \chi; \text{ thus,} \]
\[
\text{conv}(1_{PX}) = e_X^c.
\]

Since $1_{PX}$ is finitely additive, it is a fixpoint of clos $\cdot$ conv, so the previous equality yields $1_{PX} = \text{clos} \cdot \text{conv}(1_{PX}) = \text{clos}(e_X^c)$.

Consider now finitely additive maps $c, d : PX \rightarrow PX$, and let $\chi \in \beta X, z \in X$. Set also

\[ a := \uparrow_{PX}\{c(A) \mid A \in \chi\}, \]

which is a filter since $c$ is monotone. Then [2.2.1], together with Corollary II.1.13.3 tells us

\[ a \subseteq y \iff \exists X \in \beta X \left(m_X(X) = \chi \& X \text{ clos}\left(\text{conv}(c)\right) y\right) \]

for all $y \in \beta X$. Assume first that $\chi \text{ conv}(d \cdot c) z$. Then $a$ is disjoint from the ideal

\[ j = \{B \subseteq X \mid z \notin d(B)\}. \]

\[ \circlearrowleft \text{Applying Corollary II.1.13.5, we see that there is an ultrafilter } y \in \beta X \text{ containing } a \]

and disjoint from $j$. Hence $y \text{ conv}(d) z$, and there exists $X \in \beta \beta X$ with $X \text{ clos}\left(\text{conv}(c)\right) y$ and $m_X(X) = \chi$. We conclude that $\chi \text{ conv}(d \cdot c) z$. Assume now that $\chi \text{ conv}(d \cdot c) z$, so there is a $y \in \beta X$ such that $a \subseteq y$ and $y \text{ conv}(d) z$. From this we obtain $z \in d \cdot c(A)$ for every $A \in \chi$, that is, $\chi \text{ conv}(d \cdot c) z$.

Finally, let $r, s : X \leftrightarrow X$ be $(\beta, 2)$-relations, and consider $A \subseteq X, z \in X$. If $z \in \text{clos}(s \circ r)(A)$, then we have $X \in \beta \beta X$ and $y \in \beta X$ with

\[ A^\beta \subseteq X, \quad X \text{ clos}\left(r \circ s\right), \quad y \text{ s z} ; \]

and it follows that $z \in \text{clos}(s)(r(A^\beta)) = \text{clos}(s)(\text{clos}(r)(A))$. Thus, $\text{clos}(s \circ r) \leq \text{clos}(s) \cdot \text{clos}(r)$. For the other inequality, we use that $\text{conv}(d \cdot c) = \text{conv}(d) \circ \text{conv}(c)$ for fixpoints $c, d$ of clos $\cdot$ conv, and in particular for $c = \text{clos}(r), d = \text{clos}(s)$, and $c \cdot d$:

\[ \text{clos}(s) \cdot \text{clos}(r) = \text{clos} \cdot \text{conv}(\text{clos}(s) \cdot \text{clos}(r)) = \text{clos}(\text{conv}(\text{clos}(s) \circ \text{conv}(\text{clos}(r)))) = \text{clos}(s \circ r) \leq \text{clos}(s \circ r) \]

because $\text{conv} \cdot \text{clos} \leq 1$ (by Lemma 2.2.1).

\[ \circlearrowleft \text{2.2.3 Lemma.} \text{ Every } (\beta, 2) \text{-relation } r : X \leftrightarrow X \text{ satisfies conv } \cdot \text{clos}(r) = e_X^c \circ r. \]
Proof. On one hand,
\[ \text{conv}(\text{clos}(r)) = \text{conv}(1\text{clos}(r)) = e_X^c \circ \text{conv}(\text{clos}(r)) \geq e_X^c \circ r. \]

On the other hand, consider \( \chi \in \beta X \), \( y \in X \), and suppose that \( \chi \) \( \text{conv}(\text{clos}(r)) \) \( y \). Then for all \( A \in \chi \), we have \( y \in \text{clos}(r)(A) = r(A) \), which implies that there is an \( X \in \beta \beta X \) with \( m_X(x) = \chi \) and \( X (\beta r) \chi(y) \). Hence, \( \chi \) \( (e_X^c \circ r) a \), which shows that \( \text{conv}(\text{clos}(r)) \) \( e_X^c \circ r. \)

As a consequence, we obtain \( r = \text{conv} \cdot \text{clos}(r) \) for every left unitary \((\beta, 2)\)-relation \( r : X \leftrightarrow X \), and in particular for every reflexive and transitive \((\beta, 2)\)-relation.

2.2.4 Proposition. The fixpoints of the adjunction \( \text{clos} \dashv \text{conv} \) are on one hand the \((\beta, 2)\)-relations \( c : PX \to PX \), and on the other hand, the left unitary relations \( r : \beta X \leftrightarrow X \).

Proof. This follows from Lemmata 2.2.1 and 2.2.3.

2.2.5 Theorem. There is an isomorphism
\[ (\beta, 2)\text{-Cat} \cong \text{Top} \]
that commutes with the underlying-set functors.

Proof. By Proposition 2.2.4, \( \text{clos} \) and \( \text{conv} \) define a one-to-one correspondence between finitely additive maps \( c : PX \to PX \) and left unitary \((\beta, 2)\)-relations \( a : X \leftrightarrow X \). If \( c \) is a topological closure operation, so that \( 1_{PX} \leq c \) and \( c \cdot c = c \), then \( a := \text{conv}(c) \) satisfies
\[ e_X^c = \text{conv}(1_{PX}) \leq \text{conv}(c) = a \quad \text{and} \quad a \circ a = \text{conv}(c) \circ \text{conv}(c) = \text{conv}(c \cdot c) = \text{conv}(c) = a \]
by Proposition 2.2.2, that is, \((X, a : \beta X \leftrightarrow X)\) is a \((\beta, 2)\)-category. Likewise, from \( e_X^c \leq a \) and \( a \circ a = a \) one gets \( 1_{PX} \leq c \) and \( c \cdot c = c \) for \( c := \text{clos}(a) \), and therefore \( \text{clos} \) and \( \text{conv} \) actually define a bijective correspondence between \((\beta, 2)\)-categorical structures and topological closure operations on \( X \). An easy verification shows that a \((\beta, 2)\)-functor preserves the corresponding closure operation—and is therefore continuous—while a continuous map preserves the corresponding \((\beta, 2)\)-categorical structure—and is therefore a \((\beta, 2)\)-functor.

Spelled out, the previous Theorem states that a topological space \((X, \mathcal{O}X)\) can be equivalently described as a pair \((X, a)\), with \( a : \beta X \leftrightarrow X \) a relation representing convergence which, when we denote both \( a \) and \( \beta a \) by \( \rightarrow \), satisfies
\[ x \rightarrow y \& y \rightarrow z \Rightarrow \sum X \rightarrow z \quad \text{and} \quad x \rightarrow x, \]
for all \( z \in X \), \( y \in \beta X \), and \( X \in \beta \beta X \); here \( x \rightarrow y \Leftrightarrow x \supseteq a^c[y] \), and \( \sum \) is the Kowalsky sum restricted to ultrafilters (see II.1.12 and II.3.1.1(5)). In this context, the continuous maps \( f : (X, a) \leftrightarrow (X, b) \) are exactly the convergence-preserving maps, that is, the maps \( f : X \to Y \) such that
\[ \chi \rightarrow y \Rightarrow f[\chi] \rightarrow f(y) \]
for all \( y \in X \), and \( \chi \in \beta X \).
2.3 Compact Hausdorff spaces. Because the Barr extension of $\beta$ to $\text{Rel}$ is flat, it is possible to exploit Theorem 2.2.5 to obtain an elegant description of the category of $\beta$-algebras (that is, of Eilenberg–Moore algebras associated to $\beta$, see II.3.2).

Let us recall that a topological space $X$ is compact if every open cover of $X$ has a finite subcover, that is, for every open cover $\mathcal{A}$ there exists a finite subset $\mathcal{F} \subseteq \mathcal{A}$ with $\bigcup \mathcal{F} = X$.

In the following results we freely exploit that a topological space $X$ can equivalently be described via a set $\mathcal{O}X$ of open sets, a closure operation $c : PX \to PX$, or a convergence relation $a : \beta X \to X$ (Exercises II.1.F, II.1.G and Theorem 2.2.5). In fact, we can avoid the closure operation by translating the definition of the convergence relation given in 2.2 as

$$\chi \rightarrow x \iff \forall A \in \mathcal{O}X \ (x \in A \implies A \in \chi)$$  \hspace{1cm} (2.3.i)

(this easily follows from Lemma II.1.13.1 and $\mathcal{O}X = \{(c(B))^\uparrow \mid B \in PX\}$).

© 2.3.1 Proposition. The following statements are equivalent for a topological space $X$:

(i) $X$ is compact;

(ii) every ultrafilter on $X$ converges, that is, $1_{\beta X} \leq a^\circ \cdot a$.

Proof. Assume first that $X$ is compact. Let $\chi \in \beta X$ and set

$$\mathcal{A} = \{A \subseteq X \mid A \text{ open}, A \notin \chi\}.$$ 

The set $\mathcal{A}$ cannot cover $X$ since otherwise there would exist $A_1, \ldots, A_n \in \mathcal{A}$ with $A_1 \cup \ldots \cup A_n = X \in \chi$, and therefore $A_k \in \chi$ for some $k \in \{1, \ldots, n\}$ by Lemma II.1.13.1 contradicting the definition of $\mathcal{A}$. Thus, there exists $x \in X$ such that every open set $A$ with $x \in A$ belongs to $\chi$. This implies that $\chi \rightarrow x$ by (2.3.i).

Assume now that any ultrafilter $\chi \in \beta X$ converges. Let $\mathcal{A}$ be a set of open subsets of $X$ with the property that no finite subset of $\mathcal{A}$ covers $X$. Then

$$f = \uparrow_{PX}\{(\bigcup \mathcal{F})^\uparrow \mid \mathcal{F} \subseteq \mathcal{A}, \mathcal{F} \text{ finite}\}$$

is a proper filter on $X$, and therefore contained in an ultrafilter $\chi$ (Proposition II.1.13.2) that converges to some $x \in X$ by hypothesis. Hence $x \notin A$ for any $A \in \mathcal{A}$ (by (2.3.i) and the definition of $\chi \supseteq f$), that is, $\mathcal{A}$ does not cover $X$. \hfill $\Box$

© 2.3.2 Proposition. The following statements are equivalent for a topological space $X$:

(i) $X$ is Hausdorff;

(ii) every ultrafilter on $X$ has at most one convergence point, that is, $a \cdot a^\circ \leq 1_X$. 

Proof. Assume that $X$ is Hausdorff and let $\chi \in \beta X$ and $x, y \in X$ with $\chi \rightarrow x$ and $\chi \rightarrow y$. By [2.3.i], for all open subsets $A \subset X$ and $B \subset X$ with $x \in A$ and $y \in B$ one has $A \cap B \neq \emptyset$, which is possible only if $x = y$.

Assume now that every ultrafilter on $X$ converges to at most one point, and let $x, y \in X$. If every open neighborhood of $x$ intersects every open neighborhood of $y$, then there exists an ultrafilter $\chi$ containing all neighborhoods of $x$ and all neighborhoods of $y$, hence converging to both $x$ and $y$, which, by hypothesis, implies $x = y$. □

2.3.3 Theorem. There is an isomorphism

\[ \text{Set}^{\beta} \cong \text{CompHaus} \]

that commutes with the respective forgetful functors to $\text{Set}$.

Proof. Combining both propositions above, we see that a topological space $X$ is compact Hausdorff if and only if its convergence relation $a : \beta X \rightarrow X$ is actually a map $a : \beta X \rightarrow X$, which therefore satisfies $a \cdot \beta a = a \cdot m_X$. (Since $\beta a = \beta a$ because $a$ is a map, the condition $a \cdot \beta a \leq a \cdot m_X$ is an inclusion of graphs of functions with same domain, hence necessarily an equality.) Furthermore, a continuous map $f : X \rightarrow Y$ between compact Hausdorff spaces with convergence structures $a : \beta X \rightarrow X$ and $b : \beta Y \rightarrow Y$ respectively satisfies $f \cdot a \leq b \cdot Tf$, but since all relations involved are functions, this condition is actually an equality. □

2.4 Approach spaces. Approach spaces provide a common framework for both topological and metric structures. More precisely, an approach space is a pair $(X, \delta)$ consisting of a set $X$ and a function $\delta : X \times PX \rightarrow [0, \infty]$ subject to the conditions:

1. $\delta(x, \{x\}) = 0$,
2. $\delta(x, \emptyset) = \infty$,
3. $\delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}$,
4. $\delta(x, A) \leq \delta(x, A^{(u)}) + u$,

for all $x \in X$, $A, B \subseteq X$, $u \in [0, \infty]$, and where

$A^{(u)} = A^{(u)}_\delta = \{x \in X \mid \delta(x, A) \leq u\}$.

A function $\delta : X \times PX \rightarrow [0, \infty]$ satisfying the axioms (1)–(4) is called an approach distance on $X$. A morphism $f : (X, \delta) \rightarrow (Y, \delta')$ of approach spaces is given by a non-expansive map $f : X \rightarrow Y$, so that $f$ satisfies

$\delta'(f(x), f(A)) \leq \delta(x, A)$

for all $x \in X$ and $A \subseteq X$. Approach spaces and non-expansive maps are the objects and morphisms of the category $\text{App}$. 

2.4.1 Examples.

(1) Every metric space \((X, a)\) (in the sense of Example 1.3.1(2)) becomes an approach space \((X, \delta)\) when one puts
\[
\delta(y, A) := \inf\{a(x, y) \mid x \in A\}.
\]
In fact, this defines a full embedding
\[
\text{Met} \hookrightarrow \text{App}
\]
that exhibits \text{Met} as a coreflective subcategory of \text{App} (this follows from Proposition 3.4.2 together with Theorem 2.4.5 below).

(2) Every topological space \((X, c)\) (as in Exercises II.1.F and II.1.G) becomes an approach space \((X, \delta)\) when one sets
\[
\delta(y, A) := \begin{cases} 
0 & \text{if } y \in c(A), \\
\infty & \text{otherwise}.
\end{cases}
\]
In this way, one obtains a full embedding
\[
\text{Top} \hookrightarrow \text{App}
\]
that exhibits \text{Top} as a coreflective subcategory of \text{App} (see 3.6 together with Theorems 2.2.5 and 2.4.5 below).

We now show that approach spaces can be seen as “numerified topological spaces”, that is, we prove the existence of an isomorphism
\[
\mathsf{App} \cong (\mathcal{P}, \mathbb{P})-\text{Cat}
\]
for a suitable extension \(\mathcal{P} : \mathbb{P}-\text{Rel} \to \mathbb{P}-\text{Rel}\) of \(\beta : \mathsf{Set} \to \mathsf{Set}\) (defined below). In order to keep the proof similar to the one for topological spaces as presented in 2.2, we think of maps
\[
\delta : X \times PX \to [0, \infty]
\]
as of morphisms “from \(X\) to \(X\)”. Given also \(\gamma : X \times PX \to [0, \infty]\), we define the composite \(\gamma \cdot \delta : X \times PX \to [0, \infty]\) by
\[
\gamma \cdot \delta(z, A) := \inf\{\gamma(z, A^{(u)}) + u \mid u \in [0, \infty]\},
\]
for all \(A \subseteq X\) and \(z \in X\). One easily verifies that this composition preserves the order in both variables, and that the map \(\varepsilon_X : X \times PX \to [0, \infty]\), sending \((x, A)\) to 0 if \(x \in A\) and to \(\infty\) otherwise, is neutral with respect to this composition. We say that \(\delta : X \times PX \to \mathbb{P}\) is finitely additive if
\[
\delta(y, \emptyset) = \infty \quad \text{and} \quad \delta(y, A \cup B) = \min\{\delta(y, A), \delta(y, B)\},
\]
for all \( y \in X \) and \( A, B \in PX \). With this notation, \( \delta : X \times PX \to [0, \infty) \) is an approach distance if and only if \( \delta \) is finitely additive and
\[
\delta \cdot \delta \geq \delta, \quad \varepsilon_X \geq \delta.
\]
The former inequality is actually an equality thanks to the latter.

We saw in 2.2 that \( \beta : \text{Set} \to \text{Set} \) can be extended to a 2-functor \( \beta : \text{Rel} \to \text{Rel} \) by setting
\[
\chi (\beta r) y \iff r(\chi) \subseteq y \quad \text{for every relation } r : X \to Y.
\]
We now go one step further and extend the ultrafilter monad \( \beta \) to a lax monad \((\beta, m, e)\) on \( \mathcal{P}_+ - \text{Rel} \) as follows. Let us recall from \( \textbf{I} \) II.10 that \( \leq \) always refers to the natural order of \([0, \infty]\). For \( r : X \times Y \to \mathcal{P}_+ \) and \( v \in \mathcal{P}_+ \), we consider the relation \( r_v : X \to Y \) defined by
\[
x r_v y \iff r(x, y) \leq v.
\]
Given another \( \mathcal{P}_+\)-relation \( s : Y \to Z \) and \( u \in \mathcal{P}_+ \), one easily verifies \( s_u \cdot r_v \leq (s \cdot r)_{u+ v} \). We now consider the Barr extension of \( \beta \) to \( \mathcal{P}_+ - \text{Rel} \):
\[
\overline{\beta r} := \overline{\beta}_+ r : \beta X \times \beta Y \to \mathcal{P}_+
\]
\[
(\chi, y) \mapsto \inf \{ v \in \mathcal{P}_+ \mid \chi \overline{\beta} (r_v) y \},
\]
which can be equivalently expressed by
\[
\overline{\beta r}(\chi, y) = \sup_{A \in \chi, B \in y} \inf_{x \in A, y \in B} r(x, y),
\]
for all \( \mathcal{P}_+\)-relations \( r : X \to Y \), \( \chi \in \beta X \), and \( y \in \beta Y \) (see Exercise 2.8). Furthermore, \( \overline{\beta} (r_u) \leq (\overline{\beta} r)_u \) for any \( u \in \mathcal{P}_+ \); and \( (\overline{\beta} r)_u \leq \overline{\beta} (r_v) \) whenever \( u < v \) in \( \mathcal{P}_+ \).

\subsection*{2.4.2 Lemma}
Let \( r : X \to Y \) be a \( \mathcal{P}_+\)-relation, \( f \) a filter on \( X \) and \( y \in \beta Y \). Then there exists an ultrafilter \( \chi \in \beta X \) such that \( f \subseteq \chi \) and
\[
\overline{\beta} r(\chi, y) = \sup_{A \in \chi, B \in y} \inf_{x \in A, y \in B} r(x, y).
\]

\textbf{Proof.} Certainly, for every ultrafilter \( \chi \in \beta X \) with \( f \subseteq \chi \) we have
\[
\overline{\beta} r(\chi, y) \geq \sup_{A \in \chi, B \in y} \inf_{x \in A, y \in B} r(x, y).
\]
Let us set \( v := \sup_{A \in \chi, B \in y} \inf_{x \in A, y \in B} r(x, y) \). If \( v = \infty \), then any \( \chi \supseteq f \) has the desired property. If \( v < \infty \), we consider
\[
j = \{ A \subseteq X \mid \sup_{B \in y} \inf_{x \in A, y \in B} r(x, y) > v \}.
\]
Of course, \( f \cap j = \emptyset \), and it is not hard to see that \( j \) is an ideal. By Corollary \( \textbf{I} \) II.13.5 there exists an ultrafilter \( \chi \in \beta X \) with \( f \subseteq \chi \) and \( \chi \cap j = \emptyset \), and therefore \( \overline{\beta} r(\chi, y) \leq v \).

\subsection*{2.4.3 Proposition}
The Barr extension \( \overline{\beta} = (\overline{\beta}, m, e) \) is a flat associative lax extension to \( \mathcal{P}_+ - \text{Rel} \) of the ultrafilter monad \( \beta = (\beta, m, e) \). Moreover, \( \overline{\beta} (r^o) = (\overline{\beta} r)^o \) for every \( \mathcal{P}_+\)-relation \( r \).
Proof. We show only
\[ \bar{\beta}s \cdot \bar{\beta}r = \bar{\beta}(s \cdot r) \quad \text{and} \quad m_Y \cdot \bar{\beta}r = \bar{\beta}r \cdot m_X, \]
for all \( \mathbb{P}_+ \)-relations \( r : X \to Y \) and \( s : Y \to Z \), as the other verifications are straightforward. In order to show the inequality \( \bar{\beta}s \cdot \bar{\beta}r \geq \bar{\beta}(s \cdot r) \), assume \( \bar{\beta}s \cdot \bar{\beta}r(x, z) < u \) for \( x \in \beta X \) and \( z \in \beta Z \). Therefore, there is some \( y \in \beta Y \) satisfying
\[ \bar{\beta}r(x, y) + \bar{\beta}s(y, z) < u. \]
Consider \( u_1, u_2 \in \mathbb{P}_+ \) such that
\[ \bar{\beta}r(x, y) < u_1, \quad \bar{\beta}s(y, z) < u_2, \quad u_1 + u_2 = u. \]
Hence, we have \( x \in \beta B(r_{u_1}) \) \( y \in \beta B(s_{u_2}) \) \( z \), so that \( x \in \beta B(s_{u_2} \cdot r_{u_1}) z \). Since \( s_{u_2} \cdot r_{u_1} \leq (s \cdot r)_{u_1+u_2} \), we conclude \( \bar{\beta}(s \cdot r)(x, z) \leq u_1 + u_2 = u. \)
To see \( \bar{\beta}s \cdot \bar{\beta}r \leq \bar{\beta}(s \cdot r) \), let \( x \in \beta X, z \in \beta Z \) and \( u \in \mathbb{P}_+ \) with
\[ u > \bar{\beta}(s \cdot r)(x, z) = \sup_{A \in X, C \in \mathbb{P}_+} \inf_{x \in A, z \in C} s \cdot r(x, z). \]
Hence, for every \( A \in \chi \) and every \( C \in z \), there exist \( x \in A, y \in Y \) and \( z \in C \) with \( r(x, y) + s(y, z) \leq u \), that is
\[ B_{A,C} := \{ y \in Y \mid \exists x \in A, z \in C : r(x, y) + s(y, z) \leq u \} \neq \emptyset. \]
Since \( B_{A \cap A', C \cap C'} \subseteq B_{A,C} \cap B_{A',C'} \), the set
\[ \{ B_{A,C} \mid A \in \chi, C \in z \} \]
\( \odot \) is a filter base; let \( y \in \beta Y \) be any ultrafilter containing it. Then
\[ \bar{\beta}r(x, y) = \sup_{A \in \chi, B \in y} \inf_{x \in A, y \in B} s \cdot r(x, y) \leq u \]
because for any \( A \in \chi \) and \( B \in y \), one has \( B \cap B_{A,Z} \neq \emptyset \). Similarly, \( \beta s(y, z) \leq u \). Consider \( \varepsilon > 0 \) and set \( u_0 = \beta r(x, y) \); if \( u_0 = 0 \), we are done, so we can assume \( u_0 > 0 \). Thus, there is some \( A \in \chi \) with \( r_{u_0-\varepsilon}[A] \notin y \), and therefore its complement
\[ B_0 := \{ y \in Y \mid \forall x \in A : r(x, y) > u_0 - \varepsilon \} \]
belongs to \( y \). We show that \( \beta s(y, z) \leq (u - u_0) + \varepsilon. \) To this end, let \( B \in y \) and \( C \in z \). Then \( B \cap B_0 \cap B_{A,C} \neq \emptyset \), which implies that there are \( x \in A, y \in B \) and \( z \in C \) with
\[ r(x, y) + s(y, z) \leq u \quad \text{and} \quad r(x, y) > u_0 - \varepsilon, \]
therefore \( s(y, z) \leq (u - u_0) + \varepsilon. \) Consequently,
\[ \beta s(y, z) \leq \sup_{B \in y, C \in \mathbb{P}_+} \inf_{y \in B, z \in C} s(y, z) \leq (u - u_0) + \varepsilon. \]
To verify $m_Y \cdot \beta r \geq \beta r \cdot m_X$, let $u, u' \in P_+$ and $X \in \beta \beta X$, $Y \in \beta \beta Y$ be such that

$$\beta r(X, Y) < u < u'.$$

Hence, $X \beta r(u)$ and therefore $X \beta r(u')$. This implies $m_X(X) \beta (r(u)) m_Y(Y)$, that is, $\beta r(m_X(X), m_Y(Y)) \leq u'$.

Finally, we show $m_Y \cdot \beta r \leq \beta r \cdot m_X$. Let $X \in \beta \beta X$ and $y \in \beta X$, and assume

$$\beta r(m_X(X), y) < u'' < u' < u.$$

Since the Barr extension of $\beta$ to $\text{Rel}$ is associative, there is some $Y \in \beta \beta Y$ with $X \beta r(u') \rightarrow Y$, so that one has $X \beta r(u') \rightarrow Y$ and $X \beta r(u') \rightarrow Y$, that is, $\beta r(X, Y) \leq u$.

Finally, since $\beta$ commutes with the involution on $P_+ - \text{Rel}$, $m^\circ : \beta \rightarrow \beta$ is a natural transformation too (see also Exercise 1.4.4). Proposition 1.9.4 then yields that $\beta$ is associative.

Every $P_+ - \text{relation } r : \beta X \times X \rightarrow P_+$ defines a finitely additive function

$$\text{clos}(r) : X \times PX \rightarrow [0, \infty]$$

$$(y, A) \mapsto \inf \{ r(x, y) \mid x \in \beta A \},$$

and every function $\delta : X \times PX \rightarrow [0, \infty]$ yields a $P_+ - \text{relation }$ $\text{conv}(\delta) : \beta X \times X \rightarrow P_+$

$$(x, y) \mapsto \sup \{ \delta(y, A) \mid A \in x \}.$$}

As in 2.2, the following proposition contains the key facts about $\text{clos}$ and $\text{conv}$.

**2.4.4 Proposition.** The operations $\text{conv}$ and $\text{clos}$ satisfy

$$\text{clos}(s \circ r) = \text{clos}(s) \cdot \text{clos}(r), \quad \text{conv}(\gamma \cdot \delta) = \text{conv}(\gamma) \circ \text{conv}(\delta)$$

$$\text{clos}(e_X^\circ) = e_X, \quad \text{conv}(e_X) = e_X^\circ$$

for all finitely additive functions $\delta, \gamma : X \times PX \rightarrow [0, \infty]$, and $(\beta, P_+)$-relations $r, s : X \leftrightarrow X$.

**Proof.** We show only $\text{clos}(s \circ r) = \text{clos}(s) \cdot \text{clos}(r)$, as the verification of the other statements are either very similar or straightforward.

Assume first that $\text{clos}(s) \cdot \text{clos}(r)(z, A) < v$, with $z \in Z$, $A \subseteq X$ and $v \in [0, \infty]$. Then, for some $u \in [0, \infty]$,

$$\text{clos}(s)(z, A_{\text{clos}(r)}^{(u)}) + u < v.$$

Consider $u' \in P_+$ such that

$$\text{clos}(s)(z, A_{\text{clos}(r)}^{(u)}) < u' \quad \text{and} \quad u' + u = v.$$
Hence, there exists \( y \in \beta X \) with \( A_{\text{clos}(r)}^{(u)} \in y \) and \( r(y, z) < u' \). For every \( B \in y \), there is an element \( y \in B \) satisfying

\[
\inf_{y \in B} r(y, y) = \text{clos}(r)(y, A) \leq u .
\]

Therefore, we obtain

\[
\sup_{B \in y} \inf_{y \in B} \inf_{y \in B} r(y, y) \leq u ,
\]

and Lemma 2.4.2 guarantees the existence of some \( X \in \beta \beta X \) with \( A^{\beta} = \{ x \in \beta X \mid A \in x \} \in X \) and \( \beta r(X, y) \leq u \). Since

\[
A \in m_X(x) \quad \text{and} \quad s \circ r(m_X(x), z) \leq u' + u = v ,
\]

we deduce \( \text{clos}(s \circ r)(z, A) \leq v \).

Assume now \( \text{clos}(s \circ r)(z, A) < v \). Hence, for some \( \chi \in \beta A \), we have \( s \circ r(\chi, z) < v \), and there exist \( X \in \beta \beta X \), \( y \in \beta X \) with

\[
m_X(x) = \chi \quad \text{and} \quad \beta r(X, y) + s(y, z) < v .
\]

Let \( u \in [0, \infty] \) with \( \beta r(X, y) < u \) and \( u + s(y, z) = v \). We have

\[
A_{\text{clos}(r)}^{(u)} \supseteq r_u(A^{\beta}) \in y ,
\]

and therefore \( \text{clos}(s)(z, A_{\text{clos}(r)}^{(u)}) + u \leq v \).

From the identities

\[
\text{clos}(\text{conv}(\delta))(y, A) = \inf_{x \in \beta A} \text{conv}(\delta)(x, y) = \inf_{x \in \beta A} \sup_{B \in x} \delta(y, B) ,
\]

we obtain \( \delta(x, A) \leq \text{clos}(\text{conv}(\delta))(x, A) \), for all \( \delta : X \times PX \to [0, \infty] \), \( A \subseteq X \) and \( y \in X \). Moreover, equality holds if \( \delta(y, A) = \infty \). If \( \delta(y, A) < \infty \), we can consider

\[
j = \{ B \subseteq X \mid \delta(y, B) > \delta(y, A) \} ;
\]

when \( \delta \) is finitely additive, \( j \) is an ideal with \( A \notin j \), and there exists an ultrafilter \( \chi \in \beta X \) with \( A \in \chi \) and \( \chi \cap j = \emptyset \). Therefore, \( \delta = \text{clos}(\text{conv}(\delta)) \) if and only if \( \delta \) is finitely additive. On the other hand,

\[
\text{conv}(\text{clos}(r))(\chi, y) = \sup_{A \in \chi} \text{clos}(r)(y, A) = \sup_{A \in \chi} \inf_{y \in \beta A} r(y, y) ,
\]

so that \( \text{conv}(\text{clos}(r)) = e_X^* \circ r \) (see Exercise 2.4), for every \( P_+ \)-relation \( r : \beta X \to X \). Hence, \( r = \text{conv}(\text{clos}(r)) \) if and only if \( r \) is unitary.

\[\textcircled{2.4.5 \text{Theorem.}} \text{There is an isomorphism } (\beta, P_+)-\text{Cat} \cong \text{App} \text{ that commutes with the underlying-set functors.}\]
Proof. To every approach space \((X, \delta)\) we associate the \((\beta, P_+)\)-category \((X, \text{conv}(\delta))\), and to every \((\beta, P_+)\)-category \((X, r)\) corresponds the approach space \((X, \text{clos}(r))\). Then non-expansive maps preserve the corresponding \((\beta, P_+)\)-categorical structure, and \((\beta, P_+)\)-functors become non-expansive. All said, we obtain functors
\[
\text{App} \rightarrow (\beta, P_+)\text{-Cat} \quad \text{and} \quad (\beta, P_+)\text{-Cat} \rightarrow \text{App}
\]
that are inverse to one another. \(\square\)

Thus, as in the case of topological spaces, we obtain an alternate description of approach spaces. Here, the relational arrow for a convergence relation \(\rightarrow\) is replaced by a numerified “degree of convergence”, so that an approach space can be seen as a pair \((X, a)\), with \(a: \beta X \times X \rightarrow [0, \infty]\) a map satisfying
\[
\overline{\beta}a(X, y) + a(y, z) \geq a(\sum X, z) \quad \text{and} \quad a(\dot{x}, x) = 0,
\]
for all \(z \in X, y \in \beta X\), and \(X \in \beta \beta X\), and where
\[
\overline{\beta}a(X, y) = \sup_{A \in X, B \in y} \inf_{x \in A, y \in B} a(x, y).
\]

With this description, the non-expansive maps \(f: (X, a) \rightarrow (Y, b)\) between approach spaces are those satisfying
\[
a(x, y) \geq b(f[x], f(y))
\]
for all \(y \in X\), and \(x \in \beta X\), that is, the maps that improve the “degree of convergence”. The embedding \(\text{Top} \hookrightarrow \text{App}\) of 2.4.1(2) now describes topological spaces as those approach spaces whose measure of convergence \(a: \beta X \times X \rightarrow [0, \infty]\) takes its values in \(\{0, \infty\}\).

Hence, ultrafilter convergence in \(X\) is defined by
\[
x \rightarrow y \iff a(x, y) = 0
\]
for all \(x \in \beta X, y \in X\).

2.5 Closure spaces. Example 1.6.4(2) shows that when the powerset monad \(P\) is equipped with the lax extension \(\hat{P}: \text{Rel} \rightarrow \text{Rel}\)
\[
A (\hat{P}r) B \iff B \subseteq r(A), \quad (2.5.i)
\]
(for all \(A \in PX, B \in PY\), and relations \(r: X \rightarrow Y\)), \((P, 2)\)-category structures \(a: PX \rightarrow X\) are in one-to-one correspondence with a closure operation \(c: PX \rightarrow PX\) via
\[
x \in c(A) \iff A a x
\]
for all \(A \in PX, x \in X\). Under this correspondence, a \((P, 2)\)-functor \(f: (X, a) \rightarrow (X, b)\) is equivalently described as a continuous map \(f: (X, c_X) \rightarrow (Y, c_Y)\), and one obtains an isomorphism
\[
(P, 2)\text{-Cat} \cong \text{Cls}
\]
that commutes with the underlying-set functors. The extension formula (2.5.i) can also be used with the finite-powerset monad \( P_{\text{fin}} \) whose functor \( P_{\text{fin}} : \text{Set} \to \text{Set} \) sends a set \( X \) to its set \( P_{\text{fin}} \) of finite subsets; the multiplication and unit of the monad are just the appropriate restrictions of those of \( P \). By the same procedure as in the powerset case, one can identify a \((P_{\text{fin}},2)\)-category with a finitary closure space (also called an algebraic closure space), that is, a closure space \((X,c)\) whose closure operation \( c : PX \to PX \) is finitary:

\[
c(A) = \bigcup_{B \in P_{\text{fin}} A} c(B) .
\]

Denoting by \( \text{Cls}_{\text{fin}} \) the full subcategory of \( \text{Cls} \) whose objects are finitary closure spaces, one obtains an isomorphism

\[
(P_{\text{fin}},2)\text{-Cat} \cong \text{Cls}_{\text{fin}} .
\]

that commutes with the underlying-set functors.

**Exercises**

2.A *The probabilistic and metric quantales.* The unit interval \([0,1]\) with its natural order and multiplication yields a quantale

\[
([0,1], \cdot, 1)
\]

that is isomorphic to the quantale \( P_+ = ([0,\infty]^\text{op}, +, 0) \).

2.B *Ultrametric spaces.* An ultrametric is a map \( a : X \times X \to [0,\infty] \) satisfying

\[
\max\{a(x,y), a(y,z)\} \geq a(x,z) \quad \text{and} \quad 0 = a(x,x)
\]

for all \( x, y, z \in X \), and an ultrametric space is a pair \((X,a)\) composed of a set \( X \) and an ultrametric \( a : X \times X \to [0,\infty] \); a non-expansive map \( f : (X,a) \to (X,b) \) is, as in 1.3.1(2), a map \( f : X \to Y \) such that

\[
a(x,y) \geq b(f(x), f(y)) .
\]

With the quantale \( P_{\text{max}} = ([0,\infty]^\text{op}, \max, 0) \) (see II.10.1(3)), the category \( \text{UltraMet} \) of ultrametric spaces with non-expansive maps is the category of \( P_{\text{max}} \)-categories and \( P_{\text{max}} \)-functors:

\[
(\text{Id}, P_{\text{max}})\text{-Cat} = P_{\text{max}}\text{-Cat} = \text{UltraMet} .
\]

© 2.C *The underlying order via ultrafilter convergence.* Use Exercise II.1.F and the correspondence between convergence of ultrafilters and closure operations of Theorem 2.4.5 to show that the underlying order of a \((\beta,2)\)-category \((X,a)\) is given by

\[
x \leq y \iff \hat{x} \to y
\]

for all \( x, y \in X \).
2.D Bitopological spaces. When $V = 2^2$ is the diamond lattice of Exercise II.H, the Barr extension to $2^2$-Rel of the ultrafilter functor $\beta$ is defined by

$$\overline{\beta} r(\chi, y) = \bigvee \{ w \in 2^2 \mid \chi \overline{\beta} r_w \} = \bigwedge_{x \in \xi, y \in \eta} \bigvee_{x \in A, y \in B} r(x, y),$$

for all $2^2$-relations $r : X \rightarrow Y$, $\chi \in \beta X$, and $y \in \beta Y$. Verify that this yields a flat lax extension of the ultrafilter monad $\beta$ from $\text{Set}$ to $2^2$-Rel.

A bitopological space is a triple $(X, O_1, O_2)$, with $(X, O_1)$ and $(X, O_2)$ topological spaces, and a bicontinuous map $f : (X, O_1, O_2) \rightarrow (Y, O'_1, O'_2)$ between bitopological spaces is a map $f : X \rightarrow Y$ that is continuous as $f : (X, O_1) \rightarrow (Y, O'_1)$ and $f : (X, O_2) \rightarrow (Y, O'_2)$. Show that the category of $(\beta, 2^2)$-categories obtained from the previous Barr extension is isomorphic to the category $\text{Bitop}$ of bitopological spaces with bicontinuous maps:

$$\beta, 2^2 \text{-Cat} \cong \text{BiTop}.$$  

2.E An alternative description of $\overline{\beta} : P_+\text{-Rel} \rightarrow P_+\text{-Rel}$. The Barr extension to $P_+\text{-Rel}$ of the ultrafilter functor (see Proposition 2.4.3) can be equivalently expressed as

$$\overline{\beta} r(\chi, y) = \sup_{x \in X, y \in Y} \inf_{x \in A, y \in B} r(x, y),$$

for every $P_+$-relation $r : X \rightarrow Y$, $\chi \in \beta X$ and $y \in \beta Y$.

2.F Unitary $(\beta, P_+)$-relations. Show

$$e_X \circ r(\chi, y) = \sup_{x \in X, y \in Y} \inf_{\beta A} r(y, y),$$

for every $(\beta, P_+)$-relation $r : X \rightarrow X$ (recall from 2.2 that $A^\beta = \{ y \in \beta X \mid A \in y \}$).

2.G Metric closure spaces. A metric closure space is a pair $(X, c)$ composed of a set $X$ and a metric closure operation $c$, that is, a map $c : X \times PX \rightarrow [0, \infty]$ such that

1. $c(x, A) = 0$,
2. $A \subseteq B \implies c(x, B) \leq c(x, A),$
3. $c(x, A) \leq c(x, A(u)) + u,$

for all $x \in X$, $A, B \in PX$, $u \in [0, \infty]$, and where $A^{(u)} = \{ x \in X \mid c(x, A) \leq u \}$. A morphism $f : (X, c) \rightarrow (Y, c')$ of metric closure spaces is a non-expansive map, that is, a map $f : X \rightarrow Y$ that satisfies

$$c'(f(x), f(A)) \leq c(x, A)$$

for all $x \in X$ and $A \subseteq X$. Metric closure spaces with non-expansive maps form the category $\text{MetCls}$. By equipping the powerset monad $P$ with the lax extension given by

$$\hat{P} r : PX \times PY \rightarrow P_+ : (A, B) \mapsto \sup_{x \in A} \inf_{y \in B} r(x, y),$$
(for all $x \in X$, $y \in Y$, and $\mathcal{V}$-relations $r : X \rightarrowtail Y$), one obtains an isomorphism

$$(\mathcal{P}, \mathcal{P}_+)-\text{Cat} \cong \text{MetCls}.$$ 

**2.H** *An alternative description of approach spaces.* For a map $\delta : X \times PX \rightarrow [0, \infty]$ satisfying conditions (1)–(3) of 2.4, condition (4) is equivalent to

$$\sup_{x \in X} |\delta(x, A) - \delta(x, B)| \leq \min\{\sup_{y \in A} \delta(y, B), \sup_{z \in B} \delta(z, A)\}$$

for all $A, B \subseteq X$. 
3. Categories of lax algebras

In this subsection we prove topologicity of the underlying-set functor \((\mathbb{T}, \mathcal{V})\text{-Cat} \rightarrow \text{Set}\) and describe how \((\mathbb{T}, \mathcal{V})\text{-Cat}\) varies functorially under appropriate changes of the parameters \(\mathbb{T}\) and \(\mathcal{V}\), where \(\mathcal{V}\) is a quantale and \(\mathbb{T}\) a monad on \(\text{Set}\) with a lax extension \(\hat{\mathbb{T}}\) to \(\mathcal{V}\text{-Rel}\).

3.1 Initial structures. Our first goal is to show that the forgetful functor \(O : (\mathbb{T}, \mathcal{V})\text{-Cat} \rightarrow \text{Set}\) is topological (see \[1.5.8\]). It is in fact easy to give an explicit description of the \(O\)-initial lifting of sources, as follows.

3.1.1 Proposition. The \(O\)-initial lifting of a source \((f_i : X \rightarrow Y_i)_{i \in I}\), where \((Y_i, b_i)\) is a family of \((\mathbb{T}, \mathcal{V})\)-categories, is provided by the structure \(a := \bigwedge_{i \in I} f_i^\circ \cdot b_i \cdot T f_i\) on the set \(X\) or, in pointwise notation, by

\[
a(\chi, y) = \bigwedge_{i \in I} b_i(T f_i(\chi), f_i(y))
\]

for all \(\chi \in TX\) and \(y \in X\).

Proof. Let us first verify that the \(\mathcal{V}\)-relation \(a : TX \rightarrow X\) is reflexive and transitive. Since each \(b_i\) is reflexive, one has

\[
1_X \leq f_i^\circ \cdot f_i \leq f_i^\circ \cdot b_i \cdot e_{Y_i} \cdot f_i = f_i^\circ \cdot b_i \cdot T f_i \cdot e_X,
\]

so that \(1_X \leq a \cdot e_X\) by taking the infimum on all \(i \in I\). One also observes

\[
a \cdot \hat{T}a \leq \bigwedge_{i \in I}(f_i^\circ \cdot b_i \cdot T f_i) \cdot \bigwedge_{j \in J} \hat{T}(f_j^\circ \cdot b_j \cdot T f_j) \quad (\hat{T}\text{ monotone on hom-sets})
\]

\[
\leq \bigwedge_{i \in I,j \in J} f_i^\circ \cdot b_i \cdot T f_i \cdot (T f_j)^\circ \cdot \hat{T}b_j \cdot TT f_j \quad (\text{Corollary 1.4.4})
\]

\[
\leq \bigwedge_{i \in I} f_i^\circ \cdot b_i \cdot T f_i \cdot (T f_i)^\circ \cdot \hat{T}b_i \cdot TT f_i \quad (T f_i \cdot (T f_i)^\circ \leq 1_{TY_i})
\]

\[
\leq \bigwedge_{i \in I} f_i^\circ \cdot b_i \cdot m_{Y_i} \cdot TT f_i \quad (b_i \text{ transitive})
\]

\[
= a \cdot m_X \quad (m\text{ natural}).
\]

Thus, \((X, a)\) is a \((\mathbb{T}, \mathcal{V})\)-category, and every \(f_i : X \rightarrow Y_i\) is a \((\mathbb{T}, \mathcal{V})\)-functor because, by definition of \(a\), one has

\[
a(\chi, y) \leq b_i(T f_i(\chi), f_i(y))
\]

for all \(\chi \in TX\) and \(y \in X\). To prove that the source \((f_i : (X, a) \rightarrow (Y_i, b_i))_{i \in I}\) is \(O\)-initial, consider a source \((h_i : (Z, c) \rightarrow (Y_i, b_i))\) with a map \(g : Z \rightarrow X\) satisfying \(h_i = f_i \cdot g\) for all \(i \in I\). One then has

\[
g \cdot c \leq \bigwedge_{i \in I} f_i^\circ \cdot f_i \cdot g \cdot c \leq \bigwedge_{i \in I} f_i^\circ \cdot b_i \cdot T f_i \cdot T g = a \cdot T g,
\]

as desired. \[\square\]
3.1.2 Examples. Rewriting the formula of Proposition 3.1.1 for the main example categories considered previously, we obtain:

1. \( x \leq y \iff \forall i \in I \ (f_i(x) \leq f_i(y)) \) in \( \text{Ord} = (\mathbb{1}, 2)\)-Cat.

2. \( a(x, y) = \sup_{i \in I} b_i(f_i(x), f_i(y)) \) in \( \text{Met} = (\mathbb{1}, \mathbb{P})\)-Cat.

3. \( x \rightarrow y \iff \forall i \in I (f_i[x] \rightarrow f_i(y)) \) in \( \text{Top} \cong (\mathbb{2}, 2)\)-Cat.

4. \( a(x, y) = \sup_{i \in I} [f_i(x), f_i(y)] \) in \( \text{App} \cong (\mathbb{P}, \mathbb{P})\)-Cat.

3.1.3 Theorem. For a quantale \( V \) and a lax extension \( \hat{T} \) of a \( \text{Set}\)-functor \( T \) to \( V\text{-Rel} \), the forgetful functor \( O : (\mathbb{T}, V)\text{-Cat} \rightarrow \text{Set} \) is topological. It therefore admits initial and final liftings, is transportable, has both a fully faithful left and a fully faithful right adjoint, and makes \( (\mathbb{T}, V)\text{-Cat} \) a small-complete, small-cocomplete, well-powered and cowell-powered category with a generator and a cogenerator.

Proof. Proposition 3.1.1 shows that every source \( (f_i : X \rightarrow O(Y_i, b_i)) \) admits an \( O \)-initial lifting, so that \( O \) is topological. Theorem II.5.9.1 takes care of the final liftings, transportability, and the existence of adjoints. Corollary II.5.8.4 and its dual prove the statements on completeness and cocompleteness, since \( \text{Set} \) is both small-complete and small-cocomplete. Well- and cowell-poweredness of \( (\mathbb{T}, V)\text{-Cat} \) are also consequences of topologicity, see Exercise II.5.F. To obtain a generator in \( (\mathbb{T}, V)\text{-Cat} \), one can apply the left adjoint of \( O \) to a generator of \( \text{Set} \), and proceed dually for a cogenerator.

3.1.4 Remark. The proof of Proposition 3.1.1 makes it easy to describe explicitly the limit \( (L, a) \) of a diagram \( D : J \rightarrow (\mathbb{T}, V)\text{-Cat} \): just form the limit \( L \) of \( OD \) in \( \text{Set} \) and then use the formula for \( a \) as given in 3.1.1 where the \( f_i (i \in \text{ob } J) \) are the limit projections in \( \text{Set} \). There is unfortunately no easy general formula for the construction of \( O \)-final liftings, and consequently for colimits in \( (\mathbb{T}, V)\text{-Cat} \). Under additional hypotheses coproducts may be described easily; see 4.3.

3.2 Discrete and indiscrete lax algebras. The topological functor \( O : (\mathbb{T}, V)\text{-Cat} \rightarrow \text{Set} \) has both a left and right adjoint (Theorem 3.1.3) which we now proceed to describe explicitly. We first consider the functor \( (-)_d : \text{Set} \rightarrow (\mathbb{T}, V)\text{-Cat} \) that is the identity on maps, and sends a set \( X \) to \( X_d = (X, 1_X^d) \), where the discrete structure \( 1_X^d \) on \( X \) is described by

\[
1_X^d = e_X^d \cdot \hat{T}1_X
\]

(see 1.7). The reflexivity condition for a \( (\mathbb{T}, V)\)-category structure \( a \) is equivalent to \( 1_X^d \leq a \) by (1.8.i) of 1.6, so one obtains a \((\mathbb{T}, V)\)-functor \( \varepsilon_X : (X, 1_X^d) \rightarrow (X, a) \) whose underlying map is the identity on \( X \). Thus, the natural transformations \( 1 : 1_{\text{Set}} \rightarrow O(-)_d \equiv 1_{\text{Set}} \) and \( \varepsilon : (-)_d O \rightarrow 1_{(\mathbb{T}, V)\text{-Cat}} \) trivially satisfy the triangular identities of an adjunction (see II.2.5), and one concludes that \((-)_d \) is left adjoint to \( O \):

\[
(-)_d \xrightleftharpoons{\varepsilon} O : (\mathbb{T}, V)\text{-Cat} \rightarrow \text{Set}.
\]
As is the case for every topological functor (see Theorem \[ II.5.9.1 \]), the left adjoint \((-)_d\) embeds \(\text{Set}\) as a full coreflective category of \((\mathcal{T}, \mathcal{V})\)-\text{Cat}.

The right adjoint \((-)_i : \text{Set} \to (\mathcal{T}, \mathcal{V})\)-\text{Cat} of \(O\) provides a set \(X\) with the \(O\)-initial structure of the empty source with domain \(X\) (see \[ II.5.8 \]). Hence, \((-)_i\) sends a set \(X\) to the \((\mathcal{T}, \mathcal{V})\)-category \(X_i = (X, \top_X)\), where \(\top_X : TX \to X\) is the indiscrete structure given by \(\top_X(x, y) = \top\) for all \(x \in TX, y \in X\). This describes a full reflective embedding of \(\text{Set}\) into \((\mathcal{T}, \mathcal{V})\)-\text{Cat}.

As the discrete and indiscrete structures described above are determined by adjunctions, they correspond to the respective structures described for \(\text{Ord} = 2\)-\text{Cat} and \(\text{Top} \cong (\beta, 2)\)-\text{Cat} in Example \[ II.2.5.1(3) \].

### 3.3 Induced orders.

Given a lax extension \(\hat{T}\) to \(\mathcal{V}\)-\text{Rel} of a monad \(\mathcal{T}\) on \(\text{Set}\), the structure of a \((\mathcal{T}, \mathcal{V})\)-category is a reflexive and transitive \(\mathcal{V}\)-relation. This terminology not only extends the usual concept used for ordinary relations, but it also suggests that the structure induces a natural order on the underlying set. Indeed, since a \((\mathcal{T}, \mathcal{V})\)-category structure \(a : TX \to X\) is left unitary, one has

\[
e^\circ_X \cdot \hat{T}a \cdot e_{TX} \leq e^\circ_X \cdot m^\circ_X = e^\circ_X \circ a = a .
\]

The inequality in the other direction is just the expression of oplaxness of the unit \(e : 1_{\mathcal{V}\text{-Rel}} \to \hat{T}\), so we have

\[
a = e^\circ_X \cdot \hat{T}a \cdot e_{TX} \quad (3.3.i)
\]

for any \((\mathcal{T}, \mathcal{V})\)-category structure \(a : TX \to X\). This identity is used to prove the following result.

### 3.3.1 Proposition. Let \(\hat{T}\) be a lax extension of a monad \(\mathcal{T}\) on \(\text{Set}\) to \(\mathcal{V}\)-\text{Rel}. If \(a : TX \to X\) is a \((\mathcal{T}, \mathcal{V})\)-category structure, then the relation

\[
x \leq y \iff k \leq a(e_X(x), y)
\]

(for all \(x, y \in X\)) defines an order on \(X\), called the underlying order induced by \(a\) (or sometimes simply the induced order). The structure \(a\) is then monotone in its second variable with respect to this order:

\[
x \leq y \implies a(\chi, x) \leq a(\chi, y)
\]

for all \(x, y \in X, \chi \in TX\).

**Proof.** For the given relation \(\leq\) on \(X\), one immediately has \(x \leq x\) since \(k \leq a(e_X(x), x)\) by reflexivity of \(a\). By transitivity of \(a\) and the identity \((3.3.i)\), if \(x \leq y\) and \(y \leq z\), then

\[
k \leq a(e_X(x), y) \otimes a(e_X(y), z) = \hat{T}a(e_{TX}(e_X(x)), e_X(y)) \otimes a(e_X(y), z) \leq a(e_X(x), z) ,
\]

where \(\hat{T}a\) is the oplax transformation, and \(\otimes\) denotes the tensor product in \(\mathcal{V}\)-\text{Rel}. This completes the proof.
that is, \( x \leq z \), so the relation \( \leq \) is also transitive. Finally, if \( x \leq y \) then transitivity of \( a \) also yields

\[
a(\chi, x) = \hat{T}a(e_{TX}(\chi), e_X(x)) \leq \hat{T}a(e_{TX}(\chi), e_X(x)) \otimes a(e_X(x), y) \leq a(\chi, y),
\]
so that \( a \) is monotone with respect to this order.

**3.3.2 Corollary.** The order induced by a \((\mathbb{T}, \mathcal{V})\)-category structure yields a functor

\[
(\mathbb{T}, \mathcal{V})\text{-Cat} \to \text{Ord}
\]
that makes \((\mathbb{T}, \mathcal{V})\)-\text{Cat} an ordered category. Hence, as in \text{Ord}, hom-sets in \((\mathbb{T}, \mathcal{V})\)-\text{Cat} are ordered pointwise:

\[
f \leq g \iff \forall x \in X (f(x) \leq g(x))
\]
for \( f, g : (X, a) \to (Y, b) \).

**Proof.** The order given by Proposition 3.3.1 makes every \((\mathbb{T}, \mathcal{V})\)-functor \( f : (X, a) \to (Y, b) \) monotone since, for \( x, y \in X, x \leq y \) implies

\[
k \leq a(e_X(x), y) \leq b(Tf \cdot e_X(x), f(y)) = b(e_Y \cdot f(x), f(y)),
\]
that is, \( f(x) \leq f(y) \). Since the hom-sets in the ordered category \text{Ord} are ordered pointwise, the order on the hom-sets in \((\mathbb{T}, \mathcal{V})\)-\text{Cat} induced by \((\mathbb{T}, \mathcal{V})\)-\text{Cat} \to \text{Ord} is also pointwise.

**3.3.3 Corollary.** The following conditions are equivalent for \((\mathbb{T}, \mathcal{V})\)-functors \( f, g : (X, a) \to (Y, b) \):

(i) \( f \leq g \);

(ii) \( \forall y \in TY, x \in X \ (b(y, f(x)) \leq b(y, g(x))) \);

(iii) \( \forall \chi \in TX, x \in X \ (a(\chi, x) \leq b(Tf(\chi), g(x))) \).

**Proof.** (i) \( \Rightarrow \) (ii) follows from 3.3.1.

For (ii) \( \Rightarrow \) (iii), observe that \( a(\chi, x) \leq b(Tf(\chi), f(x)) \leq b(Tf(\chi), g(x)) \).

Finally, \( k \leq a(e_X(x), x) \leq b(Tf(e_X(x)), g(x)) = b(e_Y(f(x)), g(x)) \) proves (iii) \( \Rightarrow \) (i).

**3.3.4 Remark.** If \( \hat{T} \) is associative, then the following condition is also equivalent to (i)–(iii) of Corollary 3.3.3:

(iv) \( \forall \chi \in TX, y \in Y \ (b(Tg(\chi), y) \leq b(Tf(\chi), y)) \).

A proof using Kleisli convolution is indicated in Exercise 3.E.
3.3.5 Examples.

1. For the identity lax extension of \( I \) to \( 2\text{-Rel} \), we have \( 2\text{-Cat} = \text{Ord} \) (Example 1.3.1(1)), and the underlying order on an ordered set \((X, a)\) induced by \( a \) returns the original order on \( X \). For the identity lax extension of \( I \) to \( \text{P}_\text{+}-\text{Rel} \), we had \( \text{P}_\text{+}-\text{Cat} = \text{Met} \) (Example 1.3.1(2)), and the underlying order induced on a metric space \((X, a)\) is given by

\[ x \leq y \iff a(x, y) = 0 \]

for all \( x, y \in X \).

2. For the Barr extension of the ultrafilter monad \( \beta \) to \( 2\text{-Rel} \), Theorem 2.2.5 states that \((\beta, 2\text{-Cat}) \cong \text{Top}\). Here, the underlying order on \((X, a)\) is given by (when we write \( a \) as \( \rightarrow \))

\[ x \leq y \iff \dot{x} \rightarrow y . \]

By Exercise 2.C, this is precisely the underlying order of topological spaces (Example 1.1.9). The underlying order on an approach space \((X, a)\) in the guise of a \((\beta, \text{P}_\text{+})\)-category (see Theorem 2.4.5) is defined by

\[ x \leq y \iff a(\dot{x}, y) = 0 \]

for all \( x, y \in X \). In terms of the set-point distance \( \delta : X \times PX \rightarrow \text{P}_\text{+} \), this means

\[ x \leq y \iff \delta(y, \{x\}) = 0 . \]

For a \((T, \mathcal{V})\)-category \((X, a)\) we wish to define an order on the set \( TX \) such that—as one would expect of a hom-functor—\( a \) becomes order-reversing in its first variable. The strategy should be to provide \( TX \) with a “natural” \((T, \mathcal{V})\)-structure and then to consider the order induced via Proposition 3.3.1. Unfortunately, the free \( T \)-algebra structure \( m_X \) on \( TX \) will generally not provide a \((T, \mathcal{V})\)-category structure (unless \( \hat{T} \) is flat, see Proposition 1.6.5), but there is a \((T, \mathcal{V})\)-category structure \( \tilde{m}_X \geq m_X \) on \( TX \) satisfying our requirements, namely \( \tilde{m}_X := \hat{T}1_X \cdot m_X \), as we show next.

3.3.6 Proposition. There is a functor \( \tilde{T} : \text{Set} \rightarrow (T, \mathcal{V})\text{-Cat} \) sending \( X \) to \((TX, \tilde{m}_X)\) that makes

\[
\begin{array}{ccc}
(\mathbb{T}, \mathcal{V})\text{-Cat} & \xrightarrow{T} & \text{Set} \\
\tilde{T} \downarrow & & \downarrow \tau \\
\text{Set} & & \text{Set}
\end{array}
\]

commute. The order on \( TX \) induced by \( \tilde{m}_X \) is given by

\[ x \leq y \iff k \leq \hat{T}1_X(x, y) , \]

for all \( x, y \in TX \). Any \((T, \mathcal{V})\)-category structure \( a \) on \( X \) reverses this order in its first variable:

\[ x \leq y \implies a(y, z) \leq a(x, z) \]

for all \( x, y \in TX, z \in X \), so that \( a : TX \rightarrow X \) becomes a module of ordered sets.
Proof. Reflexivity and transitivity of \( \tilde{m}_X \) are easily verified:
\[
1_{TX} \leq \hat{T}1_X = \hat{T}1_X \cdot m_X \cdot e_{TX} = \tilde{m}_X \cdot e_{TX} ,
\]
and
\[
\tilde{m}_X \cdot \hat{T}\tilde{m}_X = \hat{T}1_X \cdot m_X \cdot \hat{T}\hat{T}1_X \cdot Tm_X \leq \hat{T}1_X \cdot m_X \cdot Tm_X = \tilde{m}_X \cdot m_{TX} .
\]

Also, for any map \( f : X \to Y, Tf : \hat{T}X \to \hat{T}Y \) is a \((\mathbb{T}, \mathbb{V})\)-functor:
\[
Tf \cdot \tilde{m}_X = Tf \cdot \hat{T}(e_X^o) \cdot m_X \cdot m_X \leq \hat{T}(f \cdot e_X^o) \cdot m_X \cdot m_X \\
\leq \hat{T}(e_Y^o \cdot Tf) \cdot m_X \cdot m_X = \hat{T}(e_Y^o) \cdot TTf \cdot m_X \cdot m_X \leq \hat{T}(e_Y^o) \cdot m_Y \cdot Tf \cdot m_X \\
= \hat{T}1_Y \cdot m_Y \cdot TTf = \tilde{m}_X \cdot TTf \quad \text{(Lemma 1.7.2)}.
\]

By Proposition 3.3.1, the order induced by \( \tilde{m}_X \) on \( TX \) is given by the \( \mathbb{V} \)-relation \( \hat{T}1_X \cdot m_X \cdot e_{TX} = \tilde{m}_X \cdot e_{TX} \) (Corollary 1.4.4), which gives the description in the claim. Finally, if \( a \) is a \((\mathbb{T}, \mathbb{V})\)-category structure, then it is right unitary so that \( k \leq \hat{T}1_X(x, y) \) yields
\[
a(y, z) \leq \hat{T}1_X(x, y) \otimes a(y, z) \leq a \cdot \hat{T}1_X(x, z) = a(x, z) ,
\]
which means that \( x \leq y \) implies \( a(y, z) \leq a(x, z) \), as claimed. \( \square \)

We will see in 5.4 below that \( \hat{T} \) factors through the left adjoint of \( O \), see Exercise 5.K. Also, alternative orders on \( TX \) will be considered (see in particular Examples 5.3.7).

### 3.4 Algebraic functors

Consider lax extensions \( \hat{S}, \hat{T} \) to \( \mathcal{V}\)-\( \text{Rel} \) of monads \( S = (S, n, d) \) and \( \mathbb{T} = (T, m, e) \) on \( \text{Set} \). A morphism of lax extensions \( \alpha : (S, \hat{S}) \to (T, \hat{T}) \) is a natural transformation \( \alpha : S \to T \) that becomes an oplax transformation \( \hat{S} \to \hat{T} \), so that
\[
\alpha_Y \cdot \hat{S}r \leq \hat{T}r \cdot \alpha_X
\]
for all \( \mathcal{V} \)-relations \( r : X \to Y \). A monad morphism \( \alpha : S \to \mathbb{T} \) which is also a morphism of lax extensions \( \alpha : \hat{S} \to \hat{T} \) is denoted by \( \alpha : (S, \hat{S}) \to (\mathbb{T}, \hat{T}) \). Any such natural transformation \( \alpha \) induces a functor
\[
A_\alpha : (\mathbb{T}, \mathcal{V})\text{-}\text{Cat} \to (S, \mathcal{V})\text{-}\text{Cat} ,
\]
sending \((X,a)\) to \((X,a \cdot \alpha_X)\), and mapping morphisms identically. Indeed, one has \(1_X \leq a \cdot e_X = a \cdot \alpha_X \cdot d_X\), and
\[
a \cdot \alpha_X \cdot \hat{S}(a \cdot \alpha_X) = a \cdot \alpha_X \cdot \hat{S}a \cdot S\alpha_X \\
\leq a \cdot \hat{T}a \cdot \alpha_{TX} \cdot S\alpha_X \\
\leq a \cdot m_X \cdot \alpha_{TX} \cdot S\alpha_X \\
= a \cdot \alpha_X \cdot n_X.
\]
Moreover, a \((\mathbb{T},\mathcal{V})\)-functor \(f : (X,a) \to (Y,b)\) is an \((\mathcal{S},\mathcal{V})\)-functor \(f : (X,a \cdot \alpha_X) \to (Y,b \cdot \alpha_Y)\):
\[
f \cdot a \cdot \alpha_X \leq b \cdot Tf \cdot \alpha_X = b \cdot \alpha_Y \cdot Sf .
\]
The functor \(A_\alpha\) is called the algebraic functor associated with \(\alpha\).

Given a lax extension \(\hat{T}\) of \((T,m,e)\) to \(\mathcal{V}\)-Rel, the unit \(e : 1_{\text{Set}} \to T\) immediately yields an algebraic functor. Indeed, oplaxness of \(e\) means precisely that there is a morphism of lax extensions \(e : (1_{\text{Set}},1_{\mathcal{V}\text{-Rel}}) \to (T,\hat{T})\). As \(e\) is also a monad morphism \(e : \mathbb{I} \to T\), one obtains a functor
\[
A_e : (\mathbb{T},\mathcal{V})\text{-Cat} \to \mathcal{V}\text{-Cat}
\]
that sends \((X,a)\) to its underlying \(\mathcal{V}\)-category \((X,a \cdot e_X)\) and commutes with the underlying-set functors (recall from 1.6 that we write \(\mathcal{V}\text{-Cat}\) rather than \((\mathbb{I},\mathcal{V})\text{-Cat}\)). This functor has a left adjoint that we now proceed to describe. In 1.9, we defined the unitary \((\mathbb{T},\mathcal{V})\)-relation
\[
r_z = e_Y^\circ \cdot \hat{T}r
\]
associated to a \(\mathcal{V}\)-relation \(r : X \to Y\).

3.4.1 Lemma. The \((-)_z\) transformation defined above satisfies
\[
r \leq r' \implies r_z \leq r'_z \quad \text{and} \quad s_z \circ r_z \leq (s \cdot r)_z
\]
for all \(\mathcal{V}\)-relations \(r, r' : X \to Y\) and \(s : Y \to Z\). Moreover, if \(\hat{T}\) is associative, then the second inequality is actually an equality, and \((-)_z\) defines a 2-functor \((-)_z : \mathcal{V}\text{-Rel} \to (\mathbb{T},\mathcal{V})\text{-URel}\).

Proof. The first implication is straightforward, since composition and \(\hat{T}\) both preserve the order on hom-sets. For the second expression, we write
\[
s_z \circ r_z = e_Z^\circ \cdot \hat{T}s \cdot \hat{T}(e_Y^\circ \cdot \hat{T}r) \cdot m_X^\circ \\
= e_Z^\circ \cdot \hat{T}s \cdot (Te_Y)^\circ \cdot \hat{T}\hat{T}r \cdot m_X^\circ \\
\leq e_Z^\circ \cdot \hat{T}s \cdot (Te_Y)^\circ \cdot m_Y^\circ \cdot \hat{T}r \\
= e_Z^\circ \cdot \hat{T}s \cdot \hat{T}r \\
\leq e_Z^\circ \cdot \hat{T}(s \cdot r) \\
= (s \cdot r)_z .
\]
If \( \hat{T} \) is associative, then \( \hat{T} \) preserves composition and \( m^c \) is a natural transformation (Proposition 1.9.4), so the inequalities in the previous displayed formulas become equalities. Since \((1_X)_\sharp = 1_X^\sharp \) is the identity in \((\mathbb{T}, \mathcal{V})\)-\text{URel}, the \((-)_\sharp \) transformation defines a functor \( \mathcal{V}\text{-Rel} \rightarrow (\mathbb{T}, \mathcal{V})\text{-URel}. \)

### 3.4.2 Proposition

The algebraic functor \( A_e \) has a left adjoint

\[
A^\circ : \mathcal{V}\text{-Cat} \rightarrow (\mathbb{T}, \mathcal{V})\text{-Cat}
\]

which associates to a \( \mathcal{V} \)-category \((X, r)\) the \((\mathbb{T}, \mathcal{V})\)-category \((X, r_\sharp)\).

**Proof.** Let \((X, r)\) be a \( \mathcal{V} \)-category. Since \( 1_X \leq r \) and \( r \cdot r \leq r \), we deduce from Lemma 3.4.1

\[
1_X^\sharp \leq r_\sharp \quad \text{and} \quad r_\sharp \circ r_\sharp \leq (r \cdot r)_\sharp \leq r_\sharp,
\]

so that \( r_\sharp \) is both reflexive and transitive if \( r \) is. For a \( \mathcal{V} \)-functor \( f : (X, r) \rightarrow (Y, s) \), we also have

\[
f \cdot r_\sharp = f \cdot e_X^\circ \cdot \hat{T}r \leq e_X^\circ \cdot \hat{T}(f \cdot r) \leq e_X^\circ \cdot \hat{T}(s \cdot f) = e_X^\circ \cdot \hat{T} s \cdot Tf = s_\sharp \cdot Tf,
\]

which yields \((\mathbb{T}, \mathcal{V})\)-functoriality. This functor is left adjoint to \( A_e \): for a \( \mathcal{V} \)-category \((X, r)\),

\[
r \leq e_X^\circ \cdot \hat{T}r \cdot e_X = r_\sharp \cdot e_X,
\]

and for a \((\mathbb{T}, \mathcal{V})\)-category \((X, a)\),

\[
(a \cdot e_X)_\sharp = e_X^\circ \cdot \hat{T}(a \cdot e_X) = e_X^\circ \cdot \hat{T} a \cdot Te_X \leq e_X^\circ \cdot \hat{T} a \cdot m_X^c = a.
\]

### 3.4.3 Examples.

1. For \( \mathcal{V} = 2 \) and \( \mathbb{T} = \beta \), we have \( 2\text{-Cat} = \text{Ord} \) and \((\beta, 2)\text{-Cat} = \text{Top} \), and the functor \( A_e : \text{Top} \rightarrow \text{Ord} \) sends a topological space \((X, a)\) to the ordered set \((X, a \cdot e_X)\) whose order is the underlying order of the former; that is, \( A_e \) is the forgetful functor to \( \text{Ord} \) of Corollary 3.3.2. The left adjoint \( \text{Ord} \hookrightarrow \text{Top} \) provides an ordered set \((X, \leq)\) with the Alexandroff topology, that is, the topology whose open sets are generated by the down-sets \( \downarrow x \) for \( x \in X \) (Example II.5.10.5).

2. For \( \mathcal{V} = P_+ \) and \( \mathbb{T} = \beta \), we have \( P_+\text{-Cat} = \text{Met} \) and \((\beta, P_+)\text{-Cat} = \text{App} \), so \( A_e : \text{App} \rightarrow \text{Met} \) sends an approach space \((X, a)\) to the metric space \((X, a \cdot e_X)\). The left adjoint \( \text{Met} \hookrightarrow \text{App} \) sends a metric space \((X, r)\) to the approach space whose structure is given by

\[
r_\sharp(x, y) = \sup_{A \in \xi} \inf_{x \in A} r(x, y)
\]

for all \( x \in \beta X \) and \( y \in X \). This coreflective embedding has been described in terms of point-set distance in Example 2.4.1(1).
3. **Change-of-base functors.** The algebraic functors deal with monads, that is, with the first variable in \((T, V)\)-Cat. The change-of-base functors deal with the second. Consider lax extensions \(\hat{T}\) and \(\tilde{T}\) of the monad \(T = (T, m, e)\) to \(V\)-Rel and \(W\)-Rel, respectively.

Let \(\varphi : V \to W\) be a lax homomorphism of quantales (see [1.10]), so that \(\varphi\) is order preserving and
\[
\varphi(u) \otimes \varphi(v) \leq \varphi(u \otimes v) , \quad l \leq \varphi(k)
\]
for all \(u, v \in V\), and where \(k, l\) are the units of \(V, W\), respectively. Then \(\varphi\) induces a lax functor
\[
\varphi : V\text{-Rel} \to W\text{-Rel}
\]
which leaves objects unchanged and sends \(r : X \times Y \to V\) to \(\varphi r : X \times Y \to W\). Clearly, for any \(\text{Set}\)-map \(f\), we have
\[
f \leq \varphi f \quad \text{and} \quad f^o \leq \varphi(f^o)
\]
where \(f\) and \(f^o\) are considered as \(W\)-relations when appearing on the left of the inequality sign, and as \(V\)-relations on the right. Furthermore, we assume that \(\varphi\) is compatible with the respective lax extensions \(\hat{T}\) and \(\tilde{T}\) of \(T\) to \(V\)-Rel and \(W\)-Rel, that is, \(\hat{T}(\varphi r) \leq \varphi(\tilde{T} r)\) for all \(V\)-relations \(r\):

\[
\begin{array}{ccc}
\mathcal{V}\text{-Rel} & \xrightarrow{\hat{T}} & \mathcal{V}\text{-Rel} \\
\downarrow \varphi & & \downarrow \varphi \\
\mathcal{W}\text{-Rel} & \xrightarrow{\tilde{T}} & \mathcal{W}\text{-Rel}.
\end{array}
\]

Under these conditions, \(\varphi\) induces a functor
\[
B_{\varphi} : (\mathbb{T}, \mathcal{V})\text{-Cat} \to (\mathbb{T}, \mathcal{W})\text{-Cat},
\]
called the change-of-base functor associated to \(\varphi\), sending \((X, a)\) to \((X, \varphi a)\) and leaving maps unchanged. Indeed, we observe that \(e_X^\varphi \leq \varphi(e_X^o) \leq \varphi a\) holds, as well as
\[
\varphi a \circ \varphi a = \varphi a \cdot \hat{T}(\varphi a) \cdot m_X^a
\leq \varphi a \cdot \varphi(\tilde{T} a) \cdot \varphi(m_X^o)
\leq \varphi(a \cdot \tilde{T} a \cdot m_X^o)
= \varphi a.
\]

Moreover, given a \((\mathbb{T}, \mathcal{V})\)-functor \(f : (X, a) \to (Y, b)\), one can adapt Corollary 1.4.4 to \(\varphi\) to obtain the last equality in
\[
f \cdot \varphi a \leq \varphi f \cdot \varphi a \leq \varphi(f \cdot a) \leq \varphi(b \cdot Tf) = \varphi b \cdot Tf.
\]

3.5.1 **Proposition.** Adjunctions of maps become adjunctions of the corresponding change-of-base functors. More precisely, suppose that \(\varphi : \mathcal{V} \to \mathcal{W}\) and \(\psi : \mathcal{W} \to \mathcal{V}\) are lax...
homomorphisms of quantales that are compatible with the respective lax extensions of \( \mathbb{T} \) to \( \mathcal{V}\text{-Rel} \) and \( \mathcal{W}\text{-Rel} \). Then one has:

\[
\varphi \dashv \psi \implies B_\varphi \dashv B_\psi .
\]

**Proof.** The composite \( B_\psi B_\varphi \) sends a \((\mathbb{T}, \mathcal{V})\)-category \((X, a)\) to \((X, \psi \varphi a)\). If \( a \) is seen as a map \( a : TX \times X \to \mathcal{V} \), then \( \psi \varphi a \) is the map \( \psi \cdot \varphi \cdot a \) with \( a \leq \psi \cdot \varphi \cdot a \) because \( 1_\mathcal{V} \leq \psi \cdot \varphi \); therefore, the identity \( 1_X : (X, a) \to (X, \psi \varphi a) \) is a \((\mathbb{T}, \mathcal{V})\)-functor. Dually, the identity \( 1_X : (X, \psi \varphi b) \to (X, b) \) is a \((\mathbb{T}, \mathcal{W})\)-functor for a \((\mathbb{T}, \mathcal{W})\)-category \((X, b)\). These maps then yield the respective components of the unit and counit of an adjunction \( B_\varphi \dashv B_\psi \) since the triangular identities are trivially satisfied. \( \square \)

### 3.5.2 Examples.

1. The quantale homomorphism \( \iota : 2 \to \mathcal{V} \) always has a right adjoint \( p : \mathcal{V} \to 2 \) that is a lax homomorphism of quantales (with \( p(v) = \top \) if \( k \leq v \) and \( p(v) = \bot \) otherwise, see Exercise II.1.I). In fact, \( p \) is a quantale homomorphism if and only if

\[
k \leq u \otimes v \implies k \leq u \text{ and } k \leq v
\]

for all \( u, v \in \mathcal{V} \). As a monotone map, \( \iota \) has a left adjoint \( o : \mathcal{V} \to 2 \) if and only if \( k = \top \), given then by \( o(v) = \top \) if and only if \( \bot < v \); furthermore, \( o \) is a quantale homomorphism if and only if

\[
u \otimes v = \bot \implies u = \bot \text{ or } v = \bot
\]

for all \( u, v \in \mathcal{V} \). These maps are all obviously compatible with the identical lax extensions of the identity monad \( I \) to \( \text{Rel} \) and \( \mathcal{V}\text{-Rel} \), and Proposition 3.5.1 yields adjunctions

\[
\begin{array}{ccc}
\text{Ord} & \xleftarrow{B_\iota} & \mathcal{V}\text{-Cat} \\
\downarrow B_o & & \downarrow B_p \\
\mathcal{V}\text{-Cat} & \xrightarrow{B_\iota^\circ} & \text{Ord}
\end{array}
\]

In this diagram, \( B_\iota : 2\text{-Cat} \to \mathcal{V}\text{-Cat} \) is the embedding \( \text{Ord} \hookrightarrow \mathcal{V}\text{-Cat} \) and \( B_p \) is the forgetful functor \( \mathcal{V}\text{-Cat} \to \text{Ord} \) from Corollary 3.3.2 (in the \( \mathbb{T} = I \) case).

2. By (1) there is in particular a full reflective and coreflective embedding \( \text{Ord} \hookrightarrow \text{Met} \) which provides an ordered set \((X, \leq)\) with the metric \( d \) given by \( d(x, y) = 0 \) if \( x \leq y \) and \( d(x, y) = \infty \) otherwise. There is also a full embedding

\[
B_\delta : \text{Met} \xhookrightarrow{} \text{ProbMet}
\]

(see 2.1) which is induced by the *Dirac morphism*

\[
\delta : \mathcal{P}_+ = ([0, \infty]^{\text{op}}, +, 0) \to \mathcal{D} = (\mathcal{D}, \otimes, \kappa) , \quad w \mapsto \delta_w ,
\]
3. CATEGORIES OF LAX ALGEBRAS

3.6 Fundamental adjunctions.

**Set, Ord, \(\mathcal{V}\)-Cat, and \((\mathbb{T},\mathcal{V})\)-Cat.** In the general setting of \((\mathbb{T},\mathcal{V})\)-categories (as in §1.6), the composite of the algebraic functor \(A_e: (\mathbb{T},\mathcal{V})\)-Cat \(\rightarrow\) \(\mathcal{V}\)-Cat (Proposition 3.4.2) with the underlying-order functor \(B_p: \mathcal{V}\)-Cat \(\rightarrow\) Ord (see Example 3.5.2(1)) is precisely the induced-order functor of Corollary 3.3.2. This functor has a left adjoint, since both \(B_p\) and \(A_e\) have left adjoints. We may further compose this left adjoint with the discrete-order functor \(\text{Set} \rightarrow \text{Ord}\) which then gives the following decomposition of the adjunction \((-)_d \circ O: (\mathbb{T},\mathcal{V})\)-Cat \(\rightarrow\) \(\text{Set}\) described in §3.2:

\[
\begin{array}{c}
\text{Set} \xleftarrow{\perp} \text{Ord} \\
\downarrow{\perp} \downarrow{\perp} \\
\mathcal{V}\text{-Cat} \xleftarrow{\perp} (\mathbb{T},\mathcal{V})\text{-Cat}
\end{array}
\]
Ord and Met. In the case where \( \mathcal{V} = \mathbb{P}_+ \), Example 3.5.2(1) yields the embedding \( B_\iota : \text{Ord} \to \text{Met} \), together with its right and left adjoints \( B_p \) and \( B_o \), respectively. The functor \( B_\iota \) has been described in Example 3.5.2(2), and its adjoints provide a metric space \((X, a)\) with the orders given by

\[
B_o : x \leq y \iff a(x, y) < \infty \quad \text{and} \quad B_p : x \leq y \iff a(x, y) = 0.
\]

Top and App. The homomorphism \( \iota \) is compatible with the lax extensions of the ultrafilter monad to \( \text{Rel} \) and \( \mathbb{P}_+ \text{-Rel} \), and the induced change-of-base functor is the embedding \( \text{Top} \to \text{App} \) described at the end of 2.4.

The lax homomorphism \( p : \mathbb{P}_+ \to 2 \) is also compatible with the lax extensions and provides the embedding with a right adjoint \( B_p : \text{App} \to \text{Top} \). This adjoint sends an approach space \((X, a)\) to a topological space in which an ultrafilter \( \chi \) converges to a point \( x \) precisely when \( a(\chi, x) = 0 \).

Unfortunately, the lax homomorphism \( o : \mathbb{P}_+ \to 2 \) is not compatible with the ultrafilter lax extensions. Nevertheless, given an approach space \((X, a)\), one can still consider the pair \((X, oa)\) that has an ultrafilter \( \chi \) converging to \( x \) precisely when \( a(\chi, x) < \infty \). This structure satisfies the reflexivity but not the transitivity condition for topologies defined via convergence. In other words, \((X, oa)\) is just a pseudotopological space (see Exercise 3.D). But we may apply the left adjoint of the full reflective embedding \( \text{Top} \to \text{PsTop} \) to \((X, oa)\) to obtain a topological space and thereby a left adjoint \( L : \text{App} \to \text{Top} \) to the embedding \( \text{Top} \to \text{App} \).

Ord, Met, Top, and App. The following diagram relates the functors described in the preceding paragraphs with the adjunction of Proposition 3.4.2. This diagram commutes with respect to both the solid and the dotted arrows (but not the dashed arrows); moreover, the two full embeddings \( \text{Ord} \to \text{App} \) described by it coincide.

Exercises

3.A The initial lax extension. The pair \((1_{\text{Set}}, 1_{\mathcal{V}_-\text{Rel}})\) is the initial object in the metacategory \( \mathcal{V}_-\text{LXT} \) of lax extensions of \( \text{Set} \)-functors to \( \mathcal{V}_-\text{Rel} \) and morphisms of lax extensions.
3. **Categories of Lax Algebras**


The correspondence
\[(\alpha : (S, \hat{S}) \to (T, \hat{T})) \mapsto (A_\alpha : (T, V)\text{-Cat} \to (S, V)\text{-Cat})\]

of 3.4 defines a functor \((V\text{-LXT})^\op \to \text{CAT}\) from the dual of \(V\text{-LXT}\) (see Exercise 3.A) to the metacategory \(\text{CAT}\) of categories and functors.

### 3.C Functoriality of the change-of-base transformation.

Given a monad \(\mathbb{T}\) on \(\text{Set}\), the correspondence
\[(\varphi : (V, \hat{V}) \to (W, \hat{W})) \mapsto (B_\varphi : (T, V)\text{-Cat} \to (T, W)\text{-Cat})\]

of 3.5 defines a 2-functor \(\text{QNT}(\mathbb{T}) \to \text{CAT}\) from the metacategory \(\text{QNT}(\mathbb{T})\) of quantales with lax extension of \(\mathbb{T}\) and compatible lax homomorphisms to the metacategory \(\text{CAT}\) of categories and functors. As a consequence, this functor preserves adjoint pairs (see also Proposition 3.5.1).

### 3.D Pseudotopological spaces.

The category \(\text{PsTop}\) of pseudotopological spaces is defined as follows: its objects are sets equipped with a reflexive relation \(a : \beta X \to X\) representing convergence of ultrafilters to points, and its morphisms are the convergence-preserving maps. Theorem 2.2.5 shows in particular that any topological space can be regarded as a pseudotopological space. In fact, one has a full reflective embedding \(\text{Top} \hookrightarrow \text{PsTop}\).

**Hint.** Given a reflexive relation \(a : \beta X \to X\), define \(A \subseteq X\) to be open precisely when
\[
\forall x \in X, y \in \beta X (y a x \& x \in A \implies A \in y).
\]

### 3.E Order and Kleisli convolution.

For a \((\mathbb{T}, V)\)-functor \(f : (X, a) \to (X, b)\), define \((\mathbb{T}, V)\)-relations \(f_* : X \leftrightarrow Y\) and \(f^* : Y \leftrightarrow X\) by
\[
f_* := b \cdot Tf \quad \text{and} \quad f^* := f^o \cdot b.
\]

Then:

1. \(f_* \circ f^* \leq b\).
2. \(a \leq f^* \circ f_*\) if \(m\) satisfies BC.
3. \(f \leq g\) in \((\mathbb{T}, V)\text{-Cat}\) if and only if \(f^* \leq g^*\) in \(V\text{-Rel}\).
4. Condition 3.3.4(iv) holds if and only if \(g_* \leq f_*\) in \(V\text{-Rel}\).
5. If \(\hat{\mathbb{T}}\) is associative, then
\[
f^* \leq g^* \iff g_* \leq f_*,
\]
and condition 3.3.4(iv) is equivalent to 3.3.3(i)–(iii).

**Hint.** Use Kleisli convolution and statements (1) and (2).
3.F Adjunctions in $(\mathcal{T}, \mathcal{V})$-$\text{Cat}$. Use Exercise 3.E(5) to prove that if $\mathcal{F}$ is associative, then $f : (X, a) \to (Y, b)$ is left adjoint to $g : (Y, b) \to (X, a)$ in the ordered category $(\mathcal{T}, \mathcal{V})$-$\text{Cat}$ if and only if
\[
a(x, g(y)) = b(Tf(x), y)
\]
for all $x \in TX$, $y \in Y$.

3.G Tensored $\mathcal{V}$-categories. A $\mathcal{V}$-category $(X, a)$ is called tensored if for all $x \in X$ and $u \in \mathcal{V}$ there exists $z \in X$ such that
\[
\forall y \in X \ (a(z, y) = (u \otimes a(x, y))) \ .
\]
(3.6.i)

(1) Show that $(X, a)$ is tensored if and only if, for all $x \in X$, the $\mathcal{V}$-functor
\[
a(x, -) : X \to \mathcal{V}
\]
has a left adjoint in $\mathcal{V}$-$\text{Cat}$. Conclude that any element $z$ satisfying (3.6.i) is, up to order-equivalence in $X$, uniquely determined by $x$ and $u$; one writes $z = x \otimes u$.

(2) Show that the $\mathcal{V}$-category $\mathcal{V}$ is tensored and so is $\mathcal{X} = [X^{\text{op}}, \mathcal{V}]$ (see Exercise 1.H), for every $\mathcal{V}$-category $X$.

3.H Change-of-base needs lax homomorphisms. For quantales $\mathcal{V}$ and $\mathcal{W}$, let $\varphi : \mathcal{V} \to \mathcal{W}$ be monotone. The following assertions are equivalent:

(i) For every $\mathcal{V}$-category $(X, a)$, the pair $(X, \varphi a)$ forms a $\mathcal{W}$-category;

(ii) $(\mathcal{V}, \varphi h)$ is a $\mathcal{W}$-category, with $h = (-) \otimes (-)$;

(iii) $\varphi$ is a lax homomorphism of quantales.

3.I Characterizing change-of-base functors. Let $\mathcal{V}$ and $\mathcal{W}$ be quantales. We write $(\mathcal{V}, h) = (\mathcal{V}, -)$ (see Exercise 3.H).

(1) For a lax homomorphism $\varphi : \mathcal{V} \to \mathcal{W}$ show that the change-of-base functor $B_\varphi : \mathcal{V}$-$\text{Cat} \to \mathcal{W}$-$\text{Cat}$ is a 2-functor that preserves initial morphisms with respect to the underlying-$\text{Set}$ functors.

(2) Let $F : \mathcal{V}$-$\text{Cat} \to \mathcal{W}$-$\text{Cat}$ be a 2-functor preserving underlying sets and initial morphisms. Writing $(X, \tilde{a})$ for $F(X, a)$, show:

(a) $\tilde{h}(k, -) : \mathcal{V} \to \mathcal{W}$ is monotone.

(b) For every tensored $\mathcal{V}$-category $(X, a)$,
\[
\forall u \in \mathcal{V} \ \forall x, y \in X \ (\tilde{a}(x \otimes u, y) = \tilde{h}(u, a(x, y))) \ ;
\]
in particular,
\[
\forall x, y \in X \ (\tilde{a}(x, y) = \tilde{h}(k, a(x, y))) \ .
\]
(3.6.ii)
(c) Formula (3.6.ii) holds for every \( \mathcal{V} \)-category, thanks to initiality of the Yoneda functor (see Exercise 1.H(3)).

(d) \( F = B_\varphi \), for a unique lax homomorphism \( \varphi: \mathcal{V} \to \mathcal{W} \) of quantales.

3.J Many structures on 1. For the topological functor \( O: \mathcal{V}-\text{Cat} \to \text{Set} \), the complete lattice \( O^{-1} \) (see Theorem II.5.9.1) of \( \mathcal{V} \)-category structures on a singleton set is order-isomorphic to \( \{ v \in \mathcal{V} \mid k \leq v, v \otimes v \leq v \} \), which is closed under infima in \( \mathcal{V} \) but not under suprema (unless \( \mathcal{V} \) is trivial).

3.K Lax extensions of the same monad. Let \( \mathcal{T} \) and \( \mathcal{\hat{T}} \) be lax extensions to \( \mathcal{V} \)-Rel of the monad \( \mathcal{T} = (T, m, e) \) on \( \text{Set} \) with \( \mathcal{\hat{T}}r \leq \mathcal{\hat{T}}r \) for all \( \mathcal{V} \)-relations \( r \). Then there is a full and faithful algebraic functor \( (\mathcal{\hat{T}}, \mathcal{V}, \mathcal{\hat{T}})-\text{Cat} \to (\mathcal{T}, \mathcal{V}, \mathcal{\hat{T}})-\text{Cat} \).

3.L Restricting a lax extension from \( \mathcal{V} \)-Rel to \( \text{Rel} \). Let \( \mathcal{T} \) come with a lax extension \( \mathcal{\hat{T}} \) to \( \mathcal{V} \)-Rel, and assume

\[
k = \top \quad \text{and} \quad (u \otimes v = \bot \implies u = \bot \text{ or } v = \bot)
\]

(for all \( u, v \in \mathcal{V} \)), so that the left adjoint \( o \) to \( \iota: 2 \to \mathcal{V} \) becomes a quantale homomorphism.

For \( r: X \to Y \) in \( \text{Rel} = \mathcal{2} \)-Rel, define \( \mathcal{\hat{T}}r := o\mathcal{\hat{T}}(\iota r) \) and assume that

\[
\mathcal{\hat{T}}(\iota r) = \iota(\mathcal{\hat{T}}r)
\]

holds for all relations \( r \). Then \( \mathcal{\hat{T}} \) is a lax extension to \( \text{Rel} \) of \( \mathcal{T} \), and \( \iota \) induces the change-of-base functor

\[
B_\iota: (\mathcal{T}, \mathcal{2}, \mathcal{\hat{T}})-\text{Cat} \to (\mathcal{T}, \mathcal{V}, \mathcal{\hat{T}})-\text{Cat}.
\]

In particular, for \( \mathcal{V} = \mathcal{P}_+ \) and \( \mathcal{\hat{T}} = \mathcal{\bar{\beta}} \) the Barr extension of \( \beta \) to \( \mathcal{P}_+ \)-Rel, \( \mathcal{\bar{T}} \) is the Barr extension of \( \beta \) to \( \text{Rel} \).
4 Embedding lax algebras into a quasitopos

In Theorem 3.1.3 we showed that the forgetful functor $O : (\mathbb{T}, \mathcal{V})\text{-Cat} \to \text{Set}$ is topological, giving an explicit description of $O$-initial structures and consequently of limits (see Proposition 3.1.1 and Remark 3.1.4). In this subsection, we introduce a supercategory of $(\mathbb{T}, \mathcal{V})\text{-Cat}$ which remains topological over $\text{Set}$ but, unlike $(\mathbb{T}, \mathcal{V})\text{-Cat}$, also allows for an easy description of colimits. In addition, this supercategory turns out to be a quasitopos—a term that we will explain in 4.8 below—and therefore cartesian closed. For $(\mathbb{T}, \mathcal{V}) = (\beta, 2)$ this quasitopos is the category $\text{PsTop}$ of pseudotopological spaces. We also discuss in general terms the role of the intermediate category $\text{PrTop}$ of pretopological spaces, which still enjoys an important property of quasitopoi (existence of a partial-map classifier), although it fails to be cartesian closed.

We continue to work with a quantale $\mathcal{V}$ which has a $\otimes$-neutral element $k$, and with a monad $\mathbb{T} = (\mathbb{T}, m, e)$ on $\text{Set}$ which comes with a lax extension $\hat{\mathbb{T}}$ to $\mathcal{V}$-$\text{Rel}$ (see 1.5).

4.1 $(\mathbb{T}, \mathcal{V})$-graphs. A $(\mathbb{T}, \mathcal{V})$-graph $(X, a)$ is a set $X$ equipped with a reflexive $(\mathbb{T}, \mathcal{V})$-relation $a : X \rightarrow X$ (see 1.6), that is, a $\mathcal{V}$-relation $a : TX \rightarrow X$ with $e_X^X \leq a$. A morphism $f : (X, a) \rightarrow (Y, b)$ of $(\mathbb{T}, \mathcal{V})$-graphs is defined as for $(\mathbb{T}, \mathcal{V})$-categories ($f \cdot a \leq b \cdot \mathbb{T}f$) and is therefore also called a $(\mathbb{T}, \mathcal{V})$-functor in the more general context of the category $(\mathbb{T}, \mathcal{V})\text{-Gph}$ of $(\mathbb{T}, \mathcal{V})$-graphs. There is a string of full subcategories

$$(\mathbb{T}, \mathcal{V})\text{-Cat} \hookrightarrow (\mathbb{T}, \mathcal{V})\text{-UGph} \hookrightarrow (\mathbb{T}, \mathcal{V})\text{-RGph} \hookrightarrow (\mathbb{T}, \mathcal{V})\text{-Gph},$$

with $(\mathbb{T}, \mathcal{V})$-RGph the category of right unitary $(\mathbb{T}, \mathcal{V})$-graphs $(X, a)$ that must satisfy $a \circ e_X^X \leq a$, and with $(\mathbb{T}, \mathcal{V})$-UGph the category of unitary $(\mathbb{T}, \mathcal{V})$-graphs $(X, a)$ for which $a$ is also left unitary: $e_X \circ a \leq a$. Recall from Proposition 1.7.3 that right-unitariness of a $(\mathbb{T}, \mathcal{V})$-relation $a$ is equivalently expressed by $a \cdot \hat{\mathbb{T}}1_X \leq a$, so it comes for free when $\hat{\mathbb{T}}$ is flat. For a map $f : X \rightarrow Y$ between $(\mathbb{T}, \mathcal{V})$-graphs $(X, a), (Y, b)$ one therefore has

$$b \cdot Tf = b \cdot \hat{\mathbb{T}}1_X \cdot Tf = b \cdot \hat{T}f$$

when $(Y, b)$ is right unitary, so that the condition $f \cdot a \leq b \cdot \hat{T}f$ then suffices to make $f$ a $(\mathbb{T}, \mathcal{V})$-functor.

Before presenting examples, let us state these definitions elementwise:

4.1.1 Lemma. A set $X$ with a $\mathcal{V}$-relation $a : TX \rightarrow X$ is a $(\mathbb{T}, \mathcal{V})$-graph if

$$k \leq a(e_X(x), x)$$

for all $X \in X$. It is right unitary if

$$\hat{T}1_X(x, y) \otimes a(y, y) \leq a(x, y)$$
for all $y \in X$, $x, y \in TX$, and unitary if, in addition,

$$\hat{T}a(x, e_X(y)) \leq a(m_X(x), y)$$

for all $x \in TTX$, $y \in X$.

Proof. By definition,

$$(a \cdot \hat{T}1_X)(x, y) = \bigvee_{y \in TX} \hat{T}1_X(x, y) \otimes a(y, y)$$

for all $x \in TX$, $y \in X$. Furthermore, $e_X \circ a \leq a \cdot \hat{T}a \leq a \cdot m_X$ which, when expressed elementwise, gives the stated condition for $(X, a)$ being unitary.

Note also that with $1^x_X = e^x_X \cdot \hat{T}1_X$, $(X, a)$ is a left unitary $(\mathbb{T}, \mathcal{V})$-graph if and only if

$$1^x_X \leq a = 1^x_X \circ a$$

and that it becomes unitary precisely under the additional condition

$$a = a \circ 1^x_X$$

(see Proposition 1.7.3). With $f^x = f^o \cdot 1^x_Y$, $(\mathbb{T}, \mathcal{V})$-functors are equivalently described by

$$a \circ f^x \leq f^x \circ b$$

(see 1.8).

4.1.2 Remark. The definition of a $(\mathbb{T}, \mathcal{V})$-graph does not depend on the monad multiplication $m$ of $\mathbb{T} = (T, m, e)$ or the lax extension $\hat{T}$ of the functor $T$. Right-unitariness depends on $\hat{T}$ but not on $m$, whereas unitariness depends on both $m$ and $\hat{T}$.

4.1.3 Examples.

(1) $(\mathbb{I}, \mathcal{V})$-graphs, with $\mathbb{I}$ identically extended to $\mathcal{V}$-Rel, are simply sets $X$ with a reflexive $\mathcal{V}$-relation $a : X \rightarrow X$. For $\mathcal{V} = 2$ one obtains sets with a reflexive relation, and for $\mathcal{V} = P_+$ sets with a function $a : X \times X \rightarrow [0, \infty]$ which is 0 on the diagonal of $X \times X$. $(\mathbb{I}, \mathcal{V})$-graphs are automatically unitary.

(2) $(\mathbb{B}, 2)$-graphs, where $\mathbb{B}$ is equipped with its Barr extension, are sets $X$ with a relation $a : \mathbb{B}X \rightarrow X$ that is only required to satisfy $\hat{x} \rightarrow x$ for all $x \in X$. (Here we write $\chi \rightarrow x$ instead of $\chi a x$.) These are precisely the pseudotopological spaces of Exercise 3.D. Indeed, since the Barr extension is flat, being right unitary comes for free, and the only remaining condition ($\hat{x} \rightarrow x$ for all $x \in X$) is characteristic for pseudotopology; hence,

$$(\mathbb{B}, 2)$-Gph = $(\mathbb{B}, 2)$-RGph = PsTop.
Unitary \((\beta, 2)\)-graphs must satisfy the additional condition

\[(X \rightarrow \dot{y}) \Rightarrow (\sum X \rightarrow y)\]  

(4.1.i)

where \(\sum X\) is the Kowalsky sum of \(X \in \beta \beta X\), and where \(X \rightarrow \dot{y}\) amounts to

\[\forall A \in X \exists x \in A (x \rightarrow y)\].

Hence, (4.1.i) means equivalently (see Example 1.10.3(3)),

\[\{x \in \beta X | x \rightarrow y\} \in X \Rightarrow (\sum X \rightarrow y)\]  

(4.1.ii)

for all \(X \in \beta \beta X\), \(y \in X\). From Proposition 2.2.4 we obtain immediately that unitary \((\beta, 2)\)-graphs on a set \(X\) correspond bijectively to maps \(c : PX \rightarrow PX\) with

\[A \subseteq c(A) , \quad c(\emptyset) = \emptyset , \quad c(A \cup B) = c(A) \cup c(B)\],

for all \(A, B \subseteq X\). A set \(X\) equipped with such a map \(c\) is called a \textit{pretopological space}. A morphism \(f : X \rightarrow Y\) in the category \(\text{PrTop}\) must satisfy

\[f(c_X(A)) \subseteq c_Y(f(A))\]

for all \(A \subseteq X\), and it is now easy to see that there is an isomorphism

\[(\beta, 2)\text{-UGph} \cong \text{PrTop}\]

which leaves underlying sets unchanged.

(3) For the filter monad \(\mathcal{F}\) extended by

\[a (\mathcal{F}r) \bar{b} \iff a \supseteq r^\circ [\bar{b}]\]

(for all relations \(r : X \rightarrow Y\), \(a \in FX, \bar{b} \in FY\)) as in Example 1.10.3(4), a right unitary \((\mathcal{F}, 2)\)-graph \((X, \rightarrow)\) must satisfy

\[\dot{x} \rightarrow x \quad \text{and} \quad (a \supseteq \bar{b} & \bar{b} \rightarrow y \Rightarrow a \rightarrow y)\],

(4.1.iii)

and \((X, \rightarrow)\) is unitary if

\[\{a \in FX | a \rightarrow y\} \in A \Rightarrow (\sum A \rightarrow y)\]  

(4.1.iv)

for all \(A \in FXX\), \(x, y \in X\). The category \((\mathcal{F}, 2)\text{-RGph}\) contains the full subcategory \((\mathcal{F}, 2)\text{-RGph}_{Ps}\) whose objects \((X, \rightarrow)\) satisfy (4.1.iii) and the condition

\((Ps)\) if \(a\) is a filter such that for every proper filter \(\bar{b} \supseteq a\) there exists a proper filter \(c \supseteq \bar{b}\) with \(c \rightarrow y\), then \(a \rightarrow y\)

\[(Ps)\]

for all \(a \in FX, y \in X\). It is not difficult to see that \((\mathcal{F}, 2)\text{-RGph}_{Ps}\) is isomorphic to \(\text{PsTop}\) (Exercise 4.1). Moreover, the full subcategory \((\mathcal{F}, 2)\text{-UGph}\) of \((\mathcal{F}, 2)\text{-RGph}_{Ps}\) is
isomorphic to \( \text{PrTop} \) (Exercise 4.L), so using (2) above, one obtains the following diagram of full embeddings and isomorphisms:

\[
\begin{array}{c}
(F, 2)\text{-UGph} \cong \text{PrTop} \subseteq \text{PsTop} \cong (F, 2)\text{-RGph}_P_s \\
\cong \uparrow \\
(\beta, 2)\text{-UGph} \subseteq (\beta, 2)\text{-RGph} \subseteq (F, 2)\text{-RGph} \\
\cong \downarrow \\
(\beta, 2)\text{-Gph} \subseteq (F, 2)\text{-Gph}.
\end{array}
\]

The horizontal arrows on the right are obtained by extension of the convergence relation: for \( a \in FX \) and \( y \in X \),

\[(a \rightarrow y) \iff \forall \chi \in \beta X \ (\chi \supseteq a \implies \chi \rightarrow y).\]

(4) We write \( \text{PsApp} := (\beta, P_+)\text{-Gph} = (\beta, P_+)\text{-RGph} \) (with the Barr extension of \( \beta \) to \( P_+\text{-Rel} \) of 2.4) and calls its objects pseudo-approach spaces. \( \text{PsApp} \) contains the full subcategory \( \text{PrApp} := (\beta, P_+)\text{-UGph} \) of pre-approach spaces whose objects may be described equivalently as sets equipped with a finitely additive distance function \( \delta : X \times PX \rightarrow [0, \infty] \) satisfying \( \delta(x, \{x\}) = 0 \) for all \( x \in X \). The needed isomorphism is established as in Theorem 2.4.5, with Proposition 2.4.4 providing the key ingredient to the proof. (See also Exercise 4.O.)

4.1.4 Proposition. The forgetful functor \( O : (\mathbb{T}, \mathbb{V})\text{-Gph} \rightarrow \text{Set} \) is topological. \( O \)-initial liftings of sources may be formed as for \( (\mathbb{T}, \mathbb{V})\text{-categories} \) (see Proposition 3.1.1), while a sink \((f_i : (X_i, a_i) \rightarrow (Y, b))_{i \in I}\) in \( (\mathbb{T}, \mathbb{V})\text{-Gph} \) is \( O \)-final precisely when

\[b = e^\vee \vee_{i \in I} f_i \cdot a_i \cdot (T f_i)^\circ.\]

(4.1.v)

For an epi-sink \((f_i)_{i \in I}\), this formula simplifies to \( b = \bigvee_{i \in I} f_i \cdot a_i \cdot (T f_i)^\circ \).

Proof. For a family \((X_i, a_i)\) of \((\mathbb{T}, \mathbb{V})\)-graphs and \text{Set}-maps \( f_i : X_i \rightarrow Y (i \in I) \), define \( b \) by (4.1.v). Trivially, \( b \) is reflexive, and every \((f_i : (X_i, a_i) \rightarrow (Y, b))\) has trivially become a \((\mathbb{T}, \mathbb{V})\)-functor. For \( O \)-finality of \((f_i)_{i \in I}\), consider a \((\mathbb{T}, \mathbb{V})\)-graph \((Z, c)\) and a map \( h : Y \rightarrow Z \) such that

\[h \cdot f_i \cdot a_i \leq c \cdot T(h \cdot f_i)\]

for all \( i \in I \). Then

\[h \cdot f_i \cdot a_i \cdot (T f_i)^\circ \leq c \cdot Th \cdot T f_i \cdot (T f_i)^\circ \leq c \cdot Th,
\]
and therefore \( f_i \cdot a_i \cdot (Tf_i)^o \leq h^o \cdot c \cdot Th \) by adjunction (for all \( i \in I \)). Since
\[
e^o_Y \leq e^o_Y \cdot (Th)^o \cdot Th = h^o \cdot e^o_Y \cdot Th \leq h^o \cdot c \cdot Th ,
\]
one has \( b \leq h^o \cdot c \cdot Th \), and \( h \) is a \((\mathbb{T}, \mathcal{V})\)-functor. This concludes the proof that \( O \) is topological, with \( O \)-final sinks characterized by \(4.1.vi\). That \( O \)-initial sources may be described as in Proposition \(3.1.1\) follows from a direct verification (see Exercise \(4.A\)). Finally, if \((f_i)_{i \in I} \) is epic, with Proposition \(1.2.2\) one obtains \( 1_Y \leq \bigvee_{i \in I} f_i \cdot f_i^o \). Consequently,
\[
e^o_Y \leq \bigvee_{i \in I} f_i \cdot f_i^o \cdot e^o_Y = \bigvee_{i \in I} f_i \cdot e^o_Y \cdot (Tf_i)^o \leq \bigvee_{i \in I} f_i \cdot a_i \cdot (Tf_i)^o .
\]

The category \((\mathbb{T}, \mathcal{V})\)-\text{RGph} is reflective in \((\mathbb{T}, \mathcal{V})\)-\text{Gph}, with the reflection morphisms given by \((X, a) \rightarrow (X, a \cdot \hat{T}1_X)\). Indeed, any \((\mathbb{T}, \mathcal{V})\)-functor \( f : (X, a) \rightarrow (Y, b) \) with \((Y, b)\) right unitary is a \((\mathbb{T}, \mathcal{V})\)-functor \( f : (X, a \cdot \hat{T}1_X) \rightarrow (Y, b)\):
\[
f \cdot a \cdot \hat{T}1_X \leq b \cdot Tf \cdot \hat{T}1_X \leq b \cdot \hat{T}f = b \cdot \hat{T}1_Y \cdot Tf = b \cdot Tf .
\]

4.1.5 Corollary. The forgetful functor \( O : (\mathbb{T}, \mathcal{V})\)-\text{RGph} \rightarrow \text{Set} \) is topological. \( O \)-initial liftings of sources may be formed as for \((\mathbb{T}, \mathcal{V})\)-categories (see Proposition \(3.1.1\), while a sink \((f_i : (X_i, a_i) \rightarrow (Y, b))_{i \in I} \) in \((\mathbb{T}, \mathcal{V})\)-\text{RGph} is \( O \)-final precisely when
\[
b = 1_Y^o \lor \bigvee_{i \in I} f_i \cdot a_i \cdot \hat{T}(f_i^o) . \tag{4.1.vi}
\]
For an epi-sink \((f_i)_{i \in I}\), this formula simplifies to \( b = \bigvee_{i \in I} f_i \cdot a_i \cdot \hat{T}(f_i^o) \).

Proof. We may apply Theorem \(\Pi 5.10.3\). \( O \)-final liftings in \((\mathbb{T}, \mathcal{V})\)-\text{RGph} are obtained by reflecting the \( O \)-final lifting in \((\mathbb{T}, \mathcal{V})\)-\text{Gph}. With \( b_0 \) denoting the codomain structure of the \( O \)-final lifting in \((\mathbb{T}, \mathcal{V})\)-\text{Gph}, an easy computation in the quantaloid \( \mathcal{V}\text{-Rel} \) shows
\[
b = b_0 \cdot \hat{T}1_Y = (e^o_Y \cdot \hat{T}1_Y) \lor \bigvee_{i \in I} (f_i \cdot a_i \cdot (Tf_i)^o \cdot \hat{T}1_Y) = 1_Y^o \lor \bigvee_{i \in I} f_i \cdot a_i \cdot \hat{T}(f_i^o) . \tag{4.1.vi}
\]

4.1.6 Remarks.

(1) Since \((\mathbb{T}, \mathcal{V})\)-\text{Cat} \(\hookrightarrow (\mathbb{T}, \mathcal{V})\)-\text{Gph} preserves initial sources, one may use the Taut Lift Theorem \(\Pi 5.11.1\) to obtain its reflectivity; likewise for \((\mathbb{T}, \mathcal{V})\)-\text{Cat} \(\hookrightarrow (\mathbb{T}, \mathcal{V})\)-\text{RGph}. The reflector provides a \((\mathbb{T}, \mathcal{V})\)-graph \((X, a)\) with the \((\mathbb{T}, \mathcal{V})\)-category structure
\[
\wedge \{c : X \rightarrow X \mid a \leq c, c \circ c \leq c \} .
\]
In \(4.2\) below, using ordinal recursion we give an alternative description of the reflector \((\mathbb{T}, \mathcal{V})\)-\text{RGph} \(\rightarrow (\mathbb{T}, \mathcal{V})\)-\text{Cat} that turns out to be essential in \(4.3\) when describing coproducts in \((\mathbb{T}, \mathcal{V})\)-\text{Cat}.

(2) A surjective morphism \( f : X \rightarrow Y \) in \((\mathbb{T}, \mathcal{V})\)-\text{RGph} with \( X \) in \((\mathbb{T}, \mathcal{V})\)-\text{Cat} that is final with respect to \( O : (\mathbb{T}, \mathcal{V})\)-\text{RGph} \(\rightarrow \text{Set} \) will generally fail to make \( Y \) into a \((\mathbb{T}, \mathcal{V})\)-category and be final with respect to the forgetful functor \((\mathbb{T}, \mathcal{V})\)-\text{Cat} \(\rightarrow \text{Set} \), even when \( \mathbb{T} = \mathbb{I} \) and \( \mathcal{V} = \mathbb{2} \) (Exercise \(1.G(3)\)). Nevertheless, the explicit description of \( O \)-final sinks in \((\mathbb{T}, \mathcal{V})\)-\text{RGph} turns out to be useful for the computation of colimits in \((\mathbb{T}, \mathcal{V})\)-\text{Cat} in special instances (see \(4.2\)).
4.2 Reflecting \((\mathbb{T}, \mathcal{V})\)-RGph into \((\mathbb{T}, \mathcal{V})\)-Cat. For a right unitary \((\mathbb{T}, \mathcal{V})\)-graph \((X, a)\) we define recursively (see II.1.14) an ascending chain of \((\mathbb{T}, \mathcal{V})\)-relations \(a_\nu\), for every ordinal \(\nu\), as follows:

\[
\begin{align*}
a_0 &:= e^\circ_X, \\
a_{\nu+1} &:= a \circ a_\nu, \\
a_\lambda &:= \bigvee_{\nu < \lambda} a_\nu \quad (\lambda \text{ a limit ordinal}).
\end{align*}
\]

Since \(a\) is right unitary, \(a_1 = a \circ e^\circ_X = a\), and reflexivity makes the chain ascending: \(a_\nu \leq e^\circ_X \circ a_\nu \leq a \circ a_\nu = a_{\nu+1}\). Consequently, this chain in the set \(\mathcal{V}\)-\(\text{Rel}(TX, X)\) must become stationary, that is,

\[
a_\mu = a_{\mu+1}
\]

for some ordinal number \(\mu\). Next we will show that under a mild additional assumption on the lax extension of \(\mathbb{T}\), the pair \((X, a_\infty) := (X, a_\mu)\) provides a reflection of \((X, a)\) into \((\mathbb{T}, \mathcal{V})\)-Cat.

4.2.1 Proposition. For the lax extension \(\hat{T} = (\hat{T}, m, e)\) of \(\mathbb{T}\), suppose that \(m^\circ : \hat{T} \to \hat{T}\hat{T}\) is a natural transformation. Then \((\mathbb{T}, \mathcal{V})\)-Cat is reflective in \((\mathbb{T}, \mathcal{V})\)-RGph, with the reflection morphisms obtained by ordinal recursion as described above.

Proof. When \(m^\circ\) is a natural transformation, then \((t \circ s) \circ r \leq t \circ (s \circ r)\) for all \((\mathbb{T}, \mathcal{V})\)-relations \(r : X \leftrightarrow Y, s : Y \leftrightarrow Z, t : Z \leftrightarrow W\) (Exercise I.I). In particular,

\[
(a \circ a_\nu) \circ a_\mu \leq a \circ (a_\nu \circ a_\mu)
\]

for all ordinals \(\nu, \mu\). Consequently, if \(a_\mu = a_{\mu+1}\), one can easily show \(a_\nu \circ a_\mu \leq a_\mu\) for all \(\nu\) by ordinal recursion, and \(a_\mu \circ a_\mu \leq a_\mu\) follows. Since \(e^\circ_X \leq a \leq a_\mu\), one has that \((X, a_\mu)\) is a \((\mathbb{T}, \mathcal{V})\)-category, and \(1_X : (X, a) \to (X, a_\mu)\) is a \((\mathbb{T}, \mathcal{V})\)-functor. It remains to be shown that every \((\mathbb{T}, \mathcal{V})\)-functor \(f : (X, a) \to (Y, b)\) with \((Y, b)\) a \((\mathbb{T}, \mathcal{V})\)-category gives a \((\mathbb{T}, \mathcal{V})\)-functor \(f : (X, a_\mu) \to (Y, b)\), and for that is suffices to show that \(a \leq f^\circ \cdot b \cdot Tf\) implies \(a_\nu \leq f^\circ \cdot b \cdot Tf\) for all ordinals \(\nu\). We show the successor step of the ordinal recursion:

\[
\begin{align*}
a_{\nu+1} &= a \circ a_\nu \\
\leq& (f^\circ \cdot b \cdot Tf) \circ (f^\circ \cdot b \cdot Tf) \\
\leq& f^\circ \cdot b \cdot Tf \circ (Tf)^\circ \cdot \hat{T}b \cdot TTf \cdot m_X^\circ \\
\leq& f^\circ \cdot b \cdot \hat{T}b \cdot m_Y^\circ \cdot Tf \\
=& f^\circ \cdot (b \circ b) \cdot Tf = f^\circ \cdot b \cdot Tf.
\end{align*}
\]

It follows from Proposition 4.2.1 that \((\mathbb{T}, \mathcal{V})\)-Cat is reflective in \((\mathbb{T}, \mathcal{V})\)-UGph, but there is no obvious simplification in this proof when applying the ordinal recursion to a unitary \((\mathbb{T}, \mathcal{V})\)-graph. However, under additional hypotheses on the lax extension of \(\mathbb{T}\), there is an easy one-step construction for a reflector \((\mathbb{T}, \mathcal{V})\)-RGph \(\to\) \((\mathbb{T}, \mathcal{V})\)-UGph, as follows.
**4.2.2 Proposition.** Suppose that the lax extension \( \hat{T} = (\hat{T}, m, e) \) of \( T \) to \( \mathcal{V}\text{-Rel} \) is associative. Then

\[
1_X : (X, a) \to (X, e_X^o \circ a)
\]

is a reflection of the right unitary \((\mathcal{T}, \mathcal{V})\)-graph \((X, a)\) into \((\mathcal{T}, \mathcal{V})\)-UGph.

**Proof.** By Proposition 1.9.4, \( e_X^o \circ a \) is right unitary because \( a \) is:

\[
(e_X^o \circ a) \circ 1^X = e_X^o \circ (a \circ 1^X) = e_X^o \circ a .
\]

Similarly, \( e_X^o \circ a = 1^X \circ a \) is also left unitary:

\[
1^X \circ (1^X \circ a) = (1^X \circ 1^X) \circ a = 1^X \circ a .
\]

Furthermore, if \( f : (X, a) \to (Y, b) \) is a \((\mathcal{T}, \mathcal{V})\)-functor with \((Y, b)\) unitary, \( f : (X, e_X^o \circ a) \to (Y, b) \) is also a \((\mathcal{T}, \mathcal{V})\)-functor:

\[
e_X^o \circ a \leq e_X \circ (f^o \cdot b \cdot Tf)
= e_X^o \cdot (f^o \cdot b \cdot Tf) \cdot m_X^o
= e_X^o \cdot (Tf)^o \cdot (\hat{T}b) \cdot TTf \cdot m_X^o
\leq f^o \cdot e_X^o \cdot (\hat{T}b) \cdot m_X^o \cdot Tf
\leq f^o \cdot (e_X^o \cdot b) \cdot Tf = f^o \cdot b \cdot Tf .
\]

\( \square \)

**4.3 Coproducts of \((\mathcal{T}, \mathcal{V})\)-categories.** In the topological category \((\mathcal{T}, \mathcal{V})\)-RGph over Set, the coproduct of a family \((X_i, a_i)\) of right unitary \((\mathcal{T}, \mathcal{V})\)-graphs is given by the disjoint union

\[
t_i : X_i \to X = \bigsqcup_{i \in I} X_i \quad (i \in I)
\]

endowed with the final structure \( a = \bigsqcup_{i \in I} t_i \cdot a_i \cdot \hat{T}(t_i^o) \) (see Corollary 4.1.5). Since

\[
t_j^o \cdot t_i = \begin{cases} 1_{X_j} & \text{if } j = i, \\ \bot & \text{otherwise,} \end{cases}
\]

each \( t_j \) is not just a \((\mathcal{T}, \mathcal{V})\)-functor but satisfies the equation

\[
t_j^o \cdot a = \bigsqcup_{i \in I} t_j^o \cdot t_i \cdot a_i \cdot \hat{T}(t_i^o) = a_j \cdot \hat{T}(t_j^o) .
\]

In what follows we use the following terminology on which we will elaborate further in Chapter [\( \nabla \nabla \nabla \nabla \)]

**4.3.1 Definition.** A morphism \( f : (X, a) \to (Y, b) \) in \((\mathcal{T}, \mathcal{V})\)-RGph is nearly open if \( f^o \cdot b \leq a \cdot \hat{T}(f^o) \), and open if \( f^o \cdot b \leq a \cdot (Tf)^o \).
Since every \((\mathbb{T}, \mathcal{V})\)-functor satisfies \(a \cdot (Tf)^{\circ} \leq f^{\circ} \cdot b\), near openness of \(f\) means equivalently \(f^{\circ} \cdot b = a \cdot \hat{T}(f^{\circ})\), and openness \(f^{\circ} \cdot b = a \cdot (Tf)^{\circ}\). Clearly, openness implies near openness, and the converse statement holds if
\[
\hat{T}(f^{\circ}) = \hat{T}1_{X} \cdot (Tf)^{\circ}
\] (4.3.i) for all maps \(f : X \rightarrow Y\).

4.3.2 Lemma. Let \(f : (X, a) \rightarrow (Y, b)\) be a \((\mathbb{T}, \mathcal{V})\)-functor from a \((\mathbb{T}, \mathcal{V})\)-category to a right unitary \((\mathbb{T}, \mathcal{V})\)-graph. Then near openness of \(f\) implies near openness of \(f : (X, b_{\infty}) \rightarrow (Y, b_{\infty})\) with \(b_{\infty} = \bigvee_{\nu} b_{\nu}\) (as in 4.2), provided that \(\hat{T}\) preserves \(\mathcal{V}\)-relational composition.

Proof. Proceeding to show \(f^{\circ} \cdot b_{\infty} \leq a \cdot \hat{T}(f^{\circ})\) by ordinal recursion, we consider the successor step and assume \(f^{\circ} \cdot b_{\nu} \leq a \cdot \hat{T}(f^{\circ})\). If \(\hat{T}\) preserves composition of \(\mathcal{V}\)-relations, we then obtain
\[
f^{\circ} \cdot b_{\nu+1} = f^{\circ} \cdot (b \circ b_{\nu}) = f^{\circ} \cdot b \cdot \hat{T}b_{\nu} \cdot m_{\nu}^{\mu} \leq a \cdot \hat{T}(f^{\circ}) \cdot \hat{T}b_{\nu} \cdot m_{\nu}^{\mu} \leq a \cdot \hat{T}(f^{\circ}) \cdot b_{\nu} \cdot m_{\nu}^{\mu} \leq a \cdot \hat{T}(a \cdot \hat{T}(f^{\circ})) \cdot m_{\nu}^{\mu} = a \cdot \hat{T}a \cdot \hat{T}(f^{\circ}) \cdot m_{\nu}^{\mu} \leq a \cdot m_{X} \cdot m_{\mu}^{\nu} \cdot \hat{T}(f^{\circ}) = a \cdot \hat{T}(f^{\circ}) .\]

4.3.3 Theorem. If \(\hat{T}\) is associative, then \((\mathbb{T}, \mathcal{V})\)-Cat is closed under coproducts in \((\mathbb{T}, \mathcal{V})\)-RGph.

Proof. For \((\mathbb{T}, \mathcal{V})\)-categories \((X, a_{i})\), one forms the coproduct \((t_{i} : (X, a_{i}) \rightarrow (X, a))_{i \in I}\) in \((\mathbb{T}, \mathcal{V})\)-RGph, with \(a = \bigvee_{i \in I} t_{i} \cdot a_{i} \cdot \hat{T}(t_{i}^{\circ})\), and one obtains the coproduct in \((\mathbb{T}, \mathcal{V})\)-Cat by applying the reflector \((X, a) \rightarrow (X, a_{\infty})\). Hence, we must show \(a_{\infty} = a\). But, by Lemma 4.3.2, every \(t_{i} : (X, a_{i}) \rightarrow (X, a_{\infty})\) is nearly open. Since \((t_{i})_{i \in I}\) is epic, near openness yields
\[
a_{\infty} = \bigvee_{i \in I} t_{i} \cdot t_{i}^{\circ} \cdot a_{\infty} = \bigvee_{i \in I} t_{i} \cdot a_{i} \cdot \hat{T}(t_{i}^{\circ}) = a .\]

4.3.4 Remarks.

1. Near openness of every \(t_{i} : (X, a_{i}) \rightarrow (X, a)\) \((i \in I)\) is characteristic for \(O\)-finality of the sink \((t_{i})_{i \in I}\) (see Exercise 4.B).

2. Since \(T\) preserves injections (see Exercise 4.P), every open injection \(f : (X, a) \rightarrow (Y, b)\) in \((\mathbb{T}, \mathcal{V})\)-Gph is \(O\)-initial. Indeed, \(f^{\circ} \cdot b \cdot Tf = a \cdot (Tf)^{\circ} \cdot Tf = a\) since \(Tf\) is injective. We refer to such \((\mathbb{T}, \mathcal{V})\)-functors as \textit{open embeddings}. 
(3) With \( \hat{T} \) an associative lax extension to \( \mathcal{V}\text{-Rel} \) satisfying \((4.3.i)\), a coproduct \((t_i : (X_i, a_i) \to (X, a))_{i \in I} \) in \((\mathcal{T}, \mathcal{V})\text{-Cat}\) satisfies

\[
a = \bigvee_{i \in I} t_i \cdot a_i \cdot (Tt_i)^\circ.
\]

Hence, for all \( \chi \in TX, y \in X \), one has

\[
a(\chi, y) = \begin{cases} a_i(\chi, y) & \text{if } \chi \in TX_i, y \in X_i \text{ for some } i \in I, \\ \bot & \text{else;} \end{cases}
\]

here we write \( \chi \in TX_i \) instead of \( \chi = Tt_i(\chi_i) \) for some \( \chi_i \in TX_i \).

(4) In Lemma \((4.3.2)\), the hypotheses that \( \hat{T} \) preserves \( \mathcal{V}\)-relational composition may be traded for the assumption that \( \hat{T}(r \cdot g) = \hat{T}r \cdot (Tg)^\circ \) for all \( r : X \to Y \) and \( g : Z \to X \). In fact, in this case a nearly open map \( f : (X, a) \to (Y, b) \) is open, since

\[
a \cdot \hat{T}(f^\circ) = a \cdot \hat{T}1_X \cdot (Tf)^\circ = a \cdot (Tf)^\circ.
\]

**4.3.5 Definition.** A functor \( T : \text{Set} \to \text{Set} \) is *taut* if it sends every BC-square

\[
\begin{array}{ccc}
M & \xrightarrow{g} & N \\
\downarrow m & & \downarrow n \\
X & \xrightarrow{f} & Y
\end{array}
\]

with \( n \) monic to a BC-square; that is, \( n^\circ \cdot f = g \cdot m^\circ \) with \( n \) monic implies \((Tn)^\circ \cdot Tf = Tg \cdot (Tm)^\circ\).

Clearly, a BC-square \((4.3.ii)\) with \( n \) monic is actually a pullback diagram: since the canonical morphism \( c : M \to X \times_Y N \) factors through the monomorphism \( m \), it is injective, but also surjective (by Lemma \((1.11.1)\)).

Trivially, a functor satisfying BC is taut, and every taut functor preserves monomorphisms, since \( m \) is a monomorphism if and only if

\[
\begin{array}{ccc}
M & \xrightarrow{1} & M \\
\downarrow 1 & & \downarrow m \\
M & \xrightarrow{m} & X
\end{array}
\]

is a BC-square (and then necessarily a pullback square).

**4.3.6 Corollary.** A Set-functor \( T \) is taut if and only if \( T \) preserves pullback squares \((4.3.ii)\) with \( n \) monic.

**Proof.** This is immediate from the previous discussion. \( \square \)
In what follows, in addition to tautness of $T$, we also need that the underlying ordered set of the quantale $V$ is *cartesian closed*, so that both $\otimes$ and $\land$ distribute over suprema in $V$. The status of this assumption for our purposes is clarified by the following Lemma.

4.3.7 Lemma. The following conditions are equivalent for a quantale $V$:

(i) $V$ is cartesian closed;

(ii) the underlying ordered set of $V$ is a frame;

(iii) the right Frobenius law

\[(r \land s \cdot f) \cdot f^\circ = r \cdot f^\circ \land s\]

holds in $V$-$\text{Rel}$ for all $f : X \to Y$, $r : X \to Z$, $s : Y \to Z$;

(iv) the left Frobenius law

\[f \cdot (r \land f^\circ \cdot s) = f \cdot r \land s\]

holds in $V$-$\text{Rel}$ for all $f : X \to Y$, $r : Z \to X$, $s : Z \to Y$;

Proof. Straightforward verifications.

4.3.8 Proposition. Let $V$ be a cartesian closed quantale. If $T$ is taut, then open embeddings are stable under pullback in $(T, V)$-$\text{RGph}$, and if $T$ satisfies BC, all open maps are stable under pullback.

Proof. Consider a pullback diagram

\[
\begin{array}{ccc}
(P, d) & \xrightarrow{q} & (Y, b) \\
\downarrow{p} & & \downarrow{g} \\
(X, a) & \xrightarrow{f} & (Z, c)
\end{array}
\]

in $(T, V)$-$\text{RGph}$ with $g$ open. Then $d$ is the $O$-initial structure with respect to $(p, q)$, that is, $d = (p^\circ \cdot a \cdot Tp) \land (q^\circ \cdot b \cdot Tq)$, and $g^\circ \cdot c = b \cdot (Tg)^\circ$. If $T$ satisfies BC, with Lemma 4.3.7 one obtains openness of $p$ as follows:

\[
p^\circ \cdot a = p^\circ \cdot (a \land a) \\
\leq p^\circ \cdot (a \land (f^\circ \cdot c \cdot Tf)) \\
\leq p^\circ \cdot a \land (p^\circ \cdot f^\circ \cdot c \cdot Tf) \\
= p^\circ \cdot a \land (q^\circ \cdot g^\circ \cdot c \cdot Tf) \\
= p^\circ \cdot a \land (q^\circ \cdot b \cdot (Tg)^\circ \cdot Tf) \\
= p^\circ \cdot a \land (q^\circ \cdot b \cdot Tq \cdot (Tp)^\circ) \quad \text{(BC)} \\
= (p^\circ \cdot a \cdot Tp) \land (q^\circ \cdot b \cdot Tq) \cdot (Tp)^\circ \quad \text{($V$ cartesian closed)} \\
= d \cdot (Tp)^\circ .
\]

Tautness of $T$ suffices in lieu of BC when $g$ is injective.

\[\square\]
One says that a coproduct \((t_i : X_i \to X)_{i \in I}\) in a category is universal (or stable under pullback) if for all morphisms \(f : Y \to X\) and pullback diagrams

\[
\begin{array}{ccc}
  Y_i & \xrightarrow{s_i} & Y \\
  f_i \downarrow & & \downarrow f \\
  X_i & \xrightarrow{t_i} & X
\end{array}
\]

the family \((s_i : Y_i \to Y)_{i \in I}\) is a coproduct.

**4.3.9 Theorem.** Suppose that \(\mathcal{V}\) is cartesian closed and \(T\) is taut. Then coproducts in \((\mathbb{T}, \mathcal{V})\)-RGph are universal if \(\hat{T}(f^\circ) = \hat{T}1_X \cdot (Tf)^\circ\) for all \(f : X \to Y\). If moreover \(\hat{T}\) is associative, coproducts are also universal in \((\mathbb{T}, \mathcal{V})\)-Cat.

*Proof.* Under the stated hypotheses, coproduct injections in \((\mathbb{T}, \mathcal{V})\)-RGph are open and stable under pullback. This implies universality of coproducts by Remark 4.3.4(1). The additional hypothesis makes sure that \((\mathbb{T}, \mathcal{V})\)-Cat is closed under coproducts in \((\mathbb{T}, \mathcal{V})\)-RGph. \(\square\)

A coproduct \((t_i : X_i \to X)_{i \in I}\) in a category is disjoint if for all \(i \neq j\), the pullback of \(t_i\) and \(t_j\) is given by an initial object in the category. This is certainly true in \(\text{Set}\) and, since there is only one structure on \(\emptyset\) in \((\mathbb{T}, \mathcal{V})\)-RGph and \((\mathbb{T}, \mathcal{V})\)-Cat, likewise in these categories. A category with coproducts (of small families of objects) and pullbacks is extensive if its coproducts are universal and disjoint (see also Exercise 4.E and V.5.1).

**4.3.10 Corollary.** Suppose that \(\mathcal{V}\) is cartesian closed, that \(T\) is taut, and that \(\hat{T}\) is associative. If \(\hat{T}(f^\circ) = \hat{T}1_X \cdot (Tf)^\circ\) for all \(f : X \to Y\), then \((\mathbb{T}, \mathcal{V})\)-RGph and \((\mathbb{T}, \mathcal{V})\)-Cat are extensive categories.

*Proof.* The result restates Theorem 4.3.9. \(\square\)

**4.4 Interlude on partial products and local cartesian closedness.** A finitely complete category \(\mathbb{C}\) is locally cartesian closed if its comma categories \(\mathbb{C}/Z\) are cartesian closed for all \(Z \in \text{ob} \mathbb{C}\). By definition of cartesian closedness, this means that for all \(f : X \to Z\) in \(\mathbb{C}\) the functor

\[
(\_ \times_Z X) : \mathbb{C}/Z \to \mathbb{C}/Z , \quad (Y, g) \mapsto (Y \times_Z X, g \cdot \pi_1 = f \cdot \pi_2)
\]

(with \(\pi_1\) and \(\pi_2\) the pullback projections) has a right adjoint. Since the domain functor \(\text{dom}_Z : \mathbb{C}/Z \to \mathbb{C}\) always has a right adjoint, given by

\[
W \mapsto (W \times Z, W \times Z \to Z)
\]

local cartesian closedness of \(\mathbb{C}\) makes the composite functor

\[
\text{dom}_Z((-) \times_Z X) : \mathbb{C}/Z \to \mathbb{C} , \quad (Y, g) \mapsto Y \times_Z X
\]
have a right adjoint.

Let us spell out the condition for a $C$-object $Y$ to admit a $(\text{dom}_Z((-) \times_Z X))$-couniversal arrow. Such an arrow is given by an object $(P, p : P \to Z)$ in $C/Z$ and a morphism $\varepsilon : P \times_Z X \to Y$ in $C$ such that for any $(Q, q : Q \to Z)$ in $C/Z$ and any $g : Q \times_Z X \to Y$ in $C$, there is a uniquely determined morphism $h : Q \to P$ with $p \cdot h = q$ and $\varepsilon \cdot (h \times_Z 1_X) = g$ in $C$:

One calls $P = P(Y, f)$ together with the projection $p$ and the evaluation $\varepsilon$ a partial product of $Y$ over $f$. The category $C$ has partial products if $P(Y, f)$ exists for all $(Y, f)$.

Note that for $f = 1_X$, $P(Y, 1_X) \cong Y \times X$ is simply a product. If $Z$ is a terminal object, the existence of $P(Y, ! : X \to Z)$ means precisely the existence of the exponential $Y^X$ in $C$, that is, the existence of a $((-) \times X)$-couniversal arrow for $Y$ in $C$ (see II.4.4). In particular, local cartesian closedness of $C$ implies cartesian closedness of $C$.

4.4.1 Examples.

1. The existence of $P = P(Y, f)$ means, by definition, the existence of a natural bijective correspondence

$$Q \longrightarrow P \cong \Pi_{z \in Z} \ Y^{f^{-1}z}.$$ 

Hence, when $Q$ is a terminal object, we see that in $C = \text{Set}$ the partial product $P$ must be, up to isomorphism,

$$P = \{ (j, z) \mid z \in Z, j : f^{-1}z \to Y \} \cong \Pi_{z \in Z} \ Y^{f^{-1}z},$$

with $p : (j, z) \mapsto z$ and $\varepsilon$ the evaluation map. In fact, one can write $P \times_Z X$ as

$$P \times_Z X = \{ (j, x) \mid x \in X, j : f^{-1}f(x) \to Y \} \cong \Pi_{x \in X} \ Y^{f^{-1}f(x)}$$

with $\varepsilon : (j, x) \mapsto j(x)$. In this way one sees that $\text{Set}$ has partial products.

2. $\text{Ord}$ is cartesian closed but fails to have partial products (see Exercise 4.F).

3. $\text{Top}$ fails to be cartesian closed (see Exercise 4.G).

4.4.2 Lemma. For a category $A$ with equalizers, a functor $F : A \to C/X$ has a right adjoint if $\text{dom}_X F : A \to C$ has a right adjoint.
Proof. Let $\kappa : \text{dom}_X \to \Delta X$ be the natural transformation with $\kappa_{(W,t)} = t$ for every object $(W, t)$ in $C/X$, and let $J$ be right adjoint to $\text{dom}_Z F$. Then by adjunction, $\kappa F : \text{dom}_X \to \Delta X$ corresponds to a natural transformation $\sigma : 1_A \to \Delta JX$. Given $(W, t)$, one forms the equalizer

$$G(W, t) \xrightarrow{Jt \sigma_JW} JW \xrightarrow{} JX$$

whose domain defines a right adjoint $G$ of $F$. Indeed, for every $A$-object $A$, one obtains the equalizer diagram in $\text{Set}$ given by the top row of

$$A(A, G(W, t)) \xrightarrow{} A(A, JW) \xrightarrow{A(A, Jt)} A(A, JX) \xrightarrow{} C(\text{dom}_X FA, W) \xrightarrow{\phi} C(\text{dom}_X FA, X)$$

in which the map $A(A, \sigma_JW)$ has constant value $\sigma_A$ by naturality of $\sigma$. Consequently, the map $\phi$ that makes the diagram commute (in the obvious sense) must send every $C$-morphism to $\kappa_{FA}$, which corresponds to $\sigma_A$ by adjunction. The equalizer of the two lower maps in $\text{Set}$ is

$$\{h : \text{dom}_X FA \to W \mid t \cdot h = \kappa_{FA}\} \cong C/X(FA, (W, t)),$$

which is therefore isomorphic to $A(A, G(W, t))$, naturally in $A$. \hfill \square

4.4.3 Proposition. For a morphism $f : X \to Z$ of a finitely complete category $C$, the following assertions are equivalent and characterize $f$ as an exponentiable object of $C/Z$:

(i) $f$ is exponentiable in $C/Z$, that is, $(-) \times_Z X : C/Z \to C/Z$ has a right adjoint;

(ii) for all $C$-objects $Y$, the partial product $P(Y, f)$ exists in $C$;

(iii) the pullback functor

$$f^* : C/Z \to C/X, \quad (Y, g) \mapsto (Y \times_Z X, \pi_2 : Y \times_Z X \to X)$$

has a right adjoint.

Proof. (i) $\implies$ (ii) follows from the definitions. Lemma 4.4.2 yields (ii) $\implies$ (iii) by commutativity of the diagram.
For (iii) ⇒ (i), consider the commutative diagram

\[
\begin{array}{ccc}
C/X & \xrightarrow{f^*} & C/Z \\
\downarrow f_1 & & \downarrow (-) \times_Z X \\
C/Z & \xrightarrow{-} & C/Z
\end{array}
\]

with \( f_1 : (W, t) \mapsto (W, f \cdot t) \). Actually, one easily shows \( f_1 \vdash f^* \) (so that \((-) \times_Z X\) is the functor part of a comonad on \( C/Z \)). Hence, if \( f^* \) has a right adjoint, then the composite functor \( f_1 f^* \) has a right adjoint. \( \square \)

4.4.4 Example. In \( C = \mathbf{Set} \), the right adjoint \( f_* : C/X \to C/Z \) of \( f^* \) sends \( (W, t) \) in \( C/X \) to \( (P, p) \), with

\[
P = \{ (j, z) \mid z \in Z, j : f^{-1} z \to W, t \cdot j = (f^{-1} \hookrightarrow X) \}
\]

and \( p : (j, z) \mapsto z \). One may write

\[
P \times Z X = \{ (j, x) \mid x \in X, j : f^{-1} f(x) \to W, t \cdot j = (f^{-1} f(x) \hookrightarrow X) \},
\]

with evaluation map \( \varepsilon : (j, x) \mapsto j(x) \); this \( \varepsilon \) is a \( C/X \)-morphism since \( t \cdot \varepsilon = \pi_2 \). When setting \( (W, t) = (Y \times X, Y \times X \to X) \), one retrieves the partial product \( P(Y, f) \) from this construction.

4.5 Local cartesian closedness of \((T, V)\)-Gph. Throughout this subsection we assume that

- \( V \) is cartesian closed and integral;
- \( T \) satisfies the Beck–Chevalley condition.

(In fact, the tensor product of the quantale \( V \) and the multiplication \( m \) of the monad \( T \) play no role in this subsection: it suffices that \( V \) is a frame and \( T \) an endofunctor equipped with a natural transformation \( e : 1_{\mathbf{Set}} \to T \). In particular, no lax extension of \( T \) to \( V\)-\mathbf{Rel} is needed.)

Our goal is to construct partial products in \((T, V)\)-Gph and thereby show local cartesian closedness of this category. For that, it is useful to point out that the ordered set \( V \) (considered as a category) is locally cartesian closed. Indeed, for \( \alpha, \beta, \gamma, v \in V \), one has the equivalence

\[
\begin{align*}
v \leq \gamma & \land (\alpha \to \beta) \\
v \leq \gamma & , \ v \land \alpha \leq \beta
\end{align*}
\]

where \( \alpha \to \beta \) denotes the internal hom on \( V \) with its cartesian structure \( \land \). Hence, if \( \alpha \leq \gamma \), the partial product \( P(\beta, \alpha \leq \gamma) \) exists in \( V \) and is given by \( \gamma \land (\alpha \to \beta) \).
Let us now consider \( f : (X,a) \to (Z,c) \) and \( (Y,b) \) in \((\mathbb{T},\mathcal{V})\)-\( \text{Gph} \), and form the set
\[
P = \{(j,z) \mid z \in Z, j : (X_z,a_z) \to (Y,b) \text{ a } (\mathbb{T},\mathcal{V})\text{-functor}\},
\]
where \( X_z = f^{-1}z \) and \( a_z \) is the restriction of \( a \). We can write
\[
W = P \times_Z X = \{(j,x) \mid x \in X, j : (X_{f(x)},a_{f(x)}) \to (Y,b)\}
\]
and define \( p : P \to X \) and \( \varepsilon : W \to Y \) as at the level of sets (see Example [4.4.1(1); but note that \( P \) is a subset of the partial product of \( Y \) over \( f \) in \( \text{Set} \)). The \( \mathcal{V}\)-relation \( d : TP \to P \) is defined by
\[
d(p,(j,z)) = c(Tp(p), z) \wedge \bigwedge_{w,x} (a(T\pi_2(w), x) \to b(T\varepsilon(w), j(x))) \quad (4.5.i)
\]
(for \( p \in TP, (j,z) \in P \)), with the meet ranging over all \( w \in TW \) with \( T\pi_1(w) = p \) and \( x \in X \) with \( f(x) = z \).

4.5.1 Theorem. The pair \( (P,d) \) is a partial product of \( (Y,b) \) over \( f : (X,a) \to (Z,c) \) in \((\mathbb{T},\mathcal{V})\)-\( \text{Gph} \).

Proof. We show reflexivity of \( d \), \((\mathbb{T},\mathcal{V})\)-functoriality of \( p \) and \( \varepsilon \), and verify the universal property of \( (P,d) \).

- \( d \) is reflexive. For \( (j,z) \in P \) one has
\[
k \leq c(e_Z(z),z) = c(e_Z \cdot p(j,z),z) = c(Tp(e_P(j,z)),z).
\]

We now consider the pullback diagrams
\[
\begin{array}{ccc}
X_z & \xrightarrow{s} & W & \xrightarrow{\pi_2} & X \\
\downarrow & & \downarrow & \swarrow f & \\
1 & \xrightarrow{r} & P & \xrightarrow{p} & Z
\end{array}
\]
in \( \text{Set} \), with \( r \) the constant map to \( (j,z) \) and \( s \) the map \((x' \mapsto (j,x'))\). Since for every \( w \in TW \) with \( T\pi_1(w) = e_P(j,z) \) and every \( x \in X \) with \( f(x) = z \) one has
\[
T\pi_1(w) = e_P \cdot r \cdot 1(x) = Tr(e_1((X)) \),
\]
and since \( T \) satisfies BC, there is \( \chi \in TX_z \) with \( T!(\chi) = e_1((X)) \) and \( Ts(\chi) = w \). Considering \( T(\pi_2 \cdot s) \) as a subset inclusion, one obtains by \((\mathbb{T},\mathcal{V})\)-functoriality of \( j : (X_z,a_z) \to (Y,b) \)
\[
a(T\pi_2(w),x) = a(T\pi_2 \cdot Ts(\chi),x) \\
= a(\chi, x) \\
\leq b(Tj(\chi), j(x)) \\
= b(T(\varepsilon \cdot s)(\chi), j(x)) = b(T\varepsilon(w), j(x)).
\]
Consequently, \( k \leq a(T\pi_2(w), x) \rightarrow b(T\varepsilon(w), j(x)) \) which implies \( k \leq d(e_p(j, z), (j, z)) \).

- \( p \) and \( \varepsilon \) are \((\mathbb{T}, \mathbb{V})\)-functors. Since \( d(p, (j, z)) \leq c(T\varepsilon(p), z) \) by definition of \( d \), \( p \) is a morphism in \((\mathbb{T}, \mathbb{V})\)-\textbf{Gph}. Denoting the structure of \( W \) by \( m \), for all \( w \in TW \) and \( (j, x) \in W \), one has by definition of \( d \)

\[
m(w, (j, x)) = d(T\pi_1(w), (j, f(x))) \land a(T\pi_2(w), x)
\leq (a(T\pi_2(w), x) \rightarrow b(T\varepsilon(w), j(x))) \land a(T\pi_2(w), x)
\leq b(T\varepsilon(w), j(x)) \land b(T\varepsilon(w), \varepsilon(j, x)).
\]

- \((P, d)\) satisfies the universal property. For morphisms \( q : (Q, l) \rightarrow (Z, c) \), \( g : (Q \times_Z X, n) \rightarrow (Y, b) \) in \((\mathbb{T}, \mathbb{V})\)-\textbf{Gph}, we must show that the map

\[
h : Q \rightarrow P \quad \text{ with } \quad j_t : X_{q(t)} \rightarrow Y \quad \text{ and } \quad x \mapsto g(t, x)
\]

is a well-defined \((\mathbb{T}, \mathbb{V})\)-functor. For \( t \in Q \) consider the pullback diagrams

\[
\begin{array}{ccc}
X_{q(t)} & \xrightarrow{a} & U \\
\downarrow{!} & & \downarrow{\pi_2} \\
1 & \xrightarrow{t} & Q \\
\end{array}
\]

in \textbf{Set}, with \( U = Q \times_Z X \) and \( u \) the map \((x \mapsto (t, x))\). Since \( k = \mathbb{T} \) one has \( t : (1, \mathbb{T}) \rightarrow (Q, l) \). Consequently, \( u : (X_{q(t)}, a_{q(t)}) \rightarrow (U, n) \) and

\[
j = g \cdot u : (X_{q(t)}, a_{q(t)}) \rightarrow (Y, b)
\]

are \((\mathbb{T}, \mathbb{V})\)-functors. Hence, \((j_t, q(t)) \in P\). Finally, for \( q \in TQ \), \( t \in Q \), we first note

\[
l(q, t) \leq c(Tq(q), q(t)) = c(Tp(Th(q)), q(t)).
\]

Exploiting BC for the left pullback diagram in

\[
\begin{array}{ccc}
U & \xrightarrow{h \times 1_X} & W \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
Q & \xrightarrow{h} & P \\
\end{array}
\]

for every \( w \in TW \) with \( T\pi_1(w) = Th(q) \) we obtain \( u \in TU \) with \( T\pi_1(u) = q \) and \( T(h \times 1_X)(u) = w \), hence \( T\pi_2(u) = T\pi_2(w) \). Now, for every \( x \in X_{q(t)} \) we have

\[
l(q, t) \land a(T\pi_2(w), x) = l(T\pi_1(u), t) \land a(T\pi_2(u), x)
= n(u, (t, x))
\leq b(Tg(u), g(t, x))
= b(T\varepsilon(T(h \times 1_X)(u)), g(t, x)) = b(T\varepsilon(w), j_t(x))
\]

Consequently, \( l(q, t) \leq a(T\pi_2(w), x) \rightarrow b(T\varepsilon(w), j_t(x)) \) and \( l(q, t) \leq d(Th(q), h(t)) \) follows.
4.5.2 Corollary. The category \((\mathbb{T}, \mathcal{V})\)-Gph is locally cartesian closed.

4.5.3 Examples.

(1) For \(\mathbb{T} = \mathbb{1}\) identically extended to \(\mathcal{V}\)-Rel, every \((\mathbb{T}, \mathcal{V})\)-graph is unitary, Corollary 4.5.2 gives that the category\n
\[\mathcal{V}\text{-Gph} := (\mathbb{1}, \mathcal{V})\text{-Gph} = (\mathbb{1}, \mathcal{V})\text{-UGph}\]

is locally cartesian closed whenever \(\mathcal{V}\) is cartesian closed with \(k = \top\). For \(\mathcal{V} = 2\), this is the category

\[\mathbb{R}\text{-Rel} = 2\text{-Gph}\]

of reflexive relations, that is, of sets endowed with a reflexive relation which, even in the absence of transitivity, we write as \(\leq\); morphisms are then “monotone” maps. Keeping the notation of 4.4, by Theorem 4.5.1 one can form the partial product \(P = P(Y, f)\) in \(\mathbb{R}\text{-Rel}\) by endowing the set

\[P = \{(j, z) \mid z \in Z, j : X_z \to Y \text{ monotone}\}\]

with the relation

\[(j, z) \leq (j', z') \iff z \leq z' \land \forall x \in X_z, x' \in X_{z'} (x \leq x' \implies j(x) \leq j'(x')).\]

While \(p\) and \(\varepsilon\) are defined as for sets (see Example 4.4.1(1)), we note that \(O : \mathbb{R}\text{-Rel} \to \text{Set}\) does not preserve partial products, not even exponentials.

(2) For general \(\mathcal{V}\) (as in (1) above), Theorem 4.5.1 constructs partial products in \(\mathcal{V}\)-Gph as follows. For \(f : (X, a) \to (Z, c)\) and \((Y, b)\) in \(\mathcal{V}\)-Gph and

\[P = \{(j, z) \mid z \in Z, j : (X_z, a_z) \to (Y, b) \text{ a } \mathcal{V}\text{-functor}\},\]

provide \((P, d) = P(Y, f)\) with the structure

\[d((j, z), (j', z')) = c(z, z') \land \sup_{x \in X_z, x' \in X_{z'}} (a(x, x') \to b(j(x), j'(x'))).\]

In particular, in the category

\[\mathbb{R}\text{NRel} := P_{\text{max}}\text{-Gph}\]

of reflexive numerical relations and non-expansive maps, the “metric” \(d\) on the partial product \(P\) of \(f\) over \(Y\) is given by

\[d((z, j), (z', j')) = \max\{c(z, z'), \sup_{x \in X_z, x' \in X_{z'}} (a(x, x') \to b(j(x), j'(x'))\}\]

(see Example II.10.1(5)).
4. EMBEDDING LAX ALGEBRAS INTO A QUASITOPOS

(3) Theorem 4.5.1 shows that
\[ \text{PsTop} = (\beta, 2)\text{-Gph} = (\beta, 2)\text{-RGph} \]
is locally cartesian closed, with the ultrafilter convergence \( \rightarrow \) on
\[ P = P(Y, f) = \{(j, z) | z \in Z, j : X_z \to Y \text{ continuous}\} \]
described as follows: \( p \to (j, z) \in P \) if and only if
- \( p(p) \to z \) in \( Z \) (with \( p: P \to Z \) the projection),
- for all \( x \in X_z, w \in \beta(P \times Z X) \) with \( \pi_1[w] = p, \pi_2[w] \to x \) in \( X \) implies \( \varepsilon[w] \to j(x) \) in \( Y \) (with \( \varepsilon: P \times Z X \to Y \) the evaluation map). With ultrafilters traded for filters, the same condition describes partial products in \( (\mathbb{T}, 2)\text{-Gph} \) (see Example 4.1.3(3)).

4.6 Local cartesian closedness of subcategories of \( (\mathbb{T}, \mathcal{V})\text{-Gph} \). Throughout this subsection we assume that
- \( \mathcal{V} \) is cartesian closed and integral;
- \( T \) satisfies the Beck–Chevalley condition.

Let us first give sufficient conditions for cartesian and local cartesian closedness of reflective or coreflective full subcategories of a cartesian or locally cartesian closed category.

4.6.1 Proposition. Let \( A \) be a full replete subcategory of a cartesian closed category with finite products.

(1) If \( A \) is reflective, with the reflector preserving binary products, then \( A \) is cartesian closed with exponentials (that is, internal hom-objects) formed as in \( X \).

(2) If \( A \) is coreflective and closed under binary products in \( X \), then \( A \) is cartesian closed with exponentials in \( A \) obtained by coreflection of exponentials formed in \( X \).

Proof.

(1) For \( A \)-objects \( A \) and \( B \), let \( B^A \) be the exponential formed in \( X \) with counit \( \varepsilon : B^A \times A \to B \), and let \( \rho : B^A \to R(B^A) \) be the reflection into \( A \). Since \( R(B^A \times A) \cong R(B^A) \times A \) by hypothesis, the morphism \( \rho \times 1_A : B^A \times A \to R(B^A) \times A \) serves as a reflection into \( A \). Hence, \( \varepsilon \) factors uniquely as \( \varepsilon = \varepsilon \cdot (\rho \times 1_A) \). The mate \( s \) of \( \varepsilon \) that makes

\[
\begin{array}{ccc}
A \times B^A & \xrightarrow{\varepsilon} & B \\
\downarrow^{1_A \times s} & & \uparrow^{\varepsilon} \\
A \times R(B^A)
\end{array}
\]

commute must be inverse to \( \rho \). Consequently, \( B^A \cong R(B^A) \) lies in \( A \) and serves there in the same capacity as in \( X \).
4.6.2 Corollary. Let $A$ be a full replete subcategory of a locally cartesian closed category with pullbacks.

1. If $A$ is reflective, with the reflector preserving pullbacks, then $A$ is locally cartesian closed with partial products formed as in $X$.

2. If $A$ is coreflective and closed under pullbacks in $X$, then $A$ is locally cartesian closed with partial products in $A$ obtained by coreflection of partial products formed in $X$.

Proof. Apply Proposition 4.6.1 to the reflective or coreflective subcategory $A/B$ of $X/B$ (for $B \in \text{ob} A$).

4.6.3 Corollary. If the functor $R : (T, V)\text{-Gph} \to (T, V)\text{-RGph}$, sending $(X, a)$ to $(X, a \cdot \hat{T}1_X)$, preserves binary products, then $(T, V)\text{-RGph}$ is cartesian closed with exponentials formed as in $(T, V)\text{-Gph}$. Similarly, if $R : (T, V)\text{-Gph} \to (T, V)\text{-RGph}$ preserves pullbacks, then $(T, V)\text{-RGph}$ is locally cartesian closed with partial products formed as in $(T, V)\text{-Gph}$.

Proof. By Corollary 4.1.5, $R$ is the reflector of $(T, V)\text{-Gph}$ onto $(T, V)\text{-RGph}$. Hence, Proposition 4.6.1 and Corollary 4.6.2 may be applied.

4.6.4 Example. With the filter monad $\mathbb{F}$ extended by $\hat{\mathbb{F}}$ (as in Example 1.10.3(4)), the reflector $R : (\mathbb{F}, 2)\text{-Gph} \to (\mathbb{F}, 2)\text{-RGph}$ extends the reflexive filter convergence relation $\rightarrow$ on a set $X$ to

$$a \rightarrow x \iff \exists b \in FX (a \supset b \& b \rightarrow x)$$

for all $x \in X$, $a \in FX$. It is not difficult to show that $R$ preserves finite limits. Hence, $(\mathbb{F}, 2)\text{-RGph}$ is locally cartesian closed with partial products formed as in $(\mathbb{F}, 2)\text{-Gph}$.

In addition to the two general hypotheses on $\hat{T}$ and $V$ stated at the beginning of 4.6, we now assume that

- $\hat{T}$ is associative;
- $\hat{T}(f^\circ) = \hat{T}1_X \cdot (Tf)^\circ$ for all $f : X \to Y$ (see 4.3.i).

Under these hypotheses, not only coproduct sinks in $(T, V)\text{-RGph}$ are universal (as proved in Theorem 4.3.9), but so are all $O$-final epi-sinks, where $O : (T, V)\text{-RGph} \to \text{Set}$ is the forgetful functor.

4.6.5 Proposition. $O$-final epi-sinks are stable under pullback in $(T, V)\text{-RGph}$.
Proof. For an O-final epi-sink \((g_i : (Y_i, b_i) \rightarrow (Z, c))_{i \in I}\) in \((\mathbb{T}, \mathbb{V})\)-RGph, the formula given in Corollary 4.1.5 simplifies under the additional hypotheses to \(c = \bigvee_{i \in I} g_i \cdot b_i \cdot (Tg_i)\). When for \(f : (X, a) \rightarrow (Z, c)\) we form the pullbacks
\[
\begin{array}{c}
(P_i, d_i) \xrightarrow{p_i} (X, a) \\
q_i \downarrow \hspace{1cm} f \\
(Y_i, b_i) \xrightarrow{g_i} (Z, c)
\end{array}
\]
we have \(d_i = (p_i^\circ \cdot a \cdot Tp_i) \land (q_i^\circ \cdot b_i \cdot Tq_i)\) for all \(i \in I\) and need to show \(a = \bigvee_{i \in I} p_i \cdot d_i \cdot (Tp_i)^\circ\):

\[
\begin{align*}
\bigvee_{i \in I} p_i \cdot d_i \cdot (Tp_i)^\circ &= \bigvee_{i \in I} p_i \cdot \left((p_i^\circ \cdot a \cdot Tp_i) \land (q_i^\circ \cdot b_i \cdot Tq_i)\right) \cdot (Tp_i)^\circ \\
&= \bigvee_{i \in I} a \land (p_i \cdot q_i^\circ \cdot b_i \cdot Tq_i \cdot (Tp_i)^\circ) \\
&= a \land \bigvee_{i \in I} f^\circ \cdot g_i \cdot b_i \cdot (Tg_i)^\circ \cdot Tf \\
&= a \land (f^\circ \cdot c \cdot Tf) = a .
\end{align*}
\]

\(\square\)

4.6.6 Corollary. Colimits are stable under pullback in \((\mathbb{T}, \mathbb{V})\)-RGph.

Proof. Colimit sinks in \((\mathbb{T}, \mathbb{V})\)-RGph are those O-final epi-sinks which are colimits in Set.\(\square\)

For \(f : (X, a) \rightarrow (Z, c)\) in \((\mathbb{T}, \mathbb{V})\)-RGph we now consider the commutative diagram
\[
\begin{array}{cc}
\mathbb{T}, \mathbb{V}\text{-RGph}/(Z, c) & \xrightarrow{f^\ast} \mathbb{T}, \mathbb{V}\text{-RGph}/(X, a) \\
\xrightarrow{O_Z} & \xrightarrow{O_X}
\end{array}
\]
\[
\begin{array}{cc}
\text{Set}/Z & \xrightarrow{\text{set}} \text{Set}/X \\
\xrightarrow{\text{dom}_X} & \xrightarrow{\text{dom}_X}
\end{array}
\]
\(4.6.\text{i})

Since \(O\) is topological, \(O_X\) is also topological (and so is \(O_Z\)), with \(O_X\)-final structures formed like \(O\)-final structures, that is, the upper row domain functor \(\text{dom}_X\) (see 4.4) transforms \(O_X\)-final structures into \(O\)-final sinks (see Exercise 4.H). More importantly, by Proposition 4.6.5, the upper row pullback functor \(f^\ast\) (see Proposition 4.4.3) sends \(O_Z\)-final epi-sinks to \(O_X\)-final epi-sinks. In particular, \(f^\ast\) preserves all colimits. In fact, \(f^\ast\) has a right adjoint by virtue of the Special Adjoint Functor Theorem (see Exercise 15.1).

4.6.7 Theorem. Suppose that \(\mathbb{T}\) is associative and \(\mathbb{T}(f^\circ) = \mathbb{T}1_X \cdot (Tf)^\circ\) for all \(f : X \rightarrow Y\). Then the category \((\mathbb{T}, \mathbb{V})\)-RGph is locally cartesian closed and therefore has all partial products.

Proof. We must verify that the sufficient conditions of the dual of the Special Adjoint Functor Theorem are satisfied by the functor \(f^\ast\). But the topological category \((\mathbb{T}, \mathbb{V})\)-RGph over Set inherits the relevant properties from Set and passes them on to its comma categories. In fact, for every category \(\mathbb{C}\) and \(\mathbb{C}\)-object \(X\), one easily verifies the following facts:

1. if \(\mathbb{C}\) is locally small, then \(\mathbb{C}/X\) is locally small (because \(\text{dom}_X : \mathbb{C}/X \rightarrow \mathbb{C}\) is faithful);
(2) if $C$ is wellpowered, then $C/X$ is wellpowered (because $\text{dom}_X$ preserves and reflects isomorphisms);

(3) if $C$ is small-complete, then $C/X$ is small-complete (see Exercise II.2.J);

(4) if $C$ is locally small and $\mathcal{G}$ is generating in $C$, then the class of arrows $\bigcup_{G \in \mathcal{G}} C(G, X)$ is generating in $C/X$.

4.6.8 Remark. It is important to notice that in general $f^*$ will not transform all $O_Z$-final sinks into $O_X$-final sinks, but only those that are epic. Otherwise, we could have applied the Taut Lift Theorem (II.5.11) to (4.6.i), with the result that the lower-row right adjoints would have been lifted along the vertical topological functors. But a partial product in $(\mathbb{T}, \mathbb{V})$-$\text{Gph}$ is generally not formed by endowing the partial product of the underlying $\text{Set}$-data with a $(\mathbb{T}, \mathbb{V})$-graph structure. However, one can apply the Generalized Taut Lift Theorem (Exercise II.5.U) as an alternate method of proof of Theorem 4.6.7.

4.7 Interlude on subobject classifiers and partial-map classifiers. Throughout this subsection we consider a class $\mathcal{M}$ of morphisms in a category $C$ with pullbacks such that:

- $\text{SplitMono} C \subseteq \mathcal{M} \subseteq \text{Mono} C$;
- $\mathcal{M}$ is closed under composition with isomorphisms;
- $\mathcal{M}$ is stable under pullback.

We adopt the notation of II.5.2 and write $\text{sub} X = \text{sub}_{\mathcal{M}} X := \mathcal{M}/X$ for the full subcategory of $C/X$ whose objects lie in $\mathcal{M}$, and assume throughout that the separated reflection (see II.1.3)

$$\text{sub} X \rightarrow \text{sub}^\simeq X := \text{sub} X/ \simeq$$

comes with a section in $\text{Set}$, so that it is a split epimorphism in $\text{Set}$. Since the pullback functors $f^*: C/Y \rightarrow C/X$ restrict to $f^*: \text{sub} Y \rightarrow \text{sub} X$ and satisfy $(1_X)^* \simeq 1_{\text{sub} X}$ and $(g \cdot f)^* \cong f^*g^*$ (for $f : X \rightarrow Y$, $g : Y \rightarrow Z$ in $C$), one has a functor

$$\text{sub}^\simeq : \text{C}^{\text{op}} \rightarrow \text{Set}.$$ 

An $\mathcal{M}$-subobject classifier is a $C$-object $\Omega$ that represents $\text{sub}^\simeq$, so that $C(\_ , \Omega) \simeq \text{sub}^\simeq$. Hence, $\Omega$ is characterized by the existence of maps

$$\psi_X : C(X, \Omega) \rightarrow \text{sub} X$$

that are pseudo-natural in $X$ (with respect to the order of sub $X$) and are such that for every subobject $m : M \rightarrow X$ there is a unique morphism $f : X \rightarrow \Omega$ with $\psi_X(f) \simeq m$;
one calls $f$ the characteristic morphism of $m$. Here is the hands-on characterization of $\mathcal{M}$-subobject classifiers:

**4.7.1 Proposition.** An object $\Omega$ is an $\mathcal{M}$-subobject classifier in $C$ if and only if there is an $\mathcal{M}$-subobject $t : 1 \to \Omega$ (with 1 a terminal object of $C$) such that, for any $\mathcal{M}$-subobject $m : M \to X$ there is a unique morphism $f : X \to \Omega$ making

$$
\begin{array}{ccc}
M & \longrightarrow & 1 \\
\downarrow m & & \downarrow t \\
X & \underset{f}{\longrightarrow} & \Omega
\end{array}
$$

(4.7.i)
a pullback diagram.

**Proof.** For an $\mathcal{M}$-subobject classifier $\Omega$, one puts $t := \psi_\Omega(1_\Omega) : Z \to \Omega$. Then, using pseudo-naturality of $\psi$, to obtain the characteristic morphism $f$ of $m : M \to X$ in $\mathcal{M}$ we can chase $1_\Omega \in C(\Omega, \Omega)$ in two ways around the diagram

$$
\begin{array}{ccc}
C(\Omega, \Omega) & \longrightarrow & \text{sub } \Omega \\
\downarrow C(\psi, \Omega) & & \downarrow f^* \\
C(\psi, X, \Omega) & \longrightarrow & \Omega
\end{array}
$$

yielding $f^*(t) \simeq m$, so that (4.7.i) is a pullback diagram with 1 replaced by $Z$. We now show that $Z$ must necessarily be terminal in $C$. Considering $m = 1_X$ for any object $X$, we obtain a pullback diagram

$$
\begin{array}{ccc}
X & \underset{u}{\longrightarrow} & Z \\
\downarrow 1_X & & \downarrow t \\
X & \underset{f}{\longrightarrow} & \Omega
\end{array}
$$

For any other $v : X \to Z$,

$$
\begin{array}{ccc}
X & \underset{v}{\longrightarrow} & Z \\
\downarrow 1_X & & \downarrow t \\
X & \underset{t \cdot v}{\longrightarrow} & \Omega
\end{array}
$$

is a pullback diagram since $t$ is a monomorphism. Hence, $t \cdot v$ is a characteristic morphism for $1_X$, and we must have $t \cdot v = f = t \cdot u$, which implies $u = v$. The sufficiency condition is left to the reader. \qed

**4.7.2 Remarks.**

1. The proof of Proposition 4.7.1 shows that a category $C$ with an $\mathcal{M}$-subobject classifier necessarily has a terminal object and, in the presence of pullbacks, is therefore finitely complete. Furthermore, if $C$ is locally small, $C$ must be $\mathcal{M}$-wellpowered.
(2) Since \( m \) in (4.7.1) is a pullback of the split monomorphism \( 1 \to \Omega \), one necessarily has \( \mathcal{M} \subseteq \text{RegMono}_C \) if \( C \) has an \( \mathcal{M} \)-subobject classifier; we speak of a \textit{regular-subobject classifier} if \( \mathcal{M} = \text{RegMono}_C \), while \textit{subobject classifier} refers to the case \( \mathcal{M} = \text{Mono}_C \). Hence, the existence of a subobject classifier in \( C \) forces \( \text{Mono}_C = \text{RegMono}_C \).

4.7.3 Examples.

(1) Every two-element set \( 2 \) is a subobject classifier in \( \text{Set} \), since \( 2 \) represents the powerset functor \( \mathcal{P} \) of \( \text{Set} \). Less trivially, every functor category \( \text{Set}^{\text{op}} \) has a subobject classifier for a small category \( \mathcal{D} \) (see [Johnstone, 1977], [Mac Lane and Moerdijk, 1994]).

(2) If \( U : A \to X \) is topological with right adjoint \( I \), and \( \Omega \) is a regular-subobject classifier in \( X \), then \( I\Omega \) is a regular-subobject classifier in \( A \), see Exercise [4.J]. In particular, a two-element set will assume that role both in \( \text{Ord} \) and in \( \text{Top} \) when provided with its respective indiscrete structure.

An \( \mathcal{M} \)-\textit{partial map} from \( X \) to \( Z \) in \( C \) consists of an \( \mathcal{M} \)-subobject \( m : M \to X \) and a morphism \( g : M \to Z \). The (possibly large) class
\[
\text{part}(X, Z) = \text{part}_\mathcal{M}(X, Z)
\]
of all \( \mathcal{M} \)-partial maps from \( X \) to \( Z \) is ordered by
\[
(m, g) \leq (m', g') \iff \exists h \ (m' \cdot h = m \& g' \cdot h = g) .
\]
For \( f : X \to Y \) in \( C \) there is now a pullback functor
\[
f^* : \text{part}(Y, Z) \to \text{part}(X, Z) , \quad (n, g) \mapsto (f^*(n), g \cdot f_n) ,
\]
(with \( n : N \to Y \) and \( g : N \to Z \)) defined by the pullback diagram
\[
\begin{array}{ccc}
  f_n & \rightarrow & N \\
  \downarrow f^*(n) & \nearrow \downarrow n \\
X & \rightarrow & Y
\end{array}
\]
As for \textit{sub}, when taking isomorphism classes of \( \mathcal{M} \)-partial maps, \( \text{part}(\_ , Z) \) becomes a functor
\[
\text{part}^\cong(\_ , Z) : C^{\text{op}} \to \text{SET} .
\]
Note that for a terminal object \( Z = 1 \), the functor \( \text{part}^\cong(\_ , Z) \) is isomorphic to \( \text{sub}^\cong \). An \( \mathcal{M} \)-\textit{partial-map classifier} for \( Z \) is an object \( Z^* \) that represents \( \text{part}^\cong(\_ , Z) \), so that \( C(\_ , Z^*) \cong \text{part}^\cong(\_ , Z) \). Hence, \( Z^* \) is characterized by the existence of maps
\[
\phi_X : C(X, Z^*) \to \text{part}(X, Z)
\]
that are pseudo-natural in $X$ and are such that for all $(m, g) \in \text{part}(X, Z)$, there is a unique morphism $f : X \to Z^*$ with $\phi_X(f) \simeq (m, g)$.

4.7.4 Proposition. Let $Z$ be an object in $C$. An object $Z^*$ is an $\mathcal{M}$-partial-map classifier for $Z$ if and only if there is a morphism $t_Z : Z \to Z^*$ in $\mathcal{M}$ such that, for any $\mathcal{M}$-partial map $(m, g)$ from $X$ to $Z$, there is a unique morphism $f : X \to Z^*$ making

\[
\begin{array}{ccc}
M & \xrightarrow{g} & Z \\
m \downarrow & & \downarrow t_Z \\
X & \xleftarrow{f} & Z^*
\end{array}
\]

a pullback diagram.

Proof. If $Z^*$ is an $\mathcal{M}$-partial-map classifier for $Z$, one sets

$$(t_Z : W_Z \to Z^*, s_Z : W_Z \to Z) := \phi_{Z^*}(1_{Z^*}) .$$

Given $(m, g)$ from $X$ to $Z$, let $f : X \to Z^*$ be such that $\phi_X \simeq (m, g)$. Chasing $1_{Z^*}$ around the pseudo-naturality diagram

\[
\begin{array}{ccc}
C(Z^*, Z^*) & \xrightarrow{\phi_{Z^*}} & \text{part}(Z^*, Z) \\
C(f, Z^*) \downarrow & & \downarrow f^* \\
C(X, Z^*) & \xrightarrow{\phi_X} & \text{part}(X, Z)
\end{array}
\]

one obtains $(m, g) \simeq (f^*(t_Z), s_Z \cdot f_{t_Z})$, as in the diagram

\[
\begin{array}{ccc}
W_Z & \xrightarrow{s_Z} & Z \\
t_Z \downarrow & & \downarrow t_Z \\
X & \xrightarrow{f} & Z^*. 
\end{array}
\]

A comparison with (4.7.ii) shows that if $s_Z$ is an isomorphism, then the necessity part of the statement is proved. When we specialize $(m, g) = (1_Z, 1_Z)$, one obtains $s_Z \cdot f_{t_Z}$ for the corresponding morphism $f : Z \to Z^*$. Furthermore, chasing $f \in C(Z, Z^*)$ both ways through

\[
\begin{array}{ccc}
C(Z^*, Z^*) & \xrightarrow{\phi_{Z^*}} & \text{part}(Z, Z) \\
C(s_Z, Z^*) \downarrow & & \downarrow s_Z^* \\
C(W_Z, Z^*) & \xrightarrow{\phi_{W_Z}} & \text{part}(W_Z, Z)
\end{array}
\]

we obtain $\phi_{W_Z}(f \cdot s_Z) \simeq (1_{W_Z}, s_Z)$. Since $\phi_{W_Z}(t_Z) \simeq (1_{W_Z}, s_Z)$ trivially, we conclude $f \cdot s_Z \simeq t_Z$. Hence, the split epimorphism $s_Z$ is a monomorphism, and therefore an isomorphism.

The proof of the sufficiency of the stated condition is left to the reader. $\square$
As for $\mathcal{M}$-subobject classifiers, the term *partial-map classifier* refers to the case $\mathcal{M} = \text{Mono}_C$, and the term *regular-partial-map classifier* to the case $\mathcal{M} = \text{RegMono}_C$. Actually, as we shall see below in 4.7.6, the general hypotheses on $\mathcal{M}$ leave no choice for $\mathcal{M}$ when there is an $\mathcal{M}$-partial-map classifier.

4.7.5 Examples.

(1) A partial-map classifier for $Z$ in $\text{Set}$ is $Z^* = Z + 1$. More generally, such classifiers exist in every functor category $\text{Set}^{C^{\text{op}}}$ with $C$ small (see Johnstone [1977], Mac Lane and Moerdijk [1994]).

(2) Unlike subobject classifiers, partial-map classifiers may not be lifted along topological functors. Indeed, the category $\text{Ord}$ fails to have a regular-partial-map classifier for the two-chain $2 = \{0 < 1\}$, see Exercise 4.K. A similar statement holds for $\text{Top}$. $\text{Top}$ does have $\mathcal{M}$-partial-map classifiers for all spaces when $\mathcal{M}$ is the class of open embeddings or the class of closed embeddings. However, for these choices of $\mathcal{M}$ the general hypothesis that $\mathcal{M}$ contain all split monomorphisms fails, while all others are satisfied.

4.7.6 Corollary. In a finitely complete category $C$ with an $\mathcal{M}$-partial-map classifier, $\mathcal{M}$ is precisely the class of equalizers in $C$.

Proof. Setting $g = m \in \mathcal{M}$ in (4.7.ii), one sees that $m$ is an equalizer of $t_X$ and $f$ for some $f : X \to X^*$. Conversely, if $m : M \to X$ is the equalizer of some pair $g, h : X \to Y$ of morphisms, then $m$ is a pullback of the split monomorphism $\langle 1_X, h \rangle$ along $\langle 1_X, g \rangle$ and must therefore be a morphism in $\mathcal{M}$:

$$
\begin{array}{ccc}
M & \xrightarrow{m} & X \\
\downarrow{m} & & \downarrow{(1_X,h)} \\
X & \xrightarrow{(1_X,g)} & X \times Y.
\end{array}
$$

Consequently, when $C$ has an $\mathcal{M}$-partial-map classifier and cokernel pairs of regular monomorphisms, so that every regular monomorphism is an equalizer (Exercise II.2.D), then $\mathcal{M} = \text{RegMono}_C$.

4.7.7 Theorem. For a finitely complete category $C$, the following assertions are equivalent:

(i) $C$ has partial products and an $\mathcal{M}$-subobject classifier;

(ii) $C$ is locally cartesian closed and has an $\mathcal{M}$-subobject classifier;

(iii) $C$ is cartesian closed and has $\mathcal{M}$-partial-map classifiers for all objects.
Proof. The equivalence of (i) and (ii) follows from Proposition 4.4.3. For (i) \(\Rightarrow\) (iii) one forms the partial product \(P = P(Z, t : 1 \to \Omega)\) and considers the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\varepsilon} & P \times_\Omega 1 \\
\downarrow_{1_Z} & \searrow_{(h,1)} & \downarrow_{t} \\
\downarrow_{1_Z} & \searrow_{h} & \downarrow_{\tau} \\
Z & \xrightarrow{f} & \Omega
\end{array}
\]

where \(f\) is the characteristic morphism for \(1_Z\). With the induced morphism \(h\), the evaluation morphism \(\varepsilon\) becomes a split epimorphism. In addition, the partial-product property guarantees \(h \cdot \varepsilon = \tau\), which makes \(\varepsilon\) a monomorphism since \(\tau\) is one. Hence \(\varepsilon\) is an isomorphism and we may assume \(\varepsilon = 1_Z\). Now it is easy to see that \(\tau : Z \to P\) classifies \(\mathcal{M}\)-partial maps to \(Z\).

For (iii) \(\Rightarrow\) (i) we prove that \(f^* : C/Z \to C/X\) has a right adjoint (for every \(f : X \to Z\) in \(C\), see 4.4.3) by factoring \(f\) as

\[
f = (X \xrightarrow{(f,1_X)} Z \times X \xrightarrow{\pi_1} Z)
\]

and showing that both \(\pi_1^*\) and \((f,1_X)^*\) have right adjoints. Since \(\pi_1^*\) makes the diagram

\[
\begin{array}{ccc}
C/Z & \xrightarrow{\pi_1^*} & C/Z \times X \\
\downarrow_{\text{dom}_Z} & \downarrow_{\text{dom}_{Z \times X}} & \downarrow_{\text{dom}_Z} \\
C & \xrightarrow{(-) \times X} & C
\end{array}
\]

(with \(\pi_1^*(W, u) = (W \times X, u \times 1_X)\)) commute, and \((-) \times X\) is left adjoint by cartesian closedness of \(C\), the functor \(\pi_1^*\) is also left adjoint by 4.4.3.

The split monomorphism \(m := (f, 1_X) : X \to Z \times X\) lies in \(\mathcal{M}\) by our general hypothesis on \(\mathcal{M}\), so the \(\mathcal{M}\)-partial morphism \((m, 1_X)\) is represented by \(h : Z \times X \to X^*\). For \((W, w)\) in \(C/X\) we construct the value \(m_*(w)\), to obtain an adjunction \(m^* \dashv m_* : C/X \to C/Z \times X\), as follows. For \(g : W^* \to X^*\) representing the \(\mathcal{M}\)-partial morphism \((t_W, w)\), one obtains \(m_*(w)\) as a pullback of \(g\) along \(h\), and the counit \(\varepsilon_w : m^*(m_*(w)) \to w\) by the pullback property of \(W = X \times X^*\):
That \( \varepsilon_w \) assumes the alleged role as a counit is an easy diagrammatic exercise. \qed

### 4.7.8 Examples

Set satisfies the equivalent conditions of Theorem [4.7.7]. Ord is cartesian closed and has a regular-subobject classifier, but not all regular-partial-map classifiers. The category \( \text{Set} \) of pointed sets and pointed maps has all partial-map classifiers but fails to be cartesian closed (Exercise 4.K), where a pointed set \( (X, x_0) \) is a set \( X \) with \( x_0 \in X \), and a pointed map \( f : (X, x_0) \to (Y, y_0) \) is a map \( f : X \to Y \) with \( f(x_0) = y_0 \). Top has a regular-subobject classifier but none of the other properties of 4.7.7, and Cat is cartesian closed but not locally so, and does not have a regular-subobject classifier.

The following Theorem gives in particular a necessary and sufficient condition for a topological category over \( \text{Set} \) to have regular-partial-map classifiers with underlying sets as in \( \text{Set} \).

### 4.7.9 Theorem

Let \( U : A \to X \) be a topological functor, with \( X \) finitely complete. The following assertions are equivalent:

(i) \( A \) has regular-partial-map classifiers and \( U \) preserves them;

(ii) \( X \) has regular-partial-map classifiers, and \( U \)-final sinks in \( A \) are stable under pullback along regular monomorphisms.

**Proof.** (ii) \( \implies \) (i): In order to construct a regular-partial-map classifier for \( B \) in \( A \), one takes a regular-partial-map classifier \( t_Z : Z \to Z^* \) for \( Z = UB \) in \( X \) and considers the (possibly large) family of all regular partial maps

\[
(m_i : M_i \to A_i, g_i : M_i \to B)_{i \in I}
\]  

in \( A \) with codomain \( B \). Since a morphism in \( A \) is a regular monomorphism if and only if it is \( U \)-initial and its \( U \)-image is a regular monomorphism in \( X \) (Exercise II.5.D), for every \( i \in I \) one obtains a unique morphism \( h_i : UA_i \to Z^* \) in \( X \) which makes

\[
\begin{array}{ccc}
UM_i & \xrightarrow{Ug_i} & Z \\
\downarrow{Um_i} & & \downarrow{t_Z} \\
UA_i & \xrightarrow{h_i} & Z^*
\end{array}
\]  

(a pullback diagram. We let \( (f_i : A_i \to B^*)_{i \in I} \) be a \( U \)-final lifting of \( (h_i)_{i \in I} \). Since the family [4.7.iii] contains in particular \( (1_B, 1_B) \), there is \( j \in I \) with \( m_j = g_j = 1_B \) and \( h_j = t_Z \). We claim that \( s_B := f_j : B \to B^* \) is a regular-partial-map classifier. Since \( Us_B = t_Z \) is a regular mono, we must show that \( s_B \) is \( U \)-initial. Let \( n : C \to B^* \) be a \( U \)-initial lifting of \( t_Z : Z \to UB^* \). Then the diagrams [4.7.iv] with \( h_i = Uf_i \) show that there are morphisms \( \tilde{g}_i : M_i \to C \) with \( U\tilde{g}_i = Ug_i \), by \( U \)-initiality of \( n \), and the \( U \)-initiality of every \( m_i \) shows...
are pullback diagrams in $\mathbf{A}$. By hypothesis (ii) the sink $(\tilde{g}_i)_{i \in I}$ is $U$-final, so that $U\tilde{g}_i = Ug_i$ for all $i \in I$ yields $C \leq B$ in $U^{-1}Z$, with $B \leq C$ holding trivially by $U$-initiality of $n$. It follows that $s_B$ is a regular monomorphism, and the diagrams (4.7.v) show that it has the required universal property in $\mathbf{A}$. Furthermore, $Us_B = t_Z$ is a regular-partial-map classifier in $\mathbf{X}$.

$(i) \implies (ii)$: For the existence of regular-partial-map classifiers in $\mathbf{X}$, see Exercise 4.N. Now consider a family of pullback diagrams

\[
\begin{array}{ccc}
M_i & \xrightarrow{g_i} & N \\
\downarrow m_i & & \downarrow n \\
A_i & \xrightarrow{f_i} & B
\end{array}
\]

$(i \in I)$ in $\mathbf{A}$, with $n$ a regular monomorphism and $(f_i)_{i \in I}$ a $U$-final morphism, and consider $k : UN \to UC$ in $\mathbf{X}$ such that $k \cdot Ug_i = Uh_i$ with $h_i : M_i \to C$ in $\mathbf{A}$ for all $i \in I$. In order to confirm $U$-finality of $(g_i)_{i \in I}$ we must lift $k$ to a morphism $N \to C$ in $\mathbf{A}$. But with $s_C$ a regular-partial-map classifier for $C$ in $\mathbf{A}$, for every $i \in I$ there is a unique morphism $t_i : A_i \to C^*$ in $\mathbf{A}$ exhibiting $m_i$ as a pullback of $s_C$ along $t_i$ which restricts to $h_i$. Furthermore, since $U$ preserves this classifier, there is a unique morphism $j : UB \to UC^*$ in $\mathbf{X}$ exhibiting $Un$ as pullback of $Us_C$ along $j$ which restricts to $k$. The diagram

\[
\begin{array}{ccc}
UM & \xrightarrow{U\tilde{g}_i} & UN \\
\downarrow Um_i & & \downarrow Un \\
UA & \xrightarrow{Uf_i} & UB \\
\downarrow U\tilde{t}_i & & \downarrow Uj \\
UC & & \xrightarrow{j} UC^*
\end{array}
\]

makes it obvious that $j$ lifts to an $\mathbf{A}$-morphism $B \to C^*$ by $U$-finality of $(f_i)_{i \in I}$, and that $k$ then lifts to an $\mathbf{A}$-morphism $N \to C$ by $U$-initiality of $s_C$. \qed

4.8 The quasitopos $(\mathbb{T}, \mathcal{V})\text{-Gph}$. A category $\mathbf{C}$ is a quasitopos if

- $\mathbf{C}$ is finitely complete and finitely cocomplete;
• C is locally cartesian closed;
• C has a regular-subobject classifier.

A quasitopos is a topos if it has a subobject classifier. Hence, a topos is a quasitopos in which all monomorphisms are regular. Set and, more generally, every functor category $\text{Set}^{\text{op}}$ (with D small) is a topos.

As a topological category over Set, the category $(\mathbb{T}, \mathcal{V})$-$\text{Gph}$ is small-complete and small-cocomplete, and Theorem 4.5.1 gives sufficient conditions on $\mathbb{T}$ and $\mathcal{V}$ for $(\mathbb{T}, \mathcal{V})$-$\text{Gph}$ to be locally cartesian closed. The existence of a regular-subobject classifier is easily established, not only in $(\mathbb{T}, \mathcal{V})$-$\text{Gph}$ but also in its relevant subcategories.

4.8.1 Proposition. The categories $(\mathbb{T}, \mathcal{V})$-$\text{Gph}$, $(\mathbb{T}, \mathcal{V})$-$\text{RGph}$, $(\mathbb{T}, \mathcal{V})$-$\text{UGph}$ and $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$ all have a regular-subobject classifier.

Proof. Simply put the indiscrete structure on the subobject classifier 2 of Set (see Exercise 4.J).

As a consequence, $(2, \mathbb{T})$ is a regular-subobject classifier in all four categories.

4.8.2 Corollary. The category $(\mathbb{T}, \mathcal{V})$-$\text{Gph}$ is a quasitopos provided that $\mathcal{V}$ is cartesian closed and integral, and $T$ satisfies BC. If, in addition, $\mathbb{T}$ is associative and $\hat{T}(f^\circ) = \hat{T}1_X \cdot (Tf)^\circ$ for all $f : X \rightarrow Y$, then $(\mathbb{T}, \mathcal{V})$-$\text{RGph}$ is also a quasitopos.

Proof. The statement follows immediately from the previous discussion, with Theorems 4.5.1 and 4.6.7.

Theorem 4.7.9 applied to $O : (\mathbb{T}, \mathcal{V})$-$\text{RGph} \rightarrow \text{Set}$ provides simpler conditions for $(\mathbb{T}, \mathcal{V})$-$\text{RGph}$ to have regular-partial-map classifiers.

4.8.3 Proposition. For $T$ taut, let $\hat{T}$ satisfy $\hat{T}t = Tt \cdot \hat{T}1_Z$ for every injective function $t : Y \rightarrow Z$. Then $(\mathbb{T}, \mathcal{V})$-$\text{RGph}$ has regular-partial-map classifiers preserved by $O$.

Proof. We must show that $O$-final sinks in $(\mathbb{T}, \mathcal{V})$-$\text{RGph}$ are hereditary, that is: stable under pullback along embeddings. Hence, consider a sink $(f_i : (X_i, a_i) \rightarrow (Y, b))$ in $(\mathbb{T}, \mathcal{V})$-$\text{RGph}$ with

$$b = 1_Y^1 \vee \bigvee_{i \in I} f_i \cdot a_i \cdot \hat{T}(f_i^\circ) = e_Y^\circ \cdot \hat{T}1_Y \vee \bigvee_{i \in I} f_i \cdot a_i \cdot (Tf_i)^\circ \cdot \hat{T}1_Y$$

(see Proposition 4.1.4), an embedding $t : (Z, c) \rightarrow (Y, b)$ with $c = t^\circ \cdot b \cdot Tt$, and let $\tilde{f}_i : (f_i^{-1}(Z), d_i) \rightarrow (Z, c)$ be the restriction of $f_i$ with $d_i = s_i^\circ \cdot a_i \cdot Ts_i$, $s_i \cdot f_i^{-1}(Z) \subseteq X_i$. From $t^\circ \cdot e_Z^\circ = e_Z^\circ \cdot (Tt)^\circ$ one obtains $e_Z^\circ = t^\circ \cdot e_Z^\circ \cdot Tt$ since $Tt$ is injective. With the hypothesis on $T$ and $\hat{T}$,

$$c = t^\circ \cdot e_Z^\circ \cdot \hat{T}1_Y \cdot Tt \vee \bigvee_{i \in I} t^\circ \cdot f_i \cdot a_i \cdot (Tf_i)^\circ \cdot \hat{T}1_Y \cdot Tt$$
$$= t^\circ \cdot e_Y^\circ \cdot Tt \cdot \hat{T}1_Z \vee \bigvee_{i \in I} t^\circ \cdot f_i \cdot a_i \cdot (Tf_i)^\circ \cdot Tt \cdot \hat{T}1_Z$$
$$= e_Z^\circ \cdot \hat{T}1_Z \vee \bigvee_{i \in I} \tilde{f}_i \cdot s_i^\circ \cdot a_i \cdot Ts_i \cdot (T\tilde{f}_i)^\circ \cdot \hat{T}1_Z$$
$$= 1_Z^1 \vee \bigvee_{i \in I} \tilde{f}_i \cdot d_i \cdot \hat{T}(\tilde{f}_i^\circ)$$,
4. EMBEDDING LAX ALGEBRAS INTO A QUASITOPOS

so that \((\tilde{f}_i)_{i \in I}\) is \(O\)-final, as claimed.

4.8.4 Remark. Examining the proof of Theorem 4.7.9, one sees that the hypothesis \(\hat{T}t = Tt \cdot \hat{T}1_Z\) is used only for \(t : Z \leftrightarrow Z' = Z \cup \{\ast\}\) with \(\ast \notin Z\). Since \(\hat{T}t = T1_{Z'} \cdot Tt\) always holds, one can in fact show that it suffices that \(\hat{T}\) satisfy

\[ \chi \in TZ, \perp < \hat{T}1_{Z'}(\chi, y) \implies y \in TZ \]  

for Proposition 4.8.3 to hold true.

4.8.5 Examples.

(1) The categories \(RRel, RNRel, PsTop, PsApp\) are quasitopoi (see Examples 4.1.3 and 4.5.3), but none of them is a topos (since monomorphisms need not be regular in these categories).

(2) Condition (4.8.i) fails for the lax extension \(\hat{F}\) of the monad \(F\), but it is satisfied for the lax extension \(\hat{F}\) of 1.10.3(4). Since \((F, \hat{F}, 2)\)-RGph is cartesian closed (Example 4.6.4), it is a quasitopos.

(3) \(PrTop \cong (F, 2)\)-UGph (see 4.1.3(3)) has a regular-partial-map classifier which may be constructed via Theorem 4.7.9 or Proposition 4.8.3, but \(PrTop\) is not cartesian closed (see [Herrlich et al., 1991]).

4.9 Final density of \((T, V)\)-Cat in \((T, V)\)-Gph. Let us finally show that \((T, V)\)-Gph is “not too big” an extension of \((T, V)\)-Cat, in the sense that every \((T, V)\)-graph is the codomain of a small \(O\)-final sink with domains in \((T, V)\)-Cat, where \(O : (T, V)\)-Gph \(\rightarrow\) Set is the forgetful functor. Here we assume that the functor \(T\) of the monad \(T\) preserves disjointness, so that \(X \cap Y = \emptyset\) implies \(TX \cap TY = \emptyset\); more precisely,

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & TX \\
\downarrow & & \downarrow \\
TY & \longrightarrow & T(X + Y)
\end{array}
\]

is a pullback diagram for all sets \(X\) and \(Y\). This is certainly the case when \(T\emptyset = \emptyset\) and \(T\) is taut, or when \(T\) preserves binary coproducts.

We commence by constructing a \((T, V)\)-category structure \(c\) on the set

\[ Z = X + 1 \]

depending on a given set \(X\) and elements \(\chi \in TX\) and \(v \in V\). Writing \(1 = \{\ast\}\) and using the disjointness hypothesis we define \(c = c_{\chi, v}\) by

\[
c(z, z) = \begin{cases} 
  k & \text{if } z = e_Z(z), \\
  v & \text{if } z = \chi \text{ and } z = \ast, \\
  \perp & \text{otherwise},
\end{cases}
\]
for all $z \in Z, z \in TZ$. We observe that the $\mathcal{V}$-relation $c : TZ \twoheadrightarrow Z$ has finite fibres, that is,

$$c^\circ(z) = \{z \in TZ \mid \bot < c(z, z)\}$$

is finite for all $z \in Z$. We say that the lax natural transformation $e^\circ : \hat{T} \twoheadrightarrow 1$ is fi:\n
\[\begin{array}{cccc}
TX & \xrightarrow{\hat{T}_r} & TY \\
\varepsilon_X & \downarrow & \varepsilon_Y \\
X & \xrightarrow{r} & Y
\end{array}\]

commutes for every $\mathcal{V}$-relation $r$ with finite fibres (that is, for those $r : X \twoheadrightarrow Y$ with $r^\circ(y) = \{x \in X \mid \bot < r(x, y)\}$ finite for all $y \in Y$).

4.9.1 Proposition. Suppose that $T$ preserves disjointness, $\hat{T}$ is flat, and $c^\circ$ is finitely strict. Then

$$(X + 1, c_{x,v})$$

is a $(\mathbb{T}, \mathcal{V})$-category for any set $X$ and all $x \in TX, v \in \mathcal{V}$.

Proof. As $c$ is reflexive by definition, we must only show transitivity:

$$\hat{T}c(z, z) \otimes c(z, z) \leq c(m_Z(z), z)$$

for all $z \in Z = X + 1, z \in TZ, z \in TTZ$. With $i : X \hookrightarrow Z$ denoting the inclusion one

trivially has $i^\circ \cdot c = i^\circ \cdot e_Z^\circ$, hence $(T^i)^\circ \cdot \hat{T}c = (T^i)^\circ \cdot (Te_Z)^\circ$ since $\hat{T}$ is flat. This means in pointwise terms

$$\hat{T}_x(z, z) = \begin{cases} k & \text{if } z = Te_Z(z), \\ \bot & \text{otherwise}, \end{cases}$$

whenever $z \in TX$. In proving transitivity of $c$, we can disregard cases where $c(z, z) = \bot$ or

$$\hat{T}c(z, z) = \bot;$$

hence, we are left to consider the following cases.

Case 1: $z \in X, z = e_Z(z), Z = Te_Z(z)$. Then all three terms appearing in the transitivity condition are equal to $k$, and the inequality is actually an equality.

Case 2: $z = *, z = x, Z = Te_Z(z)$. Then, similarly to the previous case, one obtains

$$\hat{T}c(z, z) \otimes c(z, z) = k \otimes v = v = c(m_Z(z), z).$$

Case 3: $z = *, z = e_Z(*)$. Since $e^\circ$ is finitely strict, one has $e_Z^\circ \cdot \hat{T}c = e_T^\circ \otimes e_Z^\circ$, so that

$$\hat{T}c(z, z) = \bigvee_{w \in e_T^\circ(z)} c(w, *) .$$

As this value is assumed not to be $\bot$, there is $w \in TZ$ with $\bot < c(w, *)$ and $e_T^\circ(w) = Z$, and for every such $w$ one has $c(m_Z(z), z) = c(w, z)$. Consequently,

$$\hat{T}c(z, z) \otimes c(z, z) = c(m_Z(z, z)) \otimes k = c(m_Z(z), z),$$

which concludes the last case. \qed
4.9.2 Theorem. Suppose that \( T \) preserves disjointness, \( \hat{T} \) is flat, and \( e^o \) is finitely strict. Then \((\mathbb{T}, \mathbb{V})\)-Cat is finally dense in \((\mathbb{T}, \mathbb{V})\)-Gph.

Proof. For \((X, a)\) a \((\mathbb{T}, \mathbb{V})\)-graph, \(x \in X\) and \(\chi \in TX\), we claim that
\[ f_{\chi, x} : (X + 1, c_{\chi, a(\chi, x)}) \to (X, a) \]
is a \((\mathbb{T}, \mathbb{V})\)-functor, with the map \(f_{\chi, x}\) induced by \(1_X : X \to X\) and \(x : 1 \to X\). Indeed, for \(z \in Z = X + 1\) and \(z \in TZ\) with \(\perp < c(z, z)\) (where \(c = c_{\chi, a(\chi, x)}\)), one has \(z = e_Z(z)\) or \(z = \chi, z = \ast \in 1\). In the former case,
\[ c(z, z) = k \leq a(e_X(f_{\chi, x}(z)), f_{\chi, x}(z)) = a(Tf_{\chi, x}(z), f_{\chi, x}(z)) \]
since \(a\) is reflexive, and in the latter case
\[ c(z, z) = a(\chi, x) = a(Tf_{\chi, x}(z), f_{\chi, x}(z)) \]
since \(f_{\chi, x}|_X = 1_X\). Furthermore, this last equation also shows
\[ a(\chi, x) = (f_{\chi, x} \cdot c_{\chi, a(\chi, x)}) \cdot (Tf_{\chi, x})^o(\chi, x) , \]
so that the epi-sink \((f_{\chi, x})_{\chi \in TX, x \in X}\) must be \(O\)-final for the forgetful functor \(O : (\mathbb{T}, \mathbb{V})\)-Gph \(\to\) Set (see Proposition 4.1.4).

Since the \(O\)-final sink just considered is small, we may state Theorem 4.9.2 more strongly as follows.

4.9.3 Corollary. Suppose that \( T \) preserves disjointness, \( \hat{T} \) is flat, and \( e^o \) is finitely strict. Then every \((\mathbb{T}, \mathbb{V})\)-graph is a quotient of a coproduct of \((\mathbb{T}, \mathbb{V})\)-categories. If, moreover, \( \hat{T} \) is associative, then every \((\mathbb{T}, \mathbb{V})\)-graph is a quotient of a \((\mathbb{T}, \mathbb{V})\)-category.

Proof. The small \(O\)-final epi-sink used in the proof of Theorem 4.9.2 factors through a coproduct of \((\mathbb{T}, \mathbb{V})\)-categories followed by a quotient map. That coproduct is again a \((\mathbb{T}, \mathbb{V})\)-category in case \( \hat{T} \) is associative and flat by Theorem 4.3.3.

This Corollary can be applied to Examples 4.1.3 with the following consequence.

4.9.4 Corollary. Every pseudotopological space is a quotient of a topological space, and every pseudo-approach space is a quotient of an approach space.

Exercises

4.A Initial sources in \((\mathbb{T}, \mathbb{V})\)-RGph and \((\mathbb{T}, \mathbb{V})\)-UGph. For the forgetful functor \(O : (\mathbb{T}, \mathbb{V})\)-RGph \(\to\) Set, \(O\)-initial sources are characterized as in Proposition 3.1.1. Likewise for \((\mathbb{T}, \mathbb{V})\)-UGph \(\to\) Set.
4.B Sinks of nearly open morphisms. Let \((f_i : (X_i, a_i) \rightarrow (Y, b))_{i \in I}\) be an epi-sink of morphisms in \((\mathbb{T}, \mathcal{V})\text{-RGph}\), with all \(f_i\) nearly open for all \(i \in I\). Then \((f_i)_{i \in I}\) is \(O\)-final, with \(O : (\mathbb{T}, \mathcal{V})\text{-RGph} \rightarrow \text{Set}\).

4.C Taut functors.

(1) The \(\text{Set}\)-functor \(T\) with \(T\emptyset = \emptyset\), and \(TX = 1\) if \(X \neq \emptyset\), preserves monomorphisms but is not taut.

(2) The filter functor \(F\) is weakly terminal amongst all taut functors, that is: for every taut functor \(T : \text{Set} \rightarrow \text{Set}\), there is a natural transformation \(\alpha : T \rightarrow F\).

\textbf{Hint.} Since \(T\) preserves monomorphisms, one may assume \(TA \subseteq TX\) for \(A \subseteq X\), and define \(\alpha_X(\chi) = \{A \subseteq X \mid \chi \in TA\}\) for \(\chi \in TX\).

4.D Powerset graphs. With respect to both lax extensions of the powerset monad \(\hat{\mathcal{P}}\) and \(\hat{\mathcal{P}}\) (see Example 1.4.2(2)), show

\((\mathcal{P}, 2)\text{-Gph} = (\mathcal{P}, 2)\text{-RGph}\)

and identify \((\mathcal{P}, 2)\text{-UGph}\) as a subcategory of \(\text{Ord}\) and \(\text{Cls}\), respectively (see Examples 1.6.4).

4.E Extensive Categories. For a category \(\mathcal{C}\) with small-indexed coproducts and pullbacks, the following statements are equivalent.

(i) \(\mathcal{C}\) is extensive (that is, coproducts are universal and disjoint);

(ii) coproducts are universal, and for all small families \((f_i : X_i \rightarrow Y_i)_{i \in I}\), the diagrams

\[
\begin{array}{ccc}
X_j & \longrightarrow & \amalg_{i \in I} X_i \\
\downarrow f_j & & \downarrow \amalg_{i \in I} f_i \\
Y_j & \longrightarrow & \amalg_{i \in I} Y_i
\end{array}
\]

are pullback diagrams;

(iii) for all small families \((Y_i)_{i \in I}\) of \(\mathcal{C}\)-objects, the functor

\[
\amalg_{i \in I} \mathcal{C}/Y_i \rightarrow \mathcal{C} / \amalg_{i \in I} Y_i, \quad (f_i : X_i \rightarrow Y_i)_{i \in I} \mapsto \amalg_{i \in I} f_i
\]

is an equivalence of categories.

4.F \(\text{Ord}\) is not locally cartesian closed. \(\text{Ord}\) is cartesian closed but not locally so: the partial product \(P = (\{0, 1\}, \{0, 1\} \mapsto \{0, \frac{1}{2}, 1\})\) does not exist in \(\text{Ord}\).
4. **Top** is not cartesian closed.

(1) Consider the topological spaces $X, Y, W, Z$ whose non-empty open sets are illustrated by the diagram

\[
\begin{array}{ccc}
X & \xleftarrow{q'} & Y \\
\downarrow{q} & & \\
W & \xleftarrow{0} & Z
\end{array}
\]

Conclude that quotient maps fail to be stable under pullback in **Top**, and that **Top** fails to be locally cartesian closed.

(2) Find a topological space $X$ and a quotient map $q : Y \to Z$ such that $q \times 1_X : Y \times X \to Z \times X$ fails to be a quotient map. Conclude that $X$ is not exponentiable.

**Hint.** Consider $X = \mathbb{Q}$, the rational numbers, as a subspace of $\mathbb{R}$.

4. **Slicing topological functors.** Consider a topological functor $U : A \to X$ and an $A$-object $A$. The induced functor $U_A : A/A \to X/U A$ is also topological, with the domain functor $\text{dom}_A : A/A \to A$ sending $U_A$-final sinks to $U$-final sinks.

4. **Pseudotopological spaces via filter convergence.** With the help of the Axiom of Choice, condition (Ps) of Example 4.1.3(3) can equivalently be expressed as

\[
(\forall \chi \in \beta X (\chi \supseteq a \implies \chi \to y)) \implies (a \to y)
\]

for all $a \in FX, y \in X$. Use this to show that the categories **PsTop** and $(\mathbb{F}, 2)$-**RGph** of Example 4.1.3(3) are isomorphic. Describe the reflector **PsTop** $\to$ **PrTop**.

4. **Lifting regular-subobject classifiers.** If $I$ is right adjoint to the topological functor $U : A \to X$, and if $\Omega$ is a regular-subobject classifier of $X$, then $I \Omega$ is a regular-subobject classifier of $A$ (see Example 4.7.3(2) and Exercise II.5.D).

4. **Partial-map classifiers.** The categories **Ord** and **Top** fail to have partial-map classifiers. The category **Set** of pointed sets has partial-map classifiers but fails to be cartesian closed.

4. **Pretopological spaces via filter convergence.** The multiplication and unit of the filter monad $\mathbb{F}$ satisfy

\[
\Sigma \mathcal{A} = \bigcap \mathcal{A} \quad \text{and} \quad (X \supseteq \mathcal{A}) \implies (\Sigma X \supseteq \bigcap \mathcal{A})
\]

for all $\mathcal{A} \subseteq FX$ and $X \in FFX$. Hence, a $(\mathbb{F}, 2)$-graph $(X, a)$ is unitary if and only if it is right unitary and

$$
\nu(y) \to y
$$
holds for all \( y \in X \), where \( \nu(y) = \bigcap \{ a \in FX \mid a \to y \} \). Deduce that unitary graphs on a set \( X \) are in bijective correspondence with maps \( c : PX \to PX \) satisfying

\[
c(\emptyset) = \emptyset, \quad A \subseteq c(A), \quad c(A \cup B) = c(A) \cup c(B),
\]

for all \( A, B \subseteq X \). The categories \( \PrTop \) and \( (\mathbb{F}, 2)\)-\text{UGph} (see Example 4.1.3(3)) are therefore isomorphic. Describe the reflector \( \PrTop \to \Top \).

*Hint.* Consider the correspondence between maps \( \nu : X \to PPX \) and \( c : PX \to PX \) given by

\[
c(A) = \{ y \in X \mid B \in \nu(y) \implies A \cap B \neq \emptyset \} \quad \text{and} \quad \nu(y) = \{ B \in PX \mid y \in c(B^\complement) \},
\]

where \( B^\complement = X \setminus B \) denotes the complement of \( B \) in \( X \).

4.M Partial-map classifiers in \( \mathcal{V}\text{-Cat} \). If \( k = \top \) in \( \mathcal{V} \), then an object \( (Z, c) \) in \( \mathcal{V}\text{-Cat} \) may have a regular-partial-map classifier only when \( Z \) is indiscrete, that is, when \( c(x, y) = \top \) for all \( x, y \in Z \). Hence, \( \mathcal{V}\text{-Cat} \) is not a quasitopos, even when \( \otimes = \land \) in \( \mathcal{V} \).

4.N Inheriting regular-partial-map classifiers. Let \( A \) be a finitely complete category with regular-partial-map classifiers \( s_A : A \to A^* \).

1. If \( B \) is a full, replete and reflective subcategory of \( A \) with monic reflection morphisms \( r_A : A \to RA \) and the reflector \( R \) preserving regular monomorphisms, then \( r_B^* \cdot s_B \) is a regular-partial-map classifier for \( B \) in \( B \).

2. If \( U : A \to X \) is topological, then \( X \) has regular-partial-map classifiers.

*Hint.* Recall Exercise II.5.D and apply (1) to the subcategory of indiscrete objects in \( A \).

4.O \( \Top \) versus \( \App \). There are full embeddings

\[
\begin{array}{cccc}
\Top & \to & \PrTop & \to & \PsTop \\
\App & \to & \PrApp & \to & \PsApp \\
\end{array}
\]

all of which are reflective, and the vertical ones are also coreflective (via \( a(\chi, x) = 0 \) if and only if \( \chi \to x \), as in 3.6). Describe the reflectors and coreflectors.
5. REPRESENTABLE LAX ALGEBRAS

5 Representable lax algebras

In this subsection we show that the \( \mathsf{Set} \)-monad \( T \) with its lax extension \( \hat{T} \) to \( \mathcal{V}\)-Rel may be lifted, first to \( \mathcal{V}\)-Cat and then to \( (\mathbb{1}, \mathcal{V})\)-Cat. The respective Eilenberg–Moore categories over \( \mathcal{V}\)-Cat and \( (\mathbb{1}, \mathcal{V})\)-Cat are topological over the Eilenberg–Moore category \( \mathsf{Set} \) \( T \) and, somewhat surprisingly, turn out to be isomorphic. These categories lead us to interesting classes of so-called ordered and metric compact Hausdorff spaces, respectively.

5.1 The monad \( \mathbb{1} \) on \( \mathcal{V}\)-Cat. We continue to work with a quantale \( \mathcal{V} \) that is assumed to be non-trivial as in 1.2. As shown in 3.3, every lax extension \( \hat{T} = (\hat{T}, m, e) \) of a monad \( T = (T, m, e) \) from \( \mathsf{Set} \) to \( \mathcal{V}\)-Rel induces an order relation on \( TX \), namely

\[
\chi \leq y \iff k \leq \hat{T}1X(\chi, y), \tag{5.1.i}
\]

for \( \chi, y \in TX \). In fact, the \( \mathcal{V}\)-relation \( \hat{T}1X : TX \to TX \) makes \( TX \) into a \( \mathcal{V}\)-category \( (TX, \hat{T}1X) \) since

\[
1_{TX} = T(1_X) \leq \hat{T}1X \quad \text{and} \quad \hat{T}1X \cdot \hat{T}1X \leq \hat{T}(1_X \cdot 1_X) = \hat{T}1X,
\]

and its underlying order is given by (5.1.i). More generally, the same computation shows that, for every \( \mathcal{V}\)-category \( (X, a_0) \),

\[
T(X, a_0) = (TX, \hat{T}a_0), \tag{5.1.ii}
\]

is a \( \mathcal{V}\)-category. This construction is the object part of a functor

\[
T : \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}
\]

as \( T \) is a \( \mathcal{V}\)-functor \( Tf : T(X, a_0) \to T(Y, b_0) \), for every \( f : (X, a_0) \to (Y, b_0) \). Furthermore, oplaxness of \( m \) and \( e \) with respect to \( \hat{T} \) imply that

\[
m_X : TT(X, a_0) \to T(X, a_0) \quad \text{and} \quad e_X : (X, a_0) \to T(X, a_0)
\]

are \( \mathcal{V}\)-functors, for every \( \mathcal{V}\)-category \( (X, a_0) \). Hence, the monad \( \mathbb{1} = (T, m, e) \) on \( \mathsf{Set} \) lifts to a monad on \( \mathcal{V}\)-Cat, which we denote by \( \mathbb{1} = (T, m, e) \) again.

5.1.1 Proposition. A lax extension \( \hat{T} \) of a monad \( \mathbb{1} \) on \( \mathsf{Set} \) induces a \( 2\)-monad \( \mathbb{1} \) on \( \mathcal{V}\)-Cat via (5.1.ii), that is, a monad \( \mathbb{1} = (T, m, e) \) with \( T \) a \( 2\)-functor.

Proof. We are left to show that \( T : \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat} \) is a \( 2\)-functor. To this end, let \( f, g : (X, a_0) \to (Y, b_0) \) be \( \mathcal{V}\)-functors with \( f \leq g \), that is, \( 1_X \leq g \circ b_0 \cdot f \). Then

\[
1_{TX} \leq \hat{T}1X \leq \hat{T}(g \circ b_0 \cdot f) = (Tg)^\circ \cdot \hat{T}b_0 \cdot Tf,
\]

that is \( Tf \leq Tg \). \qed
5.1.2 Examples.

(1) The identity monad on \( \mathbb{S}et \) extended to the identity monad on \( \mathcal{V} \text{-Rel} \) yields the identity monad on \( \mathcal{V} \text{-Cat} \).

(2) For the powerset monad \( \mathbb{P} = (P, \{-\}, \cup) \) with its extensions \( \hat{\mathbb{P}}, \check{\mathbb{P}} \) and \( \mathbb{P} \) to \( \text{Rel} \) (see Example 1.5.2(2) and 1.12), for an ordered set \( (X, \leq) \) and \( A, B \subseteq X \) we find

\[
A (\hat{\mathbb{P}} \leq) B \iff A \subseteq \downarrow B ,
\]

\[
A (\check{\mathbb{P}} \leq) B \iff B \subseteq \uparrow A ,
\]

\[
A (\mathbb{P} \leq) B \iff A \subseteq \downarrow B \& B \subseteq \uparrow A .
\]

Less formally, \( A (\hat{\mathbb{P}} \leq) B \) whenever every element of \( A \) is covered by an element of \( B \), \( A (\check{\mathbb{P}} \leq) B \) whenever every element of \( B \) covers an element of \( A \), and \( A (\mathbb{P} \leq) B \) whenever both conditions are satisfied. Note that in all three cases the order relation on \( PX \) need not be antisymmetric even if the order on \( X \) is so. For instance, for \( X = \mathbb{N} \) with its natural order \( \leq \), and \( E, O \) the sets of even and odd numbers respectively, one has

\[
O (\hat{\mathbb{P}} \leq) E \quad \text{and} \quad E (\check{\mathbb{P}} \leq) O .
\]

(3) For the list monad \( \mathbb{L} = (L, m, e) \) and its Barr extension \( \mathbb{L} \) to \( \text{Rel} \), given by

\[
(x_1, \ldots, x_n) \mathbb{L}r (y_1, \ldots, y_m) \iff n = m \& x_1 r y_1 \& \ldots \& x_n r y_n
\]

for all \( (x_1, \ldots, x_n) \in LX, (y_1, \ldots, y_m) \in LY \), and relations \( r : X \rightarrow Y \) (see also V.1.4), one has

\[
L(X, \leq) \cong \bigsqcup_{n \in \mathbb{N}} (X, \leq)^n
\]

for every ordered set \( (X, \leq) \). Under this identification \( Lf \) corresponds to \( \bigsqcup_{n \in \mathbb{N}} f^n \).

(4) With the lax extension \( \hat{\mathbb{P}} \) of \( \mathbb{P} \) to \( P_+-\text{Rel} \) given in Exercise 2.G, the powerset monad induces a 2-monad \( \mathbb{P} \) on \( P_+-\text{Cat} \cong \text{Met} \). In particular, for a metric space \( (X, d) \), the sets \( (PX, \hat{\mathbb{P}}d) \) become metric spaces with the (non-symmetric) Hausdorff metric

\[
\hat{\mathbb{P}}d(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y) ,
\]

for all \( A, B \subseteq X \).

(5) We now consider the ultrafilter monad \( \beta \) on \( \mathbb{S}et \), together with the Barr-extension \( \beta \) to \( \text{Rel} \cong 2\text{-Rel} \) (see 1.10.3). Then, for an ordered set \( (X, \leq) \), the order relation on \( \beta X \) is given by

\[
\chi \leq y \iff \forall A \in \chi, B \in y \exists x \in A, y \in B (x \leq y) ,
\]

for \( \chi, y \in \beta X \).

Finally, we consider the quantale \( P_+ \) and the extension of the ultrafilter monad \( \beta \) to \( P_+-\text{Rel} \) described in 2.4. Here, for a metric space \( (X, d) \) and ultrafilters \( \chi, y \in \beta X \), the distance between \( \chi \) and \( y \) is

\[
\sup_{A \in \chi, B \in y} \inf_{x \in A, y \in B} d(x, y) .
\]
5.2 \( \mathbb{T} \)-algebras in \( \mathcal{V}\text{-Cat} \). Let \( \mathbb{T} \) be a monad lifted from \( \text{Set} \) to \( \mathcal{V}\text{-Cat} \) as in \( \ref{section:representable-lax-algebras} \). Following the notation of \( \ref{subsection:monad-homomorphisms} \), we denote the category of \( \mathbb{T} \)-algebras and \( \mathbb{T} \)-homomorphisms by \( (\mathcal{V}\text{-Cat})^\mathbb{T} \).

We also recall that there is an adjunction

\[
(\mathcal{V}\text{-Cat})^\mathbb{T} \xrightarrow{\mathcal{F}_\mathbb{T}} \mathcal{V}\text{-Cat} \xleftarrow{\mathcal{G}_\mathbb{T}} (\mathcal{V}\text{-Cat})^\mathbb{T},
\]

that \( (\mathcal{V}\text{-Cat})^\mathbb{T} \) is complete and that \( \mathcal{G}_\mathbb{T} : (\mathcal{V}\text{-Cat})^\mathbb{T} \rightarrow \mathcal{V}\text{-Cat} \) preserves and creates limits.

An object of \( (\mathcal{V}\text{-Cat})^\mathbb{T} \) can be described as a triple \( (X,a_0,\alpha) \) where \( (X,a_0) \) is a \( \mathcal{V} \)-category and \( \alpha : TX \rightarrow X \) is simultaneously a \( \mathbb{T} \)-algebra structure on the set \( X \) and a \( \mathcal{V} \)-functor \( \alpha : \mathcal{T}(X,a_0) \rightarrow (X,a_0) \).

For \( \mathbb{T} \)-algebras \( (X,a_0,\alpha) \) and \( (Y,b_0,\beta) \), a map \( f : X \rightarrow Y \) is a \( \mathbb{T} \)-homomorphism \( f : (X,a_0,\alpha) \rightarrow (Y,b_0,\beta) \) if and only if \( f : (X,a_0) \rightarrow (Y,b_0) \) is a \( \mathcal{V} \)-functor and \( f : (X,\alpha) \rightarrow (Y,\beta) \) is a morphism of \( \mathbb{T} \)-algebras.

5.2.1 Examples.

(1) For the monad \( \mathbb{P} \) on \( \text{Ord} = 2\text{-Cat} \) obtained from the extension \( \hat{\mathbb{P}} \) of \( \mathbb{P} \) to \( \text{Rel} \), an Eilenberg–Moore algebra is an ordered set \( (X,\leq) \) together with a monotone map \( \alpha : PX \rightarrow X \) satisfying

\[
\alpha(\{\alpha(A) \mid A \in \mathcal{A}\}) = \alpha(\bigcup \mathcal{A}) \quad \text{and} \quad \alpha(\{x\}) = x, \quad (5.2.i)
\]

for all \( x \in X \) and \( \mathcal{A} \subseteq PX \). We show that \( \alpha(A) \) is a supremum of \( A \subseteq X \) in \( (X,\leq) \). Clearly, \( \{x\} \leq A \) in \( (PX,\leq) \) for any \( x \in A \), so that \( x \leq \alpha(A) \). Furthermore, for \( y \in X \) with \( x \leq y \) for all \( x \in A \), one has \( A \leq \{y\} \) and therefore \( \alpha(A) \leq y \). Hence, \( \text{Ord}_{\mathbb{P}} \) can be equivalently described as the category of complete ordered sets with chosen suprema \( \alpha : PX \rightarrow X \) satisfying \( (5.2.i) \), and a morphism is a monotone map which preserves these chosen suprema. Of course, suprema are unique if \( X \) is separated, and the full subcategory of \( \text{Ord}_{\mathbb{P}} \) defined by the separated complete ordered sets is precisely the category \( \text{Sup} \) (see \( \ref{subsection:complete-ordered-sets} \)); moreover, \( \text{Sup} \) is a reflective subcategory of \( \text{Ord}_{\mathbb{P}} \).

Dually, if we consider the extension \( \hat{\mathbb{P}} \) of \( \mathbb{P} \) to \( \text{Rel} \), the same reasoning applies with \( \alpha : PX \rightarrow X \) now representing chosen infima of an order relation on \( X \). Hence, the Eilenberg–Moore category \( \text{Ord}_{\mathbb{P}} \) has as objects complete ordered sets with chosen infima \( \alpha : PX \rightarrow X \) satisfying \( (5.2.i) \), and as morphisms those monotone maps which preserve the chosen infima. Furthermore, the category \( \text{Inf} \) of separated complete ordered sets and inf-maps is a full reflective subcategory of \( \text{Ord}_{\mathbb{P}} \).

(2) For the list monad \( \mathbb{L} = (L,m,e) \) with its Barr extension to \( \text{Rel} \) (see Example \( \ref{subsection:representable-lax-algebras} \)), the category \( \text{Ord}_{\mathbb{L}} \) is isomorphic to the category of ordered monoids, that is, of monoids in the cartesian closed category \( \text{Ord} \).
(3) An object in $\text{Ord}^\beta$ (here we consider the Barr extension $\overline{\beta}$ of $\beta$ to $\text{Rel}$) is a triple $(X, \leq, \alpha)$ where $(X, \leq)$ is an ordered set and $\alpha : \beta X \to X$ is the convergence relation of a compact Hausdorff topology on $X$ (see [2.3]); moreover, $\alpha : \beta(X, \leq) \to (X, \leq)$ is monotone. We write $R \subseteq X \times X$ for the graph of the order relation $\leq$ and $\pi_1 : R \to X$, $\pi_2 : R \to X$ for the projection maps. Recall from [1.10] that, for $\chi, y \in \beta X$,

$$\chi \ (\overline{\beta}(\leq)) \ y \iff \exists w \in \beta R \ (\beta \pi_1(w) = \chi \& \beta \pi_2(w) = y).$$

Therefore, $\alpha : \beta X \to X$ is monotone if and only if $(\alpha(\beta \pi_1(w)), \alpha(\beta \pi_2(w))) \in R$ for every ultrafilter $w \in \beta R$; that is, if and only if $R$ is a closed subset of $X \times X$ with respect to the product topology.

(4) Considering the extension $\overline{\beta}$ of $\beta$ to $\text{P}_+\text{-Rel}$ of [2.4], one obtains a monad $\mathcal{B}$ on $\text{Met} = \text{P}_+\text{-Cat}$. The objects of $\text{Met}^\mathcal{B}$ are triples $(X, d, \alpha)$ where $(X, d)$ is a metric space and $\alpha$ is the convergence relation of a compact Hausdorff topology on $X$ such that $\alpha : \beta(X, d) \to (X, d)$ is non-expansive. We call these spaces $\text{metric compact Hausdorff spaces}$, and denote the category of metric compact Hausdorff spaces and their morphisms by $\text{MetCompHaus}$.

Besides sitting over $\text{V-Cat}$, the category $(\text{V-Cat})^\mathcal{T}$ also admits a forgetful functor

$$\bar{O} : (\text{V-Cat})^\mathcal{T} \to \text{Set}^\mathcal{T}, \quad (f : (X, a_0, \alpha) \to (Y, b_0, \beta)) \mapsto (f : (X, \alpha) \to (Y, \beta)).$$

to $\text{Set}^\mathcal{T}$ which can be seen as a lifting of the forgetful functor $O : \text{V-Cat} \to \text{Set}$. Recall from Example [1.5.6.1](3) that every mono-source in $\text{Set}^\mathcal{T}$ is initial with respect to the forgetful functor $G^\mathcal{T} : \text{Set}^\mathcal{T} \to \text{Set}$, and therefore a mono-source $(f_i : (X, a_0, \alpha) \to (X_i, a_{0i}, \alpha_i))_{i \in I}$ in $(\text{V-Cat})^\mathcal{T}$ is initial with respect to the forgetful functor $(\text{V-Cat})^\mathcal{T} \to \text{Set}$ if and only if $(f_i : (X, a_0) \to (X_i, a_{0i}))_{i \in I}$ is initial with respect to the forgetful functor $O : \text{V-Cat} \to \text{Set}$.  

**5.2.2 Proposition.** The functor $\bar{O} : (\text{V-Cat})^\mathcal{T} \to \text{Set}^\mathcal{T}$ is topological.
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Proof. For a family \((f_i : (X, \alpha) \to (X_i, \alpha_i))_{i \in I}\) of morphisms of \(\mathcal{T}\)-algebras where \((X_i, a_{0i}, \alpha_i)\) is in \((\mathcal{V}\text{-}\mathcal{C}at)^\mathcal{T}\), let \((X, a_0)\) be the initial lift of \((f : X \to X_i)_{i \in I}\) in \(\mathcal{V}\text{-}\mathcal{C}at\) (see Theorem 3.1.3). Then \((X, a_0, \alpha)\) is indeed an object of \((\mathcal{V}\text{-}\mathcal{C}at)^\mathcal{T}\), since

\[
\alpha \cdot \hat{T}a_0 = \alpha \cdot \hat{T}(\land_{i \in I} f_i^o \cdot a_{0i} \cdot f_i) \\
\leq \land_{i \in I}(\alpha \cdot \hat{T}(f_i^o \cdot a_{0i} \cdot f_i)) \\
= \land_{i \in I}(\alpha \cdot (Tf_i^o \cdot \hat{T}a_{0i} \cdot Tf_i)) \\
\leq \land_{i \in I}(f_i^o \cdot \alpha_i \cdot \hat{T}a_{0i} \cdot Tf_i) \\
\leq \land_{i \in I}(f_i^o \cdot a_{0i} \cdot f_i \cdot \alpha) \\
= (\land_{i \in I} f_i^o \cdot a_{0i} \cdot f_i) \cdot \alpha = a_0 \cdot \alpha,
\]

with the penultimate equality holding by right adjointness of \((-) \cdot \alpha\) (recall that \(\alpha \vdash \alpha^\circ\) in \(\mathcal{V}\text{-}\mathcal{R}el\)).

As a consequence, the functor \(\hat{O} : (\mathcal{V}\text{-}\mathcal{C}at)^\mathcal{T} \to \text{Set}^\mathcal{T}\) has both a right adjoint and a left adjoint given by the \(\hat{O}\)-initial liftings of the empty family and the “all-family”, respectively (see Theorem II.5.9.1). In summary, in the commutative diagram

\[
\begin{array}{ccc}
(\mathcal{V}\text{-}\mathcal{C}at)^\mathcal{T} & \xrightarrow{G^\mathcal{T}} & \mathcal{V}\text{-}\mathcal{C}at \\
\hat{O} \downarrow & & \downarrow \hat{O} \\
\text{Set}^\mathcal{T} & \xrightarrow{G^\mathcal{T}} & \text{Set}
\end{array}
\]

the vertical arrows are topological and therefore have left and right adjoints; the horizontal arrows are monadic and therefore have left adjoints.

For a flat extension \(\hat{\mathcal{T}}\) of \(\mathcal{T}\), the left adjoint \((-)_d : \text{Set}^\mathcal{T} \to (\mathcal{V}\text{-}\mathcal{C}at)^\mathcal{T}\) has a simple description as \((X, \alpha)_d = (X, 1_X, \alpha)\). If, moreover, \(\mathcal{V}\) is integral so that \(k = T\) is the top element of \(\mathcal{V}\), then \((-)_d : \text{Set}^\mathcal{T} \to (\mathcal{V}\text{-}\mathcal{C}at)^\mathcal{T}\) preserves limits and therefore has a left adjoint \(\pi_0 : (\mathcal{V}\text{-}\mathcal{C}at)^\mathcal{T} \to \text{Set}^\mathcal{T}\) (see Exercise II.3.K and also Exercises 3.A and 3.B).

5.3 Comparison with lax algebras. The two structures \(a_0 : X \to X\) and \(\alpha : TX \to X\) of an object \((X, a_0, \alpha)\) in \((\mathcal{V}\text{-}\mathcal{C}at)^\mathcal{T}\) can be combined to yield a single structure \(a = a_0 \cdot \alpha : TX \to X\) turning \(X\) into a \((\mathcal{T}, \mathcal{V})\)-category \((X, a)\). In fact, one has

\[
a \cdot e_X = a_0 \cdot \alpha \cdot e_X = a_0 \geq 1_X
\]

and

\[
a \cdot \hat{T}a = a_0 \cdot \alpha \cdot \hat{T}(a_0 \cdot \alpha) = a_0 \cdot \alpha \cdot \hat{T}(a_0) \cdot T(\alpha) \leq a_0 \cdot a_0 \cdot \alpha \cdot T(\alpha) = a_0 \cdot \alpha \cdot m_X = a \cdot m_X.
\]

Note also that the underlying \(\mathcal{V}\)-category of \((X, a)\) is \((X, a_0)\). Furthermore, every morphism \(f : (X, a_0, \alpha) \to (Y, b_0, \beta)\) in \((\mathcal{V}\text{-}\mathcal{C}at)^\mathcal{T}\) becomes a \((\mathcal{T}, \mathcal{V})\)-functor \(f : (X, a) \to (Y, b)\) (where \(b = \beta \cdot b_0\)). More generally, one has the following result.
5.3.1 Lemma. Let \((X, a_0, \alpha)\), \((Y, b_0, \beta)\) be objects in \((\mathcal{V} \text{-Cat})^\mathbb{T}\) with corresponding \((\mathbb{T}, \mathcal{V})\)-categories \((X, a)\) and \((Y, b)\), and let \(f : (X, a_0) \to (Y, b_0)\) be a \(\mathcal{V}\)-functor. Then \(f\) is a \((\mathbb{T}, \mathcal{V})\)-functor \(f : (X, a) \to (Y, b)\) if and only if \(\beta \cdot Tf \leq f \cdot \alpha\) in \(\mathcal{V}\text{-Cat}\), that is, \(\beta \cdot Tf(\chi) \leq f \cdot \alpha(\chi)\) in the \(\mathcal{V}\)-category \((Y, b_0)\), for all \(\chi \in TX\).

Proof. First note that \(\beta \cdot Tf \leq f \cdot \alpha\) in \(\mathcal{V}\text{-Cat}\) is equivalent to \(b_0 \cdot f \cdot \alpha \leq b_0 \cdot \beta \cdot Tf\) in \(\mathcal{V}\text{-Rel}\). Since \(f \cdot a_0 \leq b_0 \cdot f\) one obtains \(f \cdot a = f \cdot a_0 \cdot \alpha \leq b_0 \cdot f \cdot \alpha \leq b_0 \cdot \beta \cdot Tf = b \cdot Tf\). Conversely, assuming \(f \cdot a \leq b \cdot Tf\), one derives \(f \cdot \alpha \leq b_0 \cdot \beta \cdot Tf\) and therefore \(b_0 \cdot f \cdot \alpha \leq b_0 \cdot b \cdot \beta \cdot Tf \leq b_0 \cdot \beta \cdot Tf\).

The previous constructions define a functor

\[
K : (\mathcal{V}\text{-Cat})^\mathbb{T} \to (\mathbb{T}, \mathcal{V})\text{-Cat}, \quad (X, a_0, \alpha) \mapsto (X, a_0 \cdot \alpha).
\]

It is worth noting that \(K\) is a 2-functor.

5.3.2 Example. We apply the functor \(K\) to the Examples 5.2.1(3) and (4). The ordered compact Hausdorff space \(2\) gives rise to the Sierpiński space \(2 = \{0, 1\}\) having \(\{0\}\) as its only non-trivial open subset. The metric compact Hausdorff space \([0, \infty], \mu, \xi\) (see Example 5.2.1(4)) induces the approach space \([0, \infty]\) with convergence \(P_+\)-relation \(\lambda : \beta[0, \infty] \leftrightarrow [0, \infty]\) defined by

\[
\lambda(u, v) = u \cup (\sup_{A \in u} \inf_{v \in A} v).
\]

5.3.3 Proposition. The functor \(K : (\mathcal{V}\text{-Cat})^\mathbb{T} \to (\mathbb{T}, \mathcal{V})\text{-Cat}\) sends initial sources with respect to \(\hat{O} : (\mathcal{V}\text{-Cat})^\mathbb{T} \to \text{Set}^\mathbb{T}\) to initial sources with respect to \(O : (\mathbb{T}, \mathcal{V})\text{-Cat} \to \text{Set}\).

Proof. Assume that \((f_i : (X, a_0, \alpha) \to (X_i, a_{0i}, \alpha_i))_{i \in I}\) in \((\mathcal{V}\text{-Cat})^\mathbb{T}\) is initial with respect to \(\hat{O} : (\mathcal{V}\text{-Cat})^\mathbb{T} \to \text{Set}^\mathbb{T}\). Then

\[
a = a_0 \cdot \alpha = (\bigwedge_{i \in I} f_i^* \cdot a_{0i} \cdot f_i) \cdot \alpha
\]

\[
= \bigwedge_{i \in I} (f_i^* \cdot a_{0i} \cdot f_i \cdot \alpha)
\]

\[
= \bigwedge_{i \in I} (f_i^* \cdot a_{0i} \cdot f_i \cdot Tf_i) = \bigwedge_{i \in I} (f_i^* \cdot a_i \cdot Tf_i).
\]

Consequently, applying the Taut Lift Theorem \(\llbracket 5.11.1 \rrbracket\) to the diagram

\[
\begin{array}{ccc}
\text{Set}^\mathbb{T} & \xrightarrow{G^\mathbb{T}} & \text{Set} \\
\downarrow{\hat{O}} & & \downarrow{O} \\
(\mathcal{V}\text{-Cat})^\mathbb{T} & \xrightarrow{K} & (\mathbb{T}, \mathcal{V})\text{-Cat}
\end{array}
\]

we obtain the following corollary:

5.3.4 Corollary. The functor \(K : (\mathcal{V}\text{-Cat})^\mathbb{T} \to (\mathbb{T}, \mathcal{V})\text{-Cat}\) has a left adjoint \(M : (\mathbb{T}, \mathcal{V})\text{-Cat} \to (\mathcal{V}\text{-Cat})^\mathbb{T}\). In particular, \(K\) preserves limits.
Analyzing the proof of the Taut Lift Theorem II.5.11.1, we find that the underlying Set-object of $M(X,a)$ can be chosen as $(TX,m_X)$, for a $(\mathbb{T},\mathcal{V})$-category $X = (X,a)$, and then $Mf = Tf$ for a $(\mathbb{T},\mathcal{V})$-functor $f$. The unit $X \to KM(X)$ is given by $e_X : X \to TX$ and, for every $Y = (Y,b_0,\beta)$ in $(\mathcal{V}\text{-Cat})^\mathbb{T}$, the counit $MK(Y) \to Y$ is given by $\beta : TY \to Y$. We will now give an explicit description of the $\mathcal{V}$-category structure of $M(X,a)$, under some conditions on the lax extension $\hat{T}$.

5.3.5 Theorem. Assume that the lax extension $\hat{T}$ of $\mathbb{T}$ is associative. Then the left adjoint of $K : (\mathcal{V}\text{-Cat})^\mathbb{T} \to (\mathbb{T},\mathcal{V})\text{-Cat}$ is given by

$$M : (\mathbb{T},\mathcal{V})\text{-Cat} \to (\mathcal{V}\text{-Cat})^\mathbb{T}, \quad (X,a) \mapsto (TX,\hat{T}a \cdot m^\circ_X, m_X), \quad f \mapsto Tf.$$ 

Proof. According to Theorem II.5.11.1 and Proposition 5.2.2, one has $M(X,a) = (TX, \hat{a}, m_X)$ where $\hat{a}$ is the initial $\mathcal{V}$-category structure on $TX$ with respect to the family of maps

$$TX \xrightarrow{Tf} TY \xrightarrow{\beta} (Y,b_0)$$

running over all $(\mathbb{T},\mathcal{V})$-functors $f : (X,a) \to (Y,b_0 \cdot \beta)$ with $(Y,b_0,\beta)$ in $(\mathcal{V}\text{-Cat})^\mathbb{T}$. We show that

$$\hat{a} = \hat{T}a \cdot m^\circ_X.$$ 

Let $(X,a)$ be a $(\mathbb{T},\mathcal{V})$-category. Firstly, $(TX,\hat{T}a \cdot m^\circ_X)$ is a $\mathcal{V}$-category since

$$\hat{T}a \cdot m^\circ_X \geq \hat{T}e_X \cdot m^\circ_X \geq 1_{TX}$$

and

$$\hat{T}a \cdot m^\circ_X \cdot \hat{T}a \cdot m^\circ_X = \hat{T}a \cdot \hat{T}a \cdot m^\circ_{TX} \cdot m^\circ_X$$

$$= \hat{T}a \cdot \hat{T}a \cdot (Tm_X)^\circ \cdot m^\circ_X$$

$$\leq \hat{T}a \cdot \hat{T}a \cdot \hat{T}(m^\circ_X) \cdot m^\circ_X$$

$$\leq \hat{T}(a \cdot \hat{T}a \cdot m^\circ_X) \cdot m^\circ_X = \hat{T}a \cdot m^\circ_X.$$ 

Furthermore,

$$m_X \cdot \hat{T}(\hat{T}a \cdot m^\circ_X) \leq m_X \cdot \hat{T}\hat{T}a \cdot Tm^\circ_X \cdot \hat{T}1_X$$

$$\leq \hat{T}a \cdot m^\circ_{TX} \cdot \hat{T}1_X$$

$$\leq \hat{T}a \cdot m^\circ_X \cdot m_X \cdot \hat{T}1_X$$

$$\leq \hat{T}(a \cdot \hat{T}1_X) \cdot m^\circ_X \cdot m_X = \hat{T}a \cdot m^\circ_X \cdot m_X$$

since $m_X \cdot m_{TX} = m_X \cdot Tm_X$ implies $m_{TX} \cdot Tm^\circ_X \leq m^\circ_X \cdot m_X$; therefore $(TX,\hat{T}a \cdot m^\circ_X, m_X)$ is indeed an object of $(\mathcal{V}\text{-Cat})^\mathbb{T}$. The map $e_X : X \to TX$ is actually a $(\mathbb{T},\mathcal{V})$-functor

$$e_X : (X,a) \to (TX,\hat{T}a \cdot m^\circ_X, m_X),$$
since $\hat{T}a \cdot m^o_X \cdot m_X \cdot T e_X = \hat{T}a \cdot m^o_X \geq \hat{T}a \cdot e_{TX} \geq e_X \cdot a$; hence, the identity map $1_{TX} = m_X \cdot T e_X$ on $TX$ is a $\mathcal{V}$-functor $(TX, \hat{a}) \to (TX, \hat{T}a \cdot m^o_X)$, so that $\hat{a} \leq \hat{T}a \cdot m^o_X$. Secondly, for any $(\mathbb{T}, \mathcal{V})$-functor $f : (X, a) \to (Y, b_0 \cdot \beta)$ with $(Y, b_0, \beta)$ in $(\mathcal{V}\text{-Cat})^T$ we find that

$$\beta \cdot Tf \cdot \hat{T}a \cdot m^o_X \leq \beta \cdot \hat{T}(f \cdot a) \cdot m^o_X$$

$$\leq \beta \cdot \hat{T}b_0 \cdot T \beta \cdot TTf \cdot m^o_X$$

$$\leq b_0 \cdot \beta \cdot T \beta \cdot TTf \cdot m^o_X$$

$$= b_0 \cdot \beta \cdot m_Y \cdot TTf \cdot m^o_X$$

$$= b_0 \cdot \beta \cdot T f \cdot m_X \cdot m^o_X \leq b_0 \cdot \beta \cdot T f .$$

Consequently, $\beta \cdot Tf : (TX, \hat{T}a \cdot m^o_X) \to (Y, b_0)$ is a $\mathcal{V}$-functor and we conclude that $\hat{a} \geq \hat{T}a \cdot m^o_X$. \qed

5.3.6 Remark. Under the conditions of the theorem above, the unit $e_X : (X, a) \to KM(X, a)$ of $M \dashv K$ at $(X, a) \in (\mathbb{T}, \mathcal{V})\text{-Cat}$ is even $\mathcal{O}$-initial (for the forgetful functor $\mathcal{O} : (\mathbb{T}, \mathcal{V})\text{-Cat} \to \text{Set}$) since $a$ is left unitary:

$$e_X^o \cdot \hat{T}a \cdot m^o_X \cdot m_X \cdot T e_X = e_X^o \cdot \hat{T}a \cdot m^o_X = e_X^o \circ a = a .$$

The same computation shows that the $(\mathbb{T}, \mathcal{V})$-structure $a$ on $X$ may be recovered from $\hat{a} = \hat{T}a \cdot m^o_X$:

$$e_X^o \cdot \hat{a} = e_X^o \cdot \hat{T}a \cdot m^o_X = a .$$

5.3.7 Examples.

@ (1) For the Barr extension $\beta$ of the ultrafilter monad $\beta$ to $\text{Rel}$ and a topological space $X$ with convergence relation $a : \beta X \rightrightarrows X$ (see 2.2), the order $(\leq) = (\beta a \cdot m^o_X)$ on $\beta X$ is given by

$$\chi \leq \chi' \iff \text{every closed set } A \in \chi \text{ belongs to } \chi'$$

$$\iff \text{every open set } A \in \chi' \text{ belongs to } \chi .$$

In fact, $\chi \leq \chi'$ if and only if there is some $X \in \beta \beta X$ with $m_X(X) = \chi$ and $X (\beta a) \chi'$, that is,

$$\forall A \in \chi (A^a \in X) \quad \text{and} \quad \forall A \in X (a(A) \in \chi') ,$$

where $A^a = \{a \in \beta X \mid A \in a\}$ (see Example 1.10.3(3)). Since $a(A^a) = \overline{A}$ is the closure of $A$, from $\chi \leq \chi'$ it follows that every closed set $A \in \chi$ belongs to $\chi'$. Conversely, this condition guarantees that the filter base

$$\{A^a \mid A \in \chi\}$$
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is disjoint from the ideal
\[ \{ B \subseteq \beta X \mid a[B] \notin \chi' \} , \]
so that by Corollary 11.1.13.3 there is an ultrafilter \( X \in \beta \beta X \) with \( m_X(X) = \chi \) and \( X(\beta a) \chi' \).

(2) Similarly, for the Barr extension \( \beta \) of \( \beta \) to \( P_+\text{-Rel} \) and an approach space \( X \) with \( \circ \) convergence \( P_+\text{-relation} \( a : \beta X \rightarrow X \) and corresponding approach distance \( \delta : X \rightarrow PX \) (see 2.4), the metric \( \hat{a} = \beta a \cdot m_X \) on \( \beta X \) can be written in terms of the approach distance \( \delta \) as
\[
\hat{a}(\chi, \chi') = \inf\{ u \in [0, \infty] \mid \forall A \in \chi \ (A(a) \in \chi') \} .
\]

To see this, let
\[
v := \hat{a}(\chi, \chi') = \inf\{ \beta a(X, \chi') \mid X \in \beta \beta X \ (m_X(X) = \chi) \}
\]
and
\[
w := \inf\{ u \in [0, \infty] \mid \forall A \in \chi \ (A(u) \in \chi') \} .
\]

Since
\[
v \geq \sup_{A \in \chi} \inf_{a \in A} a(a, y) ,
\]
for every \( \epsilon > 0 \), \( A \in \chi \), and \( B \in \chi' \), there exist \( a \in A^\beta \) and \( y \in B \) with \( a(a, y) \leq v + \epsilon \); hence, \( \hat{a}(y, A) \leq v + \epsilon \). Therefore, \( A^{(v+\epsilon)} \cap B \neq \emptyset \), and we conclude that \( A^{(v+\epsilon)} \in \chi' \) for every \( A \in \chi \) and \( \epsilon > 0 \). This proves \( w \leq v \). For the reverse inequality, note that \( A^{(w+\epsilon)} \cap B \neq \emptyset \), for every \( \epsilon > 0 \), \( A \in \chi \) and \( B \in \chi' \); this implies that
\[
\sup_{A \in \chi} \inf_{a \in A} a(a, y) \leq w + \epsilon .
\]

Hence, by Lemma 2.4.2, there is some \( X \in \beta \beta X \) with
\[
\{ A^\beta \mid A \in \chi \} \subseteq X \quad \text{and} \quad \beta a(X, \chi') \leq w + \epsilon ,
\]
so that \( v \leq w \).

5.4 The monad \( \mathbb{T} \) on \( (\mathbb{T}, \mathcal{V})\text{-Cat} \). In order to be able to apply Theorem 5.3.5 throughout this subsection we assume that

- the lax extension \( \hat{\mathbb{T}} \) of \( \mathbb{T} \) is associative.

The adjunction \( M \dashv K \) of the previous subsection induces a monad on \( (\mathbb{T}, \mathcal{V})\text{-Cat} \), whose functor sends a \( (\mathbb{T}, \mathcal{V})\)-category \( (X, a) \) to \( (TX, \hat{T}a \cdot m_X^\beta \cdot m_X) \) and a \( (\mathbb{T}, \mathcal{V})\)-functor \( f \) to \( Tf \). Since the multiplication and unit are given by \( m \) and \( e \) respectively, this monad constitutes a lifting of the \( \text{Set}\)-monad \( \mathbb{T} \) to \( (\mathbb{T}, \mathcal{V})\text{-Cat} \). We therefore denote this monad again by \( \mathbb{T} = (T, m, e) \).
The induced comparison functor is denoted by

\[ \tilde{K} : (\mathcal{V}\text{-Cat})^\top \to \text{(}(\mathbb{T}, \mathcal{V})\text{-Cat)}^\top ; \]

here \( \tilde{K}(X, a_0, \alpha) = (X, a_0 \cdot \alpha, \alpha) \) and \( \tilde{K} f = f \).

The algebraic functor \( A_\varepsilon : (\mathbb{T}, \mathcal{V})\text{-Cat} \to \mathcal{V}\text{-Cat} \) (defined by \( (X, a) \mapsto (X, a \cdot e_X) \)), see [3.4] has a left adjoint

\[ A^\circ : \mathcal{V}\text{-Cat} \to (\mathbb{T}, \mathcal{V})\text{-Cat} , \quad (X, a_0) \mapsto (X, e_X^0 \cdot \hat{T}a_0) , \]

and composing the adjunction \( A^\circ \dashv A_\varepsilon \) with

\[ ((\mathbb{T}, \mathcal{V})\text{-Cat})^\top \xrightarrow{G^\top} (\mathbb{T}, \mathcal{V})\text{-Cat} \]

yields a new adjunction

\[ ((\mathbb{T}, \mathcal{V})\text{-Cat})^\top \xrightarrow{G^\top} \mathcal{V}\text{-Cat} . \]

A direct computation shows that \( F_0^\top \vdash G_0^\top \) induces the monad \( \mathbb{T} \) on \( \mathcal{V}\text{-Cat} \), so we obtain the comparison functor

\[ \hat{A}_\varepsilon : ((\mathbb{T}, \mathcal{V})\text{-Cat})^\top \to (\mathcal{V}\text{-Cat})^\top \]

that sends \( (X, a, \alpha) \) to \( (X, a \cdot e_X, \alpha) \). Clearly, \( \hat{A}_\varepsilon \hat{K} = 1 \), and that \( \hat{A}_\varepsilon \) and \( \tilde{K} \) are in fact inverse to each other follows from a very pleasant property of the monad \( \mathbb{T} \) on \( (\mathbb{T}, \mathcal{V})\text{-Cat} \), as we show next.

**5.4.1 Theorem.** Assume that the lax extension \( \hat{T} \) of \( \mathbb{T} \) is associative. Then the monad \( \mathbb{T} \) on \( (\mathbb{T}, \mathcal{V})\text{-Cat} \) is of Kock–Zöberlein type.

**Proof.** Firstly, if \( f \leq g \) for \( f, g : (X, a) \to (Y, b) \) in \( (\mathbb{T}, \mathcal{V})\text{-Cat} \), then \( Tf \leq Tg \); indeed, it follows from \( f^\circ \cdot b \leq g^\circ \cdot b \) that

\[ (Tf)^\circ \cdot \hat{T}b \cdot m_Y^0 \cdot m_Y = \hat{T}(f^\circ \cdot b) \cdot m_Y^0 \cdot m_Y \leq \hat{T}(g^\circ \cdot b) \cdot m_Y^0 \cdot m_Y = (Tg)^\circ \cdot \hat{T}b \cdot m_Y^0 \cdot m_Y . \]

Secondly, we show that \( e_{TTX} : TX \to TTX \) is right adjoint to \( m_X : TTX \to TX \), for every \( (\mathbb{T}, \mathcal{V})\)-category \( X = (X, a) \). Clearly, \( m_X \cdot e_{TX} = 1_{TX} \). To see \( 1_{TTX} \leq e_{TX} \cdot m_X \), we recall that the \( (\mathbb{T}, \mathcal{V})\)-category structure on \( TX \) is given by \( \hat{T}a \cdot m_X^0 \cdot m_X \), and the \( \mathcal{V}\)-category structure on \( TTX \) by \( c = \hat{T} \hat{a} \cdot Tm_X \cdot m_{TX}^0 \), where \( \hat{a} = \hat{T}a \cdot m_X^0 \). Using the fact that \( \hat{a} \cdot m_X \) is unitary, we then compute:

\[ m_X^0 \cdot e_{TX} \cdot c = m_X^0 \cdot e_{TX} \cdot \hat{T}(\hat{a} \cdot m_X) \cdot m_{TX}^0 = m_X^0 \cdot \hat{a} \cdot m_X \geq m_X^0 \cdot m_X \geq 1_{TTX} . \]

Hence, for all \( x \in TTX \), we have \( c(x, e_{TX} \cdot m_X(x)) \geq k \); and therefore \( 1_{TTX} \leq e_{TX} \cdot m_X \) in \( (\mathbb{T}, \mathcal{V})\text{-Cat} \). \( \square \)
5. REPRESENTABLE LAX ALGEBRAS

For \( X = (X, a) \) in \((\mathcal{T}, \mathcal{V})\)-Cat, an Eilenberg–Moore structure \( \alpha : TX \to X \) in \((\mathcal{T}, \mathcal{V})\)-Cat gives rise to an adjunction \( \alpha \dashv e_X \) in \((\mathcal{T}, \mathcal{V})\)-Cat (see Proposition \[\text{[II]}\,\text{4.9.1}]). Since the algebraic functor \( A_e : (\mathcal{T}, \mathcal{V})\)-Cat \to \mathcal{V}\)-Cat is a 2-functor, the underlying \( \mathcal{V}\)-functor

\[
e_X : A_e(X, a) = (X, a_0) \to A_e T(X, a) = (TX, \hat{T}a \cdot m_X^0 \cdot m_X \cdot e_{TX}) = (TX, \hat{T}a \cdot m_X^0)
\]

is right adjoint to \( \alpha : (TX, \hat{T}a \cdot m_X^0) \to (X, a_0) \) as well. Hence, with Exercise 3.F applied to \( \mathcal{V}\)-Cat we obtain from the adjunction \( \alpha \dashv e_X \) in \( \mathcal{V}\)-Cat the equation

\[
a_0(\alpha(\chi), x) = \hat{T}a \cdot m_X^0(\chi, e_X(x)) \quad \text{(5.4.i)}
\]

for all \( \chi \in TX \) and \( x \in X \); this means \( a_0 \cdot \alpha = e_X^0 \cdot \hat{T}a \cdot m_X^0 = e_X^0 \circ a = a \). As a consequence, \( \hat{K} \) and \( A_e \) are inverse to each other:

5.4.2 Corollary. If the lax extension \( \hat{T} \) of \( \mathcal{T} \) is associative, then \(((\mathcal{T}, \mathcal{V})\text{-Cat})\^{\hat{T}} \simeq (\mathcal{V}\text{-Cat})\^{\hat{T}}\).

The following diagram summarizes the situation exhibited so far:

\[
\begin{array}{ccc}
((\mathcal{T}, \mathcal{V})\text{-Cat})\^{\hat{T}} & \xrightarrow{\hat{A}_e} & (\mathcal{V}\text{-Cat})\^{\hat{T}} \\
\downarrow{G^\mathcal{T}} & & \downarrow{G^\mathcal{T}} \\
(\mathcal{T}, \mathcal{V})\text{-Cat} & \xrightarrow{A_e} & \mathcal{V}\text{-Cat} \\
\end{array}
\]

\[
\begin{array}{cccc}
\xrightarrow{\hat{O}} & \xrightarrow{\hat{O}} & \xrightarrow{\hat{O}} \\
\downarrow{G^\mathcal{T}} & & \downarrow{G^\mathcal{T}} \\
\text{Set} & \xrightarrow{\text{Set}} & \text{Set} \\
\end{array}
\]

One important consequence of Theorem 5.4.1 is that a \((\mathcal{T}, \mathcal{V})\)-category \( X \) admits up to equivalence at most one \( \mathcal{T} \)-algebra structure \( \alpha : TX \to X \), since necessarily \( \alpha \dashv e_X \) in \((\mathcal{T}, \mathcal{V})\)-Cat (see Proposition \[\text{[II]}\,\text{4.9.1}]).

5.4.3 Definitions.

(1) A \((\mathcal{T}, \mathcal{V})\)-category \( X \) is representable if \( e_X : X \to TX \) has a left adjoint in \((\mathcal{T}, \mathcal{V})\)-Cat. Since \( \mathcal{T} \) is of Kock–Zöberlein type, a \((\mathcal{T}, \mathcal{V})\)-functor \( \alpha : TX \to X \) is a left adjoint of \( e_X \) if and only if \( \alpha \cdot e_X \simeq 1_X \). We hasten to remark that a representable \((\mathcal{T}, \mathcal{V})\)-category does not need to be a \( \mathcal{T} \)-algebra since \( \alpha \dashv e_X \) only implies \( \alpha \cdot e_X \simeq 1_X \) and \( \alpha \cdot T\alpha \simeq \alpha \cdot m_X \). Of course, if \( X \) is separated, in the sense that its underlying order is separated (see also \[\text{[V]}\,\text{2.1}]), then \( \alpha \) is a \( \mathcal{T} \)-algebra structure.

(2) A \((\mathcal{T}, \mathcal{V})\)-functor \( f : X \to Y \) between representable \((\mathcal{T}, \mathcal{V})\)-categories \( X \) and \( Y \), with left adjoints \( \alpha : TX \to X \) and \( \beta : TY \to Y \) respectively, is a pseudo-homomorphism whenever

\[
\beta \cdot Tf \simeq f \cdot \alpha .
\]

As before, if \( Y \) is separated, then we have equality above. We note that this condition does not depend on the particular choice of the left adjoints \( \alpha \) and \( \beta \).
CHAPTER III. LAX ALGEBRAS

(3) The category of representable \((\mathbb{T}, \mathcal{V})\)-categories and pseudo-homomorphism will be denoted as
\[
(\mathbb{T}, \mathcal{V})\text{-RepCat}.
\]

(4) A \((\mathbb{T}, \mathcal{V})\)-category \(X\) is \(\mathbb{T}\)-cocomplete if the \(\mathcal{V}\)-functor \(e_X : A_vX \to A_vTX\) has a left adjoint in \(\mathcal{V}\text{-Cat}\). While we will not elaborate on this notion in this book, here we note that, by (5.4.i), \(X = (X, a)\) is \(\mathbb{T}\)-cocomplete if and only if \(a\) can be written as \(a = a_0 \cdot \alpha\) (with \(a_0 = a \cdot e_X\)), for some map \(\alpha : TX \to X\). Put differently, for every \(\chi \in TX\) there must exist a tacitly chosen generic point \(x_0 \in X\) so that
\[
a(\chi, x) = a_0(x_0, x)
\]
for all \(x \in X\), and such a generic point is unique up to equivalence.

5.4.4 Proposition. Assume that the lax extension \(\hat{T}\) of \(\mathbb{T}\) is associative. Then a \((\mathbb{T}, \mathcal{V})\)-category \(X = (X, a)\) is representable if and only if \(X\) is \(\mathbb{T}\)-cocomplete and \(a \cdot \hat{T}a = a \cdot m_X\).

Proof. Clearly, a representable \((\mathbb{T}, \mathcal{V})\)-category \(X = (X, a)\) is \(\mathbb{T}\)-cocomplete and, with \(a = a_0 \cdot \alpha\),
\[
a \cdot \hat{T}a = a_0 \cdot \alpha \cdot \hat{T}a_0 \cdot T\alpha \geq a_0 \cdot a_0 \cdot \alpha \cdot T\alpha = a_0 \cdot \alpha \cdot m_X = a \cdot m_X.
\]
Assume now that \(X = (X, a)\) is \(\mathbb{T}\)-cocomplete and that \(a \cdot \hat{T}a = a \cdot m_X\). Since \(\mathbb{T}\) is of Kock–Zöberlein type, it is enough to verify that the map \(\alpha : TX \to X\) is a \((\mathbb{T}, \mathcal{V})\)-functor. In fact,
\[
\alpha \cdot \hat{T}a \cdot m_X^\circ \cdot m_X \leq a \cdot \hat{T}a \cdot \hat{T}a_0 \cdot m_X \cdot m_X = a \cdot m_X \cdot m_X^\circ \cdot m_X \\
\leq a \cdot m_X = a \cdot \hat{T}a = a \cdot \hat{T}a_0 \cdot T\alpha = a \cdot T\alpha.
\]

5.5 Dualizing \((\mathbb{T}, \mathcal{V})\)-categories. In 1.3 we introduced the dual \(X^\text{op}\) of a \(\mathcal{V}\)-category \(X = (X, a)\) as \(X^\text{op} = (X, a^\circ)\), for a commutative quantale \(\mathcal{V}\). This definition cannot be used directly for \((\mathbb{T}, \mathcal{V})\)-categories in general since \(a^\circ : X \to TX\) does not have the correct type, and in this subsection we will discuss one possibility to deal with this problem. Roughly speaking, we consider only those \((\mathbb{T}, \mathcal{V})\)-categories \(X = (X, a)\) where \(a = a_0 \cdot \alpha\), for some \(\alpha : TX \to X\) and where \(a_0 = a \cdot e_X\) is the underlying \(\mathcal{V}\)-category structure, then dualize just \((X, a_0)\) and combine the result with \(\alpha\); hence the structure of \(X^\text{op}\) is given by \(a_0^\circ \cdot \alpha\). This defines, however, in general only a \((\mathbb{T}, \mathcal{V})\)-graph (see Proposition 5.5.3 below). We therefore consider the concept in this context.

Throughout this subsection we assume that

- \(\mathcal{V}\) is commutative.
A \((\mathcal{T}, \mathcal{V})\)-graph \(X = (X, a)\) is called dualizable whenever \(a_0 = a \cdot e_X\) is transitive and \(a = a_0 \cdot \alpha\), for some map \(\alpha : TX \to X\). For a dualizable \((\mathcal{T}, \mathcal{V})\)-graph \(X = (X, a)\), we write \(X_0\) to denote its underlying \(\mathcal{V}\)-category \(X_0 = (X, a_0)\). We consider \(TX\) as a discrete \(\mathcal{V}\)-category, so that \(\alpha : TX \to X_0\) is a \(\mathcal{V}\)-functor. With this notation, \(a_0 \cdot \alpha = \alpha_*\) (see \[1.3\]) and, if \(\alpha_* = a = \beta_*\), also \(\alpha^* = \beta^*\) and therefore
\[
a_0 \cdot \alpha = (\alpha^*)_0 = (\beta^*)_0 = a_0 \cdot \beta.
\]

**5.5.1 Lemma.** Let \(X = (X, a)\) be a dualizable \((\mathcal{T}, \mathcal{V})\)-graph. Then \((X, a_0^* \cdot \alpha)\) is a dualizable \((\mathcal{T}, \mathcal{V})\)-graph as well, where the underlying \(\mathcal{V}\)-category of \((X, a_0^* \cdot \alpha)\) is \((X_0)^{\text{op}}\).

**Proof.** It suffices to show \(a_0^* = a_0^* \cdot \alpha \cdot e_X\). From \(\alpha = a_0 \cdot \alpha\) we infer \(a_0 = a_0 \cdot \alpha \cdot e_X = (\alpha \cdot e_X)_*,\) hence \(a_0 = (\alpha \cdot e_X)^*\) and therefore \(a_0^* = a_0^* \cdot \alpha \cdot e_X\).

The dual \((\mathcal{T}, \mathcal{V})\)-graph of a dualizable \((\mathcal{T}, \mathcal{V})\)-graph \(X = (X, a)\) is then defined as \(X^{\text{op}} = (X, a_0^* \cdot \alpha)\). This definition is independent of the choice of \(\alpha : TX \to X\) by the calculation given before Lemma 5.5.1.

Every \(\mathcal{T}\)-cocomplete \((\mathcal{T}, \mathcal{V})\)-category, seen as a \((\mathcal{T}, \mathcal{V})\)-graph, is dualizable. In particular, every \(\mathcal{V}\)-category is dualizable and its dual in the sense above is just the usual dual. In 5.8 we will see another important example of a dualizable \((\mathcal{T}, \mathcal{V})\)-graph.

**5.5.2 Lemma.** For a \((\mathcal{T}, \mathcal{V})\)-functor \(f : (X, a) \to (Y, b)\) between dualizable \((\mathcal{T}, \mathcal{V})\)-graphs with \(a = a_0 \cdot \alpha\) and \(b = b_0 \cdot \beta\), the map \(f : X \to Y\) defines also a \((\mathcal{T}, \mathcal{V})\)-functor \(f^{\text{op}} : X^{\text{op}} \to Y^{\text{op}}\) if and only if \(f \cdot \alpha \simeq \beta \cdot Tf\).

**Proof.** As for Lemma 5.3.1.

**5.5.3 Proposition.** Assume that the lax extension \(\hat{\mathcal{T}}\) of \(\mathcal{T}\) is associative and let \(X = (X, a)\) be a \(\mathcal{T}\)-cocomplete \((\mathcal{T}, \mathcal{V})\)-category. Then the following assertions are equivalent:

(i) the \((\mathcal{T}, \mathcal{V})\)-graph \(X^{\text{op}}\) is a \((\mathcal{T}, \mathcal{V})\)-category;

(ii) \(X\) satisfies \(a \cdot \hat{T}a = a \cdot m_X\);

(iii) \(X\) is representable.

**Proof.** By Proposition 5.4.4 (iii) \(\implies\) (ii); and the implication (iii) \(\implies\) (i) can be shown as in 5.3. Assume now that \(X^{\text{op}}\) is a \((\mathcal{T}, \mathcal{V})\)-category. Since \(X\) is a \((\mathcal{T}, \mathcal{V})\)-category,
\[
(\alpha \cdot T\alpha)_* = a_0 \cdot \alpha \cdot T\alpha \leq a_0 \cdot \alpha \cdot \hat{T}a_0 \cdot T\alpha \leq a_0 \cdot \alpha \cdot m_X = (\alpha \cdot m_X)_*;
\]

similarly, since \(X^{\text{op}}\) is a \((\mathcal{T}, \mathcal{V})\)-category,
\[
a_0^* \cdot \alpha \cdot T\alpha \leq a_0^* \cdot \alpha \cdot m_X
\]
and therefore \((\alpha \cdot T\alpha)^* \leq (\alpha \cdot m_X)^*\). Consequently, \((\alpha \cdot T\alpha)_* = (\alpha \cdot m_X)_*\) by Exercise [II.4.E], hence \(a \cdot \hat{T}a \geq a \cdot m_X\). The inequality \(a \cdot \hat{T}a \leq a \cdot m_X\) we get from \(X\) being a \((\mathcal{T}, \mathcal{V})\)-category, therefore \(a \cdot \hat{T}a = a \cdot m_X\).
In conclusion, \((f : X \to Y) \mapsto (f^{\text{op}} : X^{\text{op}} \to Y^{\text{op}})\) defines a functor
\[
(-)^{\text{op}} : (\mathcal{T}, \mathcal{V})\text{-RepCat} \to (\mathcal{T}, \mathcal{V})\text{-RepCat}
\]
which makes the diagram
\[
\begin{array}{ccc}
(\mathcal{T}, \mathcal{V})\text{-RepCat} & \xrightarrow{(-)^{\text{op}}} & (\mathcal{T}, \mathcal{V})\text{-RepCat} \\
\downarrow (-)_0 & & \downarrow (-)_0 \\
\mathcal{V}\text{-Cat} & \xrightarrow{(-)^{\text{op}}} & \mathcal{V}\text{-Cat}
\end{array}
\]
commutative.

5.6 The ultrafilter monad on \(\text{Top}\). We consider the ultrafilter monad \(\beta\) with its Barr extension \(\beta\) to \(\text{Rel}\), so that \((\beta, 2)\text{-Cat} \cong \text{Top}\). By 5.4 \(\beta\) extends to a monad on \(\text{Top}\), again denoted by \(\beta\). For a topological space \(X\), \(\beta X\) is the space of all ultrafilters of the set \(X\) where, for \(x \in \beta \beta X\) and \(\chi \in \beta X\), one has \(x \to \chi\) precisely when \(m_X(x) \leq \chi\) (see Examples 5.3.7), that is, precisely when for every open set \(A \in \chi\), one has \(A^\beta = \{a \in \beta X \mid A \in a\} \in x\). Therefore:

\[\boxplus 5.6.1\text{ Lemma.} \quad \text{The sets } A^\beta \text{ (with } A \subseteq X \text{ open) form a base of the topology of } \beta X. \]

\[\text{Important note.} \quad \text{For a topological space } X \text{ (usually assumed to be completely regular), the space } \beta X \text{ of ultrafilters on } X \text{ should not be confused with the Čech–Stone compactification of } X. \text{ However, in the discussion following Proposition 5.6.2 we show how the Čech–Stone compactification of } X \text{ may be obtained from the space } \beta X \text{ defined here.} \]

Diagram (5.4.ii) specializes to
\[
\begin{array}{ccc}
\text{Top}^\beta & \xrightarrow{\cong} & \text{OrdCompHaus} \\
\downarrow & & \downarrow \\
\text{Top} & \xrightarrow{\beta} & \text{OrdHaus} \xrightarrow{\beta} \text{Set}.
\end{array}
\]

Here the category \(\text{OrdCompHaus}\) of ordered compact Hausdorff spaces and their morphisms (see Examples 5.2.1) appears as the category of Eilenberg–Moore algebras for the ultrafilter monad \(\beta\) on both \(\text{Ord}\) and \(\text{Top}\).

The inclusion functor \(\text{CompHaus} \to \text{Top}\) factors as
\[
\text{CompHaus} \xrightarrow{(-)^d} \text{Ord} \xrightarrow{\beta} \text{Top},
\]
so its left adjoint, usually called \(\check{\text{Cech–Stone compactification}}\), can be taken as \(\pi_0 \cdot M\) where \(M \dashv K\) (see Theorem 5.3.5) and \(\pi_0 \dashv (-)^d\). Recall that for an ordered compact Hausdorff
space $X = (X, \leq, \alpha)$, the graph $R \subseteq X \times X$ of the order relation $\leq$ is closed and therefore compact Hausdorff. The reflection $q : X \rightarrow \pi_0(X)$ is actually the coequalizer

$$
R \xrightarrow{p_1} X \xrightarrow{p_2} \pi_0(X)
$$

(5.6.i)

in $\text{CompHaus}$ which is usually different from the coequalizer in $\text{Set}$ (see Exercise 5.A). However, if the graph $E \subseteq X \times X$ of the equivalence relation induced by $\leq$ is closed, then $E$ is compact Hausdorff, and the coequalizer (5.6.i) in $\text{CompHaus}$ can be constructed as in $\text{Set}$. This is certainly the case when $\leq$ is confluent: see Exercise 5.A.

5.6.2 Proposition. For a topological space $X = (X, a)$, the order relation $\hat{\alpha} = \beta a \cdot m_X \circ \tau$ on $\beta X$ is confluent if and only if for all disjoint closed subsets $A, B \subseteq X$ there exist open subsets $U, V \subseteq X$ with $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Proof. Assume first that $\hat{\alpha} = \beta a \cdot m_X \circ \tau$ is confluent. Let $A, B \subseteq X$ be closed subsets of $X$ with the property that every open neighborhood of $A$ intersects every open neighborhood of $B$. Hence, by Corollary III.1.13.3, there is an ultrafilter $\chi \in \beta X$ with

$$
\{U \cap V \mid U, V \in \mathcal{O}X, A \subseteq U, B \subseteq V\} \subseteq \chi.
$$

By definition, the filter generated by the filter base $\{A\}$ is disjoint from the ideal generated by $\{U \in \mathcal{O}X \mid U \not\in \chi\}$, so Corollary III.1.13.5 guarantees the existence of an ultrafilter $a \in \beta X$ with $A \subseteq a$ and $\chi \leq a$. A similar argument yields an ultrafilter $b \in \beta X$ with $B \subseteq b$ and $\chi \leq b$. By hypothesis, there is some $y \in \beta X$ with $a \leq y$ and $b \leq y$. Hence, $a \in y$ and $b \in y$, so that $A \cap B \neq \emptyset$.

Assume now that the condition on disjoint closed subsets is satisfied. Let $\chi, a, b \in \beta X$ with $\chi \leq a$ and $\chi \leq b$. Let $A \subseteq a$ and $B \subseteq b$ be closed. Then $U \in \chi$ and $V \subseteq x$, and therefore $U \cap V \neq \emptyset$, for all open subsets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$. Consequently, $A \cap B \neq \emptyset$. Hence, by Corollary III.1.13.3, there is an ultrafilter $y \in \beta X$ with

$$
\{A \cap B \mid A, B \subseteq X \text{ closed}, A \subseteq a, B \subseteq b\} \subseteq y,
$$

so that $a \leq y$ and $b \leq y$. □

A topological space with the property described in Proposition 5.6.2 is called normal. By the discussion preceding this Proposition, we have:

5.6.3 Corollary. The Čech–Stone compactification of a normal space $X$ is isomorphic to the space of connected components of $\beta X$ with respect to the order relation of Example 5.3.7(1).

A topological space $X$ is $\beta$-cocomplete if every ultrafilter $\chi$ has a generic convergence point $x_0 \in X$, so that

$$
\chi \rightarrow x \iff x_0 \leq x
$$
for all \( x \in X \), or equivalently, \( \lim x = \{x_0\} \) where \( \lim x \) denotes the set of limit points of \( x \). A subset of the form \( \{x\} \), for \( x \in X \), is a trivial example of an irreducible closed subset, that is, of a non-empty closed subset \( A \subseteq X \) with the property that, whenever \( A \subseteq A_1 \cup A_2 \) for closed subsets \( A_1, A_2 \subseteq X \), then \( A \subseteq A_1 \) or \( A \subseteq A_2 \). A topological space \( X \) is called sober if every irreducible closed subset \( A \subseteq X \) is of the form \( A = \{x\} \), for a unique \( x \in X \); without this uniqueness requirement \( X \) is weakly sober. By Exercise 5.D, every irreducible closed subset \( A \subseteq X \) is the set \( A = \lim x \) of limit points of some ultrafilter \( x \in \beta X \). Hence, we obtain the following result.

\[ 5.6.4 \text{ Proposition.} \quad \] A topological space \( X \) is \( \beta \)-cocomplete if and only if

1. for every \( x \in \beta X \), \( \lim x \) is irreducible, and
2. \( X \) is weakly sober.

\[ \text{Proof.} \quad \] The result follows immediately from the definitions.

Note that (1) implies in particular that \( X \) is compact.

To better understand the condition \( a \cdot \beta a = a \cdot m_X \), we introduce the way-below relation on the lattice of open subsets of \( X \): for \( A \subseteq X \) open, \( A \) is way-below \( B \), written as \( A \ll B \), if every open cover of \( B \) has a finite sub-cover of \( A \); that is, whenever \( B \subseteq \bigcup_{i \in I} B_i \) with open subsets \( B_i \), then there exists a finite subset \( K \subseteq I \) such that \( A \subseteq \bigcup_{i \in K} B_i \). This relation can be equivalently expressed in the language of ultrafilter convergence: \( A \ll B \) precisely when every ultrafilter \( x \) on \( A \) has a limit point in \( B \) (Exercise 5.E). Furthermore, \( X \) is called core-compact if, for every point \( x \in X \) and every open neighborhood \( B \) of \( x \), there exists an open neighborhood \( A \) of \( x \) with \( A \ll B \).

\[ 5.6.5 \text{ Remark.} \quad \] We will see in Theorem 5.8.5 below that the core-compact spaces are precisely the exponentiable objects (see II.4.4) in the category \( \text{Top} \).

\[ 5.6.6 \text{ Proposition.} \quad \] A topological space \( X \) with convergence relation \( a : \beta X \rightarrow X \) is core-compact if and only if \( a \cdot \beta a = a \cdot m_X \).

\[ \text{Proof.} \quad \] Assume first that \( X \) is core-compact. Since \( a \cdot \beta a \leq a \cdot m_X \) for every topological space, we only have to show \( a \cdot \beta a \geq a \cdot m_X \). Let \( x \in \beta \beta X \) and \( x \in X \) where \( x \) is a limit point of \( m_X(x) \). Hence, \( B^\beta \in X \) for every open neighborhood \( B \) of \( x \). Let now \( A \) be an open neighborhood of \( x \), and choose any open neighborhood \( B \) of \( x \) with \( B \ll A \). Then

\[ \lim^{-1} A = \{x \in \beta X \mid \lim x \cap A \neq \emptyset\} \supseteq B^\beta \in X, \]

where \( \lim x \) denotes the set of all limit points of \( x \). Hence, the neighborhood filter \( f \) of \( x \) is disjoint from the ideal \( j = \{A \subseteq X \mid \lim^{-1} A \notin x\} \); therefore, by Corollary II.1.13.5, there exists some \( x \in \beta X \) disjoint from \( j \) (so that \( \beta a x \), see Example 1.10.3(3)) and \( x \) contains every open neighborhood of \( x \), that is, \( x a x \). Assume now that \( X \) is not core-compact,
that is, there is some $x \in X$ and some open neighborhood $B$ of $x$ so that for every open neighborhood $A$ of $x$ there is some ultrafilter $y \in \beta X$ with $A \in y$ and $\lim y \cap B = \emptyset$. Consequently, there is some $X \in \beta \beta X$ containing $\{ y \in \beta X \mid \lim y \cap B = \emptyset \}$ and $A^\beta$, for every open neighborhood $A$ of $x$. Then $m_X(x)$ converges to $x$, but $X \not\rightarrow \chi$ implies that $B \notin \chi$, so $\chi$ cannot converge to $x$.

5.6.7 Remark. It is often easier to check core-compactness of a topological space $X$ by just looking at the elements of a subbase for $\mathcal{O}X$ (that is, a subset of $\mathcal{O}X$ whose set of finite intersections forms a base for $\mathcal{O}X$, see II.1.9). Given a set $X$ and a subset $\mathcal{B} \subseteq \mathcal{P}X$ of the powerset of $X$ (with no further axioms), one defines core-compactness and convergence $a : \beta X \leftrightarrow X$ for $(X, \mathcal{B})$ in the same way as one does for topological spaces, and we note that the topology $\langle \mathcal{B} \rangle$ generated by $\mathcal{B}$ has the same convergence as $\mathcal{B}$. We wish to conclude that $a \cdot \beta a = a \cdot m_X$ implies that $(X, \mathcal{B})$ is core-compact; however, the proof above uses that the collection of all neighborhoods of $x \in X$ forms a filter, a fact that is not necessarily true if $\mathcal{B}$ is just any subset of $\mathcal{P}X$. For the filter $f$ generated by all $\mathcal{B}$-neighborhoods of $x$, we still have $f \cap j = \emptyset$ if the convergence $a$ satisfies the following condition: every ultrafilter has a smallest convergence point with respect to the order relation $a \cdot e_X$. Hence, under this condition, a topological space $X$ is core-compact if $X$ is core-compact with respect to a subbase since

$$(X, \mathcal{B}) \text{ is core-compact} \iff a \cdot \beta a = a \cdot m_X \text{ for the convergence } a \text{ of } \mathcal{B}$$

$$\iff a \cdot \beta a = a \cdot m_X \text{ for the convergence } a \text{ of } \langle \mathcal{B} \rangle$$

$$\iff (X, \langle \mathcal{B} \rangle) \text{ is core-compact.}$$

In fact, this argument works for any other property of a topological space which can be equivalently expressed in terms of opens and in terms of ultrafilter convergence, without using the axioms of a topology. Another important example is compactness: a topological space $X$ is compact if $X$ is compact with respect to a subbase. This result is known as Alexander’s Subbase Lemma.

Combining Propositions 5.4.4, 5.6.4 and 5.6.6 gives

5.6.8 Proposition. A topological space $X$ is representable if and only if

(1) $X$ is core-compact,

(2) for every $\chi \in \beta X$, $\lim \chi$ is irreducible, and

(3) $X$ is weakly sober.

5.7 Representable topological spaces. In this subsection we will present a more detailed analysis of representable topological spaces.
First we will see that a representable topological space is not only core-compact but even
locally compact; a topological space is \textit{locally compact} if the neighborhood filter of every
point \( x \in X \) has a base formed by compact neighborhoods of \( x \).

\subsection*{5.7.1 Lemma} \textit{Every representable topological space is locally compact.}

\textit{Proof.} By Lemma \ref{lemma:core-compactness}, the topology on \( \beta X \) is generated by all sets of the form
\[ A^\beta = \{ a \in \beta X \mid A \in a \} , \]
where \( A \subseteq X \) is open. Furthermore, for any ultrafilter \( X \in \beta \beta X \) with \( A^\beta \in X \), we have
\( m_X(X) \in A^\beta \), and therefore \( A^\beta \) is compact. For any \( x \in \beta X \),
\[ \{ A^\beta \mid A \in x \} \]
is a base of the neighborhood filter of \( x \); and we conclude that \( \beta X \) is locally compact. If \( X \)
is representable, then \( X \) is a split subobject of \( \beta X \) (since \( \alpha : \beta X \to X \) can be chosen such
that \( \alpha(e_X(x)) = x \) and hence also locally compact (Exercise 5.F). \hfill \square

For a core-compact space \( X \), condition (1) of Proposition \ref{proposition:core-compactness} is equivalent
\textit{to the following stability property of the way-below relation} \( \ll \) : for open subsets \( U_1, \ldots, U_n \) and \( V_1, \ldots, V_n \)
\((n \in \mathbb{N})\) of \( X \) with \( U_i \ll V_i \) for each \( 1 \leq i \leq n \), also \( \bigcap_i U_i \ll \bigcap_i V_i \) (Exercise 5.H). Note that
it is enough to consider the cases \( n = 0 \) and \( n = 2 \), and for \( n = 0 \) this condition reads as
\( X \ll X \) which just means that \( X \) is compact. Saying that a topological space \( X \) with this
stability property is \textit{stable}, we obtain the following result.

\subsection*{5.7.2 Theorem} \textit{A topological space \( X \) is representable if and only if \( X \) is locally compact, weakly sober and stable.}

We remark that representable T0-spaces are also called \textit{stably compact} in the literature
where, however, the stability condition on the way-below relation is usually replaced by the
requirement that the compact down-closed subsets of \( X \) are closed under finite intersection.
To see that these conditions are indeed equivalent, first note that a representable space \( X \) is
compact, and the binary intersections of pairs of compact down-closed subsets is compact:
if \( A, B \subseteq X \) are compact and down-closed and \( A \cap B \in x \in \beta X \), then any smallest
convergence point of \( x \) belongs to both \( A \) and \( B \) and therefore also to \( A \cap B \). Consequently,
in a representable space the finite intersection of compact down-closed subsets is again
compact. For the converse implication, we use the following description of the way-below
relation on the lattice of opens of a locally compact space.

\subsection*{5.7.3 Lemma} \textit{Let \( X \) be locally compact and \( U, V \subseteq X \) open. Then \( U \ll V \) if and only if \( U \subseteq K \subseteq V \) for some compact and down-closed \( K \subseteq X \).}

\textit{Proof.} Assume first that \( U \ll V \). For every \( x \in V \) there is a compact neighborhood \( K \)
of \( x \) with \( K \subseteq V \). Since \( V = \bigcup \{ K \subseteq V \mid K \text{ is a compact neighborhood of some } x \in V \} \)
and $U \ll V$, there is some compact $K$ with $U \subseteq K \subseteq V$. Furthermore, if $A \subseteq W$ for some $A \subseteq X$ and some open subset $W \subseteq X$, then also $A \subseteq W$ since open subsets are down-closed. In particular, the down-closure of a compact subset is compact and therefore $K$ above can be chosen down-closed. Conversely, if $U \subseteq K \subseteq V$ for some compact $K \subseteq X$, then every open cover of $V$ contains a finite sub-cover of $K$ and hence also of $U$. □

5.7.4 Remark. If the compact open subsets of a locally compact space $X$ form a base of the topology of $X$, then $K$ in the proof of Lemma 5.7.3 can be chosen as a compact open subset of $X$.

From Lemma 5.7.3 we deduce at once that, for a locally compact space, stability of the way-below relation under finite intersection follows from stability of compact down-sets under finite intersection.

5.7.5 Theorem. A topological space is representable if and only if it is locally compact, weakly sober, and if the finite intersection of compact down-closed subsets is compact.

We turn now our attention to pseudo-homomorphisms. Recall from 5.4 that a pseudo-homomorphism between representable topological spaces is a continuous map $f : X \rightarrow Y$ that preserves the smallest convergence points of ultrafilters.

5.7.6 Proposition. Let $f : X \rightarrow Y$ be a continuous map between representable topological spaces. Then the following assertions are equivalent:

(i) $f$ is a pseudo-homomorphism;

(ii) for every compact down-closed subset $K \subseteq Y$, $f^{-1}(K)$ is compact;

(iii) for all open subsets $U, V \subseteq Y$, $U \ll V$ implies $f^{-1}(U) \ll f^{-1}(V)$.

Proof. Assume first (i) and let $K \subseteq Y$ be compact, $\chi \in \beta X$ with $f^{-1}(K) \in \chi$ and $x$ be a smallest convergence point of $\chi$. Then $f(x)$ is a smallest convergence point of $\beta f(\chi)$ and, since $K$ is compact and $K \in \beta f(\chi)$, we have $f(x) \in K$. Therefore, $x \in f^{-1}(K)$, and we have shown that $f^{-1}(K)$ is compact, that is, (i) $\implies$ (ii). The implication (ii) $\implies$ (iii) follows from Lemma 5.7.3. Assume now (iii) and let $x \in X$ be a smallest convergence point of $\chi \in \beta X$. Assume that $\beta f(\chi) \rightarrow y \in Y$. Let $V \subseteq Y$ be any open neighborhood of $y$ and choose some open $U \subseteq X$ with $y \in U \ll V$. Then $f^{-1}(U) \ll f^{-1}(V)$ and $f^{-1}(U) \in \chi$, hence $x \in f^{-1}(V)$ and therefore $f(x) \in V$. We conclude that $f(x) \leq y$, so (iii) $\implies$ (i). □

5.7.7 Corollary. Let $X$ be a representable topological space, and 2 as in Examples 5.3.2. A continuous map $\varphi : X \rightarrow 2$ is a pseudo-homomorphism if and only if the open set $\varphi^{-1}(0) \subseteq X$ is compact.

5.7.8 Proposition. Let $(X, \leq, \alpha)$ be an ordered compact Hausdorff space and $a = (\leq) \cdot \alpha$ its induced topology. A subset $A \subseteq X$ is open in $(X, \alpha)$ if and only if $A$ is down-closed and open in the compact Hausdorff space $(X, \alpha)$. 
(2) Let $X$ be a representable space, $\chi \in \beta X$ and $x_0 \in X$ be a smallest convergence point of $\chi$. For any $x \in X$, $x \leq x_0$ if and only if $\chi$ contains all complements of compact down-sets $B$ with $x \notin B$.

Proof. To see (1), let $(X, \leq, \alpha)$ be an ordered compact Hausdorff space and $A \subseteq X$. Let $\varphi : X \to 2$ be the characteristic map of the complement $X \setminus A$ of $A$. Then

$$A \text{ is open } \iff \varphi : (X, (\leq) \cdot \alpha) \to 2 \text{ is continuous}$$

$$\iff \varphi : (X, \leq) \to 2 \text{ is monotone and } \varphi : (X, \alpha) \to 2 \text{ is continuous (by 5.3.1)}$$

$$\iff A \text{ is down-closed in } (X, \leq) \text{ and open in } (X, \alpha).$$

To see (2), let first $x \leq x_0$. Then $\chi$ cannot contain any compact down-sets $B$ with $x \notin B$. Assume now that $\chi$ contains these subsets. Take a neighborhood $B$ of $x_0$ where $B$ is a compact down-set. Then $x \in B$ since otherwise $B \in \chi$ and $X \setminus B \in \chi$. 

\[\Box\]

© 5.7.9 Corollary. Let $X$ be a representable space. Then the topology of $X^{\text{op}}$ is generated by the complements of compact down-sets $B$ of $X$. Furthermore, the ultrafilter convergence of the the topology generated by the opens and the complements of compact down-sets of $X$ is given by taking smallest convergence points of ultrafilters of $X$.

© 5.7.10 Corollary. Let $(X, \leq, \alpha)$ be an ordered compact Hausdorff space with separated order. The topology of $(X, \alpha)$ is generated by the open subsets and the complements of compact down-closed subsets of the representable space $(X, (\leq) \cdot \alpha)$.

5.8 Exponentiable topological spaces. In Section 4 we have seen that the category Top can be fully embedded into the (locally) cartesian closed category PsTop (see Examples 4.1.3 and Corollary [4.5.2]), and that Top itself is not cartesian closed (see Exercise 4.G). In this subsection we will provide a characterization of exponentiable topological spaces, that is, of those spaces $X$ where $(-) \times X : \text{Top} \to \text{Top}$ has a right adjoint.

Let $X$ be a topological space with convergence $a : \beta X \to X$ (but we write more intuitively $\chi \to x$ instead of $\chi ; a x$) and let 2 be the Sierpiński space (see Examples 5.3.2). We form the exponential $2^X$ in PsTop, and write $\pi_1 : 2^X \times X \to 2^X$ and $\pi_2 : 2^X \times X \to X$ for the projection maps and $\varepsilon : 2^X \times X \to 2$ denotes the evaluation map. In the sequel we will think of the elements of $2^X$ as closed subsets of $X$. For any subset $V \subseteq X$, we put

$$V^\circ = \{ A \subseteq X \mid A \text{ closed, } A \cap V \neq \emptyset \} .$$

The topology of $X$ we denote as $\mathcal{O}X$, and $\mathcal{O}(x)$ stands for the collection of open neighborhoods of $x \in X$.

5.8.1 Proposition. For every topological space $X$, the pseudotopological space $2^X$ is dualizable. The underlying order of $2^X$ is subset inclusion; and $p \rightarrow A \iff \mu(p) \subseteq A$ for $p \in \beta(2^X)$ and $A \subseteq X$ closed, where $\mu(p) = \bigcap_{A \in p} \bigcup A$. 

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Proof. Recall from Examples 4.5.3 that the convergence structure of \(2^X\) is given by

\[
P \rightarrow A \iff \begin{cases}
\forall x \in X, \ w \in \beta(2^X \times X) \text{ with } \beta \pi_1(w) = p, \ \chi := \beta \pi_2(w) : \\
\chi \rightarrow x \implies (\beta \varepsilon(w) = 1 \implies x \in A)
\end{cases}
\]

\[
\iff \begin{cases}
\forall x \in X, \ w \in \beta(2^X \times X) \text{ with } \beta \pi_1(w) = p, \ \chi := \beta \pi_2(w) : \\
(\chi \rightarrow x \& \beta \varepsilon(w) = 1) \implies x \in A
\end{cases}
\]

\[
\iff \begin{cases}
\forall x \in X : \\
p(a \cdot (\beta \varepsilon)) x \rightarrow x \in A
\end{cases}
\]

for \(p \in \beta(2^X)\) and \(A \subseteq X\) closed, where in the last line we interpret \(\varepsilon : 2^X \times X \rightarrow 2\) as the membership relation \(2^X \rightarrow X\). We put

\[
\mu(p) = \{ x \in X \mid p(a \cdot (\beta \varepsilon)) x \}
\]

for \(p \in 2^X\), and then

\[
P \rightarrow A \iff \mu(p) \subseteq A.
\]

Furthermore,

\[
\mu(p) = \{ x \in X \mid \exists \chi \in \beta X : (p(\beta \varepsilon) \chi \& \chi \rightarrow x) \}
\]

\[
= \{ x \in X \mid \forall V \in \mathcal{O}(x), A \in p \exists A \in \mathcal{A}, y \in V : y \in A \}
\]

\[
= \{ x \in X \mid \forall V \in \mathcal{O}(x), A \in p V^\circ \cap A \neq \emptyset \}
\]

\[
= \{ x \in X \mid \forall V \in \mathcal{O}(x) (V^\circ \in p) \}
\]

We also note that \(V^\circ \cap A \neq \emptyset\) is equivalent to \(V \cap \bigcup A \neq \emptyset\), and therefore

\[
\mu(p) = \{ x \in X \mid \forall A \in p, V \in \mathcal{O}(x) (V \cap \bigcup A \neq \emptyset) \}
\]

\[
= \{ x \in X \mid \forall A \in p x \in \bigcup \mathcal{A} \}
\]

\[
= \bigcap_{A \in p} \bigcup \mathcal{A}.
\]

Consequently, \(\mu(p)\) is a closed subset of \(X\), for every \(p \in \beta(2^X)\); and \(\mu(p) = A\) for \(p = \hat{A}\) the principal ultrafilter generated by \(A \subseteq X\) closed. Therefore \(\hat{A} \rightarrow B \iff A \subseteq B\). □

5.8.2 Proposition. For every topological space \(X\), the pseudotopological space \((2^X)^{\text{op}}\) is topological, where the topology of \((2^X)^{\text{op}}\) is generated by the sets \(V^\circ\) with \(V \subseteq X\) open.

Proof. By definition, the convergence of \((2^X)^{\text{op}}\) is given by

\[
P \rightarrow A \iff A \subseteq \bigcap_{A \in p} \bigcup \mathcal{A}
\]

\[
\iff \forall x \in A \forall A \in p (x \in \bigcup \mathcal{A})
\]

\[
\iff \forall x \in A \forall A \in p \forall V \in \mathcal{O}(x) (V \cap \bigcup A \neq \emptyset)
\]

\[
\iff \forall x \in A \forall V \in \mathcal{O}(x) \forall A \in p (V^\circ \cap A \neq \emptyset)
\]

\[
\iff \forall V \in \mathcal{O}X (A \in V^\circ \implies V^\circ \in p);
\]
that is, it is generated by the sets
\[ V^\circ = \{ A \subseteq X \mid A \text{ closed}, A \cap V \neq \emptyset \} \quad (V \subseteq X \text{ open}) \]
and therefore it is the convergence of the topology generated by these sets.

We find it remarkable that, although $2^X$ is topological if and only if $X$ is exponentiable (see [5.8.4] below), its dual belongs always to $\text{Top}$. The topological space $VX := (2^X)^\text{op}$ is usually referred to as the lower-Vietoris space.

5.8.3 Lemma. Let $X$ be a pseudotopological space. Then $(-)^X : \text{PsTop} \to \text{PsTop}$ preserves initial sources (with respect to the canonical forgetful functor to Set).

Proof. Let $(f_i : Y \to Y_i)_{i \in I}$ be initial in $\text{PsTop}$. Let $p \in \beta(Y^X)$ and $h \in Y^X$ so that, for all $i \in I$, $\beta(f_i^X)(p) \to f^X_i(h)$. Let $w \in \beta(Y^X \times X)$ and $x \in X$ with $\beta\pi_1(w) = p$ and $\beta\pi_2(w) \to x$. Then, for all $i \in I$,

\[ \beta\pi_1(\beta(f_i^X \times 1_X)(w)) = \beta\pi_1(w) \to x \quad \text{and} \quad \beta\pi_2(\beta(f_i^X \times 1_X)(w)) = \beta(f_i^X)(p) \to f_i^X(h), \]

therefore $\beta(f_i^X \times 1_X)(w) \to (f_i \cdot h, x)$ and

\[ \beta f_i(\beta \varepsilon_Y(w)) = \beta \varepsilon_Y(\beta(\pi_i^X \times 1_X)(w)) \to f_i(h(x)). \]

Hence, by hypothesis, $\beta \varepsilon_Y(w) \to h(x)$. This proves $p \to h$.

5.8.4 Proposition. Let $X$ be a topological space. Then the following assertions are equivalent:

(i) $X$ is exponentiable in $\text{Top}$;

(ii) the pseudotopological space $2^X$ is topological;

(iii) for every topological space $Y$, the pseudotopological space $Y^X$ is topological.

Proof. Assume first that $X$ is exponentiable in $\text{Top}$. We write temporarily $[X, -]$ for the right adjoint of $(-) \times X : \text{Top} \to \text{Top}$, and $\varepsilon'$ denotes the counit of $(-) \times X \dashv [X, -]$; in particular, we consider $\varepsilon'_2 : [X, 2] \times X \to 2$. By the universal property of $\varepsilon : 2^X \times X \to 2$ in $\text{PsTop}$, there is a unique map $t : [X, 2] \to 2^X$ in $\text{PsTop}$ making

\[ \begin{array}{ccc}
[X, 2] \times X & \xrightarrow{t \times 1_X} & 2^X \times X \\
\varepsilon'_2 & \downarrow & \varepsilon_2 \\
2 & & 2
\end{array} \]
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For any $p \rightarrow A$ in $2^X$, we let $(2^X)_{p,A}$ be the topological space of all closed subsets of $X$ where, besides the principal convergence, only $p \rightarrow A$ (see Exercise 5.C). Then there exists a continuous map $s_{p,A} : (2^X)_{p,A} \rightarrow [X, 2]$ making

$$(2^X)_{p,A} \times X \xrightarrow{s_{p,A} \times 1_X} [X, 2] \times X \xrightarrow{\varepsilon_2} 2$$

commute. Hence, for all $p \rightarrow A$ in $2^X$, $t \cdot s_{p,A}$ is the identity map; and therefore $s : [X, 2] \rightarrow 2^X$ is an isomorphism. In particular, $2^X$ is topological. Assume now that $2^X$ is topological. Let $Y$ be a topological space. Then the source $\text{PsTop}(Y, 2)$ is initial. Hence, also $\text{PsTop}(Y^X, 2^X)$ is initial by Lemma 5.8.3, and therefore $Y^X$ is topological. Finally, the implication $(iii) \implies (i)$ is clear.

We can now derive a characterization of exponentiable topological spaces.

5.8.5 Theorem. Let $X$ be a topological space. Then $X$ is exponentiable if and only if $X \circ$ is core-compact (see Proposition 5.6.6).

Proof. The space $X$ is exponentiable if and only if $(VX)^{op}$ is topological, which by Proposition 5.5.3 is equivalent to $VX$ being core-compact. Finally, $VX$ is core-compact if and only if $X$ is core-compact (see Exercise 5.I).

We also note that $VX$ is $\beta$-cocomplete, for any topological space $X$, but $VX$ is only representable if $X$ is core-compact. For $X$ core-compact and $K \subseteq X$ compact, $K^\diamond$ is a compact down-set in $VX$ and therefore its complement is open in $VX^{op}$ (see Corollary 5.7.9). Assume now that $X$ is even locally compact. Then one easily verifies that the sets $$(K^\diamond)^\circ = \{A \in VX \mid A \cap K = \emptyset\} \quad (K \subseteq X \text{ compact})$$
generate the convergence of $2^X = (VX)^{op}$. Hence, if we interpret the elements of $2^X$ as open subsets of $X$, the topology of $2^X$ is generated by the sets

$$\{V \subseteq X \text{ open} \mid K \subseteq V\} \quad (K \subseteq X \text{ compact}).$$

This topology is known as the compact-open topology.

5.9 Representable approach spaces. The situation for approach spaces is quite similar to the one for topological spaces when one considers the Barr extension $\beta$ of $\beta$ to $P_+\text{-Rel}$. This extension then yields a lifting of the ultrafilter monad $\beta$ to $\text{App} \simeq (\beta, P_+)\text{-Cat}$. In this case, diagram (5.4.ii) becomes

$$\begin{array}{cccc}
\text{App}^\beta & \xrightarrow{\simeq} & \text{MetCompHaus} & \longrightarrow \text{CompHaus} \\
\downarrow & & \downarrow & \\
\text{App} & \longrightarrow \text{Met} & \longrightarrow \text{Set}.
\end{array}$$
Here \( \text{MetCompHaus} \) denotes the category of metric compact Hausdorff spaces and morphisms (see Example \( \text{5.2.1}(4) \)). An approach space \( X = (X, a) \) is representable if and only if \( a \cdot \beta a = a \cdot m_X \) and, moreover, every ultrafilter \( \chi \) has a generic convergence point \( x_0 \), with \( a_0 = a \cdot e_X \) the underlying metric, this latter condition means

\[
a(\chi, x) = a_0(x_0, x)
\]

for all \( x \in X \). Similar to the situation in \( \text{Top} \), one calls a non-expansive map \( \varphi : X \to [0, \infty] \) irreducible if \( \inf_{x \in X} \varphi(x) = 0 \) and, if for any non-expansive maps \( \varphi_1, \varphi_2 : X \to [0, \infty] \) with \( \varphi(x) \geq \min\{\varphi_1(x), \varphi_2(x)\} \) (for all \( x \in X \)) one has \( \varphi \geq \varphi_1 \) or \( \varphi \geq \varphi_2 \). Recall from Examples \( \text{5.3.2} \) that we consider \( [0, \infty] \) as an approach space with convergence \( \lambda(u, v) = u \cap (\sup_{A \in u} \inf_{x \in A} v) \), for \( u \in \beta[0, \infty] \) and \( u \in [0, \infty] \). A typical example of an irreducible non-expansive map is \( \varphi = a_0(x, -) \), for \( x \in X \) (see Exercise \( \text{5.3} \)). An approach space \( X \) is called sober whenever every irreducible non-expansive map \( \varphi : X \to [0, \infty] \) is of the form \( \varphi = a_0(x, -) \), for a unique \( x \in X \); and \( X \) is called weakly sober if such \( x \in X \) is not necessarily unique.

\( \textcircled{5.9.1} \) \textbf{Lemma.} Let \( X = (X, a) \) be an approach space and \( \varphi : X \to [0, \infty] \) be an irreducible non-expansive map. Then there exists some \( \chi \in \beta X \) with \( \varphi = a(\chi, -) \).

\textit{Proof.} We freely make use of Exercise \( \text{5.3} \) and the notation introduced there. Let \( \varphi : X \to [0, \infty] \) be irreducible. For every \( u \in [0, \infty], u > 0 \), put \( A_u = \{ x \in X \mid \varphi(x) \leq u \} \); by hypothesis, \( A_u \neq \emptyset \). Then \( \varphi_{A_u} \leq \varphi \) since, with \( A := \{ x \in X \mid \varphi_{A_u}(x) \leq \varphi(x) \} \), one has \( A_u \subseteq A \) and \( 0 < \inf_{x \in A} \varphi(x) \mid x \in X, x \notin A \} =: v \). Then \( \varphi(x) \geq \min\{\varphi_{A_u}(x), v\} \), but \( v \leq \varphi \) is not possible since \( \inf_{x \in X} \varphi(x) = 0 \), therefore \( \varphi_{A_u} \leq \varphi \). The down-directed set

\[
f = \{ A_u \mid u \in [0, \infty], u > 0 \}
\]

is disjoint from

\[
j = \{ B \subseteq X \mid \varphi_B \not\leq \varphi \},
\]

and \( j \) is in ideal since \( \varphi \) is irreducible. Hence, by Corollary \( \text{11.13.5} \) there is some ultrafilter \( \chi \in \beta X \) with \( f \subseteq \chi \) and \( \chi \cap j = \emptyset \). Then

\[
a(\chi, -) = \sup_{A \in \chi} \varphi_{A} \leq \varphi
\]

and \( \varphi \leq a(\chi, -) \) since \( \sup_{A \in \chi} \inf_{x \in A} \varphi(x) = 0 \).

Following the example of topological spaces, we call an approach space \( X = (X, a) \) core-compact whenever \( a \cdot \beta a = a \cdot m_X \) and, we call \( X \) stable whenever \( a(\chi, -) \) is irreducible, for every \( \chi \in \beta X \). With this terminology we have the following result.

\( \textcircled{5.9.2} \) \textbf{Theorem.} An approach space \( X = (X, a) \) is representable if and only if \( X \) is weakly sober, stable and core-compact.
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Proof. If $X$ is representable, then $X$ is core-compact (see Proposition 5.4.4) and, for every $\chi \in \beta X$, $a(\chi, -)$ is irreducible since $a(\chi, -) = a_0(x_0, -)$ for some $x_0 \in X$. We also conclude that $X$ is weakly sober since every irreducible non-expansive map $\varphi : X \to [0, \infty]$ is of the form $\varphi = a(\chi, -)$ for some $\chi \in \beta X$ (by Lemma 5.9.1). Assume now that $X$ is weakly sober, stable and core-compact. Then every ultrafilter $\chi \in \beta X$ has a generic convergence point since $a(\chi, -)$ is irreducible and $X$ is weakly sober. Since $X$ is core-compact, the assertion follows from Proposition 5.4.4.

Exercises

5.A Connected components of an ordered set. Show that the left adjoint $\pi_0 : \text{Ord} \to \text{Set}$ of $(-)_d : \text{Set} \to \text{Ord}$, $X \mapsto (X, =)$ sends an ordered set $X = (X, \to)$ to $X/\sim$, where $x \sim y$ precisely when there exists a path $x \to \bullet \leftarrow \cdots y$. The reflection map $q : X \to X/\sim$ is the coequalizer of $p_1, p_2 : R \to X$, where $R \subseteq X \times X$ is the graph of the order relation $\to$ of $X$. Furthermore, if $\to$ is confluent, so that for all $x \to y$, $x \to y'$ there exists $z \in X$ with $y \to z$ and $y \to z$

\[ \begin{array}{ccc} x & \to & y \\ \downarrow & & \downarrow \\ \downarrow & & \downarrow \\ x & = & y' \\ \end{array} \]

then $x \sim y$ if and only if $x \to z \leftarrow y$ for some $z \in X$.

5.B Connected components of a $\mathcal{V}$-category. Let $\mathcal{V}$ be a quantale with $k = \top$ the top element of $\mathcal{V}$. Furthermore, assume that $u \otimes v = \bot$ implies $u = \bot$ or $v = \bot$, for all $u, v \in \mathcal{V}$, so that $o : \mathcal{V} \to 2$ defined by $(o(x) = 1 \iff x > \bot)$ is a lax homomorphism of quantales (see also Exercise II.1.1). Show that the left adjoint $\pi_0 : \mathcal{V}\text{-Cat} \to \text{Set}$ of $(-)_d : \text{Set} \to \mathcal{V}\text{-Cat}$, $X \mapsto (X, 1_X)$ is the composite

\[ \mathcal{V}\text{-Cat} \xrightarrow{B_o} \text{Ord} \xrightarrow{\pi_0} \text{Set}, \]

where $B_o$ is defined as in 3.5.

5.C A topology with chosen convergence. Let $X$ be a set, $\chi_0 \in \beta X$ be an ultrafilter on $X$ and $x_0 \in X$. Define a relation $a : \beta X \to X$ by

\[ a(\chi, x) \text{ whenever } \begin{cases} \chi = \chi_0 \text{ and } x = x_0, & \text{or} \\ \chi = \hat{x}. & \end{cases} \]

Show that $a$ is the convergence of a topology on $X$, that is, $1_X \leq a \cdot e_X$ and $a \cdot \beta a \leq a \cdot m_X$.

5.D Irreducible closed sets. Let $X$ be a topological space and $A \subseteq X$ be a non-empty closed subset of $X$. Show that $A$ is irreducible if and only if for all open subsets $U, V \subseteq X$
CHAPTER III. LAX ALGEBRAS

if \( U \cap V \cap A = \emptyset \), then \( U \cap A = \emptyset \) or \( V \cap A = \emptyset \). Conclude that \( A \) is the set of limit points of some filter \( \chi \) with \( A \in \chi \). Give an example of a compact topological space \( X \) with an ultrafilter \( \chi \) where \( \lim \chi \) is not irreducible.

5.E The way-below relation via ultrafilter convergence. Let \( X \) be a topological space and \( A, B \subseteq X \) be open subsets of \( X \). Show that \( A \ll B \) if and only if every ultrafilter \( \chi \in \beta X \) with \( A \in \chi \) has a limit point in \( B \). In particular, \( X \) is compact if and only if \( X \ll X \) if and only if every ultrafilter of \( X \) converges.

5.F Split subobjects of locally compact spaces. Let \( f : X \to Y \) and \( g : Y \to X \) be continuous maps between topological spaces with \( g \cdot f = 1_X \), and assume that \( Y \) is locally compact. Show that \( X \) is locally compact.

5.G Local compactness versus core-compactness. Let \( X \) be a topological space. For \( B \subseteq X \) open, consider

\[
\lim^{-1}(B) = \{ \chi \in \beta X \mid \lim \chi \cap B \neq \emptyset \}
\]

where \( \lim \chi \) denotes the set of convergence points of \( \chi \). Show that \( \lim^{-1}(B) \) is open in \( \beta X \) if and only if, for every \( x \in B \), there is some open neighborhood \( U \) of \( x \) with \( U \ll B \). Hence, the following statements hold.

1. For a core-compact space \( X \), the subspace

\[ \lim^{-1}(X) = \{ \chi \in \beta X \mid \chi \text{ converges to some } x \in X \} \]

of \( \beta X \) is locally compact.

2. If \( X \) is core-compact and every convergent ultrafilter \( \chi \in \beta X \) has a smallest convergence point, then the map \( \lim^{-1}(X) \to X \) that associates to every convergent ultrafilter a (tacitly chosen) smallest convergence point is continuous; therefore, \( X \) is locally compact.

3. If \( X \) is Hausdorff, then

\[ X \text{ is core-compact} \iff X \text{ is locally compact} \iff \text{every point of } X \text{ has a compact neighborhood.} \]

5.H Stable spaces. Let \( X \) be a topological space. Show that

1. if \( \lim \chi \) is irreducible for every \( \chi \in \beta X \), then \( X \) is stable, and

2. if \( X \) is stable and core-compact, then \( \lim \chi \) is irreducible for every \( \chi \in \beta X \).

5.I Local compactness of \( VX \). Consider the space \( VX \) of 5.8 for a topological space \( X \). Let \( x \in X \) and let \( U, U_i \subseteq X \) (\( i \in I \)) be open. Then the following hold:

1. \( \overline{\{x\}} \in U^\circ \) if and only if \( x \in U \);
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(2) \((\bigcup_{i \in I} U_i)^\circ = \bigcup_{i \in I} U_i^{\circ}\);

(3) \(X\) is core-compact if and only if \(\mathcal{V}X\) is core-compact.

Hint. For (3), use Remark 5.6.7.

5.J The approach space \([0, \infty]\).

Consider the Barr extension \(\mathcal{B}\) of the ultrafilter monad \(\mathcal{B}\) to \(\mathcal{P}_+-\text{Rel}\).

(1) Show that \((\mathcal{B}[0, \infty], \mu, \xi)\) is a metric compact Hausdorff space where \(\mu(u, v) = v \ominus u\) and \(\xi : \mathcal{B}[0, \infty] \to [0, \infty], u \mapsto \sup_{A \in u} \inf_{x \in A} u\).

(2) Let \(X = (X, a)\) be an approach space (= \((\mathcal{B}, \mathcal{P}_+)-\text{category})\). Show:

(a) \(a(\chi, -) : X \to [0, \infty]\) is non-expansive, for every \(\chi \in \beta X\).

(b) \(\varphi_A : X \to [0, \infty], x \mapsto \inf\{a(\chi, x) \mid \chi \in \beta X, A \in X\}\), is non-expansive, for every \(A \subseteq X\). Furthermore, \(\varphi_{A \cup B}(x) = \min\{\varphi_A(x), \varphi_B(x)\}\), for all \(A, B \subseteq X\) and \(x \in X\).

(c) \(a(\hat{x}, -) : X \to [0, \infty]\) is irreducible, for every \(x \in X\).

(3) Let \(X = (X, a)\) be an approach space, \(\chi \in \beta X\) and \(x \in X\). Show

\[ a(\chi, x) = \sup_{A \in \chi} \varphi_A(x) \] .

Conclude that

\[ a(\chi, -) = \sup\{\varphi : X \to [0, \infty] \mid \text{non-expansive}, \sup_{A \in \chi} \inf_{x \in X} \varphi(x) = 0\} \] .

5.K Lifting \(\tilde{T}\). Show that the functor \(\tilde{T} : \text{Set} \to (\mathbb{T}, \mathcal{V})\text{-Cat}\) of Proposition 3.3.6 factors through the functor \(T : (\mathbb{T}, \mathcal{V})\text{-Cat} \to (\mathbb{T}, \mathcal{V})\text{-Cat}\) of 5.4.

5.L Fundamental adjunctions revisited. With the notations of 3.6 and 5.4, show that the diagrams commute (where \(I\) denotes the induced-order functor, see 3.3). Describe \(J := A^\circ B_1\) and show

\[ G^\mathbb{T}M J(X, =) = (TX, \hat{1}_X) = (TX, \hat{T}1_X) \] (see the Proof of 5.3.5).

\[ K F^\mathbb{T}B_1(X, =) = \hat{T}X \] (see 3.3.6).
Notes on Chapter III

The Notes on Chapter IV give some information on the history of the axiomatization of convergence in topology which culminated in the Manes–Barr characterization of a topological space in terms of an abstract ultrafilter convergence relation (see Manes [1969, 1974], Barr [1970]). With the motivation taken from these papers and from Lawvere’s landmark contribution Lawvere [1973], the theory of $(T, V)$-categories as presented in this Chapter started with Clementino and Hofmann [2003] and Clementino and Tholen [2003]. Whereas the term lax extension of a monad $T$ to $V$-Rel was first understood in a more restrictive sense than the one used here (dealing only with flat extensions), the set of axioms presented in 1.4 and 1.5 was shaped in Seal [2005]. The construction of the lax extension of a functor and monad to Rel of respectively 1.10 and 1.12 stems from Barr [1970]. The deep connection between extensions of functors to categories of internal relations and the preservation of weak pullbacks was first exposed in Trnková [1977]. The Beck–Chevalley condition belongs to the folklore domain of higher-level category theory and is credited independently to both name givers. The weaker notion of taut monad appears first (under the name Alexandrov monad) in Môbus’ thesis Môbus [1981] and was re-introduced in Manes [2002]. The double-categorical presentation of lax extensions presented in 1.13 appears in Cruttwell and Shulman [2010] following a suggestion by Paré.

The quantaloid $V$-Rel described in 1.1 was introduced in a more general form in Betti, Carboni, Street, and Walters [1983] and extensively used in a particular case by Rosebrugh and Wood [2002]. In these papers, the quantale $V$ is allowed to be a monoidal category or even a bicategory, and the term $V$-matrix is used instead of $V$-relation. This term was adopted also in the first studies of lax algebras for a Set-monad laxly extended to $V$-Rel, as given in Clementino and Tholen [2003] (for $V$ a symmetric monoidal closed category), as well as in the subsequent paper Clementino, Hofmann, and Tholen [2004b] (for $V$ a commutative and unital quantale). The latter paper also introduced the Kleisli convolution of Definition 1.7.1 (under the name co-Kleisli composition). The associativity criterion of Proposition 1.9.4 for this operation is original, while the identification of maps (in Lawvere’s sense) in the 2-category $V$-Rel as given in Proposition 1.2.1 goes back to Freyd and Scedrov [1990] and Clementino and Hofmann [2009].

Of course, for a symmetric monoidal closed category $V$, the notion and theory of $(T, V)$-categories as initiated in Clementino and Tholen [2003] builds on the theory of $V$-categories, as introduced in Eilenberg and Kelly [1966] and developed further in Kelly [1982], after their significance in the context of the subject of this book had been emphasized in Lawvere [1973]. The concept of module (called bimodule by Lawvere) was originally introduced (under the name distributor, but often also called profunctor) by Bénabou, see Bénabou [2000].

The presentation of metric spaces as enriched categories (Example 1.3.1(2)) is due to Lawvere [1973]. This description motivated numerous works on the reconciliation of order, metric and category theory; see in particular the work of Flagg and his coauthors on continuity spaces Flagg [1992, 1997], Flagg and Kopperman [1997] and Flagg, Sümmer, and Wagner [1996] (and metric generalizations of domain theory as in Bonsangue, van Breugel, and Rutten [1998], Wagner [1994]). The probabilistic metric spaces presented in 2.1 where introduced in Menger [1942]; for more information see Schweizer and Sklar [1983]. They were recognized as enriched categories in Flagg [1992] and as such further investigated in Chai [2009].

The monadicity of compact Hausdorff spaces exposed in 2.3.3 is due to Manes [1969], and the ensuing presentation of topological spaces as relational algebras (Theorem 2.2.5) was established in Barr [1970]. Approach spaces were introduced in Lowen [1989], and a comprehensive presentation of their theory can be found in Lowen [1997]. The description of approach spaces as lax algebras (Theorem 2.4.5) was established in Clementino and Hofmann [2003]. The lax-algebraic description of closure spaces of 2.5 together with the introduction of their metric version of Exercise 2.G appeared first in Seal [2005].

The easily-established but important property of topologicity of categories of lax algebras over Set along
with the investigation of algebraic functors, change-of-base functors and induced orders was already present in the initial papers on the subject (see Clementino and Hofmann, 2003, Clementino and Tholen, 2003, Clementino, Hofmann, and Tholen, 2004b, Seal, 2009). Of course, these types of functors were previously studied in the monad and enriched-category contexts. Universality of coproducts in these categories was recognized in Mahmoudi, Schubert, and Tholen, 2006; for the study of quotient structures, see Hofmann, 2005.

The notions of pseudotopological and pretopological spaces were introduced in Choquet, 1948 (Example 4.1.3.2). Partial products of topological spaces first appeared in Pasynkov’s paper 1965 and were studied in Dyckhoff, 1984. The categorical notion and its linkage with exponentiable morphisms (studied in the general categorical as well as the topological realm by Niefsfield, 1982) was established in Dyckhoff and Tholen, 1987. The notion of quasitopos (see 4.8) was given by Penon in 1973. Machado proved in 1973 that $\mathbf{PsTop}$ is cartesian closed and Wyler in 1976 that $\mathbf{PsTop}$ is a quasitopos (in fact, the quasi-topos hull of $\mathbf{Top}$, see Example 4.8.5.1). The corresponding facts about $\mathbf{PsApp}$ can be found in Colebunders and Lowen, 1988, 1989. Under the conditions of Corollary 4.8.2, but in the broader context of a monoidal category $\mathcal{V}$, the quasitopos property of $(\mathbb{T}, \mathcal{V})$-$\mathbf{Gph}$ was established in Clementino et al., 2003a. Final density of $(\mathbb{T}, \mathcal{V})$-$\mathbf{Cat}$ in this category as stated in Theorem 4.9.2 originates with Clementino and Hofmann, 2012.

The theme of Section 5 is motivated by the equivalence between Nachbin’s ordered compact Hausdorff spaces (introduced in Nachbin, 1950) and stably compact spaces which was first described in Gierz, Hofmann, Keimel, Lawson, Mislove, and Scott, 1980. Another source of inspiration is Hermida’s work 2000, 2001 on representable multicategories which established a similar correspondence in the context of multicategories.

The $(\mathbb{T}, \mathcal{V})$-framework presented in the first subsections stems largely from Tholen, 2009 and is augmented by crucial ingredients from Clementino and Hofmann, 2009, such as the functor $M$ of Theorem 5.3.5 that in essence facilitates the notion of representability of a $(\mathbb{T}, \mathcal{V})$-category. Representable $\mathbb{T}_0$-spaces are known as stably compact spaces; for more information we refer to Gierz, Hofmann, Keimel, Lawson, Mislove, and Scott, 1980. Another source of inspiration is Hermida’s work 2000, 2001 on representable multicategories which established a similar correspondence in the context of multicategories.

The notion of dual stably compact spaces goes back to de Groot, 1967, de Groot, Strecker, and Wattel, 1967, Hochster, 1969 (see Corollary 5.7.9). The Čech–Stone compactification of a completely regular topological space, briefly referred to in 5.6, was introduced in Čech, 1937 and Stone, 1937 building on Tychonoff, 1930. The characterization of normal topological spaces in terms of convergence (Proposition 5.6.2) was first obtained in Möbus, 1981. The characterization of exponentiable topological spaces as precisely the core-compact ones (Theorem 5.8.5) is due to Day and Kelly, 1970; for more information see Isbell, 1966. The characterization of core-compactness via convergence (Proposition 5.6.6) stems from Möbus, 1981, 1983 and Pisani, 1999. However, the approach taken in this book is quite distinct from the one in these sources as it makes essential use of the Vietoris construction which has its roots in Vietoris, 1922. The notion of sobriety for approach spaces featured in the characterization of representable approach spaces 5.9 was introduced in Banaschewski, Lowen, and Van Olmen, 2006 and further studied in Van Olmen, 2005 which uses the term approach prime map for what is called (in resemblance to the topological counterpart) irreducible non-expansive map in this book.

This Chapter revolves around an alternate presentation of \((\mathcal{T}, \mathcal{V})\)-Cat as the category \(\mathcal{T}\)-Mon of monoids in the hom-set of a Kleisli category that has the advantage of avoiding explicit use of relations or lax extensions. Our role model is given by the filter monad \(\mathcal{F}\) for which \(\mathcal{F}\)-Mon \(\cong\) Top. After obtaining an isomorphism
\[
\mathcal{T}\text{-Mon} \cong (\mathcal{T}, 2)\text{-Cat}
\]
in Section 1, in Section 2 we use the isomorphisms
\[
(\beta, 2)\text{-Cat} \cong \text{Top} \cong (\mathcal{F}, 2)\text{-Cat}
\]
as role models to compare \((\mathcal{S}, \mathcal{V})\)-categories with \((\mathcal{T}, \mathcal{V})\)-categories for a monad morphism \(\alpha : \mathcal{S} \to \mathcal{T}\) and a general \(\mathcal{V}\) in lieu of the embedding \(\beta \to \mathcal{F}\) and \(\mathcal{V} = 2\). In Section 3 we prove that any \((\mathcal{T}, \mathcal{V})\)-category obtained from an associative lax extension can also be presented as a \((\mathcal{F}, 2)\)-category, in effect passing all needed information provided by \(\mathcal{V}\), \(\mathcal{T}\) and its lax extension to \(\mathcal{V}\text{-Rel}\) into a new monad \(\mathbb{I} = \mathbb{I}(\mathcal{T}, \mathcal{V})\). In Section 4 we identify the injective \((\mathcal{T}, 2)\)-categories as precisely the \(\mathcal{T}\)-algebras by exploiting the fact that the forgetful functor Set\(^{\mathcal{F}} \to (\mathcal{T}, 2)\text{-Cat}\) is monadic of Kock-Zöberlein type. Finally, in Section 5 we focus on the filter monad and investigate the interplay between \((\mathcal{F}, 2)\)-categories and \(\mathcal{F}\)-algebras in the context of ordered sets.

1 Kleisli monoids and lax algebras

In II.1.9 a topological space is defined as a set equipped with a collection of subsets closed under finite intersections and arbitrary unions, and in Exercise II.1.G an equivalent description in terms of a set with a finitely additive closure operation is given. In III.2.2 convergence of ultrafilters is used as a defining structure. In this subsection, we present two filter-based counterparts that avoid the Axiom of Choice: the first focuses on neighborhood filters (Proposition 1.1.1) and serves as a model for the Kleisli monoids introduced in 1.3; the second concentrates on filter convergence (Corollary 1.5.4) and is facilitated by a new general construction of a lax extension in 1.4 namely the Kleisli extension of a monad.
1.1 Topological spaces via neighborhood filters. A topological space can be entirely defined in terms of its neighborhood systems by way of the filter monad $\mathcal{F} = (F, m, e)$ described in Example II.3.1.1(5). From a categorical viewpoint, it is convenient for the map $\tau_X : PX \to FX$ that sends a set $A$ to the principal filter $\hat{A}$ (see II.1.12) to be monotone; hence, the set $FX$ of filters on $X$ is ordered by the refinement order:

$$x \leq y \iff x \supseteq y,$$

for all $x, y \in FX$. A filter $x$ is finer than $y$, or $y$ is coarser than $x$, if $x \supseteq y$.

Given a topology $\mathcal{O}_X$ and $x \in X$, the collection of all open sets that contain $x$ spans the neighborhood filter $\nu(x)$ of $x$:

$$A \in \nu(x) \iff \exists U \in \mathcal{O}_X \ (x \in U \subseteq A),$$

for all $A \subseteq X$. This defines a map $\nu : X \to FX$ that sends a point of a topological space to its neighborhood filter and is such that $e_X(x) = \{A \subseteq X \mid x \in A\}$ contains $\nu(x)$ for all $x \in X$, that is,

$$e_X \leq \nu \quad (1.1.i)$$

in the pointwise refinement order. To relate $\nu$ with the filter monad multiplication, we define for $A \subseteq X$ the set $A^F$ of filters that contain $A$,

$$A^F := \{a \in FX \mid A \in a\},$$

and recall that an open set is a neighborhood of each of its points (see Exercise II.1.Q); thus, in particular, for all $x \in X$ and $A \subseteq X$, one has

$$A \in \nu(x) \iff \exists B \in \nu(x) \ \forall y \in B \ (A \in \nu(y))$$

$$\iff \exists B \in \nu(x) \ \forall y \in B \ (\nu(y) \in A^F)$$

$$\iff \exists B \in \nu(x) \ (B \subseteq \nu^{-1}(A^F))$$

$$\iff \nu^{-1}(A^F) \in \nu(x)$$

$$\iff A^F \in F \nu \cdot \nu(x)$$

$$\iff A \in m_X \cdot F \nu \cdot \nu(x).$$

This last expression is simply the Kleisli composition of $\nu : X \to FX$ with itself, so the previous equivalences show that

$$\nu \circ \nu \leq \nu \quad (1.1.ii)$$

By (1.1.i) and (1.1.ii), a topology on $X$ determines a monoid in the ordered hom-set $\text{Set}_F(X, X)$ of the Kleisli category $\text{Set}_F$. Consider now a continuous map $f : X \to Y$ between topological spaces, with $\nu : X \to FX$ and $\mu : Y \to FY$ the corresponding neighborhood filter maps. If $B \subseteq Y$ is a neighborhood of $f(x)$, then there exists an open set $U \subseteq B$ containing $f(x)$; thus, $f^{-1}(U)$ is an open set with $x \in f^{-1}(U) \subseteq f^{-1}(B)$, and $f^{-1}(B)$ is an element of $\nu(x)$. Thanks to the equivalence

$$f^{-1}(B) \in \nu(x) \iff B \in Ff \cdot \nu(x)$$
(for all $B \subseteq Y$ and $x \in X$), we deduce $\mu \cdot f(x) \subseteq Ff \cdot \nu(x)$ for all $x \in X$, that is,

$$Ff \cdot \nu \leq \mu \cdot f .$$

Instead of considering a map $f : X \to Y$, one can look at its image $f_\sharp = e_Y \cdot f : X \to FY$ under the left adjoint $\text{Set} \to \text{Set}_F$ (1.3.6), and this last condition becomes

$$f_\sharp \circ \nu \leq \mu \circ f_\sharp$$

by naturality of $e$. Not only do the neighborhood filters of topological spaces have properties nicely expressible in the language of the Kleisli category of $F$, but the conditions (1.1.i), (1.1.ii) and (1.1.iii) are sufficient to describe topological spaces and continuous maps.

1.1.1 Proposition. The category $\text{Top}$ of topological spaces and continuous maps is isomorphic to the category $\mathbb{F}$-$\text{Mon}$ whose objects are pairs $(X, \nu)$, with $\nu : X \to FX$ a monoid in $\text{Set}_F(X, X)$:

$$\nu \circ \nu \leq \nu , \quad e_X \leq \nu ,$$

and whose morphisms $f : (X, \nu) \to (Y, \mu)$ are maps $f : X \to Y$ such that

$$f_\sharp \circ \nu \leq \mu \circ f_\sharp .$$

Proof. The previous discussion shows that the neighborhood filters of a topological space define a map $\nu : X \to FX$ satisfying the required properties. Conversely, given a monoid $\nu : X \to FX$ in $\text{Set}_F(X, X)$, we define open sets as those $U \subseteq X$ that are neighborhoods of each of their points:

$$U \in \mathcal{O}X \iff \forall x \in X (x \in U \implies U \in \nu(x)) .$$

Straightforward verifications show that the set $\mathcal{O}X$, ordered by inclusion, is closed under arbitrary suprema as well as under finite infima, and that $U \in \mathcal{O}Y$ implies $f^{-1}(U) \in \mathcal{O}X$ when $f : X \to Y$ satisfies $Ff \cdot \nu \leq \mu \cdot f$. One therefore has two functors

$$\text{Top} \to \mathbb{F}$-$\text{Mon}$ and $\mathbb{F}$-$\text{Mon} \to \text{Top}$$

whose composites are routinely verified to be the identities on $\text{Top}$ and $\mathbb{F}$-$\text{Mon}$.

1.2 Power-enriched monads. The ultrafilter-convergence presentation of topological spaces of III.2.2 uses the algebra of relations in an essential way. In turn, relations are precisely the morphisms of the Kleisli category associated to the powerset monad: $\text{Set}_P = \text{Rel}$. The passage from neighborhood to convergence structure presented further on in III.5 exploits the interaction of filters and relations via the principal filter monad morphism $\tau : \mathbb{P} \to \mathbb{F}$ whose components $\tau_X : PX \to FX$ send a set $A \in PX$ to the principal filter $\hat{A} \in FX$. This monad morphism allows us to place the study of neighborhood systems, appearing in Proposition 1.1.1 as morphisms of the Kleisli category $\text{Set}_P$, in a more general context. The
CHAPTER IV. KLEISLI MONOIDS

following Proposition recalls that a monad morphism \( \tau : \mathcal{P} \to \mathbb{T} \) relates \( \text{Set}_\mathbb{T} \) with both \( \text{Rel} \) and \( \text{Sup} \) via functors 
\[ \text{Rel} \to \text{Set}_\mathbb{T} \to \text{Sup} \]

(Exercises \[\text{II.3.H}\] and \[\text{II.3.I}\]).

1.2.1 Proposition. For a monad \( \mathbb{T} = (T, m, e) \) on \( \text{Set} \), one has a one-to-one correspondence between:

(i) monad morphisms \( \tau : \mathcal{P} \to \mathbb{T} \);

(ii) extensions \( E \) of the functor \( F_\mathbb{T} : \text{Set} \to \text{Set}_\mathbb{T} \) along the functor \( (-)_\mathbb{T} : \text{Set} \to \text{Rel} \) of \[\text{III.1.2}\]:

\[ \text{Rel} \xrightarrow{E} \text{Set}_\mathbb{T} \]
\[ \text{(-)_\mathbb{T}} \xrightarrow{F_\mathbb{T}} \text{Set} \]

(iii) liftings \( L \) of the functor \( G^\mathbb{T} : \text{Set}^\mathbb{T} \to \text{Set} \) along the forgetful functor \( \text{Sup} \to \text{Set} \):

\[ \text{Set}^\mathbb{T} \xrightarrow{L} \text{Sup} \]
\[ \text{G^\mathbb{T}} \downarrow \]
\[ \text{Set} \]

(iv) complete lattice structures on \( TX \) such that \( Tf : TX \to TY \) and \( m_X : TTX \to TX \) are sup-maps for all maps \( f : X \to Y \) and sets \( X \).

Proof. To simplify the proof, we identify \( \text{Rel} \) with \( \text{Set}_\mathcal{P} \) (Example \[\text{II.3.6.2}\]), and \( \text{Sup} \) with \( \text{Set}_\mathcal{P} \) via the isomorphism \( \text{Set}_\mathcal{P} \cong \text{Sup} \) of Example \[\text{II.3.2.2(2)}\].

(i) \( \iff \) (ii): This is a direct consequence of Exercise \[\text{II.3.1}\]. Here, the functor \( E \) sends a \( \text{Set}_\mathcal{P} \)-morphism \( r : X \to PY \) to the map \( Er = \tau_Y \cdot r : X \to TY \).

(i) \( \iff \) (iii): The equivalence follows from Exercise \[\text{II.3.H}\]. Note that \( L \) sends a map \( f : X \to TY \) to the \( \mathcal{P} \)-homomorphism \( m_Y \cdot Tf : (TX, m_X \cdot \tau_TX) \to (TY, m_Y \cdot \tau_{TY}) \).

(iii) \( \iff \) (iv): The functor \( G_\mathbb{T} \) of (iii) sends a map \( g : X \to TY \) to \( m_Y \cdot Tg : TX \to TY \), so that (with \( g = 1_{TY} \) or \( g = e_{TY} \cdot f \)) condition (iv) is just an element-wise restatement of (iii). \( \square \)

For a morphism \( \tau : \mathcal{P} \to \mathbb{T} \) of monads on \( \text{Set} \), condition (iii) equips the underlying set \( TX \) of a free \( \mathbb{T} \)-algebra with the separated order given by

\[ \chi \leq y \iff m_X \cdot \tau_TX (\{\chi, y\}) = y \]  \hspace{1cm} (1.2.i)

for all \( \chi, y \in TX \). The hom-sets \( \text{Set}(X, TY) \) become separated ordered sets via the induced pointwise order:

\[ f \leq g \iff \forall x \in X \left( f(x) \leq g(x) \right) \]
for all \( f, g : X \to TY \). Composition on the right is always monotone, but composition on the left \((-)^\mathbb{T} : f : \text{Set}_\mathbb{T}(Y, Z) \to \text{Set}_\mathbb{T}(X, Z)\) may fail to be so, see Exercise [1.C] (here, \((-)^\mathbb{T} = m_Y \cdot T(\_\_\_)\) denotes the monad extension operation of [1.3.7]). To remedy this and therefore make \(\text{Set}_\mathbb{T}\) into a separated ordered category, it suffices that \((-)^\mathbb{T}\) be monotone:

\[
f \leq g \implies f^\mathbb{T} \leq g^\mathbb{T},
\]

for all \( f, g : X \to TY \). If this condition is satisfied, then the functors \( E : \text{Rel} \to \text{Set}_\mathbb{T} \) and \( L : \text{Set}_\mathbb{T} \to \text{Sup} \) of Proposition [1.2.1] become 2-functors between ordered categories.

1.2.2 Definition. A power-enriched monad is a pair \((\mathbb{T}, \tau)\) composed of a monad \(\mathbb{T}\) on \(\text{Set}\) and a monad morphism \(\tau : \mathbb{P} \to \mathbb{T}\) such that

\[
f \leq g \implies f^\mathbb{T} \leq g^\mathbb{T},
\]

for all \( f, g : X \to TY \). A morphism \(\alpha : (\mathbb{S}, \sigma) \to (\mathbb{T}, \tau)\) of power-enriched monads is a monad morphism \(\alpha : \mathbb{S} \to \mathbb{T}\) such that \(\tau = \alpha \cdot \sigma\):

\[
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{\sigma} & \mathbb{T} \\
\downarrow \tau & & \downarrow \tau \\
\mathbb{S} & \xrightarrow{\alpha} & \mathbb{T}
\end{array}
\]

so the category of power-enriched monads is a full subcategory of the comma category \(\mathbb{P}/\text{MND}_{\text{Set}}\) (see also Exercise [1.A]). When working with power-enriched monads \((\mathbb{T}, \tau)\), we will often assume a fixed choice of \(\tau\), and speak of “the power-enriched monad \(\mathbb{T}\”).

1.2.3 Examples.

(1) There are two trivial monads on \(\text{Set}\) (Exercise [1.3.A]), but only one is power-enriched, namely the terminal monad \(1\) whose functor sends all sets to a singleton \(\{\star\}\); the components of its structure morphism \(\mathbb{P} \to 1\) are the unique maps \(!_X : PX \to \{\star\}\). The other monad does not even have a structure morphism, as there is no map from \(P\varnothing = \{\star\}\) to \(\varnothing\).

(2) The powerset monad \(\mathbb{P}\) with the identity structure \(1_{\mathbb{P}} : \mathbb{P} \to \mathbb{P}\) is power-enriched. Hence, \((\mathbb{P}, 1_{\mathbb{P}})\) is an initial object in the category of power-enriched monads and their morphisms. The order on the sets \(PX\) coming from (1.2.i) is simply subset inclusion because the supremum operation is given by arbitrary union.

(3) The filter monad \(\mathbb{F}\) is power-enriched via the principal filter natural transformation \(\tau : \mathbb{P} \to \mathbb{F}\) which yields a monad morphism \(\tau : \mathbb{P} \to \mathbb{F}\). The order on \(FX\) defined by (1.2.i) is the refinement order introduced in 1.1 and suprema in \(FX\) are given by intersections.

(4) The ultrafilter monad \(\beta\) is not power-enriched: for the set \(X = \varnothing\), one observes that \(\beta X = \varnothing\) cannot be a complete lattice.
The up-set monad has at least two different structure morphisms \( \sigma, \tau : \mathcal{P} \to \mathcal{U} \), defined componentwise for \( A \in PX \) by

\[
\sigma_X(A) = \{ B \subseteq X \mid A \cap B \neq \emptyset \} \quad \text{and} \quad \tau_X(A) = \{ B \subseteq X \mid A \subseteq B \}
\]

(\( \tau \) is just the extension of the principal filter natural transformation). The order induced on \( UX \) by \( \sigma \) is given by subset inclusion, while the one induced by \( \tau \) is opposite, that is, \( \tau \) induces the refinement order on up-sets: for all \( x, y \in UX \),

\[
x \leq y \iff x \supseteq y.
\]

These morphisms demonstrate that the morphism \( \mathcal{P} \to \mathcal{T} \) given with a power-enriched monad is indeed a structure and not a property of the monad.

(6) Both monad morphisms of the previous example can be extended to the double-powerset monad to give \( \sigma, \tau : \mathcal{P} \to \mathcal{P}^2 \). However, neither of these satisfy the condition (1.2.ii) (see Exercise 1.C).

1.3 \( \mathcal{T} \)-monoids. Motivated by Proposition 1.1.1, we introduce the category of monoids in the hom-sets of a Kleisli category. reflex

1.3.1 Definition. Let \( \mathcal{T} = (T, m, e) \) be a monad on a category \( X \) whose Kleisli category \( X_\mathcal{T} \) is a separated ordered category. The category \( \mathcal{T}-\text{Mon} \) of \( \mathcal{T} \)-monoids (or Kleisli monoids) has as objects pairs \( (X, \nu) \), where \( X \) is an \( X \)-object, and its structure \( \nu : X \to TX \) is a transitive and reflexive \( X_\mathcal{T} \)-morphism:

\[
\nu \circ \nu \leq \nu, \quad e_X \leq \nu
\]

(where \( \circ \) is composition of the Kleisli category \( X_\mathcal{T} \)); a morphism \( f : (X, \nu) \to (Y, \mu) \) is an \( X \)-morphism \( f : X \to Y \) satisfying:

\[
Tf \cdot \nu \leq \mu \cdot f \quad \text{or equivalently} \quad f_\sharp \circ \nu \leq \mu \circ f_\sharp,
\]

where \( f_\sharp := e_Y \cdot f \). In the case where \( \mathcal{T} = (\mathcal{T}, \tau) \) is a power-enriched monad, the order on the hom-sets of \( \text{Set}_\mathcal{T} \) depends on \( \tau \); however, we will often assume that \( \tau \) is given implicitly, and denote a category of Kleisli monoids by \( \mathcal{T}-\text{Mon} \) rather than by \( (\mathcal{T}, \tau)-\text{Mon} \).

We hasten to remark that in presence of the reflexivity condition, transitivity can be expressed as an equality \( \nu \circ \nu = \nu \), since

\[
\nu = \nu \circ e_X \leq \nu \circ \nu \leq \nu.
\]

Idempotent structures are also preserved by the functor \( G_\mathcal{T} = (-)^\mathcal{T} : X_\mathcal{T} \to X \):

\[
\nu^\mathcal{T} \cdot \nu^\mathcal{T} = (\nu \circ \nu)^\mathcal{T} = \nu^\mathcal{T}.
\]
1.3.2 Examples.

(1) For $T = 1$ the terminal monad, Kleisli monoids are simply pairs $(X, !_X : X \to \{\star\})$, and morphisms are maps $f : X \to Y$. In other words, the category of Kleisli monoids is isomorphic to $\text{Set}$:

$$1\text{-Mon} \cong \text{Set}.\]$$

(2) In the case of the powerset monad (together its identity structure $1_P$), $P\text{-Mon}$ is the category of ordered sets. Indeed, a map $\nu : X \to PX$ is precisely a relation on $X$, and the transitivity and reflexivity conditions translate as reflexivity and transitivity of $\nu$; because the set $PX$ is ordered by set-inclusion, $\nu$ is the down-set map $\downarrow_X : X \to PX$ (see [II.1.7]). A map $f : X \to Y$ is a morphism of $P\text{-Mon}$ if and only if it preserves the relations, that is, if and only if $f$ is a monotone map. Hence,

$$P\text{-Mon} \cong \text{Ord}.\]$$

(3) Proposition [1.1.1] and Example [1.2.3(3)] show that when $F$ is equipped with the principal filter morphism $\tau : P \to F$, $F\text{-Mon}$ is the category of topological spaces and continuous maps:

$$F\text{-Mon} \cong \text{Top}.\]$$

(4) With the principal filter morphism $\tau : P \to U$ of Example [1.2.3(5)], the category of $U$-monoids is isomorphic to the category of interior spaces:

$$U\text{-Mon} \cong \text{Int}$$

(one can proceed as in the proof of Proposition [1.1.1] or more syntactically as in Exercise [1.D]). In fact, the monad morphism $\sigma : P \to U$ of Example [1.2.3(5)] yields

$$U\text{-Mon} \cong \text{Cls},$$

so that the structures $\tau$ and $\sigma$ return isomorphic categories of $U$-monoids.

1.3.3 Proposition. A morphism of power-enriched monads $\alpha : (S, \sigma) \to (T, \tau)$ induces a functor

$$S\text{-Mon} \to T\text{-Mon}$$

that sends $(X, \nu)$ to $(X, \alpha_X \cdot \nu)$ and commutes with the underlying-set functors.

Proof. Thanks to the functor $\text{Set}_\alpha : \text{Set}_S \to \text{Set}_T$ that sends $\nu$ to $\alpha_X \cdot \nu$ (Exercise [II.3.1]), the claim easily follows from the fact that $\alpha_X$ is monotone (Exercise [1.A]).

$\square$
1.4 The Kleisli extension. Categories of lax algebras depend upon the lax extension of a monad \( \mathbb{T} \) on \( \mathbf{Set} \) to \( \mathcal{V} \)-\( \text{Rel} \). The Barr extension \([\text{III} \; \text{1.10}]\) provides a construction of a lax extension by viewing a relation \( r : X \rightarrow Y \) as a composite \( r = q \cdot p^\circ \) (where \( p, q \) are projection maps); the Kleisli extension introduced below exploits relations as morphisms of the Kleisli category \( \text{Set}_{\mathbb{T}} = \text{Rel} \) (Example \([\text{II} \; \text{3.6.2}]\)). Hence, we will often be working with maps \( r : X \rightarrow PY \) representing relations \( r : X \rightarrow Y \), and will indifferently use the notations \( P = (P, \cup, \{-\}) \) or \( (P, (-)^p, \{-\}) \) for the powerset monad, and \( \mathbb{T} = (T, m, e) \) or \( (T, (-)^\mathbb{T}, e) \) for an arbitrary monad on \( \mathbf{Set} \), together with the corresponding expressions for monad morphisms \( \tau : \mathbb{P} \rightarrow \mathbb{T} \) (see \([\text{II} \; \text{3.7}]\).

Let us denote by \((-)^\flat : \text{Rel}^{\text{op}} \rightarrow \text{Set}_{\mathbb{T}} \) the functor that is identical on sets and sends a relation \( r : X \rightarrow Y \) to the map \( r^\flat : Y \rightarrow PX \) representing the opposite relation \( r^\circ : Y \rightarrow X \):

\[
  x \cdot r \cdot y \iff x \in r^\flat(y) .
\]

By composition with the functors \( E : \text{Set}_{\mathbb{T}} \rightarrow \text{Set}_{\mathbb{T}} \) and \( L : \text{Set}_{\mathbb{T}} \rightarrow \text{Sup} \) of Proposition \([\text{I} \; \text{2.1}]\), one obtains a functor

\[
  \tau : \text{Rel}^{\text{op}} \xrightarrow{(-)^\flat} \text{Set}_{\mathbb{T}} \xrightarrow{E} \text{Set}_{\mathbb{T}} \xrightarrow{L} \text{Sup}
\]

that sends a set \( X \) to \( TX \), and a relation \( r : X \rightarrow Y \) to the map \( r^\tau : TY \rightarrow TX \) defined by

\[
  r^\tau := m_X \cdot T(\tau_X \cdot r^\flat) = (\tau_X \cdot r^\flat)^\mathbb{T} .
\]

1.4.1 Definition. Given a power-enriched monad \( (\mathbb{T}, \tau) \), the Kleisli extension \( \mathbb{T} \) of \( \mathbb{T} \) to \( \text{Rel} \) (with respect to \( \tau \)) is described by the functions \( \mathbb{T} = \mathbb{T}_{X,Y} : \text{Rel}(X, Y) \rightarrow \text{Rel}(TX, TY) \) (indexed by sets \( X \) and \( Y \)), with

\[
  \chi (\mathbb{T} r) y \iff \chi \leq r^\tau(y) \quad (1.4.i)
\]

for all relations \( r : X \rightarrow Y \), and \( \chi \in TX \), \( y \in TY \), or, equivalently,

\[
  (\mathbb{T} r)^\flat = \downarrow_{TX} \cdot r^\tau : TY \rightarrow PTX .
\]

1.4.2 Examples.

(1) In the case of the terminal power-enriched monad \( (\mathbb{1}, !) \), the Kleisli extension of a relation \( r : X \rightarrow Y \) is \{\*\} \rightarrow \{\*\} with constant value \( \mathbb{T} \).

(2) To obtain an explicit description of the Kleisli extension of the powerset monad \( (\mathbb{P}, 1_\mathbb{P}) \), observe that

\[
  A \subseteq r^\flat(B) \iff A \subseteq \bigcup_X \cdot Pr^\flat(B) \iff \forall x \in A \exists y \in B \ (x \in r^\flat(y)) \iff A \subseteq r^\circ(B) ,
\]

for a relation \( r : X \rightarrow Y \), and \( A \in PX \), \( B \in PY \), where \( r^\circ(B) = \{ x \in X \mid \exists y \in B \ (x \in r^\flat(y)) \} \), as in Example \([\text{III} \; \text{1.10.3}]\). Hence,

\[
  A (\mathbb{P} r) B \iff A \subseteq r^\circ(B) .
\]

Here, we obtain the lax extension \( \mathbb{P} \) introduced in Example \([\text{III} \; \text{1.4.2}]\).
1. KLEISLI MONOIDS AND LAX ALGEBRAS

(3) Let $T = F$ be the filter monad and $\tau: \mathcal{P} \to \mathbb{T}$ the principal filter natural transformation. For a relation $r: X \to Y$, $A \subseteq X$ and $y \in FY$, we have

$$A \in m_X \cdot F(\tau_X \cdot r^\flat)(y) \iff (\tau_X \cdot r^\flat)^{-1}(A^F) \in y$$

$$\iff \{y \in Y \mid A \in \tau_X \cdot r^\flat(y)\} \in y$$

$$\iff \{y \in Y \mid r^\flat(y) \subseteq A\} \in y$$

$$\iff \exists B \in y \ (r^\circ(B) \subseteq A) .$$

This shows precisely that

$$r^\tau(y) = \uparrow_{PX} \{r^\circ(B) \mid B \in y\} ,$$

so $\chi (\hat{F}r) y \iff \chi \supseteq r^\tau(y)$ or, if we use the notation $r^\circ[y] = \{r^\circ(B) \mid B \in y\}$,

$$\chi (\hat{F}r) y \iff \chi \supseteq r^\circ[y] .$$

(1.4.ii)

The Kleisli extension of the filter monad returns the lax extension $\hat{T}$ of Example III.1.10.3(4).

(4) The Kleisli extension of the up-set monad $\mathbb{U}$, equipped with the principal filter natural transformation $\tau: \mathcal{P} \to \mathbb{U}$, is obtained as for the filter monad in the previous example, so that

$$\chi (\hat{U}r) y \iff \chi \supseteq r^\circ[y]$$

for all maps $r: X \to PY$ and up-sets $\chi \in UX$, $y \in UY$.

To prove that $\hat{T}: \text{Rel} \to \text{Rel}$ is indeed a lax extension of the $\text{Set}$-functor $T$, it is convenient to express the former as a composite of lax functors. In view of this, we remark that $\hat{T}r$ (for a relation $r: X \to Y$) can be written as

$$\hat{T}r = (r^\tau)^*: TX \Rightarrow TY ,$$

where $(-)^*: \text{Ord} \to \text{Mod}^{\text{op}}$ is the functor that sends a monotone map $f: X \to Y$ to the module $f^* = f^\circ \cdot (\leq Y): Y \Rightarrow X$ (see II.1.4 and II.2.2). The Kleisli extension is therefore a functor

$$\hat{T}: \text{Rel}^{\text{op}} \xlongrightarrow{(-)^r} \text{Sup} \xlongrightarrow{(-)^\tau} \text{Ord} \xlongrightarrow{(-)^r} \text{Mod}^{\text{op}}$$

(here, $\text{Sup} \to \text{Ord}$ is the forgetful functor). There is moreover a lax functor $\text{Mod} \to \text{Rel}$ that assigns to a module its underlying relation: composition of modules is composition of relations, identity modules are order relations and $1_X \leq (\leq X)$ for any ordered set $X$. Hence, with $E: \text{Set}_\mathbb{T} \to \text{Set}_\mathbb{T}$ and $L: \text{Set}_\mathbb{T} \to \text{Sup}$ denoting the functors from (ii) and (iii) of Proposition 1.2.1, the Kleisli extension $\hat{T}^{\text{op}}$ can be decomposed as the top line of the
Thus, if \( \mathcal{X} \) which concludes the proof.

For proving oplaxness of \( \mathcal{X} \), one then has \( \mathcal{X} \) for all \( \mathcal{X} \). The lax extension condition \( (Tf)^{\circ} \leq \hat{T}(f^{\circ}) \) can be deduced from the diagram

\[
\begin{array}{ccccccc}
\text{Rel}^{op} & \xrightarrow{(-)^{\circ}} & \text{Set} & \overset{E}{\xrightarrow{\sim}} & \text{Set} & \xrightarrow{\sim} & \text{Sup} & \overset{L}{\xrightarrow{\sim}} & \text{Ord} & \overset{(-)^{*}}{\xrightarrow{\sim}} & \text{Mod}^{op} & \xrightarrow{\sim} & \text{Rel}^{op} \\
\downarrow{(-)^{\circ}} & & \downarrow{\sim} & & \downarrow{T} & & \downarrow{\sim} & & \downarrow{\sim} & & \downarrow{\sim} & & \downarrow{\sim} & & \downarrow{\sim} \\
\text{Set} & & \text{Set} & & \text{Set} & & \text{Set} & & \text{Set} & & \text{Set} & & \text{Set} & & \text{Set} \\
\end{array}
\]  

(1.4.iii)

in which all arrows except \( \text{Mod}^{op} \to \text{Rel}^{op} \) are functors, and \( \text{Mod}^{op} \to \text{Rel}^{op} \) is a lax functor that fails only to preserve identities (the unnamed arrows are all forgetful).

**1.4.3 Proposition.** Given a power-enriched monad \((\mathbb{T}, \tau)\), the Kleisli extension \( \hat{T} \) of \( T \) to \( \text{Rel} \) yields a lax extension \( \hat{T} \in \text{Rel}^{op} \) of \( (\mathbb{T}, m, e) \) to \( \text{Rel} \).

**Proof.** The fact that \( \hat{T} : \text{Rel} \to \text{Rel} \) is a lax functor follows from its decomposition as lax functors preserving composition in the first line of (1.4.iii). The lax extension condition \( (Tf)^{\circ} \leq \hat{T}(f^{\circ}) \) can be deduced from the diagram

\[
\begin{array}{ccccccc}
\text{Rel}^{op} & \overset{(-)^{\circ}}{\xrightarrow{\sim}} & \text{Ord} & \overset{(-)^{*}}{\xrightarrow{\sim}} & \text{Mod}^{op} & \overset{\leq}{\xrightarrow{\sim}} & \text{Rel}^{op} \\
\downarrow(-)^{\circ} & & \downarrow{\sim} & & \downarrow{\sim} & & \downarrow{1_{\text{Rel}^{op}}} \\
\text{Set} & \overset{T}{\xrightarrow{\sim}} & \text{Set} & \overset{(-)^{\circ}}{\xrightarrow{\sim}} & \text{Rel}^{op} \\
\end{array}
\]

in which the first line is \( \hat{T}^{op} \). The second lax extension condition \( \hat{T}(h^{\circ} \cdot r) = (Th)^{\circ} \cdot \hat{T}r \) for all relations \( r : X \to Y \) and maps \( h : Z \to Y \) (see Proposition III.1.4.3(3)) comes from the equivalences

\[
\chi \ (\hat{T}(h^{\circ} \cdot r)) \ z \iff \chi \leq r^{\circ} \cdot (h^{\circ})^{\circ} (z) \iff \chi \leq r^{\circ} \cdot Th(z) \iff \chi \ ((Th)^{\circ} \cdot \hat{T}r) \ z
\]

for all \( \chi \in TX, \ z \in TZ \). To verify oplaxness of \( e : 1_{\text{Rel}} \to \hat{T} \), we use that \( \tau_X = m_X \cdot \tau_{TX} \cdot Pe_X = \sqcup TX \cdot Pe_X \). Given a relation \( r : X \to Y \), and \( x \in TX, y \in TY \) with \( x \ r \ y \), one then has

\[
e_X(x) \leq \bigvee_{x' \in r^{\circ}(y)} e_X(x') = \tau_X \cdot r^{\circ}(y) = (\tau_X \cdot r^{\circ})^{\circ} \cdot e_Y(y) = r^{\circ} \cdot e_Y(y),
\]

as required. For proving oplaxness of \( m : \hat{T}^{\hat{T}} \to \hat{T} \), recall that \( m_X \cdot \tau_{TX} \cdot \downarrow TX = \bigvee TX \cdot \downarrow TX = 1_{TX} \), and note that

\[
(r^{\circ})^{\top} = (r^{\circ} \cdot 1_{TY})^{\top} = ((\tau_X \cdot r^{\circ})^{\circ} \cdot 1_{TY})^{\top} = (\tau_X \cdot r^{\circ})^{\circ} \cdot 1_{TY} = r^{\circ} \cdot m_Y.
\]

Thus, if \( X \in TX \) and \( \gamma \in TTY \) are such that \( X \ (\hat{T}^{\hat{T}}r) \ \gamma \), or equivalently \( X \leq (\hat{T}r)^{\circ} (\gamma) \), then

\[
m_X(X) \leq m_X \cdot (\hat{T}r)^{\circ} (\gamma) = 1_{TX}^{\circ} \cdot (\tau_{TX} \cdot \downarrow TX \cdot r^{\circ})^{\circ} (\gamma) = (1_{TX}^{\circ} \cdot \tau_{TX} \cdot \downarrow TX \cdot r^{\circ})^{\circ} (\gamma) = (r^{\circ})^{\top} (\gamma) = r^{\circ} \cdot m_Y (\gamma),
\]

which concludes the proof.
1.4.4 Remark. Since the Kleisli extension provides the monad $\mathcal{T}$ with a lax extension, there is a natural order on $TX$ associated with $\mathcal{T}$ (see III.3.3); on $TX$ there is also the order $(1.2.i)$ induced by the monad morphism $\tau: \mathcal{P} \to \mathcal{T}$. Since the first order $\mathcal{T}1_X$ is defined via the second:

$$\chi (\mathcal{T}1_X) y \iff \chi \leq y$$

(Definition (1.4.i)), the orders are equivalent. Let us emphasize that $\mathcal{T}$ fails to preserve identity relations unless $\mathcal{T} = \mathcal{T}$ is the terminal power-enriched monad.

1.5 Topological spaces via filter convergence. In this subsection, we show that $(\mathcal{F},2)$-Cat is isomorphic to $\mathcal{F}$-Mon $\cong$ Top (Proposition 1.1.1), that is, we present topological spaces as sets equipped with a transitive and reflexive convergence relation $a: FX \to X$. The correspondence between convergence and neighborhoods can be formalized as in III.2.2 via maps

$$\text{conv}: \text{Set}(X,FX) \to \text{Rel}(FX,X) \quad \text{and} \quad \text{nbhd}: \text{Rel}(FX,X) \to \text{Set}(X,FX).$$

In fact, one can without further thought replace the filter monad $\mathcal{F}$ with a power-enriched monad $(\mathcal{T},\tau)$. By identifying $\text{Rel}(TX,X)$ with $\text{Set}(X,PTX)$, isomorphic as ordered sets, we define

$$\text{conv}(\nu) = \downarrow_{TX} \cdot \nu \quad \text{and} \quad \text{nbhd}(r) = \bigvee_{TX} \cdot r^\flat$$

for all maps $\nu: X \to TX$ and relations $r: TX \to X$. In pointwise notation, these maps may be written as

$$\chi \text{ conv}(\nu) y \iff \chi \leq \nu(y) \quad \text{and} \quad \text{nbhd}(r)(y) = \bigvee\{\chi \in TX \mid \chi \in r^\flat(y)\},$$

for all $y \in X$ and $\chi \in TX$, as a direct generalization of the fact that, in a topological space $X$, a filter $\chi$ converges to a point $y$ precisely when $\chi$ is finer than the neighborhood filter of $y$.

1.5.1 Proposition. With $\text{Set}(X,TX)$ and $\text{Rel}(TX,X)$ ordered pointwise, the monotone maps defined above form an adjunction $\text{nbhd} \dashv \text{conv}$: $\text{Set}(X,TX) \to \text{Rel}(TX,X)$ for all sets $X$.

Moreover, the fixpoints of $(\text{conv} \cdot \text{nbhd})$ are precisely the unitary relations, and $\text{conv}$ is fully faithful, so that the fixpoints of $(\text{nbhd} \cdot \text{conv})$ are the maps $\nu: X \to TX$.

Proof. The equivalence

$$\text{nbhd}(r) \leq \nu \iff r^\flat \leq \text{conv}(\nu)$$

(for all maps $\nu: X \to TX$ and relations $r: TX \to X$) follows directly from the adjunction $\bigvee_{TX} \downarrow_{TX} \cdot \downarrow_{TX}$. Similarly, from $\bigvee_{TX} \cdot \downarrow_{TX} = 1_{TX}$ follows that $\text{nbhd} \cdot \text{conv} = 1$, that is, $\text{conv}$ is fully faithful (see Corollary III.1.5.2).
Thus, suppose that $X$. Hence, if $r : TX \to X$ is unitary, then

$$r^b = (\downarrow_{TX})^r \cdot r^b$$

(r right unitary in [1.7.3])

$$= (\downarrow_{TX})^F \cdot (e^X_\ast \cdot \bar{T}r \cdot m_X^\circ)^b$$

(r left unitary)

$$= (\downarrow_{TX})^F \cdot ((m^\circ_X)^F \cdot (\bar{T}r)^b \cdot e_X$$

$$= (\downarrow_{TX} \cdot m_X^\circ)^F \cdot (\bar{T}r)^b \cdot e_X$$

$$= (\downarrow_{TX} \cdot m_X^\circ)^F \cdot (\tau_{TX} \cdot r^b)^\top \cdot e_X$$

(definition of $\bar{T}$)

$$\geq (\downarrow_{TX} \cdot m_X^\circ)^F \cdot (-)_{TTX} \cdot \tau_{TX} \cdot r^b$$

$$= \downarrow_{TX} \cdot m_X \cdot \tau_{TX} \cdot r^b$$

$$\geq \text{conv} \cdot \text{nbhd}(r)$$

$$\geq r^b$$

(nbhd $\dashv$ conv)

and we may conclude that $\text{conv} \cdot \text{nbhd}(r) = r$ (via the understood identification of $\text{Rel}(TX, X)$ with $\text{Set}(X, PTX))$. Conversely, if $r$ is a fixpoint of $\text{conv} \cdot \text{nbhd},$ then $r$ is of the form $\text{conv}(\nu)$ for some $\nu : X \to TX,$ and one sees that $r \cdot \bar{T}1_X \leq r,$ so $r$ is right unitary. To prove that $r$ is left unitary, we must verify that $e^X_\ast \cdot \bar{T}r \leq r \cdot m_X.$ Thus, suppose that $X$ ($\bar{T}r \cdot e_X(y)$ holds. By definition of $\bar{T},$ we have $X \leq r^r \cdot e_X(y),$ or equivalently $X \leq \tau_{TX} \cdot r^r(y).$ Applying $m_X$ to each side of this inequality, we obtain $m_X(X) \leq m_X \cdot \tau_{TX} \cdot r^r(y),$ which means precisely that $m_X(X) \leq \downarrow_{TX} \cdot r^r(y) = \text{nbhd}(r)(y),$ or $m_X(X)$ (conv $\cdot \text{nbhd}(r) / y$ which is $m_X(X) \cdot r$ $y$ by the fixpoint condition.

1.5.2 Proposition. The adjoint maps $\text{nbhd}$ and $\text{conv}$ defined above are monoid homomorphisms between $\text{Set}_\Sigma(X, X)$ and $(\bar{T}, \bar{V})$-$\text{UREl}^{op}(X, X),$ that is, they satisfy

$$\text{nbhd}(s \circ r) = \text{nbhd}(r) \circ \text{nbhd}(s), \quad \text{conv}(\mu) \circ \text{conv}(\nu) = \text{conv}(\nu \circ \mu)$$

for all unitary relations $r, s : TX \to X,$ and maps $\mu, \nu : X \to TX.$

Proof. The equality $\text{nbhd}(1^X_X) = e_X$ follows immediately from the definition of $1^X_X,$ as

$$\chi \cdot 1^X_X \cdot y \iff \chi \cdot (e^X_\ast \cdot \bar{T}1_X) \cdot y \iff \chi \leq e_X(y)$$

for all $\chi \in TX$ and $y \in X.$ The multiplication $m_X = 1^TX_T$ of the monad $\bar{T}$ is a sup-map and $1^TX_T \cdot \tau_{TX} = \downarrow_{TX}$ (Proposition 1.2.1), so

$$1^TX_T \cdot \tau_{TX} \cdot Pm_X \cdot \downarrow_{TTX} = \downarrow_{TX} \cdot Pm_X \cdot \downarrow_{TTX} = m_X \cdot \downarrow_{TX} \cdot \downarrow_{TTX} = 1^TX_T.$$
Therefore,

\[
\text{nbhd}(r) \circ \text{nbhd}(s) = (1^T_{TX} \cdot \tau_{TX} \cdot r^p) \cdot 1^T_{TX} \cdot \tau_{TX} \cdot s^p \\
= ((1^T_{TX} \cdot \tau_{TX} \cdot r^p)^T) \cdot \tau_{TX} \cdot s^p \\
= (1^T_{TX} \cdot (\tau_{TX} \cdot r^p)^T) \cdot \tau_{TX} \cdot s^p \\
= (1^T_{TX} \cdot \tau_{TX} \cdot Pm_X \cdot (\bar{T}r)^b)^T \cdot \tau_{TX} \cdot s^p \\
= 1^T_{TX} \cdot (\tau_{TX} \cdot Pm_X \cdot (\bar{T}r)^b)^p \cdot \tau_{TX} \cdot s^p \\
= \bigvee_{TX} (m_X^o)^p \cdot (\bar{T}r)^b \cdot s^p \\
= \bigvee_{TX} (s \cdot \bar{T}r \cdot m_X^o)^b \\
= \text{nbhd}(s \circ r).
\]

The equalities for conv then follow directly from the fact that conv and nbhd are inverse of each other on fixpoints (Proposition [1.5.1]).

1.5.3 Theorem. Given a power-enriched monad \((\mathbb{T}, \tau)\) equipped with its Kleisli extension \(\mathcal{T}\), there is an isomorphism

\[(\mathbb{T}, 2)\text{-Cat} \cong \mathbb{T}\text{-Mon}\]

that commutes with the underlying-set functors.

Proof. For a \((\mathbb{T}, 2)\)-algebra \((X, r)\), Proposition [1.5.2] implies that \((X, \text{nbhd}(r))\) is a \(\mathbb{T}\)-monoid, and conversely, if \((X, \nu)\) is a \(\mathbb{T}\)-monoid, then \((X, \text{conv}(\nu))\) is a \((\mathbb{T}, 2)\)-algebra (one also uses the fact that nbhd and conv are monotone). Moreover, Proposition [1.5.1] entails that these objects are in bijective correspondence.

We are therefore left to show that this correspondence is functorial. Consider first a \((\mathbb{T}, 2)\)-functor \(f : (X, r) \to (X, s)\), so that \(r \cdot (Tf)^o \leq f^o \cdot s\). We have

\[
Tf \cdot \text{nbhd}(r) = Tf \cdot \bigvee_{TX} r^p \\
= \bigvee_{TX} Pf \cdot r^p \\
= \bigvee_{TX} (r \cdot (Tf)^o)^b \\
\leq \bigvee_{TX} (f^o \cdot s)^b \\
= \bigvee_{TX} (s)^p \cdot f \\
= \text{nbhd}(s) \cdot f,
\]

hence, \(f\) is a morphism of \(\mathbb{T}\)-monoids. Consider now \(f : (X, \nu) \to (Y, \mu)\) satisfying \(Tf \cdot \nu \leq \mu \cdot f\). Then \(\chi \text{ conv}(\nu) y\) means \(\chi \leq \nu(y)\), so we have \(Tf(\chi) \leq Tf \cdot \nu(y) \leq \mu \cdot f(y)\), and can therefore conclude that \(Tf(\chi) \text{ conv}(\mu) y\); that is, \(f\) is a \((\mathbb{T}, 2)\)-functor between the corresponding \((\mathbb{T}, 2)\)-categories.
1.5.4 Corollary. The category \( \textbf{Top} \) of topological spaces is isomorphic to the category \((\mathcal{F}, 2)\)-\textbf{Cat} whose objects are pairs \((X, a)\), with \(a : FX \to X\) a relation representing convergence and, when \(a\) and \(\tilde{a}\) are denoted by \(\to\), satisfying
\[
X \to y \land y \to z \implies \sum X \to z \quad \text{and} \quad \dot{x} \to x ,
\]
for all \(x, z \in X, y \in FX\), and \(X \in FFX\); here \(X \to y \iff X \supseteq a^\circ[y]\). The morphisms are the convergence-preserving maps \(f : X \to Y\):
\[
\chi \to y \implies f[\chi] \to f(y)
\]
for all \(y \in X, \chi \in FX\).

Proof. Proposition 1.1.1 together with the previous Theorem yield an isomorphism between \(\textbf{Top}\) and \((\mathcal{F}, 2)\)-\textbf{Cat}, and the statement is just an explicit description of the latter category using the Kleisli extension of the filter monad (1.4.ii). \(\square\)

Naturally, the same statement holds for the category \(\textbf{Cls}\) of closure spaces, for which we give the more synthetic form below.

1.5.5 Corollary. For the up-set monad \(\mathbb{U}\) equipped with the Kleisli extension associated with the principal filter natural transformation, there is an isomorphism
\[
\textbf{Cls} \cong (\mathbb{U}, 2)\text{-Cat}
\]
that commutes with the underlying-set functors.

Proof. This is another application of Theorem 1.5.3 with Exercise 1.D in the case of the up-set monad. \(\square\)

Exercises

1.A The slice under \(\mathbb{P}\). Let \(\mathbb{S} = (S, n, d), \mathbb{T} = (T, m, e)\) be monads on \(\textbf{Set}\), and \(\alpha : \mathbb{S} \to \mathbb{T}\) a monad morphism. The following statements are equivalent for objects \(\sigma : \mathbb{P} \to \mathbb{S}\) and \(\tau : \mathbb{P} \to \mathbb{T}\) of the comma category \(\mathbb{P}/\textbf{MND}_{\textbf{Set}}\):

(i) \(\alpha\) is a morphism \(\sigma \to \tau\);

(ii) \(\alpha_X : SX \to TX\) is a sup-map for all sets \(X\).

1.B Constructing the power-enrichment. Let \(\mathbb{T} = (T, m, e)\) be a monad on \(\textbf{Set}\). Suppose that the sets \(TX\) are equipped with an order that make them complete lattices, and is such that \(Tf : TX \to TY\) and \(m_X : TTX \to TX\) are sup-maps for all maps \(f : X \to Y\) and
1. **KLEISLI MONOIDS AND LAX ALGEBRAS**

sets $X$. The monad morphism $\tau : P \to T$ of Proposition 1.2.1 is then given componentwise by

$$\tau_X(A) = \bigvee_{x \in A} e_X(x).$$

1.C **The double-powerset monad is not power-enriched.** Consider the maps $f, g : \{a, b\} \to P^2\{*\}$ given by

$$f(a) = \{\emptyset\}, \quad f(b) = \{\{\ast\}\}$$

$$g(a) = \{\emptyset\}, \quad g(b) = \{\emptyset, \{\ast\}\}.$$

One has $f(x) \subseteq g(x)$ for all $x \in \{a, b\}$, but if $x = \{\{a\}, \{b\}\} \in P^2\{a,b\}$, then $f^{P^2}(x) \not\subseteq g^{P^2}(x)$, since

$$f^{P^2}(x) = \{\emptyset, \{\ast\}\}, \quad g^{P^2}(x) = \{\{\ast\}\},$$

where $(-)^{P^2} = m_X \cdot P^2(-)$ comes from the double-powerset monad $P^2$. Hence, neither the principal filter natural transformation $\tau : P \to P^2$, nor the natural transformation $\sigma : P \to P^2$ of Example 1.2.3(6) make $P^2$ into a power-enriched monad.

1.D **Kleisli monoids from the double-powerset monad.** The adjunction $P^\bullet \dashv (P^\bullet)^{op} : \text{Set}^{op} \to \text{Set}$ of Example II.2.5.1(6) yields a one-to-one correspondence

$$\begin{array}{ccc}
X & \xrightarrow{f} & P^2Y \\
\downarrow & & \downarrow \\
PX & \xleftarrow{\nu} & PY
\end{array}$$

Both hom-sets $\text{Set}(X, P^2Y)$ and $\text{Set}(PY, PX)$ inherit a pointwise order, the first from the refinement order on $P^2Y$ and the second from the inclusion order on $PX$. If $f, g \in \text{Set}(X, P^2Y)$ and $P^\bullet f, P^\bullet g \in \text{Set}(PY, PX)$ are maps obtained via (1.5.ii), then

$$f \leq g \iff P^\bullet g \leq P^\bullet f.$$

If we denote by $\text{PSet}$ the category whose objects are sets and morphisms from $X$ to $Y$ are maps $f : PX \to PY$, then the correspondence (1.5.ii) describes an isomorphism of ordered categories:

$$\text{Set}_{P^2} \cong \text{PSet}^{\text{co}op}.$$
the set of all finitary up-sets on a set $X$ is denoted by $U_{\text{fin}}X$. Show that the components of the up-set monad $U$ restrict to such elements to yield the finitary-up-set monad $U_{\text{fin}}$ on $\text{Set}$. With every set $U_{\text{fin}}X$ ordered by subset inclusion, $U_{\text{fin}}$ is power-enriched.

The category of $U_{\text{fin}}$-monoids is isomorphic to the full subcategory $\text{Cls}_{\text{fin}}$ of closure spaces $\text{Cls}$ (see [III,2.5]):

$$U_{\text{fin}}\text{-Mon} \cong \text{Cls}_{\text{fin}}.$$  

**1.F The clique monad and closure spaces.** A clique $a$ on a set $X$ is a subset of $PX$ such that

1. $A, B \in a \implies A \cap B \neq \emptyset$,
2. $A \in a, A \subseteq B \implies B \in a$,

for all $A, B \in PX$. A clique $a$ is proper if $\emptyset \notin a$. The set of all cliques on $X$ is denoted by $CX$, and the double-powerset monad $P^2$ restricts to such sets to yield the clique monad $C$. The clique monad is power-enriched via the principal filter natural transformation $\tau : P \rightarrow C$, and there is an isomorphism

$$U\text{-Mon} \cong C\text{-Mon}.$$  

Since $U\text{-Mon}$ is isomorphic to the category $\text{Cls}$ of closure spaces, cliques provide alternative convergence structure for the study of closure spaces via the isomorphism $\text{Cls} \cong C\text{-Mon} \cong (C,2)-\text{Cat}$.

**1.G The Kleisli extension to $\text{Rel}$ is associative.** For a power-enriched monad $(T, \tau)$, the monoid homomorphisms nbhd and conv of [1.5] can be extended to yield monotone maps

$$\text{nbhd} = \text{nbhd}_{X,Y} : \text{Rel}(TX, Y) \rightarrow \text{Set}(Y, TX)$$
$$\text{conv} = \text{conv}_{Y,X} : \text{Set}(Y, TX) \rightarrow \text{Rel}(TX, Y)$$

that form an adjunction $\text{nbhd} \dashv \text{conv}$ for all sets $X, Y$. When $\hat{T}$ is equipped with its Kleisli extension, one has

$$\text{nbhd}(s \circ r) = \text{nbhd}(r) \circ \text{nbhd}(s) \quad \text{conv}(\mu) \circ \text{conv}(\nu) = \text{conv}(\nu \circ \mu)$$
$$\text{nbhd}(1^T_X) = e_X \quad \text{conv}(e_X) = 1^T_X$$

for all unitary relations $r, s : TX 
\rightarrow Y$, and maps $\mu, \nu : Y \rightarrow TX$. As a consequence, the Kleisli extension $\hat{T}$ is associative, and one can form the category $(\hat{T},2)-\text{URel}$.  

The maps nbhd and conv define functors $\text{nbhd} : \text{Set}_T \rightarrow (\hat{T},2)-\text{URel}^{\text{op}}$ and $\text{conv} : (\hat{T},2)-\text{URel}^{\text{op}} \rightarrow \text{Set}_T$ that determine a 2-isomorphism

$$\text{Set}_T \cong (\hat{T},2)-\text{URel}^{\text{op}}.$$
Unitary relations as sup-maps. For a power-enriched monad \( (T, \tau) \) and a relation \( r : TX \leftrightarrow Y \), the following statements are equivalent:

1. \( r \) is unitary;
2. \( r^\flat = \downarrow_{TX} \lor r^\flat \);
3. \( r(-, y) : TX \to 2^{\text{op}} \) is a sup-map for all \( y \in Y \).
2 Lax extensions of monads

In Corollary 1.5.4 we effectively established an isomorphism \((\mathcal{F}, 2)\text{-Cat} \to (\beta, 2)\text{-Cat}\), both categories being isomorphic to \(\text{Top}\). It turns out that this isomorphism may be thought of as induced by the monad morphism \(\beta \to \mathcal{F}\). More generally, in this subsection we seek sufficient conditions for a monad morphism \(\alpha : \mathcal{S} \to \mathcal{T}\) into a power-enriched monad \(\mathcal{T}\) to induce an isomorphism

\[ A_\alpha : (\mathcal{T}, \mathcal{V})\text{-Cat} \to (\mathcal{S}, \mathcal{V})\text{-Cat} \]

when \(\mathcal{S}\) and \(\mathcal{T}\) are equipped with adequate lax extensions; here, \(A_\alpha\) is the algebraic functor of \(\alpha\), see III.3.4. For this, we proceed in two steps: we first obtain isomorphisms

\[(\mathcal{S}, 2)\text{-Cat} \cong \mathcal{T}\text{-Mon} \cong (\mathcal{T}, 2)\text{-Cat},\]

and then consider the case where 2 is replaced by an arbitrary quantale \(\mathcal{V}\) (Theorems 2.3.3 and 2.5.3). Each of these steps is facilitated by the construction of lax extensions induced by a lax extension \(\hat{T}\) of \(\mathcal{T}\) to \(\text{Rel}\). Specifically, in 2.1 we “transfer” the lax extension from \(\mathcal{T}\) to \(\mathcal{S}\) along \(\alpha\), and in 2.4 we describe a process of generating a lax extension of \(\mathcal{T}\) to \(\mathcal{V}\)-\(\text{Rel}\) from a lax extension to \(\text{Rel}\).

2.1 Initial extensions. In III.1.10 and 1.4 two constructions of a lax extension of a monad to \(\text{Rel}\) are given: the Barr extension and the Kleisli extension. In practice, the Barr extension can be extracted from the Kleisli extension of a larger monad. For example, the Barr extension of the ultrafilter functor \(\beta : \text{Set} \to \text{Set}\):

\[ (\beta r) y \iff \chi \supseteq r^\circ[y]\]

(for all relations \(r : X \to Y\), \(\chi \in \beta X\), \(y \in \beta Y\)) is the restriction to ultrafilters of the Kleisli extension of the filter functor

\[ (\mathcal{F} r) y \iff \chi \supseteq r^\circ[y]\]

(for all \(\chi \in FX\), \(y \in FY\)).

More generally, if \(\alpha : S \to T\) is a natural transformation of \(\text{Set}\)-functors, and \(\mathcal{T}\) is a lax extension of \(T\) to \(\mathcal{V}\)-\(\text{Rel}\), the initial extension of \(S\) induced by \(\alpha\) is the lax extension \(\hat{S}\) given by

\[ \hat{S} r := \alpha_Y^\circ \cdot \mathcal{H} r \cdot \alpha_X, \]

for any \(\mathcal{V}\)-relation \(r : X \to Y\). In pointwise notation, the definition becomes

\[ \hat{S} r(\chi, y) = \mathcal{H} r(\alpha_X(\chi), \alpha_Y(y)) , \]

for all \(\chi \in SX\), \(y \in SY\). Before showing that \(\hat{S}\) is indeed a lax extension of \(S\) if \(\mathcal{H}\) is one of \(T\) (Proposition 2.1.1), we briefly discuss the “initial” terminology.
Recall from \[\text{III.3.4}\] that a morphism of lax extensions $\alpha : (S, \hat{S}) \rightarrow (T, \hat{T})$ is a natural transformation $\alpha : S \rightarrow T$ that extends to an oplax transformation $\hat{S} \rightarrow \hat{T}$:

$$\alpha_Y \cdot \hat{S} r \leq \hat{T} r \cdot \alpha_X,$$

for all $\mathcal{V}$-relations $r : X \rightarrow Y$. If $U : \mathcal{V}LXT \rightarrow \text{Set}^\text{Set}$ denotes the forgetful functor from the category of lax extensions to $\mathcal{V}$-$\text{Rel}$ (see \[\text{III.3.4}\]), then the initial extension is an $U$-initial morphism in the sense of \[\text{III.3.4}\]. Indeed, consider a natural transformation $\lambda : R \rightarrow S$ with a lax extension $\hat{R}$ of $R$; then $\lambda$ is a morphism of lax extensions if and only if $\alpha \cdot \lambda : R \rightarrow T$ is one:

$$\alpha_Y \cdot \lambda_Y \cdot \hat{R} r \leq \hat{T} r \cdot \alpha_X \cdot \lambda_X \iff \lambda_Y \cdot \hat{R} r \leq \alpha_Y \cdot \hat{T} r \cdot \alpha_X \cdot \lambda_X \iff \lambda_Y \cdot \hat{R} r \leq \hat{S} r \cdot \lambda_X$$

for all relations $r : X \rightarrow Y$.

\[\text{2.1.1 Proposition.}\] For a lax extension $\hat{T}$ to $\mathcal{V}$-$\text{Rel}$ of a Set-functor $T$, and a natural transformation $\alpha : S \rightarrow T$, the initial extension $\hat{S}$ of $S$ induced by $\alpha$ is a lax extension of $S$.

Furthermore, if $\hat{T}$ belongs to a lax extension to $\mathcal{V}$-$\text{Rel}$ of a monad $\hat{\mathbb{T}} = (T, m, e)$ and $\alpha : S \rightarrow \hat{\mathbb{T}}$ is a monad morphism, then $\hat{S}$ also belongs to a lax extension of $S = (S, n, d)$.

\[\text{Proof.}\] Since $\hat{T}$ preserves the order on the hom-sets $\mathcal{V}$-$\text{Rel}(X, Y)$, it is immediate that $\hat{S}$ does too. If $r : X \rightarrow Y$ and $s : Y \rightarrow Z$ are $\mathcal{V}$-relations, then

$$\hat{S} s \cdot \hat{S} r = \alpha_Z^0 \cdot \hat{T} s \cdot \alpha_Y \cdot \alpha_Y^0 \cdot \hat{T} r \cdot \alpha_X \leq \alpha_Z^0 \cdot \hat{T} s \cdot \hat{T} r \cdot \alpha_X \leq \alpha_Z^0 \cdot \hat{T} (s \cdot r) \cdot \alpha_X = \hat{S} (s \cdot r),$$

because $\hat{T}$ is a lax functor. As $\alpha$ is a natural transformation, we have $\alpha_Y \cdot S f = T f \cdot \alpha_X$ which can be written $S f \leq \alpha_Y^0 \cdot T f \cdot \alpha_X$, or equivalently $(S f)^\circ \leq \alpha_X^0 \cdot (T f)^\circ \cdot \alpha_Y$, in $\mathcal{V}$-$\text{Rel}$; the extension conditions for $\hat{S}$ then follow because they are satisfied for $\hat{T}$.

Finally, suppose that $\hat{T}$ yields a lax extension of the monad $\hat{\mathbb{T}}$. Since $e : 1_{\mathcal{V}-\text{Rel}} \rightarrow \hat{T}$ is oplax, then so is $d : 1_{\mathcal{V}-\text{Rel}} \rightarrow \hat{S}$: indeed, $\alpha$ is a monad morphism, so we have $\alpha \cdot d = e$, and

$$r \leq e_Y^0 \cdot \hat{T} r \cdot e_X = d_Y^0 \cdot \hat{S} r \cdot d_X,$$

as expected. To verify oplaxness of $n$, we use that $m \cdot T \alpha \cdot \alpha S = \alpha \cdot n$:

$$\hat{S} \hat{S} r = \alpha_{Sy}^0 \cdot \hat{T} (\alpha_Y^0 \cdot \hat{T} r \cdot \alpha_X) \cdot \alpha_S X \leq \alpha_{Sy}^0 \cdot (T \alpha_Y)^\circ \cdot \hat{T} \hat{T} r \cdot T \alpha_X \cdot \alpha_S X$$

as required. \[\square\]
CHAPTER IV. KLEISLI MONOIDS

From this last result one infers that in presence of the initial extension \( \hat{S} \) of \( S \), the maps \( \alpha_X : SX \to TX \) become order-embeddings with respect to the orders \( \text{III}3.3 \) induced by the lax extensions:

\[
\chi \leq y \iff \alpha_X(\chi) \leq \alpha_X(y)
\]

for all \( \chi, y \in SX \). In fact, this condition witnesses the smooth interaction of the initial and Kleisli extensions, as we will see next.

2.1.2 Proposition. A morphism \( \alpha : (S, \sigma) \to (\mathbb{T}, \tau) \) of power-enriched monads becomes a morphism \( \alpha : \hat{S} \to \hat{T} \) of the Kleisli extensions to \text{Rel}.

When the sets \( SX \) and \( TX \) are equipped with the orders \( \text{I.2.1} \) induced by \( \sigma \) and \( \tau \) respectively, the components \( \alpha_X \) are order-embeddings if and only if the initial extension of \( S \) induced by \( \alpha \) is the Kleisli extension of \( S \).

Proof. Observe that \( \alpha_X \cdot r^\sigma = r^\tau \cdot \alpha_Y \) for any relation \( r : X \to Y \). Therefore,

\[
\chi \leq r^\sigma(y) \implies \alpha_X(\chi) \leq \alpha_X \cdot r^\sigma(y) = r^\tau \cdot \alpha_Y(y)
\]

for all \( \chi \in SX, y \in SY \) (\( \alpha_X \) is monotone by Exercise \( \text{I.A} \)). This implies that \( \alpha \) is a morphism between the respective Kleisli extensions. If \( \alpha_X \) is an order-embedding, the implication above is an equivalence, so that \( \alpha \) is initial. Conversely, if the initial extension of \( S \) is the Kleisli extension, the equivalence also holds, and we can conclude that \( \alpha_X \) is an order-embedding by choosing \( r = 1_X \).

2.1.3 Examples.

(1) For every monad \( S \) on \text{Set}, there is a unique monad morphism \( ! : S \to \mathbb{1} \) into the terminal monad. When the latter is equipped with its largest lax extension \( \star^\top : \mathcal{V}\text{-Rel} \to \mathcal{V}\text{-Rel} \) (such that \( \star^\top r(\star, \star) = \top \) for all \( \mathcal{V}\)-relation \( r : X \to Y \) as in Example \( \text{III}1.4.2(3) \)), the initial extension \( \hat{S} \) of \( \star^\top \) induced by \( ! \) is given by

\[
\hat{S}r(\chi, y) = \top,
\]

for all \( \chi \in SX, y \in SY \); hence, \( \hat{S} = S^\top \).

(2) It is obvious from Example \( 1.4.2(4) \) that the Kleisli extension \( \hat{F} \) of \( F \) can be obtained as the restriction to filters of the Kleisli extension \( \hat{U} \) of the up-set functor \( U \); in this case, the components \( \alpha_X : FX \to UX \) of \( \alpha \) are the embeddings, and \( \hat{F} \) is the initial extension of \( \hat{U} \) induced by \( \alpha \). Similarly, the Barr extension of the ultrafilter functor can be obtained by restriction of the Kleisli extension \( \hat{F} \) of the filter functor, and the lax extension of the identity monad can also be seen as a restriction of the ultrafilter functor (via the principal ultrafilter natural transformation). Thus, the chain of natural transformations, whose respective components are all embeddings, as described in Example \( \text{II}3.1.1(5) \):

\[
1\text{Set} \to \beta \to F \to U
\]
yields the following chain of initial extensions:

\[ 1_{\text{Rel}} \rightarrow \beta \rightarrow \tilde{F} \rightarrow \tilde{U} \]

### 2.2 Sup-dense and interpolating monad morphisms

The isomorphism \((\mathcal{S}, 2)-\text{Cat} \cong (\mathcal{T}, 2)-\text{Cat}\) that we are aiming for requires that the monads \(\mathcal{S}\) and \(\mathcal{T}\) be sufficiently compatible. The conditions we present continue to be guided by the case where \(\mathcal{S} = \beta, \mathcal{T} = \mathcal{F}\) and \(\alpha : \beta \hookrightarrow \mathcal{F}\) is the inclusion of the set of ultrafilters into the set of filters.

Hence, consider a monad morphism \(\alpha : \mathcal{S} \rightarrow \mathcal{T}\), where \(\mathcal{T} = (T, m, e)\) is a power-enriched monad with structure \(\tau : P \rightarrow T\) and equipped with its Kleisli extension \(\tilde{T}\), and \(\mathcal{S} = (S, n, d)\) a monad equipped with its initial extension \(\hat{S}\) induced by \(\alpha\). To be able to exploit the adjunction \(\bigsqcup_{TX} \dashv \downarrow_{TX}\) as in Proposition 1.5.2, we introduce the transformation \(\alpha_Y : PS \rightarrow T\) via

\[
\alpha_Y^X := \bigvee \alpha_X = m_X \cdot \tau_{TX} \cdot P\alpha_X = \alpha_X^T \cdot \tau_{SX},
\]

or equivalently, \(\alpha_Y^X(A) = \bigvee \alpha_X(A)\) for all \(A \subseteq SX\). Each \(\alpha_Y^X\) preserves suprema, and therefore has a right adjoint, denoted by \(\alpha^{-1}_X : TX \rightarrow PSX\), so that

\[
\alpha_X^X(f) = \{x \in SX \mid \alpha_X(x) \leq f\} = \alpha^{-1}_X \cdot \downarrow_{TX}(f)
\]

for all \(f \in TX\). The maps \(\alpha_X^X\) allow for a convenient description of the initial extension \(\hat{S}\) of \(S\). Indeed, for a relation \(r : X \rightarrow Y\), we have

\[
(\hat{S}r)^\beta = \alpha_X^X \cdot r^\tau \cdot \alpha_Y,
\]

so the order relation \(\hat{S}1_X\) on \(SX\) is given by \((\hat{S}1_X)^\beta = \alpha_X^X \cdot \alpha_X\).

The monad morphism \(\alpha : \mathcal{S} \rightarrow \mathcal{T}\) is sup-dense if one has

\[
\alpha_Y \cdot \alpha_X^X = 1_{TX} ; \tag{2.2.i}
\]

in pointwise notation this says that every element of \(TX\) can be expressed as a supremum of \(\alpha_X\)-images of elements of \(SX\):

\[
\forall f \in TX \exists A \subseteq SX (f = \bigvee \alpha_X(A)).
\]

When \(\mathcal{S}\) is a submonad of \(\mathcal{T}\) and the embedding is sup-dense, we simply say that \(\mathcal{S}\) is sup-dense in \(\mathcal{T}\).

The morphism \(\alpha : \mathcal{S} \rightarrow \mathcal{T}\) is interpolating for a relation \(r : SX \rightarrow X\) if

\[
\alpha_X^X \cdot \alpha_X^Y \cdot r^\beta \leq (\downarrow_{SX} \cdot n_X)^\beta \cdot (\hat{S}r)^\beta \cdot d_Y \tag{2.2.ii}
\]

holds. This condition expands to \(\alpha_X^X \cdot \alpha_X^Y \cdot r^\beta \leq (\alpha_X^X \cdot \alpha_X \cdot n_X)^\beta \cdot \alpha_{SX}^X \cdot \tau_{SX} \cdot r^\beta\) and can be written pointwise as

\[
\alpha_X(x) \leq \bigvee \{\alpha_X(y) \mid y \in Y\} \implies \exists X \in SSX (x \leq n_X(X) \& \alpha_{SX}(X) \leq \tau_{SX} \cdot r^\beta(y))
\]
for all $\chi \in SX$, $y \in X$. If $S$ is a submonad of $T$, the previous condition naturally has a simpler expression, and may be represented graphically by

$$\chi \leq m_X \cdot \tau_{SX} \cdot r^\flat(y) \implies \exists X : X \leq \tau_{SX} \cdot r^\flat(y) \implies \chi \leq m_X(X)$$

A monad morphism $\alpha : S \to T$ is interpolating if it is interpolating for all relations $r : SX \to X$. If $S$ is a submonad of $T$ and the embedding is interpolating, we may simply say that $S$ is interpolating in $T$.

Note that $\alpha$ is interpolating whenever it is a morphism of power-enriched monads $\alpha : (S, \sigma) \to (T, \tau)$. Indeed, since $\{\}_{SSX} \leq \alpha^\flat_X \cdot \alpha_{SX} \cdot \sigma_{SX} \cdot r^\flat$, we have

$$\alpha^\flat_X \cdot \alpha^\flat_X \cdot r^\flat = \alpha^\flat_X \cdot \alpha^\flat_X \cdot \sigma_{SX} \cdot r^\flat = (\alpha^\flat_X \cdot \sigma_{SX})^\flat \cdot \{\}_{SSX} \cdot r^\flat \leq (\alpha^\flat_X \cdot \sigma_{SX})^\flat \cdot \alpha^\flat_{SX} \cdot \sigma_{SX} \cdot r^\flat = (\alpha^\flat_X \cdot \sigma_{SX})^\flat \cdot \alpha^\flat_{SX} \cdot \tau_{SX} \cdot r^\flat$$

for all relations $r : X \to X$.

2.2.1 Examples.

(1) Any power-enriched monad $T = (T, m, e)$ comes with the interpolating monad morphism $\alpha = e : \mathbb{I} \to T$. Indeed, using that $\alpha^\flat_X = 1_{TX}$ and $\{\} \leq \alpha^\flat_X \cdot \alpha_X$, we have

$$\alpha^\flat_X \cdot \alpha^\flat_X \cdot r^\flat = \alpha^\flat_X \cdot \tau_X \cdot r^\flat = (\{\})^\flat \cdot \alpha^\flat_X \cdot \tau_X \cdot r^\flat \leq (\alpha^\flat_X \cdot \alpha_X)^\flat \cdot \alpha^\flat_{SX} \cdot \tau_X \cdot r^\flat$$

for all relations $r : X \to X$.

(2) If $S = \mathbb{P}$ is the powerset monad embedded in $T = \mathbb{F}$ via the principal filter morphism $\tau : \mathbb{P} \to \mathbb{F}$, then the interpolation condition is immediate since $\tau$ is a morphism of power-enriched monads.

(3) Consider the filter monad $\mathbb{F}$ with the principal monad morphism $\tau : \mathbb{P} \to \mathbb{F}$. Every filter is the supremum (that is, the intersection) of all ultrafilters finer than it (Corollary II.1.13.4), so the ultrafilter monad $\beta$ is sup-dense in $\mathbb{F}$.

Let us verify that $\beta$ is interpolating in $\mathbb{F}$. For ultrafilters $\chi, y$ on $X$ and a relation $r : \beta X \to X$, suppose $\chi \leq \sum r^\tau(y)$ (with $\sum$ denoting the monad multiplication of $\beta$), that is,

$$\forall B \in y \ (r^\flat(B) \subseteq A^\beta \implies A \in \chi),$$

for all $A \subseteq X$ (where $A^\beta = \{ z \in \beta X \mid A \in z \}$, see [III 3.2]). If there existed $A \in \chi$ and $B \in y$ with $A^\beta \cap r^\flat(B) = \emptyset$, we would have $r^\flat(B) \subseteq (A^\beta)^\mathbb{C} = (A^\beta)^\beta$, so that
A^c \in \chi$, a contradiction. Therefore, $A^\emptyset \cap r^\emptyset(B) \neq \emptyset$ for all $A \in \chi$ and $B \in y$, and there exists an ultrafilter $X$ on $\beta X$ that refines both $\{A^\emptyset \mid A \in \chi\}$ and $r^\emptyset(y)$. In particular, $\Sigma_X(x) = \chi$. By setting $y = \hat{y}$, we observe that $r^\emptyset(y) = \tau_{\beta x} \cdot r^\emptyset(y)$, so the interpolation condition is verified.

2.3 $(S,2)$-categories as Kleisli monoids. Given a lax extension $\hat{S}$ to $\mathcal{V}$-$\text{Rel}$ of a monad $S$ on $\text{Set}$, and a non-trivial quantale $\mathcal{V}$ (see III.2), recall (see III.2) that $(S,\mathcal{V})$-$\text{UGph}$ is the category whose object are pairs $(X,r)$ with $r : SX \to X$ a unitary $\mathcal{V}$-relation, so

\[ e^\emptyset_X \cdot \hat{S}r \cdot m^\emptyset_X \leq r \quad \text{and} \quad r \cdot \hat{S}1_X \leq r , \]

and whose morphisms $f : (X,r) \to (Y,s)$ are maps $f : X \to Y$ satisfying

\[ f \cdot r \leq s \cdot Sf . \]

In addition, given a functor $T : \text{Set} \to \text{Set}$ that lifts tacitly along the forgetful functor $\text{Ord} \to \text{Set}$, we can consider the lax comma category $(1_{\text{Set}} \downarrow T)_{\leq}$ whose objects are pairs $(X,\nu)$ with a map $\nu : X \to TX$, and whose morphisms are maps $f : X \to Y$ with $Tf \cdot \nu \leq \mu \cdot f$:

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \quad \nu \\
TX \xrightarrow{Tf} TY
\end{array}
\]

We now present conditions which will give us an isomorphism between the two categories $(S,2)$-$\text{UGph}$ and $(1_{\text{Set}} \downarrow T)_{\leq}$ that restricts to an isomorphism

\[(S,2)$-$\text{Cat} \cong T$-$\text{Mon} . \]

Hypotheses. We consider a monad morphism $\alpha : S \to T$, where $T = (T,m,e)$ is a monad power-enriched by $\tau : \mathcal{P} \to T$ and equipped with its Kleisli extension $\hat{T}$, and $S = (S,n,d)$ is a monad equipped with the initial extension $\hat{S}$ induced by $\alpha$. The sets $SX$ and $TX$ are equipped with the orders $\mathcal{P}$ induced by the respective lax extensions, and hom-sets of $\text{Set}$ are ordered pointwise.

Following [1.5], we define for all sets $X$ an adjunction

\[ \text{Set}(X,TX) \xrightarrow{\text{conv}} \mathcal{R}(SX,X) , \]

\[ \text{Set}(X,TX) \xrightarrow{\text{nbhd}} \]
by exploiting the \( \text{Ord}\)-isomorphism \( \text{Rel}(SX, X) \cong \text{Set}(X, PSX) \). More concretely, we set
\[
\text{conv} : \text{Set}(X, TX) \to \text{Set}(X, PSX), \qquad \nu \mapsto \alpha_X^\downarrow \cdot \nu,
\]
\[
\text{nbhd} : \text{Set}(X, PSX) \to \text{Set}(X, TX), \qquad r^\flat \mapsto \alpha_X^\uparrow \cdot r^\flat.
\]
The adjunction \( \text{nbhd} \dashv \text{conv} \) follows from \( \alpha_X^\uparrow \dashv \alpha_X^\downarrow \); to verify this, one can use the pointwise notation of \( \text{conv} \) and \( \text{nbhd} \):
\[
\chi \cdot \text{conv}(\nu) \cdot y \iff \alpha_X(\chi) \leq \nu(y) \quad \text{and} \quad \text{nbhd}(r)(y) = \bigvee_{TX} \alpha_X(r^\flat(y)),
\]
for all relations \( r : SX \to X \), and maps \( \nu : X \to TX \). Naturally, \( \text{conv} \) and \( \text{nbhd} \) restrict to mutually inverse isomorphisms between the sets of fixpoints of \( (\text{conv} \cdot \text{nbhd}) \) and of \( (\text{nbhd} \cdot \text{conv}) \).

The fixpoints of \( (\text{nbhd} \cdot \text{conv}) \) are exactly the maps \( \nu : X \to TX \) such that:
\[
\forall y \in X \exists A \subseteq SX (\nu(y) = \bigvee \alpha_X(A)).
\]
Hence, \( (\text{nbhd} \cdot \text{conv}) = 1_{\text{Set}(X, TX)} \) precisely when \( \alpha \) is sup-dense. In turn, a relation \( r : SX \to X \) is a fixpoint of \( (\text{conv} \cdot \text{nbhd}) \) precisely when
\[
\forall y \in X (\alpha_X(\chi) \leq \bigvee \alpha_X(r^\flat(y)) \implies \chi \cdot r \cdot y).
\]

One obtains the following generalizations of Proposition 1.5.1, Proposition 1.5.2, and Theorem 1.5.3.

2.3.1 Lemma. For the adjunction \( \text{nbhd} \dashv \text{conv} : \text{Set}(X, TX) \to \text{Rel}(SX, X) \) defined above, the following hold:

1. \( \text{Fix}(\text{nbhd} \cdot \text{conv}) = \text{Set}(X, TX) \) if and only if \( \alpha \) is sup-dense;
2. a relation \( r : SX \to X \) is a fixpoint of \( (\text{conv} \cdot \text{nbhd}) \) if and only if it is unitary and \( \alpha \) is interpolating for \( r \).

Proof. The first point is immediate from the previous discussion.

For a unitary relation \( r : SX \to X \), we obtain as in the proof of Proposition 1.5.1
\[
r^\flat = (\downarrow_{SX} \cdot n_X)^p \cdot (\hat{S}r)^b \cdot d_X.
\]

If moreover \( \alpha \) is interpolating for \( r \), then
\[
\alpha_X^\downarrow \cdot \alpha_X^\uparrow \cdot r^\flat \leq (\downarrow_{SX} \cdot n_X)^p \cdot (\hat{S}r)^b \cdot d_X = r^\flat,
\]
and \( r \) is indeed a fixpoint of \( (\text{conv} \cdot \text{nbhd}) \).

Conversely, if \( r : SX \to X \) is a fixpoint of \( (\text{conv} \cdot \text{nbhd}) \), it is of the form \( \text{conv}(\nu) \) for a map \( \nu : X \to TX \), so \( r \cdot \hat{S}1_X \leq r \), and \( r \) is right unitary. Moreover, if \( X \cdot \hat{S}r \cdot d_X(y) \) holds, we can apply \( m_X \cdot T\alpha_X \) to each side of the inequality \( \alpha_{SX}(x) \leq r^\tau \cdot e_X(y) \) to conclude that \( n_X(x) \cdot r \cdot y \) by the fixpoint condition, that is, \( r \) is left unitary. As \( r \) is unitary, (2.3.1) holds and implies that \( \alpha \) is interpolating for \( r \). \( \square \)
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2.3.2 Proposition. The adjoint maps \( \text{nbhd} \) and \( \text{conv} \) defined above satisfy

\[
\text{nbhd}(s \circ r) \leq \text{nbhd}(r) \circ \text{nbhd}(s) , \quad \text{conv}(\mu) \circ \text{conv}(\nu) \leq \text{conv}(\nu \circ \mu)
\]

\[
\text{nbhd}(1^X_X) = e_X , \quad \text{conv}(e_X) = 1^X_X
\]

for all relations \( r, s : SX \leftrightarrow X \), and maps \( \mu, \nu : X \rightarrow TX \) (where \( 1^X_X = d^X_X \cdot \hat{S}1_X \)).

Moreover, if \( \alpha \) is sup-dense, then \( \text{nbhd}(s \circ r) = \text{nbhd}(r) \circ \text{nbhd}(s) \) for all relations \( r, s : TX \leftrightarrow X \). If in addition \( \alpha \) is interpolating, then \( \text{conv}(\mu) \circ \text{conv}(\nu) = \text{conv}(\nu \circ \mu) \) also holds.

Proof. The displayed equalities follow from the fact that

\[
\chi (1^X_X) y \iff \chi (d^\circ_X \cdot \alpha^\circ_X \cdot \bar{T}1_X \cdot \alpha_X) y \iff \alpha_X(\chi) \leq e_X(y)
\]

for all \( \chi \in SX \) and \( y \in X \).

To show that \( \text{nbhd}(s \circ r) \leq \text{nbhd}(r) \circ \text{nbhd}(s) \), we first note that

\[
\alpha^Y_X \cdot Pn_X = (\alpha^X_X \cdot 1^S_{SX})^T \cdot \tau_{SSX} = (\alpha^X_X \cdot \alpha_{SX})^T \cdot \tau_{SSX} = \alpha^X_X \cdot \alpha^Y_{SX} ;
\]

by composing these equalities with \( \alpha^\downarrow_{SX} \cdot r^\tau \cdot \alpha_X \) on the right, we obtain

\[
\alpha^Y_X \cdot Pn_X \cdot \alpha^\downarrow_{SX} \cdot r^\tau \cdot \alpha_X \leq \alpha^\tau_X \cdot r^\tau \cdot \alpha_Y . \tag{2.3.ii}
\]

We can now proceed as in Proposition 1.5.2

\[
\text{nbhd}(r) \circ \text{nbhd}(s) = (\alpha^Y_X \cdot r^b)^T \cdot \alpha^Y_X \cdot s^b
\]

\[
= \alpha^\tau_X \cdot (\tau_{SX} \cdot r^b)^T \cdot \alpha^\tau_X \cdot \tau_{SX} \cdot s^b \quad ((g^T \cdot f)^T = g^T \cdot f^T)
\]

\[
= (\alpha^\tau_X \cdot r^\tau \cdot \alpha_X)^T \cdot \tau_{SX} \cdot s^b \quad (g^T \cdot f^T = (g^\tau \cdot f^T))
\]

\[
\geq (\alpha^\tau_X \cdot Pn_X \cdot (\hat{S}r)^b)^T \cdot \tau_{SX} \cdot s^b \quad \text{by (2.3.ii)}
\]

\[
= \alpha^\tau_X \cdot (\tau_{SX} \cdot Pn_X \cdot (\hat{S}r)^b)^T \cdot \tau_{SX} \cdot s^b \quad ((g^\tau \cdot f)^T = g^\tau \cdot f^T)
\]

\[
= \alpha^\tau_X \cdot \tau_{SX} \cdot (Pn_X \cdot (\hat{S}r)^b)^P \cdot s^b \quad (\tau \text{ natural transformation})
\]

\[
= \alpha^\tau_X \cdot (((n^0_\chi^0)^b)^P \cdot (\hat{S}r)^b)^b \cdot s^b \quad (Pn_X = ((n^0_\chi^0)^b)^P)
\]

\[
= \alpha^\tau_X \cdot (s \cdot \hat{S}r \cdot n^0_\chi)^b \quad \text{(Rel = Set}_F\text{)}
\]

\[
= \text{nbhd}(s \circ r) .
\]

The inequality for \( \text{conv} \) follows from the adjunction \( \text{nbhd} \dashv \text{conv} \).

If \( \alpha \) is sup-dense, then the inequality in (2.3.ii) becomes an equality, so that \( \text{nbhd}(r) \circ \text{nbhd}(s) = \text{nbhd}(s \circ r) \). For maps \( \mu, \nu : X \rightarrow TX \), the \((S, 2)\)-relations \( \text{conv}(\mu) \) and \( \text{conv}(\nu) \) are unitary, and therefore so is \( \text{conv}(\mu) \circ \text{conv}(\nu) \) (Exercise III.1.N). The claim for \( \text{conv} \) then follows from \( \text{nbhd}(r) \circ \text{nbhd}(s) = \text{nbhd}(s \circ r) \) and the fact that \( \text{nbhd} \) and \( \text{conv} \) form a bijection between \( \text{Set}(X, TX) \) and the set of all unitary \((S, 2)\)-relations \( r : X \leftrightarrow X \) (see Lemma 2.3.1).
2.3.3 Theorem. Let \((\mathbb{T}, \tau)\) be a power-enriched monad together with a monad morphism \(\alpha : \mathbb{S} \to \mathbb{T}\), and suppose that \(\mathbb{T}\) is equipped with its Kleisli extension \(\tilde{T}\), and \(\mathbb{S}\) with the initial extension of \(\tilde{T}\) induced by \(\alpha\). If \(\alpha\) is sup-dense, then there is a full reflective embedding 
\[\mathbb{T}\text{-Mon} \hookrightarrow (\mathbb{S}, 2)\text{-Cat}\]
that commutes with the underlying-set functors and restricts to a full reflective embedding
\[1\text{-Set} \downarrow \tilde{T} \hookrightarrow (\mathbb{S}, 2)\text{-UGph}\]
If \(\alpha\) is also interpolating, then this functor is an isomorphism.

Proof. For every map \(\nu : X \to TX\), the relation \(\text{conv}(\nu)\) is a fixpoint of \((\text{conv} \circ \text{nbhd})\), and is therefore unitary by Lemma 2.3.1. Similarly, a unitary relation \(r : SX \to X\) yields a map \(\text{nbhd}(r) : X \to TX\). Thus, we can consider the functors
\[C : (1\text{-Set} \downarrow T) \leq \to (\mathbb{S}, 2)\text{-UGph}\]
\[N : (\mathbb{S}, 2)\text{-UGph} \to (1\text{-Set} \downarrow T) \leq\]
defined on objects by \(C(X, \nu) = (X, \text{conv}(\nu))\) and \(N(X, r) = (X, \text{nbhd}(r))\), and leaving maps untouched (the fact that \(C\) and \(N\) send morphisms to morphisms follows easily from the definitions; see also Exercise [L.G]). The adjunction \(\text{nbhd} \dashv \text{conv}\) yields an adjunction \(N \dashv C\). Lemma 2.3.1 shows that if \(\alpha\) is sup-dense, then \(C\) is a full reflective embedding, and Proposition 2.3.2 yields that \(C\) restricts to a functor \(\mathbb{T}\text{-Mon} \hookrightarrow (\mathbb{S}, 2)\text{-Cat}\). Finally, if \(\alpha\) is also interpolating, then \(C\) is an isomorphism by Lemma 2.3.1 again.

Theorem 1.5.3 now appears as a direct consequence of this more general result, since \(\alpha = 1_\mathbb{T}\) is both sup-dense and interpolating. Moreover, the category of Kleisli monoids provides a link between presentations of lax algebras.

2.3.4 Proposition. If \(\alpha : \mathbb{S} \to \mathbb{T}\) is sup-dense as in Theorem 2.3.3, then the algebraic functor
\[A_\alpha : (\mathbb{T}, 2)\text{-Cat} \to (\mathbb{S}, 2)\text{-Cat}\]
of III.3.4 is a full reflective embedding. If \(\alpha\) is also interpolating, then \(A_\alpha\) is an isomorphism.

Proof. The isomorphism \((\mathbb{T}, 2)\text{-Cat} \cong \mathbb{T}\text{-Mon}\) of Theorem 1.5.3 composed with the full reflective embedding \(\mathbb{T}\text{-Mon} \hookrightarrow (\mathbb{S}, 2)\text{-Cat}\) of Theorem 2.3.3 sends \((X, a : TX \to X)\) to \((X, a \cdot \alpha_X : SX \to X)\). Hence, this composition is precisely the algebraic functor \(A_\alpha\). When \(\alpha\) is interpolating, \(A_\alpha\) is an isomorphism by Theorem 2.3.3.

2.3.5 Examples.

(1) Depending on whether a relation \(r\) on a set \(X\) is seen as a map
\[r : X \times X \to 2, \quad r : X \to PX, \quad \text{or} \quad r : PX \times X \to 2,\]
the category \(\text{Ord}\) of ordered sets is described respectively as any of the three categories
\[2\text{-Cat} \cong \mathbb{P}\text{-Mon} \cong (\mathbb{P}, 2)\text{-Cat}.\]
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(2) Whether ultrafilter convergence, neighborhood systems, or filter convergence is chosen as defining structure, the category \( \text{Top} \) of topological spaces appears as

\[
(\beta, 2)-\text{Cat} \cong \text{F-Mon} \cong (\mathbb{F}, 2)-\text{Cat} .
\]

2.4 Strata extensions. To pass from \((\mathbb{I}, 2)\)-categories to \((\mathbb{I}, \mathcal{V})\)-categories, one views a \(\mathcal{V}\)-relation \( r : X \rightrightarrows Y \) as a family of 2-relations \( (r_v : X \rightrightarrows Y)_{v \in \mathcal{V}} \) indexed by \( \mathcal{V} \), via

\[
x r_v y \iff v \leq r(x, y)
\]

for all \( x \in X, y \in Y \), and \( v \in \mathcal{V} \). The relation \( r_v \) is referred to as the \( v\)-stratum of \( r \). Conversely, given a family \( (r_v : X \rightrightarrows Y)_{v \in \mathcal{V}} \) of relations, one can define a \(\mathcal{V}\)-relation \( r : X \rightrightarrows Y \) by setting

\[
r(x, y) := \bigvee \{v \in \mathcal{V} \mid x r_v y\} .
\]

Starting with a \( \mathcal{V}\)-relation \( r : X \rightrightarrows Y \), the \( \mathcal{V}\)-relation obtained from the family \( (r_v : X \rightrightarrows Y)_{v \in \mathcal{V}} \) is \( r \) again. The family \( (r_v)_{v \in \mathcal{V}} \) obtained from \( r \) is not arbitrary, as one has for example if \( u, v \in \mathcal{V} \)

\[
u \leq v \implies r_v \leq r_u .
\]

In fact, a \( \mathcal{V}\)-indexed family comes from a \( \mathcal{V}\)-relation precisely when the family can be identified with an inf-map \( r_{(-)} : \mathcal{V}^{\text{op}} \to \text{Rel}(X, Y) \).

2.4.1 Lemma. For any sets \( X \) and \( Y \), if the set \( \text{Inf}(\mathcal{V}^{\text{op}}, \text{Rel}(X, Y)) \) is ordered pointwise, then the preceding correspondence between \(\mathcal{V}\)-relations and \( \mathcal{V}\)-indexed families of relations describes an \( \text{Ord}\)-isomorphism

\[
\mathcal{V}\text{-Rel}(X, Y) \cong \text{Inf}(\mathcal{V}^{\text{op}}, \text{Rel}(X, Y)) .
\]

Proof. If \( (r_v)_{v \in \mathcal{V}} \) is obtained from a \( \mathcal{V}\)-relation \( r \), then one has

\[
x (\bigwedge_{v \in A} r_v) y \iff \forall v \in A \ (x r_v y) \iff \forall v \in A \ (v \leq r(x, y)) \iff \bigvee A \leq r(x, y)
\]

for all \( A \subseteq \mathcal{V} \), \( x \in X, y \in Y \), so \( r_{(-)} \) is an indeed an inf-map \( r_{(-)} : \mathcal{V}^{\text{op}} \to \text{Rel}(X, Y) \). A straightforward verification shows that if \( r \) was originally the image of a family \( (r_v')_{v \in \mathcal{V}} \) then \( (r_v)_{v \in \mathcal{V}} = (r_v')_{v \in \mathcal{V}} \), and the discussion preceding the statement allows us to conclude. \( \square \)

Suppose that a lax extension \( \hat{T} : \text{Rel} \to \text{Rel} \) of a functor \( T : \text{Set} \to \text{Set} \) is given. For every \( \mathcal{V}\)-relation \( r : X \rightrightarrows Y \), we set

\[
\hat{T}\nu r(x, y) := \bigvee \{v \in \mathcal{V} \mid x (\hat{T} r_v) y\} .
\]

The assignment

\[
\hat{T}_\nu : \mathcal{V}\text{-Rel} \to \mathcal{V}\text{-Rel} , \quad (r : X \rightrightarrows Y) \mapsto (\hat{T}_\nu r : TX \to TY) ,
\]

is called the strata extension of \( \hat{T} \) along \( \text{Rel} \to \mathcal{V}\text{-Rel} \).

2.4.2 Proposition. Let \( T \) be a \( \text{Set}\)-functor. If \( \hat{T} \) is a lax extension of \( T \) to \( \text{Rel} \), then the strata extension \( \hat{T}_\nu \) of \( \hat{T} \) is a lax extension of \( T \) to \( \mathcal{V}\text{-Rel} \).
These inequalities then yield $\hat{T} f \leq \hat{T} f \leq \hat{T} f$, $(T f)^\circ \leq \hat{T}(f^\circ) \leq \hat{T}(f^\circ)$

because both $f$ and $f^\circ$ take values in $\{\bot, k\}$. For $\mathcal{V}$-relations $r, r' : X \rightarrow Y$ with $r \leq r'$ one has $r_v \leq r'_v$, and then $\hat{T} r_v \leq \hat{T} r'_v$ for all $v \in \mathcal{V}$, which implies $\hat{T} \mathcal{V} r \leq \hat{T} \mathcal{V} r'$. To verify $\hat{T} \mathcal{V} s \cdot \hat{T} r \leq \hat{T} (s \cdot r)$ for all $\mathcal{V}$-relations $r : X \rightarrow Y$, and $s : Y \rightarrow Z$, consider $\chi \in TX$, $y \in TY$, and $z \in TZ$. If there are $u, v \in \mathcal{V}$ with $\chi \hat{T} r_u y$ and $y \hat{T} s_y z$, then $\chi \hat{T} (s_y \cdot r_u) z$

holds by the hypothesis on $\hat{T}$; but $s_y \cdot r_u \leq (r \cdot s)_{\mathcal{V} \mathcal{V} \mathcal{V}}$, so $\chi (\hat{T} (r \cdot s)_{\mathcal{V} \mathcal{V} \mathcal{V}}) z$

holds. Hence,

$$\hat{T} \mathcal{V} r(\chi, y) \otimes \hat{T} \mathcal{V} s(y, z) = \bigvee \{ u \in \mathcal{V} \mid \chi (\hat{T} r_u) y \} \otimes \bigvee \{ v \in \mathcal{V} \mid y (\hat{T} s_v) z \}$$

$$\leq \bigvee \{ u \otimes v \in \mathcal{V} \mid \chi (\hat{T} (r \cdot s)_{\mathcal{V} \mathcal{V} \mathcal{V}}) z \} \leq \hat{T} \mathcal{V} (r \cdot s)(\chi, z)$$

concludes the proof.

\[\Box\]

2.4.3 Proposition. For $\mathcal{V}$ completely distributive, the strata extension $\hat{T} \mathcal{V}$ of a lax extension $\hat{\nabla}$ of $\mathcal{T}$ to $\mathcal{V}$-$\mathcal{Rel}$ yields a lax extension of $\mathcal{T}$ to $\mathcal{V}$-$\mathcal{Rel}$.

\[\Box\]

2.4.4 Remarks.

(1) For a lax extension $\hat{T}$ of $\mathcal{T}$ to $\mathcal{Rel}$, one has for $\mathcal{V} = 2$:

$$\hat{T}_2 r(\chi, y) = \top \iff \chi \ r \ y$$

for all relations $r : X \rightarrow Y$, and $\chi \in TX$, $y \in TY$, that is, the strata extension of $\hat{T}$

to $2$-$\mathcal{Rel} = \mathcal{Rel}$ returns $\hat{T}$:

$$\hat{T}_2 = \hat{T}.$$  

(2) The strata extension to $\mathcal{V}$-$\mathcal{Rel}$ of the identity functor $1_{\mathcal{Rel}}$, which is a lax extension of $1_{\mathcal{Set}}$ to $\mathcal{Rel}$ (see Example [III 1.5.2]), returns the identity functor $1_{\mathcal{V}-\mathcal{Rel}}$.  


The Barr extension of the ultrafilter monad $\beta$ to $P_+_\text{Rel}$ given in III.2.4 is simply the strata extension of the Barr extension $\bar{\beta}$ to $\text{Rel}$ along $\text{Rel} \to P_+_\text{Rel}$. More generally, for a completely distributive $\mathcal{V}$, the Barr extension $\bar{\beta}$ of $\beta$ to $\mathcal{V}_\text{Rel}$ is given by the strata extension to $\mathcal{V}_\text{Rel}$ of its Barr extension to $\mathcal{V}_\text{Rel}$:

$$\bar{\beta} r(\chi, y) := \beta_\mathcal{V} r(\chi, y) = \bigvee \{ v \in \mathcal{V} \mid \chi (\beta r_v) y \},$$

and one obtains the corresponding equivalent expression

$$\bar{\beta} r(\chi, y) = \bigwedge_{A \in \chi, B \in y} \bigvee_{x \in A, y \in B} r(x, y)$$

for all $\mathcal{V}$-relations $r : X \to Y$, and $\chi \in \beta X$, $y \in \beta Y$. Proposition 2.4.3 then offers a generalization of Proposition III.2.4.3), as follows.

2.4.5 Corollary. For $\mathcal{V}$ completely distributive, the Barr extension $\bar{\beta} = (\bar{\beta}, m, e)$ is a flat lax extension to $\mathcal{V}_\text{Rel}$ of the ultrafilter monad $\beta = (\beta, m, e)$.

Proof. By Proposition 2.4.3, we only need to verify that the Barr extension is flat. For $v \in \mathcal{V}$ with $\perp < v$, one has

$$(1_x)_v (x, y) = \begin{cases} 1_X (x, y) & \text{if } v \leq k \\ \perp & \text{otherwise,} \end{cases}$$

for all $x, y \in X$. Moreover, if $r = \perp : X \to X$ is the constant relation to $\perp$, that is, the empty relation, then $\bar{\beta} r = \perp_X$. By definition, one therefore has $\bar{\beta}_\mathcal{V} 1_X (\chi, y) = \beta 1_X (\chi, y)$ for all $\chi, y \in \beta X$, so $\bar{\beta}_\mathcal{V}$ is flat.

For a power-enriched monad $\mathbb{T}$ and completely distributive $\mathcal{V}$, the Kleisli extension $\tilde{T}$ of $\mathbb{T}$ to $\mathcal{V}_\text{Rel}$ is given by the strata extension to $\mathcal{V}_\text{Rel}$ of its Kleisli extension to $\text{Rel}$:

$$\tilde{T} r(\chi, y) := \tilde{T}_\mathcal{V} r(\chi, y) = \bigvee \{ v \in \mathcal{V} \mid \chi \leq (r_v) \tilde{T} y \},$$

for all $\mathcal{V}$-relations $r : X \to Y$ and $\chi \in TX$, $y \in TY$. As in the case of the Barr extension above, there are alternate descriptions of the lax extensions given in Examples 1.4.2 when $\mathcal{V}$ is completely distributive; see Exercise 2.H.

The strata extension has a very good behavior with respect to the order induced on the sets $TX$ and to initial extensions.

2.4.6 Proposition. Let $\hat{T}$ be a lax extension to $\text{Rel}$ of a Set-functor $T$. The order induced on $TX$ by the strata extension $\hat{T}_\mathcal{V}$ of $\hat{T}$ is the order induced by $\hat{T}$.

Proof. Suppose first that $\chi, y \in TX$ are such that $\chi \hat{T} 1_X y$. Since $1_X = (1_X)_k$ for $1_X : X \to X$ seen as a $\mathcal{V}$-relation, we have $k \leq \hat{T}_\mathcal{V} 1_X (\chi, y)$ by definition of the strata extension to $\mathcal{V}_\text{Rel}$. If on the other hand, one has $k \leq \hat{T}_\mathcal{V} 1_X (\chi, y)$, because $\mathcal{V}$ is non-trivial (by convention, see 2.3) there exists $v \in \mathcal{V}$ such that $\perp < v$ and $\chi \hat{T} (1_X)_v x y$; as $(1_X)_v = (1_X)_k = 1_X$ because $1_X$ takes values in $\{ \perp, k \}$, we can conclude that $\chi \hat{T} 1_X y$. □
2.4.7 Proposition. If \( \alpha : (S, \hat{S}) \to (T, \hat{T}) \) is a morphism of lax extensions, then \( \alpha : (S, \hat{S}_V) \to (T, \hat{T}_V) \) is one too. Moreover, if \( \hat{S} \) is the initial extension of \( S \) to \( \mathcal{R}el \) induced by \( \alpha : S \to T \), then \( \hat{S}_V \) is the initial extension of \( S \) to \( \mathcal{V}\mathcal{R}el \) induced by \( \alpha \) (where \( T \) is equipped with its respective lax extensions \( \hat{T} \) and \( \hat{T}_V \)).

Proof. By definition of a morphism of lax extensions and of the strata extension, 
\[
\hat{S}_V r(\chi, y) = \bigvee \{v \in \mathcal{V} \mid \chi (\hat{S}_V r_u) y \leq \chi (\hat{T}_V r_u) y \} = \hat{T}_V r(\alpha_X(\chi), \alpha_Y(y))
\]
for all \( \chi \in SX, \ y \in SY \) and \( \mathcal{V}\)-relation \( r : X \to Y \). Hence, \( \alpha \) is a morphism between the respective strata extensions and if \( \hat{S} \) is initial, then \( \hat{S}_V \) is initial. \( \Box \)

2.5 \((\mathcal{S}, \mathcal{V})\)-categories as Kleisli towers. The strata extension of a lax extension \([2.4]\) constructs a lax extension \( \hat{T}_V \) to \( \mathcal{V}\mathcal{R}el \) from a lax extension \( \hat{T} \) to \( \mathcal{R}el \) of a monad \( \mathcal{T} \) on \( \mathbb{S}et \). Such a lax extension \( \hat{T}_V \) then determines a category of \((\mathcal{T}, \mathcal{V})\)-categories. In contrast, the tower construction that we describe here determines \((\mathcal{T}, \mathcal{V})\)-categories directly from a category of \((\mathcal{T}, 2)\)-categories by exploiting the isomorphism
\[
\mathcal{V}\mathcal{R}el(X, Y) \cong \text{Inf}(\mathcal{V}^{\mathcal{O}p}, \mathcal{R}el(X, Y))
\]

of Lemma \([2.4.1]\). Under this isomorphism, reflexive and transitive \((\mathcal{T}, \mathcal{V})\)-relations \( a : X \to X \) correspond to certain inf-maps \( \mathcal{V}^{\mathcal{O}p} \to \mathcal{R}el(TX, X) \) whose characterization depends only on the original lax extension \( \hat{T} \) to \( \mathcal{R}el \) (Proposition \([2.5.2]\) below). In fact, these inf-maps corestrict to fixpoints of \( \text{nbhd} \cdot \text{conv} \) so that Proposition \([1.5.1]\) allows us to consider instead inf-maps \( \mathcal{V}^{\mathcal{O}p} \to \mathbb{S}et(X, TX) \), in effect relating \((\mathcal{T}, \mathcal{V})\)-category structures with \( \mathcal{T}\)-monoid structures. In this subsection, we pursue this idea in the presence of a sup-dense and interpolating monad morphism \( \alpha : \mathcal{S} \to \mathcal{T} \), considering \( \mathcal{R}el(SX, X) \) in lieu of \( \mathcal{R}el(TX, X) \) and using Lemma \([2.3.1]\) instead of Proposition \([1.5.1]\). We then obtain Theorem \([2.5.2]\) as a generalization of Theorem \([2.3.3]\).

The following Lemma will be used throughout without explicit mention in our proofs.

2.5.1 Lemma. For all maps \( f : X \to Y, \ g : Y \to Z, \mathcal{V}\)-relations \( r : X \to Y, \ s : Y \to Z \) and \( u, v \in \mathcal{V} \),
\[
s_v \cdot f = (s \cdot f)_v, \quad g^o \cdot r_v = (g^o \cdot r)_v \quad \text{and} \quad s_u \cdot r_v \leq (s \cdot r)_{u \otimes u}.
\]

Proof. The expressions follow directly from the definition of the strata of a relation (see \([2.4]\)). \( \Box \)

2.5.2 Proposition. Let \( \hat{T} \) be a lax extension of \( \mathcal{T} \) to \( \mathcal{R}el \) and \( \mathcal{V} \) a quantale. Via the isomorphism \( \mathcal{V}\mathcal{R}el(TX, X) \cong \text{Inf}(\mathcal{V}^{\mathcal{O}p}, \mathcal{R}el(TX, X)) \) of Lemma \([2.4.1]\), there is a one-to-one correspondence between transitive and reflexive \((\mathcal{T}, \mathcal{V})\)-relations \( r : X \to X \) and inf-maps \( r_{(-)} : \mathcal{V}^{\mathcal{O}p} \to (\mathcal{T}, \mathcal{V})\mathcal{U}\mathcal{R}el(X, X) \) that satisfy
\[
r_v \circ r_u \leq r_{u \otimes u} \quad \text{and} \quad e^o_X \leq r_k
\]
for all \( u, v \in \mathcal{V} \).
Proof. For a transitive \( V \)-relation \( r : TX \to X \), we show \( r_v \cdot (\hat{T}r)_u \leq r_{u \odot v} \cdot m_X \) for all \( u, v \in V \) since \( \hat{T}r_u \leq (\hat{T}r)_u \) by definition, \( r_u \circ r_v \leq r_{v \odot u} \) will follow. If \( z \in X \) and \( X \in TTX \) are such that \( X (r_v \cdot (\hat{T}r)_u) z \), then there exists \( y \in TX \) with \( v \leq r(y, z) \) and \( u \leq \hat{T}r_v \). Therefore,

\[
u = \bigvee_{y \in TX} \hat{T}r(x, y) \odot r(y, z) = r \cdot \hat{T}r(x, z) \leq r \cdot m_X(x, z),
\]

and \( (r \cdot m_X)_{u \odot v} = r_{u \odot v} \cdot m_X \) yields the required inequality. If \( r : TX \to X \) is a reflexive \( V \)-relation, then \( e^0_X \leq r_k \) because \( e^0_X \) takes values in \( \{\bot, k\} \).

Conversely, assume that \( r_v \circ r_u \leq r_{u \odot v} \) holds for all \( u, v \in V \). One has to show that \( Tvr(x, y) \odot r(y, z) \leq r(m_X(x), z) \) for all \( z \in X \), \( y \in TX \), \( X \in TTX \). Choose elements \( z, y, X \) and set \( v := r(y, z), A := \{ u \in V \mid x (\hat{T}r_u \ y) \} \), so that \( Tvr(x, y) \odot r(y, z) = \bigvee_{u \in A} u \odot v \). Since by hypothesis \( r_v \cdot \hat{T}r_u \leq r_{u \odot v} \cdot m_X \), we have \( u \odot v \leq r(m_X(x), z) \) for all \( u \in A \). Hence, \( r \) is transitive. Reflexivity of \( r \) follows from \( e^0_X \leq r_k \leq r \). \( \square \)

Let now \((\Upsilon, \tau)\) be a power-enriched monad, so that all hom-sets \( \text{Set}_\Upsilon(X, X) \) are complete lattices when equipped with the pointwise order (see Proposition 1.2.1). To extend Theorem 2.3.3 to \( V \)-relations, we consider the category

\( (\Upsilon, \nu)-\text{Mon} \)

whose objects are \textit{Kleisli} \((\Upsilon, \nu)\)-\textit{towers}, that is, pairs \((X, \nu(-))\) with \( X \) a set and \( \nu(-) : \nu^{\text{op}} \to \text{Set}_\Upsilon(X, X) \) an inf-map satisfying

\[
u^u \circ \nu^v \leq \nu^{u \odot v} \quad \text{and} \quad e_X \leq \nu^k
\]

for all \( u, v \in V \). A morphism of \textit{Kleisli} \((\Upsilon, \nu)\)-\textit{towers} \( f : (X, \nu(-)) \to (Y, \mu(-)) \) is a map \( f : X \to Y \) such that

\[ T f \cdot \nu^v \leq \mu^v \cdot f \]

for all \( v \in V \).

#### 2.5.3 Theorem.

\( (\Upsilon, \tau) \) be a power-enriched monad, \( \alpha : S \to \Upsilon \) a sup-dense and interpolating monad morphism, and suppose that \( V \) is completely distributive. With respect to the initial extension \( \hat{S}_V \) of \( S \) induced by \( \alpha \) (where \( \Upsilon \) is equipped with its \textit{Kleisli} extension \( \hat{\Upsilon}_V \)), there is an isomorphism

\[
(\hat{S}_V, \nu)-\text{Cat} \cong (\Upsilon, \nu)-\text{Mon}
\]

that commutes with the underlying-set functors.

\( \text{Proof.} \) By Lemma 2.3.1 a \textit{Kleisli} \((\Upsilon, \nu)\)-\textit{tower} \((X, \nu(-))\) corresponds to a pair \((X, r(-))\), with \( r(-) : \nu^{\text{op}} \to (S, 2)-\text{URel}(X, X) \) an inf-map (set \( r_v := \text{conv}(\nu^v) \) so that \( \nu^v = \text{nbhd}(r_v) \)). By Proposition 2.3.2 the conditions for \( \nu(-) \) to be a \textit{Kleisli} \((\Upsilon, \nu)\) tower translate as

\[
r_v \circ r_u \leq r_{u \odot v} \quad \text{and} \quad e_X^0 \leq r_k
\]
for all \( u, v \in V \). Proposition 2.5.2 then yields a bijective correspondence between Kleisli \((T, V)\)-towers \((X, \nu^\leftarrow)\) and \((T, V)\)-categories \((X, r)\).

Consider now a morphism \( f : (X, \nu^\leftarrow) \to (Y, \mu^\leftarrow) \) of Kleisli \((T, V)\)-towers. By Theorem 2.3.3, \( f \cdot \text{conv}(\nu^v) \leq \text{conv}(\mu^v) \cdot Tf \) for all \( v \in V \); hence, by defining \( r \) and \( s \) as the \((S, V)\)-relations corresponding respectively to the families \( (\text{conv}(\nu^v) : S X \to X)_{v \in V} \) and \( (\text{conv}(\mu^v) : S Y \to Y)_{v \in V} \) via Lemma 2.4.1, one obtains that \( f : (X, r) \to (Y, s) \) is an \((S, V)\)-functor. Conversely, if \( f : (X, r) \to (Y, s) \) is an \((S, V)\)-functor, then

\[
f \cdot r_v = (f \cdot r)_v \leq (s \cdot Tf)_v = s_v \cdot Tf
\]

for all \( v \in V \). This proves that the mentioned correspondence on objects is functorial. \(\square\)

2.5.4 Corollary. Let \((T, \tau)\) be a power-enriched monad and suppose that \(V\) is completely distributive. With respect to the Kleisli extension \(\tilde{T}V\), there is an isomorphism

\[
(T, V)\text{-Cat} \cong (T, V)\text{-Mon}.
\]

Proof. Apply the Theorem to \(\alpha = 1_T\) which is both sup-dense and interpolating. \(\square\)

2.5.5 Examples.

(1) The category \(\text{Met}\) of metric spaces (see Example III.1.3.1(2)) is isomorphic to any of the categories

\[
P_+\text{-Cat} \cong (P, P_+)\text{-Mon} \cong (P, P_+)\text{-Cat},
\]

(with \(P\) equipped with its Kleisli extension to \(P_+\text{-Rel}\)).

(2) The category \(\text{App}\) of approach spaces can equivalently be described as

\[
(\beta, P_+)\text{-Cat} \cong (F, P_+)\text{-Mon} \cong (F, P_+)\text{-Cat},
\]

(with \(\beta\) equipped with its Barr extension and \(F\) with its Kleisli extension to \(P_+\text{-Rel}\)). We use here the isomorphism \((\beta, P_+)\text{-Cat} \cong \text{App}\) of Theorem III.2.4.5 and therefore the Axiom of Choice, to prove these results. Nevertheless, the isomorphism \((F, P_+)\text{-Mon} \cong \text{App}\) can be established without the use of the Axiom of Choice, see Exercise 2.K.

(3) The category of “many-valued neighborhood spaces” is obtained as

\[
(\beta, [0, 1])\text{-Cat} \cong (F, [0, 1])\text{-Mon} \cong (F, [0, 1])\text{-Cat},
\]

where the frame \([0, 1]\) is considered a quantale (see II.1.10). Analogous results hold for \(F\) replaced by \(U\), giving rise to extensions of the category of interior (or closure) spaces.

(4) \((\beta, 2^2)\text{-Cat}\) is the category \(\text{BiTop}\) of bitopological spaces and bicontinuous maps (Exercise II.2.1). This category can also be described as \((F, 2^2)\text{-Mon}\), or as \((F, 2^2)\text{-Cat}\), avoiding the use of the Axiom of Choice.
2. LAX EXTENSIONS OF MONADS

(5) Tower extensions allow us to effectively describe \((\mathbb{T}, \mathcal{V})\)-categories for different lattices \(\mathcal{V}\). For instance, \((\mathbb{T}, \{0, 1, 2\})\)-\text{Cat}, where the frame \(\{0, 1, 2\}\) is considered a quantale, is isomorphic to the category whose objects are triples \((X, \mathcal{O}_0X, \mathcal{O}_1X)\), with topologies \(\mathcal{O}_0X \subseteq \mathcal{O}_1X\), and whose morphisms are maps which are continuous with respect to both topologies.

(6) For the quantale \(3 = (\bot, k, \top)\) (see Example II.1.10.1(2)), there are isomorphisms

\[
3\text{-Cat} \cong (\mathcal{P}, 3)\text{-Mon} \cong (\mathcal{P}, 3)\text{-Cat}.
\]

Objects of \(3\text{-Cat}\) are sets \(X\) with a map \(r(-) : 3 \to \text{Rel}(X, X)\) such that \(r_\bot\) satisfies \(x \perp r_\bot x\) for all \(x \in X\), \(r_k\) is a reflexive and transitive relation, and \(r_\top\) is a relation such that \(r_\top \leq r_k, r_\top \cdot r_k \leq r_\top,\) and \(r_k \cdot r_\top \leq r_\top\); morphisms of \(3\text{-Cat}\) are maps that preserve the relations \(r_k\) and \(r_\top\). Therefore, \(3\text{-Cat}\) is isomorphic to the category whose objects are ordered sets \((X, \leq_X)\) equipped with an auxiliary relation, that is, a module \(a : X \to X\) such that \(a \leq (\leq_X)\) (see II.1.4), and whose morphisms are monotone maps preserving the auxiliary relation.

**Exercises**

2.A The category of lax extensions to \(\mathcal{V}\)-\text{Rel}. The forgetful functor \(U : \mathcal{V}\text{-LXT} \to \text{Set}^{\text{Set}}\) (that sends a lax extension \(\hat{T} : \mathcal{V}\text{-Rel} \to \mathcal{V}\text{-Rel}\) to its underlying functor \(T : \text{Set} \to \text{Set}\)) is topological: if \((T_i, \hat{T}_i)_{i \in I}\) is a family of lax extensions to \(\mathcal{V}\text{-Rel}\), then every family \((\alpha_i : S \to T_i)_{i \in I}\) of natural transformations admits a \(U\)-initial lifting \((\alpha_i : (S, \hat{S}) \to (T_i, \hat{T}_i))_{i \in I}\) with

\[
\hat{S}r := \bigwedge_{i \in I} (\alpha_i)_Y \cdot \hat{T}_i \cdot (\alpha_i)_X,
\]

or equivalently,

\[
\hat{S}r(\chi, y) = \bigwedge_{i \in I} \hat{T}_r((\alpha_i)x(\chi), (\alpha_i)y(\chi))
\]

for all \(\mathcal{V}\text{-relations} : X \to Y, \chi \in TX,\) and \(y \in TY\). Moreover, if all \(\alpha_i\) are morphisms of monads \(\alpha_i : S \to T_i\), and all \(\hat{T}_i\) are lax extensions of \(T_i = (T_i, m_i, e_i)\), then \(\hat{S}\) is a lax extension of \(S = (S, n, d)\).

2.B The Zariski topology on a set of ultrafilters. Consider the set \(\beta X\) of all ultrafilters on a set \(X\), and the closure operation \(c : P\beta X \to P\beta X\) given by

\[
\chi \in c(A) \iff \chi \leq \bigvee A \iff \chi \supseteq \bigcap A
\]

for all \(\chi \in \beta X, A \subseteq \beta X\). This closure operation defines the Zariski topology on \(\beta X\) whose closed sets can be identified with the set of filters \(F\beta X\).

A relation \(r : \beta X \to Y\) is therefore a fixpoint of \((\text{conv} \cdot \text{nbhd}) : \text{Rel}(\beta X, X) \to \text{Rel}(\beta X, X)\) exactly when \(r^\flat(y)\) is closed in this topology on \(\beta X\) for all \(y \in X\).
2.C The neighborhood map as a monoid homomorphism. For a power-enriched monad \( T \) equipped with its Kleisli extension \( \hat{T} \) to \( \text{Rel} \), a monad morphism \( \alpha : S \to T \) inducing the initial extension on \( S \), and relations \( r, s : SX \to X \), the map

\[
\text{nbhd} : \text{Set}(X, PSX) \to \text{Set}(X, TX)
\]

satisfies \( \text{nbhd}(s \circ r) = \text{nbhd}(r) \circ \text{nbhd}(s) \) if and only if one of the following equivalent conditions hold:

(i) \( \alpha_X^T \cdot r^\tau \cdot \alpha_X = \alpha_X^Y \cdot Pn_X \cdot (\hat{S}r)^b \);

(ii) \( \alpha_X^T \cdot r^\tau \cdot \alpha_X \leq \alpha_X^Y \cdot Pn_X \cdot (\hat{S}r)^b \);

(iii) \( \alpha_X^T \cdot r^\tau \cdot \alpha_X(y) \leq \bigvee \{ \alpha_X \cdot n_X(X) \mid X \in SSX : \alpha_{SX}(X) \leq r^\tau \cdot \alpha_X(y) \} \) for all \( y \in SX \).

Hint. For (ii) \( \Rightarrow \) (i), use that \( \alpha_X^T \cdot r^\tau \cdot \alpha_X = \alpha_X^Y \cdot Pn_X \cdot (\hat{S}r)^b \).

2.D Full coreflective subcategories of \( \mathcal{T}-\text{Mon} \). Consider a monad morphism \( \alpha : S \to T \) into a power-enriched monad \( (T, \tau) \) equipped with its Kleisli extension \( \hat{T} \), where \( S \) is equipped with its initial extension \( \hat{S} \) induced by \( \alpha \). If \( \alpha \) is interpolating and satisfies

\[
\alpha_X^T \cdot r^\tau \cdot \alpha_X \leq \alpha_X^Y \cdot Pn_X \cdot (\hat{S}r)^b
\]

(see Exercise 2.C) for all for all unitary relations \( r : SX \to X \), then there is a full coreflective embedding \( (S,2)\text{-UGph} \hookrightarrow (1\text{Set}\downarrow T)_{\leq} \) that commutes with the underlying-set functors and restricts to

\[
(S,2)\text{-Cat} \hookrightarrow \mathcal{T}-\text{Mon}.
\]

By composing this embedding with the isomorphism \( \mathcal{T}-\text{Mon} \cong (T,2)\text{-Cat} \), one obtains a coreflective embedding \( (S,2)\text{-Cat} \to (T,2)\text{-Cat} \) that sends \( (X, r : SX \to X) \) to \( (X, \hat{r} : TX \to X) \), where

\[
\hat{r}(f, y) = \bigwedge \{ r(x, y) \mid x \in SX : \alpha_X(x) \leq f \},
\]

for all \( f \in TX, \ y \in X \). In particular, the principal filter monad morphisms \( \mathcal{P} \to \mathcal{F} \to \mathcal{U} \) yield full coreflective embeddings

\[
\text{Ord} \subseteq \text{Top} \subseteq \text{Cls}.
\]

2.E Ultracliques and closure spaces. With the set of cliques \( CX \) of Exercise 1.F ordered by reverse inclusion, an ultraclique is a minimal proper element of \( CX \). Alternatively, an ultraclique \( \chi \) is an up-set in \( PX \) such that

\[
A \in \chi \iff A^c \notin \chi
\]

for all \( A \in PX \). Although the existence of “sufficiently many” (see below) ultracliques requires the Axiom of Choice, these structures appear to be less elusive than ultrafilters. For
example, if the set $X$ has at least three distinct elements $x, y, z \in X$, then a non-principal ultraclique is given by
\[ \uparrow_{PX} \{ \{x, y\}, \{y, z\}, \{z, x\} \} . \]
The clique monad restricts to ultracliques to form the *ultraclique monad* $\kappa = (\kappa, m, e)$. This monad is sup-dense and interpolating in the clique monad $C$, so that one obtains an isomorphism
\[ \text{Cls} \cong (\kappa, 2)-\text{Cat} . \]

2.F *A flat extension of the ultraclique monad.* Consider the Kleisli extension $C^\tau$ of the clique functor $C$, where $\tau : P \to C$ is the principal filter natural transformation. The initial extension of the ultraclique functor $\kappa$ (see Exercise 2.E) is flat, although $\kappa$ does not satisfy the Beck–Chevalley condition.

*Hint.* For the first statement, investigate the induced order on $\kappa X$. For the second, consider the maps
\[ \{x\} \xleftarrow{f} \{x, y\} \xleftarrow{g} \{x, y, z\} , \]
where $g(y) = g(z) = y$ and $g(x) = x$.

2.G *Unitary $\mathcal{V}$-relations are sup-maps.* Let $\mathcal{V}$ be a completely distributive quantale and $(\mathbb{T}, \tau)$ a power-enriched monad. For a unitary $(\mathbb{T}, \mathcal{V})$-relation $r : X \to Y$ (with respect to the Kleisli extension of $\mathbb{T}$ to $\mathcal{V}$-$\text{Rel}$), the map $r(-, y) : TX \to \mathcal{V}^{\text{op}}$ preserves suprema for all $y \in Y$ (see also Exercise 1.H).

2.H *Alternative descriptions of lax extensions.* If the lattice $\mathcal{V}$ is completely distributive, the Kleisli extensions to $\mathcal{V}$-$\text{Rel}$ of the powerset, filter, and up-set functors of Examples 1.4.2 can equivalently be expressed as
\[ \hat{P}r(A, B) = \bigwedge_{x \in A} \bigvee_{y \in B} r(x, y) , \]
\[ \hat{F}r(f, g) = \bigwedge_{B \in \mathcal{V}} \bigwedge_{A \in \mathcal{V}} \bigwedge_{x \in A} \bigvee_{y \in B} r(x, y) , \]
\[ \hat{U}r(\chi, y) = \bigwedge_{B \in \mathcal{V}} \bigwedge_{A \in \mathcal{V}} \bigwedge_{x \in A} \bigvee_{y \in B} r(x, y) , \]
respectively, where $r : X \to Y$ is a $\mathcal{V}$-relation, $A \in PX$, $B \in PY$, $f \in FX$, $g \in FY$, $\chi \in UX$ and $y \in UY$.

2.I *Monad retraction induce equivalences of categories.* Let $\hat{\mathbb{T}}$ be a lax extension of $\mathbb{T}$ and $\alpha : S \to \mathbb{T}$ a retraction of monads (so there exists a monad morphism $\rho : \mathbb{T} \to S$ such that $\alpha \cdot \rho = 1_{\mathbb{T}}$). If $\hat{S}$ is the initial extension induced by $\alpha$, then $(S, \mathcal{V})$-$\text{Cat}$ and $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$ are equivalent categories.

2.J *Topologicity of the functor $(\mathbb{T}, 2)$-$\text{UGph} \to \text{Set}.* For a lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ on $\text{Set}$, the forgetful functor $U : (\mathbb{T}, \mathcal{V})$-$\text{UGph} \to \text{Set}$ is topological.
CHAPTER IV. KLEISLI MONOIDS

2.K Approach spaces as \((F, P_+)-categories\). Consider the filter monad \(F\) with the principal filter natural transformation \(\tau: P \to F\) and the extended real half-line \(P_+\). The category of reflexive and transitive \(P_+\)-towers of \((F, 2)\)-\(U\)\(g\)\(p\) is isomorphic to the category of reflexive and transitive \(P_+\)-towers of \(\text{PSet}\) (see Exercise 1.D). Moreover, \(P_+\)-towers of \(\text{PSet}\) \((c_v: PX \to PX)_{v \in V}\) and maps \(\delta: X \times PX \to P_+\) determine each other via

\[
\delta(x, A) := \inf \{v \in [0, \infty] \mid x \in c_v(A)\} \quad \text{and} \quad c_v(A) = \{x \in X \mid \delta(x, A) \leq v\},
\]

and this correspondence induces an isomorphism \((F, P_+)-\text{Cat} \cong \text{App}\).

2.L Quantales as \((T, V)\)-categories. Let \((T, \tau)\) be a power-enriched monad, with \(T = (T, m, e)\), and \(V\) a completely distributive quantale (with totally below relation \(\ll\), see II.1.11). The map \(\alpha: V \to TV\) defined by

\[
\alpha(v) := \bigvee \{x \in TV \mid \forall u \in V \ (u \ll v \implies x \leq \tau_V(\uparrow u))\}
\]

is an inf-map \(\alpha: V^{\text{op}} \to TV\). Hence, \(\alpha\) has a left adjoint \(\xi: TV \to V^{\text{op}}\) in \(\text{Ord}\). Verify that the pair \((V, \alpha)\) is a \(T\)-monoid, and therefore \((V, \xi)\) is a \((T, 2)\)-category (see II.1.4). The map \(\alpha\) extends to \(\tilde{\alpha}: V^{\text{op}} \to \text{Set}(V, TV)\), via

\[
\tilde{\alpha}(u)(v) := \alpha(u \otimes v),
\]

so that \((V, \tilde{\alpha})\) is a \(V\)-tower of \(\text{T-Mon}\), that is, an object of \((T, V)\)-\text{Mon}. The structure \(V\)-relation of the corresponding \((T, V)\)-category \((X, r_{\tilde{\alpha}})\) is given by

\[
r_{\tilde{\alpha}}(x, v) = \xi(x) \leftarrow v
\]

(see II.1.10).
3 Lax algebras as Kleisli monoids

In this subsection, we show that an associative lax extension to $\mathcal{V}$-$\text{Rel}$ of a monad $\mathcal{T}$ always has an associated power-enriched monad $\Pi$ that, when equipped with its Kleisli extension to $\text{Rel}$, returns the same category of lax algebras:

$$\mathcal{(\mathcal{T}, \mathcal{V})}\text{-Cat} \cong \Pi\text{-Mon}$$

(Theorem 3.2.2). In 3.3, an alternate description of the monad $\Pi$ is given in the case of approach spaces, that is, when $\mathcal{T} = \beta$ and $\mathcal{V} = \mathcal{P}_+$. 

3.1 The ordered category $(\mathcal{T}, \mathcal{V})$-$\mathcal{URel}$. Given an associative lax extension to $\mathcal{V}$-$\text{Rel}$ of a monad $\mathcal{T}$ on $\text{Set}$, recall from 3.1.9 that sets with unitary $(\mathcal{T}, \mathcal{V})$-relations form the category $(\mathcal{T}, \mathcal{V})$-$\mathcal{URel}$, with Kleisli convolution as composition and the identity on $X$ given by $1^x_X = e^o_X \cdot \hat{T}1_X$. Moreover, $(\mathcal{T}, \mathcal{V})$-$\mathcal{URel}$ forms an ordered category when the hom-sets $(\mathcal{T}, \mathcal{V})$-$\mathcal{Rel}(X, Y) \subseteq \mathcal{V}$-$\mathcal{Rel}(\hat{T}X, Y)$ are equipped with the pointwise order inherited from $\mathcal{V}$-$\text{Rel}$, since the Kleisli convolution preserves this order on the left and right.

3.1.1 Lemma. Let $\hat{\mathcal{T}}$ be a lax extension of $\mathcal{T}$ to $\mathcal{V}$-$\text{Rel}$. For a family of unitary $(\mathcal{T}, \mathcal{V})$-relations $(\varphi_i : X \rightarrow Y)_{i \in I}$, the $(\mathcal{T}, \mathcal{V})$-relation $\Lambda_{i \in I} \varphi_i$ is unitary.

Proof. The statement follows from the inequalities

$$(\Lambda_{i \in I} \varphi_i) \cdot \hat{T}1_X \leq \Lambda_{i \in I} (\varphi_i \cdot \hat{T}1_X) = \Lambda_{i \in I} \varphi_i ,$$

$$e^o_Y \cdot \hat{T}(\Lambda_{i \in I} \varphi_i) \cdot m^o_X \leq \Lambda_{i \in I} (e^o_Y \cdot \hat{T}\varphi_i \cdot m^o_X) = \Lambda_{i \in I} \varphi_i$$

since the inequalities in the other direction always hold.

This Lemma implies that the ordered category $(\mathcal{T}, \mathcal{V})$-$\mathcal{URel}$ has complete hom-sets. However, in general $(\mathcal{T}, \mathcal{V})$-$\mathcal{URel}$ is not a quantaloid since $\varphi \circ (\cdot)$ typically does not preserve suprema. The situation is better for composition on the right, and when the maps $(\cdot) \circ \varphi$ preserve suprema we say that $(\mathcal{T}, \mathcal{V})$-$\mathcal{URel}$ is a right-sided quantaloid.

3.1.2 Proposition. Let $\hat{\mathcal{T}}$ be an associative lax extension of $\mathcal{T}$ to $\mathcal{V}$-$\text{Rel}$. Then, for every $(\mathcal{T}, \mathcal{V})$-relation $\varphi : X \rightarrow Y$ and every set $Z$, the monotone map

$$(\cdot) \circ \varphi : (\mathcal{T}, \mathcal{V})$-$\text{Rel}(Y, Z) \rightarrow (\mathcal{T}, \mathcal{V})$-$\text{Rel}(X, Z)$$

has a right adjoint $(-) \circ \varphi : (\mathcal{T}, \mathcal{V})$-$\text{Rel}(X, Z) \rightarrow (\mathcal{T}, \mathcal{V})$-$\text{Rel}(Y, Z)$ that sends $\psi : X \rightarrow Z$ to $\psi \circ (\hat{T}\varphi \cdot m^o_X)$ (see Exercise 3.1.1.E). Moreover, if $\varphi : X \rightarrow Y$ and $\psi : X \rightarrow Z$ are unitary, then $\psi \circ \varphi$ is also unitary, and consequently $(\mathcal{T}, \mathcal{V})$-$\mathcal{URel}$ is a right-sided quantaloid.

Proof. Consider a $(\mathcal{T}, \mathcal{V})$-relation $\varphi : X \rightarrow Y$. For all $\gamma : Y \rightarrow Z$ and $\psi : X \rightarrow Z$,

$$\gamma \circ \varphi \leq \psi \iff \gamma \cdot \hat{T}\varphi \cdot m^o_X \leq \psi \iff \gamma \leq \psi \circ (\hat{T}\varphi \cdot m^o_X) ,$$
hence, the map
\((-) \circ \varphi : (\mathbb{T}, \mathcal{V}){-}\text{Rel}(X, Z) \to (\mathbb{T}, \mathcal{V}){-}\text{Rel}(Y, Z), \quad \psi \mapsto (\psi \circ \varphi) := \psi \cdot (\hat{T} \varphi \cdot m_X^0)\)

is right adjoint to \((-) \circ \varphi\). Suppose now that \(\varphi : X \leftrightarrow Y\) and \(\psi : X \leftrightarrow Z\) are unitary. By associativity of the Kleisli convolution and Proposition \[\text{III.1.9.4], we have
\[
(1^Y_Z \circ (\psi \circ \varphi)) \circ \varphi \leq 1^Y_Z \circ \psi = \psi \quad \text{and} \quad ((\psi \circ \varphi) \circ 1^Y_Z) \circ \varphi \leq (\psi \circ \varphi) \circ \varphi \leq \psi ;
\]
therefore, \(1^Y_Z \circ (\psi \circ \varphi) \leq \psi \circ \varphi\) and \((\psi \circ \varphi) \circ 1^Y_Z \leq \psi \circ \varphi\). \(\square\)

Recall from Exercise \[\text{III.1.M} that a map \(f : X \to Y\) gives rise to a unitary \((\mathbb{T}, \mathcal{V}){-}\text{relation} \(f^\sharp : Y \leftrightarrow X\) via
\[
f^\sharp = f^\circ \cdot e_Y^0 \cdot \hat{T} l_Y = e_X^0 \cdot (T f)^\circ \cdot \hat{T} l_Y = e_X^0 \cdot \hat{T}(f^\circ) ,
\]
where \(1^X_X = (1_X)_X\) is the identity morphism on \(X\) in \((\mathbb{T}, \mathcal{V}){-}\text{URel}\).

3.1.3 Lemma. Let \(\hat{T}\) be a lax extension of \(\mathbb{T}\) to \(\mathcal{V}\)-\text{Rel}. For a unitary \((\mathbb{T}, \mathcal{V}){-}\text{relation} \(\varphi : X \leftrightarrow Y\), one has
\[
f^\sharp \circ \varphi = f^\circ \cdot \varphi
\]
for all maps \(f : Z \to Y\).

Proof. One computes \(f^\sharp \circ \varphi = f^\circ \cdot e_Y^0 \cdot \hat{T} l_Y \cdot \hat{T} \varphi \cdot m_X^0 = f^\circ \cdot (1^Y_X \circ \varphi) = f^\circ \cdot \varphi\). \(\square\)

If \(\hat{T}\) is an associative lax extension of \(\mathbb{T}\) to \(\mathcal{V}\)-\text{Rel}, then \(f^\sharp \circ g^\sharp = (g \cdot f)^\sharp\) for all maps \(f : X \to Y\) and \(g : Y \to Z\) in \(\text{Set}\):
\[
f^\sharp \circ g^\sharp = f^\circ \cdot g^\circ = f^\circ \cdot g^\circ \cdot e_Z^0 \cdot \hat{T} l_Z = (g \cdot f)^\sharp .
\]
Hence, there is a functor
\[
(\mathbb{T}, \mathcal{V}){-}\text{Rel}(-, 1) : (\mathbb{T}, \mathcal{V}){-}\text{URel}^{\text{op}} \to \text{Set} ,
\]
that maps objects identically. We now proceed to show that this functor is left adjoint to the contravariant hom-functor
\[
(\mathbb{T}, \mathcal{V}){-}\text{Rel}(\_, 1) : (\mathbb{T}, \mathcal{V}){-}\text{URel}^{\text{op}} \to \text{Set} ,
\]
where \(1 = \{\ast\}\) denotes a singleton. We identify elements \(x \in X\) with maps \(x : 1 \to X\), and with a unitary \((\mathbb{T}, \mathcal{V}){-}\text{relation} \(\psi : X \leftrightarrow Y\) we associate the map \(\psi^\circ : Y \to (\mathbb{T}, \mathcal{V}){-}\text{URel}(X, 1)\) defined by
\[
\psi^\circ(y) := y^\sharp \circ \psi = y^\circ \cdot \psi = \psi(-, y)
\]
for all \(y \in Y\) (the second equality follows from Lemma \[3.1.3\] and the third by definition of composition in \(\mathcal{V}\)-\text{Rel}); here, \(\psi(-, y)(\chi, \ast) := \psi(\chi, y)\).
The next Lemma shows that unitariness of a \((\mathbb{T}, \mathcal{V})\)-relation \(\varphi : X \leftrightarrow Y\) can be tested just by using elements of \(Y\).

3.1.4 Lemma. Let \(\hat{T}\) be an associative lax extension of \(\mathbb{T}\) to \(\mathcal{V}\)-Rel, and \(\varphi : X \leftrightarrow Y\) a \((\mathbb{T}, \mathcal{V})\)-relation. Then \(\varphi\) is unitary if and only if \(y^0 \cdot \varphi\) is unitary for all \(y \in Y\).

Proof. If \(\varphi\) is unitary, then \(y^0 \cdot \varphi = y^0 \circ e_X\) is unitary as well (see \[\text{III.1.9}\]). To verify the other implication, suppose that \(y^0 \cdot \varphi\) is unitary for all \(y \in Y\). Then one has

\[y^0 \cdot (\varphi \circ e_X) = y^0 \cdot \varphi \cdot \hat{T}1_X = y^0 \cdot \varphi\]

and

\[y^0 \cdot (e_X^0 \circ \varphi) = y^0 \cdot e_X^0 \cdot \hat{T} \varphi \cdot m_X^0 = e_X^0 \cdot (Ty)^0 \cdot \hat{T} \varphi \cdot m_X^0 = e_X^0 \cdot \hat{T}(y^0 \cdot \varphi) \cdot m_X^0 = y^0 \cdot \varphi\]

for all \(y \in Y\), so \(\varphi \circ e_X = \varphi\) and \(e_X^0 \circ \varphi = \varphi\).

\[\square\]

3.1.5 Proposition. Let \(\hat{T}\) be an associative lax extension of \(\mathbb{T}\) to \(\mathcal{V}\)-Rel.

1. For a set \(X\), the product \(1^X = \prod_{x \in X} 1_x\) in \((\mathbb{T}, \mathcal{V})\)-\(\mathcal{U}Rel\) (with \(1_x = 1\) for all \(x \in X\)) exists, and can be chosen as \(1^X = X\) with projections \(\pi_x = x^\sharp : X \leftrightarrow 1\) (\(x \in X\)).

2. The contravariant hom-functor

\[\ldbracket \mathbb{T}, \mathcal{V}\rdbra{URel}(-, 1) : (\mathbb{T}, \mathcal{V})\rightarrow \mathcal{U}Rel^{op} \rightarrow \text{Set}\]

has \((-)^\sharp : \text{Set} \rightarrow (\mathbb{T}, \mathcal{V})\rightarrow \mathcal{U}Rel^{op}\) as left adjoint. The unit and counit of the associated adjunction are given by the Yoneda maps

\[\gamma_X = (1^X)^\flat : X \rightarrow (\mathbb{T}, \mathcal{V})\rightarrow \mathcal{U}Rel(X, 1) \ , \ x \mapsto x^\sharp\]

and the evaluation relations

\[\varepsilon_X : X \rightarrow (\mathbb{T}, \mathcal{V})\rightarrow \mathcal{U}Rel(X, 1) \ , \ \varepsilon_X(\chi, \psi) = \psi(\chi, \ast)\]

respectively.

Proof.

1. For a family \((\phi_x : Y \leftrightarrow 1)_{x \in X}\) of unitary \((\mathbb{T}, \mathcal{V})\)-relations, one can define a \((\mathbb{T}, \mathcal{V})\)-relation \(\phi : Y \leftrightarrow X\) by setting \(\phi(y, x) = \phi_x(y, \ast)\) for all \(y \in TY\). Since \(x^\circ \phi = \phi_x\) is unitary for all \(x \in X\), then so is \(\phi\) by Lemma 3.1.4. Unicity of the connecting morphism \(\phi : Y \leftrightarrow X\) follows from its definition.

2. By Exercise \[\text{III.2.3}\], the functor \(H = (\mathbb{T}, \mathcal{V})\rightarrow \mathcal{U}Rel(-, 1) : (\mathbb{T}, \mathcal{V})\rightarrow \mathcal{U}Rel^{op} \rightarrow \text{Set}\) has a left adjoint \(F = 1^{(-)}\) that sends a set \(X\) to its product \(1^X = X\), and a map \(f : X \rightarrow Y\) to the unitary \((\mathbb{T}, \mathcal{V})\)-relation \(f^\sharp : Y \leftrightarrow X\). The same exercise yields that \(y\) is indeed the unit of the adjunction and \(\varepsilon\) its counit.

\[\square\]
3.2 The discrete presheaf monad. For an associative lax extension \( \hat{\tau} \) of \( \mathbb{T} \) to \( \mathcal{V}\text{-Rel} \), the adjunction described in Proposition 3.1.5 induces a monad

\[
\mathbb{P} = \mathbb{P}(\mathbb{T}, \mathcal{V}) = \mathbb{P}(\mathbb{T}, \mathcal{V}, \hat{\tau}) = (\mathbb{P}, \mathbb{m}, \mathbb{y})
\]
on \( \text{Set} \), where

\[
\mathbb{P}X = (\mathbb{T}, \mathcal{V})\text{-URel}(X, 1)
\]

\[
\mathbb{m}_X(\Psi) = \Psi \circ \varepsilon_X,
\]

\[
\mathbb{y}_X(x) = x^\sharp;
\]

for all \( x \in X, f : X \to Y \), and unitary \( (\mathbb{T}, \mathcal{V}) \)-relations \( \psi : X \leftrightarrow 1, \Psi : \mathbb{P}X \leftrightarrow 1 \). We call \( \mathbb{P} \) the discrete presheaf monad associated to \( \hat{\tau} \).

The fully faithful comparison functor \( L : \text{Set}_{\mathbb{P}} \to (\mathbb{T}, \mathcal{V})\text{-URel}^{op} \) (Proposition II.3.6.1) is bijective on objects since the left adjoint \((-)^2 : \text{Set} \to (\mathbb{T}, \mathcal{V})\text{-URel}^{op} \) is so. That is, the Kleisli category \( \text{Set}_{\mathbb{P}} \) of \( \mathbb{P} \) is isomorphic to \((\mathbb{T}, \mathcal{V})\text{-URel}^{op} \). Explicitly, \( LX = X \) for each set \( X \), and \( L \) sends a morphism \( r : X \to \Pi Y \) in \( \text{Set}_{\mathbb{P}} \) to the unitary \( (\mathbb{T}, \mathcal{V}) \)-relation \( r^\sharp \circ \varepsilon_Y : Y \to X \).

By Lemma 3.1.3, \( r^\sharp \circ \varepsilon_Y(y,x) = r(x)(y,\cdot) \) for all \( x \in X \) and \( y \in TY \), so the inverse of \( L \) sends a unitary \( (\mathbb{T}, \mathcal{V}) \)-relation \( \varphi : Y \leftrightarrow X \) to \( \varphi^\flat : X \to \Pi Y \).

Since \( (\mathbb{T}, \mathcal{V})\text{-URel} \) is a right-sided quantaloid (Proposition 3.1.2), \( \Pi X = (\mathbb{T}, \mathcal{V})\text{-URel}(X, 1) \) is a complete lattice when equipped with its pointwise order:

\[
\psi_1 \leq \psi_2 \iff \forall x \in TX \ (\psi_1(x) \leq \psi_2(x)),
\]

for all unitary \( (\mathbb{T}, \mathcal{V}) \)-relations \( \psi_1, \psi_2 : X \leftrightarrow 1 \). By Proposition 3.1.2, the map \((-) \circ \varphi : \Pi Y \to \Pi X \) preserves suprema for every unitary \( (\mathbb{T}, \mathcal{V}) \)-relation \( \varphi : X \leftrightarrow Y \); in particular, \( \Pi f = (-) \circ f^\sharp \) and \( \mathbb{m}_X = (-) \circ \varepsilon_X \) preserve suprema for every set \( X \) and every map \( f : X \to Y \). The order on the sets \( \Pi X \) therefore corresponds to a monad morphism \( \tau : P \to \mathbb{P} \) from the powerset monad \( P \) (Proposition 1.2.1). As a consequence, the hom-sets of \( \text{Set}_{\mathbb{P}} \) are complete ordered sets when ordered pointwise:

\[
f_1 \leq f_2 \iff \forall x \in X \ (f_1(x) \leq f_2(x))
\]

for all maps \( f_1, f_2 : X \to \Pi Y \). One then has for unitary \( (\mathbb{T}, \mathcal{V}) \)-relations \( \varphi_1, \varphi_2 : Y \leftrightarrow X \):

\[
\varphi_1 \leq \varphi_2 \iff \forall x \in X \ ∀ y \in TY \ (\varphi_1(y,x) \leq \varphi_2(y,x)) \iff \varphi_1^\flat \leq \varphi_2^\flat.
\]

We therefore obtain an isomorphism (see also the discussion after Lemma 3.1.3)

\[
(\mathbb{T}, \mathcal{V})\text{-URel}(Y, X) \to \text{Set}(X, (\mathbb{T}, \mathcal{V})\text{-URel}(Y, 1)), \ \varphi \mapsto (\varphi^\flat : x \mapsto \varphi(\cdot, x))
\]
in \text{Ord}. Since \( (\mathbb{T}, \mathcal{V})\text{-URel} \) is an ordered category, this shows that \( \text{Set}_{\mathbb{P}} \) is also an ordered category.

3.2.1 Proposition. Let \( \hat{\tau} \) be an associative lax extension to \( \mathcal{V}\text{-Rel} \) of a monad \( \mathbb{T} \) on \( \text{Set} \). The discrete presheaf monad \( \mathbb{P} = \mathbb{P}(\mathbb{T}, \mathcal{V}) \) is power-enriched, and the comparison functor

\[
L : \text{Set}_{\mathbb{P}} \to (\mathbb{T}, \mathcal{V})\text{-URel}^{op}
\]
is a \( 2 \)-isomorphism. Moreover, for every \( f : X \to Y \) in \( \text{Set} \) one has \((f^\sharp)^0 = y_Y \cdot f\).
3. LAX ALGEBRAS AS KLEISLI MONOIDS

Proof. The functor $L$ and its inverse have been described above. Since the Kleisli comparison functor $L$ makes the diagram

$$
\begin{array}{ccc}
\text{Set}_\Pi & \xrightarrow{L} & (\mathbb{T}, \mathcal{V})\text{-URel}^{\text{op}} \\
\downarrow F_\Pi & & \downarrow (-)^f \\
\text{Set} & \xrightarrow{(-)^f} & (\mathbb{T}, \mathcal{V})\text{-URel}^{\text{op}}
\end{array}
$$

commute, $(f^\sharp)^h = L^{-1}(f^\sharp) = F_\Pi f = y_Y \cdot f$.

3.2.2 Theorem. Given an associative lax extension $\hat{T}$ of $T$ to $\mathcal{V}\text{-Rel}$, there is an isomorphism

$$(\mathbb{T}, \mathcal{V})\text{-Cat} \cong \mathbb{P}\text{-Mon}$$

that commutes with the underlying-set functors.

Proof. Recall that every reflexive and transitive $(\mathbb{T}, \mathcal{V})$-relation $a : X \leftrightarrow X$ is also unitary. By Proposition 3.2.1, sending $a$ to $a^b$ defines a bijection between reflexive and transitive $(\mathbb{T}, \mathcal{V})$-relations $a : X \leftrightarrow X$ and maps $\nu : X \rightarrow \Pi X$ satisfying $y_X \leq \nu$ and $\nu \circ \nu \leq \nu$. Furthermore, a map $f : X \rightarrow Y$ is a $(\mathbb{T}, \mathcal{V})$-functor $f : (X, a) \rightarrow (Y, b)$ if and only if $a \circ f^\sharp \leq f^\sharp \circ b$; this condition is equivalent to $(y_Y \cdot f) \circ a^b \leq b^b \circ (y_Y \cdot f)$ in $\text{Set}_\Pi$, that is, $f : (X, a^b) \rightarrow (Y, b^b)$ is a morphism of Kleisli monoids.

Since $\mathbb{P}$ is power-enriched, we can consider the corresponding Kleisli extension $\tilde{\mathbb{P}}$ of $\mathbb{P}$ to $\text{Rel}$. Explicitly, the extension $\tilde{\mathbb{P}}$ of the $\text{Set}$-functor $\mathbb{P}$ to $\text{Rel}$ sends a relation $r : X \leftrightarrow Y$ to the relation $\tilde{\mathbb{P}}r : \Pi X \leftrightarrow \Pi Y$ defined by

$$
\psi_1 \tilde{\mathbb{P}}r \psi_2 \iff \psi_1 \leq \psi_2 \cdot \tilde{T}r
$$

for all unitary $(\mathbb{T}, \mathcal{V})$-relations $\psi_1 : X \leftrightarrow 1$, $\psi_2 : Y \leftrightarrow 1$ (Exercise 3.B). With respect to this extension of $\mathbb{P}$ we consider the category $(\tilde{\mathbb{P}}, 2)\text{-Cat}$.

3.2.3 Corollary. Given an associative lax extension $\hat{T}$ of $T$ to $\mathcal{V}\text{-Rel}$, there is an isomorphism

$$(\mathbb{T}, \mathcal{V})\text{-Cat} \cong (\tilde{\mathbb{P}}, 2)\text{-Cat}$$

that commutes with the underlying-set functors, where $\mathbb{P} = \mathbb{P}(\mathbb{T}, \mathcal{V})$ is equipped with its Kleisli extension $\tilde{\mathbb{P}}$.

Proof. Theorem 3.2.2 yields the isomorphism $(\mathbb{T}, \mathcal{V})\text{-Cat} \cong \mathbb{P}\text{-Mon}$, and Theorem 1.5.3 the isomorphism $\mathbb{P}\text{-Mon} \cong (\tilde{\mathbb{P}}, 2)\text{-Cat}$.

3.2.4 Remark. The Kleisli extension of a power-enriched monad $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}(\mathbb{T}, \mathcal{V}, \hat{T})$ is in particular associative (see Exercise 1.G). Hence, the transition from $\hat{T}$ to $\tilde{\mathbb{P}}$ preserves associativity while maintaining the same categories of lax algebras (up to isomorphism).
3.2.5 Proposition. Let \( \hat{T} \) be an associative lax extension of \( T \) to \( V\text{-Rel} \). Then the natural transformation \( Y \) defined componentwise by
\[
Y_X : TX \to \Pi X, \quad \chi \mapsto \hat{T}1_X(-, \chi)
\]
is a monad morphism \( Y : \mathbb{T} \to \Pi(\mathbb{T}, V) \).

Proof. The left adjoint functor \((-)^\sharp : \text{Set} \to (\mathbb{T}, V)\text{-Rel}^{op} \) extends to a functor
\[
(-)^\sharp : \text{Set}_T \to (\mathbb{T}, V)\text{-URel}^{op}
\]
sending \( r : X \rightsquigarrow Y \) to \( r^\sharp := r^\circ \cdot \hat{T}1_Y : Y \rightsquigarrow X \). Indeed, the identity \( 1_X : X \to X \) in \( \text{Set}_T \) (which is the map \( e_X : X \to TX \)) goes to the identity \( 1_X^\sharp = e_X^\circ \cdot \hat{T}1_X \) in \( (\mathbb{T}, V)\text{-URel} \); and for \( r : X \to Y \) and \( s : Y \to Z \) one has
\[
r^\sharp \circ s^\sharp = r^\circ \cdot \hat{T}1_Y \cdot (s^\circ \cdot \hat{T}1_Z) \cdot m_Z^\circ = r^\circ \cdot (Ts)^\circ \cdot \hat{T}1_Z \cdot m_Z^\circ = (m_Z \cdot Ts \cdot r)^\circ \cdot \hat{T}1_Z,
\]
hence \((-)^\sharp \) preserves composition. By definition, the diagram
\[
\begin{array}{ccc}
\text{Set}_T & \xrightarrow{(-)^\sharp} & (\mathbb{T}, V)\text{-URel}^{op} \\
\downarrow F_T & & \downarrow (-)^\sharp \\
\text{Set} & & \\
\end{array}
\]
commutes, and therefore \((-)^\sharp : \text{Set}_T \to (\mathbb{T}, V)\text{-URel}^{op} \) induces a monad morphism \( Y : \mathbb{T} \to \Pi(\mathbb{T}, V) \). Its component \( Y_X \) is the composite (see II.3.I)
\[
TX \xrightarrow{y_TX} (\mathbb{T}, V)\text{-URel}(TX, 1) \xrightarrow{(-)^\sharp 1_{TX}} (\mathbb{T}, V)\text{-URel}(X, 1),
\]
hence \( Y_X(\chi) = \chi^\circ \cdot e_{TX}^\circ \cdot \hat{T}1_{TX} \cdot \hat{T}1_X \cdot m_X^\circ = \chi^\circ \cdot \hat{T}1_X = \hat{T}1_X(-, \chi). \)

3.2.6 Examples.

(1) For the identity monad \( I \) on \( \text{Set} \) extended to the identity monad on \( \text{Rel} \), one has
\[
(I, 2)\text{-URel} \cong \text{Rel}
\]
and the monad \( \Pi = \Pi(I, 2) \) is isomorphic to the powerset monad \( \mathbb{P} \). Hence, \( \text{Ord} \cong (\mathbb{P}, 2)\text{-Cat} \) by 3.2.3. The monad morphism \( Y : I \to \mathbb{P} \) is necessarily given by the unit of \( \mathbb{P} \).
More generally, for the identity monad $I$ on $\mathbf{Set}$ extended to the identity monad on $\mathcal{V}$-$\mathbf{Rel}$, one has

$$(\mathbb{1}, \mathcal{V})$-$\mathbf{URel} \cong \mathcal{V}$-$\mathbf{Rel}$$

and the monad $\Pi = \Pi(\mathbb{1}, \mathcal{V})$ is isomorphic to the $\mathcal{V}$-powerset monad $\mathcal{P}_\mathcal{V}$ (see Exercise III.1.D). Hence, $\mathcal{V}$-$\mathbf{Cat} \cong (\mathcal{P}_\mathcal{V}, 2)$-$\mathbf{Cat}$ by 3.2.3. As above, the monad morphism $\mathbf{Y} : \mathbb{1} \to \mathcal{P}_\mathcal{V}$ is necessarily given by the unit of $\mathcal{P}_\mathcal{V}$.

The monoid $(\mathcal{V}, \otimes, k)$ induces a monad $\mathcal{V}$ on $\mathbf{Set}$ with functor $\mathcal{V} \times (-)$ (see Exercise II.3.B), and for each set $X$, there is a map $\alpha_X : \mathcal{V} \times X \to \mathcal{V}^X$ defined by

$$\alpha_X(u, x)(y) = \begin{cases} u & \text{if } x = y, \\ \bot & \text{else.} \end{cases}$$

One easily verifies that these maps yield a monad morphism $\alpha : \mathcal{V} \to \mathcal{P}_\mathcal{V}$ (Exercise 3.D).

It is also clear that $\alpha$ is sup-dense; therefore, when considering the $\mathcal{V}$-powerset monad $\mathcal{P}_\mathcal{V}$ with its Kleisli extension and $\mathcal{V}$ with the initial extension induced by $\alpha$, we obtain (see Proposition 2.3.4) a full reflective embedding

$$A_\alpha : \mathcal{V}$-$\mathbf{Cat} \hookrightarrow (\mathcal{V}, 2)$-$\mathbf{Cat}.$$.

However, $A_\alpha$ is not an equivalence in general. Indeed, for $\mathcal{V} = \mathcal{P}_+$ and a metric space $X = (X, a)$, one has

$$(u, x) (a \cdot \alpha_X) y \iff a(x, y) \leq u ,$$

for all $x, y \in X$ and $u \in [0, \infty]$. Consider $X = \{a, b\}$ with the relation $\rightarrow : [0, \infty] \times X \to X$ defined by

$$(u, a) \rightarrow b \iff 0 < u$$

and $(u, a) \rightarrow a$, $(u, b) \rightarrow b$ for all $u \in [0, \infty]$. Then $X$ is indeed a $(\mathcal{V}, 2)$-category but $\rightarrow$ is not induced by a metric on $X$.

(3) For the powerset monad $\mathcal{P}$ on $\mathbf{Set}$ with its Kleisli extension $\mathcal{P}_\mathcal{P}$ to $\mathbf{Rel}$, one has an isomorphism

$$(-)^\sharp : \mathbf{Set}_\mathcal{P} \to (\mathcal{P}, 2)$-$\mathbf{URel}^{\text{op}}$$

commuting with the left adjoints from $\mathbf{Set}$, and therefore $\mathbf{Y} : \mathcal{P} \to \Pi(\mathcal{P}, 2)$ is an isomorphism.

(4) Consider the ultrafilter monad $\beta$ on $\mathbf{Set}$ with its Barr extension $\mathcal{P}_\beta$ to $\mathbf{Rel}$ (Example III.1.10.3(3)). In this case, the monad $\Pi = \Pi(\beta, 2)$ is isomorphic to the filter $\odot$ monad $\mathbb{F}$ on $\mathbf{Set}$. Indeed, a unitary $(\beta, 2)$-relation $\psi : X \leftrightarrow 1$ may be identified with a set $\mathcal{A} \subseteq \beta X$ of ultrafilters on $X$ with the property that $\chi \supseteq \bigcap \mathcal{A}$ implies $\chi \in \mathcal{A}$. Therefore, the map

$$\delta_X : (\beta, 2)$-$\mathbf{URel}(X, 1) \to FX , \quad \mathcal{A} \mapsto \bigcap \mathcal{A}$$
is a bijection. Let us show that $\delta = (\delta_X)$ is indeed a monad morphism $\delta : \Pi \to \mathcal{F}$. Recall from Example II[3.1.1](4) that the filter monad on $\textbf{Set}$ is induced by the adjunction

$$
\text{SLat}^{\text{op}} \xrightarrow{\text{SLat}(\cdot, 2)} \mathcal{S} \xleftarrow{\Pi^{-}} \text{SLat}^{\text{op}}
$$

Since $\text{SLat}$ is equivalent to the category $\text{SLat}^{\text{co}}$ of join-semilattices and their homomorphisms, $\mathcal{F}$ is also induced by the adjunction

$$
\text{SLat}^{\text{co}} \xrightarrow{\text{SLat}(\cdot, 2)} \mathcal{S} \xleftarrow{\Pi^{-}} \text{SLat}^{\text{co}}
$$

In the former case, a filter $f \in FX$ corresponds to the characteristic map $\chi_f : PX \to 2$ with

$$
\chi_f(A) = 1 \iff A \in f,
$$

whereas in the latter case $f$ corresponds to the map $\chi'_f : PX \to 2$

$$
\chi'_f(A) = 1 \iff A^c /\notin f.
$$

The covariant hom-functor $(\mathcal{B}, 2)^{-}\mathbf{-URel}(1, -) : (\mathcal{B}, 2)^{-}\mathbf{-URel} \to \text{Set}$ lifts to a functor $L : (\mathcal{B}, 2)^{-}\mathbf{-URel} \to \text{SLat}^{\text{co}}$. Since, for every $\psi : X \leftrightarrow Y$ in $(\mathcal{B}, 2)^{-}\mathbf{-URel}$, the map

$$
\psi \circ (-) : PX \to PY, \ A \mapsto \{y \in Y \mid \exists x \in \beta X (A \in x \& x \psi y)\}
$$

preserves finite suprema (we use here the identification $(\mathcal{B}, 2)^{-}\mathbf{-URel}(1, X) \cong PX$). It follows from Lemma 3.1.3 that the diagram

$$
(\mathcal{B}, 2)^{-}\mathbf{-URel}^{\text{op}} \xrightarrow{L^{\text{op}}} \text{SLat}^{\text{co}} \xleftarrow{\Pi^{-}} \mathcal{S}
$$

commutes up to natural isomorphism, and therefore $L^{\text{op}}$ induces an isomorphism of monads $\Pi \to \mathcal{F}$ whose component at $X$ is precisely $\delta_X$. The composite $\delta \cdot Y : \mathcal{B} \to \Pi \to \mathcal{F}$ is the canonical monad morphism $\mathcal{B} \to \mathcal{F}$.

(5) By Corollary 3.2.3 and Theorem III[2.4.5]

$\text{App} \cong (\mathcal{B}, P_+)^{-}\mathbf{-Cat} \cong (\Pi, 2)^{-}\mathbf{-Cat}$.

In the next subsection we provide an alternative description of $\Pi = \Pi(\mathcal{B}, P_+)$. 

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3.3 Approach spaces. For an alternative description of $\Pi(\beta, P_+)$ (where the ultrafilter monad $\beta$ is provided with the extension $\beta_P$ to $P_+\text{-Rel}$ of III.2.4), we consider the category $\text{Met}_{(\vee, \bot, +)}$ of separated metric spaces $X = (X, d)$, whose underlying order given by

\[ x \leq y \iff 0 = d(x, y), \]

has finite suprema, and which admit an action $+: X \times [0, \infty] \to X$ (denoted here as a right action) satisfying

\[ d(x + u, y) = d(x, y) \ominus u, \quad (3.3.3) \]

for all $x, y \in X$ and $u \in [0, \infty]$ (recall from II.1.10.1(3) that “$\ominus$” denotes truncated subtraction); a morphism of $\text{Met}_{(\vee, \bot, +)}$ is a non-expansive map that preserves finite suprema and the action of $[0, \infty]$. The equation (3.3.3) implies immediately that the monotone map $d(x, -) : X \to P_+$ has $x + (-) : P_+ \to X$ as a left adjoint, therefore a separated metric space admits at most one such action.

The metric space $[0, \infty] = ([0, \infty], \mu)$ with $\mu(u, v) = v \ominus u$ ($u, v \in [0, \infty]$) belongs to $\text{Met}_{(\vee, \bot, +)}$ since its underlying order is the natural $\geq$, which has finite suprema, and the action of $[0, \infty]$ is given by the usual addition $+$. Similarly, for every set $X$, the set $[0, \infty]^X$ with the metric

\[ [\varphi, \varphi'] = \sup \{\mu(\varphi(x), \varphi'(x)) \mid x \in X\} \]

(for all $\varphi, \varphi' \in [0, \infty]^X$) and the action given by pointwise addition belongs to $\text{Met}_{(\vee, \bot, +)}$. In fact, in $\text{Met}_{(\vee, \bot, +)}$ one has

\[ [0, \infty]^X \cong \prod_{x \in X} [0, \infty], \]

where $[0, \infty]_x = ([0, \infty], \mu)$ for all $x \in X$. Therefore, as in Proposition 3.1.5 the contravariant hom-functor

\[ \text{Met}_{(\vee, \bot, +)}(-, [0, \infty]) : \text{Met}_{(\vee, \bot, +)}^{\text{op}} \to \text{Set} \]

has a left adjoint

\[ [0, \infty](-) : \text{Set} \to \text{Met}_{(\vee, \bot, +)}^{\text{op}}. \]

The functor $J$ of the monad $J\beta$ induced on Set by this adjunction is given by

\[ JX = \text{Met}_{(\vee, \bot, +)}([0, \infty]^X, [0, \infty]), \]

with the unit $X \to JX$ defined by evaluation $x \mapsto (\varphi \mapsto \varphi(x))$, and the multiplication $JJX \to JX$ defined by $\Psi \mapsto (\varphi \mapsto \Psi(\text{ev}_\varphi))$, with $\text{ev}_\varphi(\Phi) = \Phi(\varphi)$ for all $\Phi \in JX$, $\varphi \in [0, \infty]^X$.

We show now that $J\beta$ is isomorphic to the monad $\Pi$. To this end, we first observe that a map $\varphi : X \to [0, \infty]$ can be interpreted as a unitary $(\beta, P_+)$-relation $\varphi : 1 \hookrightarrow X$; in particular, every element $u \in [0, \infty]$ can be seen as a unitary $(\beta, P_+)$-relation $u : 1 \hookrightarrow 1$. From this
perspective, the distance \( \varphi, \varphi' \in [0, \infty] \) (where \( \varphi, \varphi' \in [0, \infty]^X \)) is precisely the lifting \( \varphi \circ \varphi' \) of \( \varphi' \) along \( \varphi \in (\beta, P_+)-\text{URel} \) (see Example II.1.8.3), and the action \( \varphi + u \) is the composite \( \varphi \circ u \). Every unitary \((\beta, P_+)-\text{relation} \psi : X \leftrightarrow Y \) defines a mapping

\[
\psi \circ (\cdot) : (\beta, P_+)-\text{URel}(1, X) \to (\beta, P_+)-\text{URel}(1, Y)
\]

that clearly preserves the action of \([0, \infty] , \) and from

\[
\psi \circ \varphi \circ (\varphi - \varphi') \geq \psi \circ \varphi'
\]

follows that \( \varphi - \varphi' \geq (\psi \circ \varphi) - (\psi \circ \varphi') \), for all \( \varphi, \varphi' : 1 \leftrightarrow X \). To see that \( \psi \circ (\cdot) \) preserves finite suprema, note that

\[
\varphi = \inf_{x \in \beta X} \psi(x, \cdot) + \xi(\beta \varphi(x)) = \psi \cdot \hat{\varphi},
\]

where \( \hat{\varphi} : 1 \leftrightarrow \beta X \) is defined as \( \hat{\varphi}(x) = \xi(\beta \varphi(x)) \) with

\[
\xi(a) := \sup_{A \in a} \inf_{u \in A} u = \inf_{A \in a} \sup_{u \in A} u,
\]

for all \( a \in \beta[0, \infty] \). Being left adjoint, \( \psi \cdot (\cdot) \) preserves all suprema, and \( \varphi \mapsto \hat{\varphi} \) also preserves binary suprema since \( \min : [0, \infty] \times [0, \infty] \to [0, \infty] \) is continuous. All said, the covariant hom-functor \( K := (\beta, P_+)-\text{URel}(1, \cdot) \) takes values in \( \text{Met}_{(\lor, \bot, +)} \). Moreover, it follows from Lemma 3.1.3 that the diagram

\[
\begin{array}{ccc}
(\beta, P_+)-\text{URel} & \xrightarrow{K^\text{op}} & \text{Met}_{(\lor, \bot, +)}^\text{op} \\
(\cdot)^2 & \downarrow \quad & \downarrow \text{Set} \\
[0, \infty][\cdot] & & \end{array}
\]

commutes up to natural isomorphism, and therefore \( K^\text{op} \) induces a monad morphism \( \delta : \Pi \to J \). A small computation shows that, for every set \( X \), the map

\[
\delta_X : \Pi X \to \text{Met}_{(\lor, \bot, +)}([0, \infty]^X, [0, \infty])
\]

sends \( \psi : X \leftrightarrow 1 \) to \( \psi \circ (\cdot) : [0, \infty]^X \to [0, \infty] \).

**3.3.1 Theorem.** \( \delta : \Pi \to J \) is an isomorphism.

*Proof.* Every subset \( A \subseteq X \) can be seen as an element of \([0, \infty]^X \), namely as the function \( A = \theta_A : X \to [0, \infty] \) sending \( x \in A \) to 0 and everything else to \( \infty \). With this interpretation, for every \( \Phi : [0, \infty]^X \to [0, \infty] \) in \( \text{Met}_{(\lor, \bot, +)} \) we set

\[
\Gamma(\Phi)(x) = \sup_{A \in x} \Phi(A).
\]

Then \( \psi = \Gamma(\Phi ) : X \leftrightarrow 1 \) is indeed unitary since one obtains

\[
\xi(\beta \psi(x)) \geq \psi(m_X(x))
\]
from the inequality Φ(A) ≤ Γ(Φ)(χ), for all A ∈ X. If Φ is of the form Φ = ψ ◦ (−) for some unitary ψ : X → 1, then
\[ \Gamma(Φ)(χ) = \sup_{A ∈ Χ} ψ ◦ A = \sup_{A ∈ Χ} ψ \cdot \hat{A} ≤ ψ \cdot \sup_{A ∈ Χ} \hat{A}. \]
Since \( \hat{A}(y) = 0 \) if \( A \in y \) and \( \hat{A}(y) = \infty \) if \( A \notin y \), one obtains \( \sup_{A ∈ Χ} \hat{A}(χ) = 0 \) and \( \sup_{A ∈ Χ} \hat{A}(y) = \infty \) for \( y \neq x \), and consequently \( \Gamma(Φ) ≤ ψ \). To see that
\[ \psi(x) ≤ \Gamma(Φ)(χ) = \sup_{A ∈ Χ} \inf_{y ∈ βA} \psi(y), \]
we use Lemma III.2.4.2, which guarantees the existence of some \( X ∈ βX \) with
\[ \{βA | A ∈ X\} ⊆ X \quad \text{and} \quad \sup_{A ∈ Χ} \inf_{y ∈ βA} \psi(y) ≥ ξ(βψ(X)). \]
Since \( ψ : X → 1 \) is unitary,
\[ ξ(βψ(X)) ≥ ψ(m_X(X)) = ψ(x). \]
Let now \( Φ : [0, \infty]^X → [0, \infty] \) in Met\((V_\vee, \perp, \top)\). We show first that \( Γ(Φ) ◦ (−) \) coincides with Φ on subsets \( B ⊆ X \) of X. Indeed,
\[ Γ(Φ) ◦ B = \inf_{x ∈ βX} \sup_{A ∈ Χ} Φ(A) + \hat{B}(χ) = \inf_{x ∈ βB} \sup_{A ∈ Χ} Φ(A) ≥ Φ(B). \]
To see that \( \inf_{x ∈ βB} \sup_{A ∈ Χ} Φ(A) ≤ Φ(B) \), we apply Corollary III.1.13.5 (if \( Φ(B) < \infty \)) to the filter base \( \{B\} \) and the ideal \( j = \{A ⊆ X | Φ(A) > Φ(B)\} \). Hence, there is some ultrafilter \( χ ∈ βX \) with \( B ∈ χ \) and \( χ \cap j = \varnothing \), and therefore
\[ \sup_{A ∈ Χ} Φ(A) ≤ Φ(B). \]
To finish the proof, we show that any \( Φ : [0, \infty]^X → [0, \infty] \) is completely determined by its restriction to subsets \( B ⊆ X \) of X. Since
\[ Φ(ϕ) = \sup_{u < \infty} \min\{Φ(ϕ), (Φ(0) + u)\} = \sup_{u < \infty} \min\{Φ(ϕ), Φ(u)\} = \sup_{u < \infty} Φ(\min\{ϕ, u\}), \]
\( Φ \) is determined by its restriction to bounded maps \( ϕ : X → [0, \infty] \). Let now \( ϕ : X → [0, \infty] \) be a bounded map, \( ε > 0 \), and \( N \) be any natural number with \( ϕ(x) < N \cdot ε \), for all \( x ∈ X \). For every natural number \( n \) with \( 0 ≤ n < N \) we set \( A_n = \{x ∈ X | n \cdot ε ≤ ϕ(x) < (n+1) \cdot ε\} \) and \( u_n = n \cdot ε \), and define
\[ ϕ_ε = \min\{A_n + u_n | 0 ≤ n < N\} \]
(note that finite suprema in \([0, \infty]^X \) are given by pointwise minima). Then \( Φ(ϕ_ε) ≤ Φ(ϕ) ≤ Φ(ϕ_ε) + ε \) and \( Φ(ϕ_ε) = \min\{Φ(A_n) + u_n | 0 ≤ n < N\} \), which proves that \( Φ \) is determined by its effect on subsets of \( X \). □
3.4 Revisiting change-of-base. In 3.2, we devised a construction that incorporates a quantale into a monad. In this subsection we extend this construction to quantale morphisms and show that certain change-of-base functors correspond to algebraic functors.

Recall from III.3.5 that every lax homomorphism $\varphi : V \to W$ of quantales induces a lax functor $\varphi : V\text{-Rel} \to W\text{-Rel}$ that sends the $V$-relation $r : X \times Y \to V$ to $\varphi r : X \times Y \to W$. Moreover, $\varphi : V\text{-Rel} \to W\text{-Rel}$ is actually a functor if $\varphi : V \to W$ is a homomorphism of quantales.

Let now $T = (T, m, e)$ be a monad on $\text{Set}$ together with associative lax extensions $\hat{T} : V\text{-Rel} \to V\text{-Rel}$ and $\check{T} : W\text{-Rel} \to W\text{-Rel}$.

A lax homomorphism of quantales $\varphi : V \to W$ is compatible with these lax extensions if $\check{T}(\varphi r) \leq \varphi(\hat{T} r)$ for all $r : X \to Y$ in $V\text{-Rel}$, and we say that $\varphi : V \to W$ is strictly compatible with $\hat{T}$ and $\check{T}$ if $\check{T}(\varphi r) = \varphi(\hat{T} r)$ for all $r : X \to Y$ in $V\text{-Rel}$.

3.4.1 Proposition. If $\varphi : V \to W$ is a lax homomorphism of quantales compatible with associative lax extensions $\hat{T}$ and $\check{T}$ of a monad $\mathbb{T} = (T, m, e)$ to $V\text{-Rel}$ and $W\text{-Rel}$ respectively, then $r \mapsto \varphi r$ defines a lax functor $\varphi : (\mathbb{T}, V)\text{-URel} \to (\mathbb{T}, W)\text{-URel}$ such that $(-)^{\sharp} \leq \varphi^{\text{op}} \cdot (-)^{\sharp}$. Moreover, $\varphi : (\mathbb{T}, V)\text{-URel} \to (\mathbb{T}, W)\text{-URel}$ is even a functor making

\[
\begin{array}{ccc}
(\mathbb{T}, V)\text{-URel} & \xrightarrow{\varphi^{\text{op}}} & (\mathbb{T}, W)\text{-URel} \\
(-)^{\sharp} \downarrow & & \downarrow (-)^{\sharp} \\
\text{Set} & & \\
\end{array}
\]

commute provided that $\varphi : V \to W$ is a homomorphism of quantales strictly compatible with the lax extensions $\hat{T}$ and $\check{T}$ of $\mathbb{T}$.

Proof. Let $r : X \to Y$ and $s : Y \to Z$ be morphisms in $(\mathbb{T}, V)\text{-URel}$. Then

\[
(\varphi s) \circ (\varphi r) = (\varphi s) \cdot \hat{T}(\varphi r) \cdot m_X^\circ \\
\leq (\varphi s) \cdot (\varphi \hat{T} r) \cdot (\varphi m_X^\circ) \\
\leq \varphi(s \circ r)
\]

with equality if $\varphi : V \to W$ is a homomorphism of quantales strictly compatible with the lax extensions $\hat{T}$ and $\check{T}$ of $\mathbb{T}$. If $f : X \to Y$ is map, then

\[
f^\sharp = e_Y^\circ \cdot \hat{T}(f^\circ) \leq \varphi(e_Y^\circ \cdot \hat{T}(f^\circ)) = \varphi f^\sharp,
\]

and, as above, we have equality if $\varphi : V \to W$ is a homomorphism of quantales strictly compatible with the lax extensions $\hat{T}$ and $\check{T}$ of $\mathbb{T}$. Note that this applies in particular to the identity $1_X^\sharp$ in $(\mathbb{T}, V)\text{-URel}$.  

\(\square\)
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3.4.2 Lemma. Let $\varphi : \mathcal{V} \to \mathcal{W}$ be a homomorphism of quantales strictly compatible with associative lax extensions $\hat{T}$ and $\bar{T}$ of $\mathbb{T} = (T, m, e)$ to $\mathcal{V}-\text{Rel}$ and $\mathcal{W}-\text{Rel}$ respectively. Then the right adjoint $\psi : \mathcal{W} \to \mathcal{V}$ of $\varphi$ is a lax homomorphism of quantales compatible with these lax extensions of $\mathbb{T}$.

Proof. For $u, v \in \mathcal{W}$,
\[ \psi(u) \otimes \psi(v) \leq \psi \varphi(\psi(u) \otimes \psi(v)) = \psi(\varphi \psi(u) \otimes \varphi \psi(v)) \leq \psi(u \otimes v) \quad \text{and} \]
\[ k \leq \psi \varphi(k) = \psi(l). \]
Similarly, for a $\mathcal{W}$-relation $r : X \to Y$ we obtain $\hat{T}(\psi r) \leq \psi \varphi \hat{T}(\psi r) = \psi \bar{T}(\varphi \psi r) \leq \psi(\bar{T}r).$ \hfill \( \square \)

3.4.3 Proposition. Every homomorphism $\varphi : \mathcal{V} \to \mathcal{W}$ of quantales strictly compatible with lax extensions $\hat{T}$ and $\bar{T}$ of $\mathbb{T} = (T, m, e)$ to $\mathcal{V}-\text{Rel}$ and $\mathcal{W}-\text{Rel}$ induces a morphism $\Pi(\mathbb{T}, \varphi) : \Pi(\mathbb{T}, \mathcal{V}) \to \Pi(\mathbb{T}, \mathcal{W})$ of power-enriched monads. The component of $\Pi(\mathbb{T}, \varphi)$ at a set $X$ is given by
\[ (\mathbb{T}, \mathcal{V})-\text{URel}(X, 1) \to (\mathbb{T}, \mathcal{W})-\text{URel}(X, 1), \quad r \mapsto \varphi r. \]

Proof. Firstly, $\varphi : \mathcal{V} \to \mathcal{W}$ induces a functor $\varphi : (\mathbb{T}, \mathcal{V})-\text{Rel} \to (\mathbb{T}, \mathcal{W})-\text{Rel}$ so that
\[ (\mathbb{T}, \mathcal{V})-\text{URel}^{\text{op}} \xrightarrow{\varphi^{\text{op}}} (\mathbb{T}, \mathcal{W})-\text{URel}^{\text{op}} \]
\[ \text{Set} \]

commutes. A quick computation shows that the $X$-component of the monad morphism $\Pi(\mathbb{T}, \varphi)$ induced by $\varphi^{\text{op}}$ has the form described above. Secondly, the right adjoint $\psi$ of $\varphi$ induces a lax functor $\psi : (\mathbb{T}, \mathcal{W})-\text{Rel} \to (\mathbb{T}, \mathcal{V})-\text{Rel}$, therefore the map $s \mapsto \psi s$ defines a right adjoint to
\[ (\mathbb{T}, \mathcal{V})-\text{URel}(X, 1) \to (\mathbb{T}, \mathcal{W})-\text{URel}(X, 1), \quad r \mapsto \varphi r. \]

Hence, each component of $\Pi(\mathbb{T}, \varphi)$ preserves suprema. \hfill \( \square \)

3.4.4 Theorem. Let $\varphi : \mathcal{V} \to \mathcal{W}$ be a homomorphism of quantales strictly compatible with the lax extensions $\hat{T}$ and $\bar{T}$ of $\mathbb{T} = (T, m, e)$ to $\mathcal{V}-\text{Rel}$ and $\mathcal{W}-\text{Rel}$ and let $\psi : \mathcal{W} \to \mathcal{V}$ be the right adjoint of $\varphi$. Then the change-of-base functor $B_\psi$ corresponds to the algebraic functor $A_{\Pi(\mathbb{T}, \varphi)}$, that is, the diagram
\[ (\mathbb{T}, \mathcal{W})-\text{Cat} \xrightarrow{A_{\Pi(\mathbb{T}, \varphi)}} (\Pi(\mathbb{T}, \mathcal{V}), 2)-\text{Cat} \]
\[ \text{Cat} \xrightarrow{B_\psi} \text{Cat} \]

commutes. Here the horizontal arrows represent the isomorphism of Corollary 3.2.3.
Proof. Let \((X,a)\) be a \((\mathbb{T},\mathcal{W})\)-category, \(\rho \in \Pi(\mathbb{T},\mathcal{V})(X)\) and \(x \in X\). Then \(\rho \rightarrow x\) in the \((\Pi(\mathbb{T},\mathcal{V}),2)\)-category obtained via the upper-right path of the diagram if and only if
\[
\varphi \rho \leq a(-,x),
\]
which is equivalent to
\[
\rho \leq \psi a(-,x);
\]
and this means precisely \(\rho \rightarrow x\) in the \((\Pi(\mathbb{T},\mathcal{V}),2)\)-category obtained via the lower-left path of the diagram.

\[\square\]

Exercises

3.A The functor \((-)\sharp : \text{Set} \rightarrow (\widehat{\mathbb{T}},\mathcal{V})\text{-URel}.\) Let \(\widehat{\mathbb{T}}\) be a lax extension to \(\mathcal{V}\text{-Rel}\) of a monad \(\mathbb{T} = (T,m,e)\) on \(\text{Set}\). There is a functor \((-)\sharp : \text{Set} \rightarrow (\widehat{\mathbb{T}},\mathcal{V})\text{-URel}\) that sends a map \(f : X \rightarrow Y\) to the unitary \((\mathbb{T},\mathcal{V})\)-relation \(f\sharp : X \rightarrow \downarrow Y\) given by
\[
f\sharp := e^\circ_Y \cdot \widehat{T}1_X \cdot Tf = e^\circ_Y \cdot \widehat{T}f.
\]
One has in particular
\[
\varphi \circ f\sharp = \varphi \cdot Tf
\]
for all unitary \((\mathbb{T},\mathcal{V})\)-relations \(\varphi : Y \rightarrow Z\). If \(\widehat{\mathbb{T}}\) is associative, then \(f\sharp\) and \(f\sharp^\sharp\) form an adjunction \(f\sharp \dashv f\sharp^\sharp\) in the ordered category \((\mathbb{T},\mathcal{V})\text{-URel}\).

3.B The Kleisli extension of \(\Pi\). For an associative lax extension \(\widehat{\mathbb{T}}\) of \(\mathbb{T}\) to \(\mathcal{V}\text{-Rel}\) and its associated power-enriched monad \(\Pi = (\Pi,\mathbf{m},y)\), the power-enrichment \(\tau : \mathcal{P} \rightarrow \Pi\) is given componentwise by
\[
\tau_X(A) = \bigvee_{x \in A} x^\sharp.
\]
For a relation \(r : X \rightarrow Y\), one has
\[
\tau_X \cdot r^\flat = (r \cdot 1^\sharp_X)^\flat,
\]
where the \((-)^\flat\) operation on the left comes from the construction of the Kleisli extension \([1,4]\) and the \((-)^\flat\) on the right is the one defined in \([3,1]\). Hence, \(r^\flat(\psi_2) = \psi_2 \circ (r \cdot 1^\sharp_X) = \psi_2 \cdot \widehat{T}r\), so that
\[
\psi_1(\Pi r) \psi_2 \iff \psi_1 \leq \psi_2 \cdot \widehat{T}r
\]
for all unitary \((\mathbb{T},\mathcal{V})\)-relations \(\psi_1 : X \rightarrow 1\), \(\psi_2 : Y \rightarrow 1\).

3.C The discrete presheaf monad for the list monad. Since the list monad \(\mathbb{L} = (L,m,e)\) on \(\text{Set}\) is cartesian (see Exercise \([1,Q]\)), the Barr extension \(\mathbb{L}\) of \(\mathbb{L}\) to \(\text{Rel}\) is associative and, moreover, every \((\mathbb{L},2)\)-relation \(\varphi : X \rightarrow Y\) is unitary. With \(\Pi = \Pi(\mathbb{L},2)\) denoting the discrete presheaf monad associated to \(\mathbb{L}\), the functor \(\Pi\) can be identified with the composite
PL of the powerset functor $P$ with $L$. Via this identification, $y_X$ sends $x \in X$ to $\{(x)\}$ and $\Pi f$ sends $A \subseteq LX$ to $\{Lf(x_1, \ldots, x_k) \mid (x_1, \ldots, x_k) \in A\}$, and the multiplication sends $A \in PLPLX$ to

$$\{((x_1,1, \ldots, x_{1,i_1}, \ldots, x_{k,1}, \ldots, x_{k,i_k})) \mid (A_1, \ldots, A_k) \in A, (x_{j,1}, \ldots, x_{j,i_j}) \in A_j\} \in PLX.$$  

3.D A monad morphism $\alpha : \mathbb{V} \rightarrow \mathbb{P}_\mathbb{V}$. The maps $\alpha_X$ of 3.2.6(2) form the components of a monad morphism $\alpha : \mathbb{V} \rightarrow \mathbb{P}_\mathbb{V}$ that is sup-dense but not interpolating.

3.E Continuous $\mathbb{P}_+$-actions on compact Hausdorff spaces. The convergence map  

$$\xi : \beta[0, \infty] \rightarrow [0, \infty], \quad x \mapsto \sup_{A \in \mathbb{x}} \inf_{x \in A} x$$

yields the standard topology on $[0, \infty]$, with respect to which addition $+ : [0, \infty] \times [0, \infty] \rightarrow [0, \infty]$ is continuous. There is a distributive law $\delta$ (see II.3.8) of the ultrafilter monad $\beta$ over the monad $\mathbb{V}$ induced by the monoid $([0, \infty], +, 0)$ (see Example 3.2.6(2)) defined by

$$\delta_X : \beta([0, \infty] \times X) \rightarrow [0, \infty] \times \beta X, \quad w \mapsto (\xi(\beta \pi_1(w)), \beta \pi_2(w)),$$

for each set $X$. The induced monad $\mathbb{w} = ([0, \infty] \times \beta, \check{m}, \check{e})$ has its multiplication given by

$$\check{m}_X : [0, \infty] \times \beta([0, \infty] \times \beta X) \rightarrow [0, \infty] \times \beta X, \quad (v, \mathcal{W}) \mapsto (v + \xi(a), m_X(X))$$

(with $a = \beta \pi_1(\mathcal{W})$, $X = \beta \pi_2(\mathcal{W})$ and $m$ the multiplication of the ultrafilter monad) and its unit by

$$\check{e}_X : X \rightarrow [0, \infty] \times \beta X, \quad x \mapsto (0, e_X(x))$$

(with $e$ the unit of the ultrafilter monad). Describe the category of $\mathbb{w}$-algebras.

3.F Approach spaces as $(\mathbb{w}, 2)$-categories. There is a monad morphism $\alpha : \mathbb{w} \rightarrow \mathbb{P}$ from the monad $\mathbb{w}$ of Exercise 3.E to the monad $\mathbb{P}$ of Example 3.2.6(5) whose component at a set $X$ is given by

$$\alpha_X(u, \chi)(y) = \begin{cases} u & \text{if } \chi = y, \\ \bot & \text{otherwise.} \end{cases}$$

This monad morphism $\alpha : \mathbb{w} \rightarrow \mathbb{P}$ is sup-dense and interpolating, so the algebraic functor

$$A_\alpha : (\mathbb{P}, 2)-\mathbf{Cat} \rightarrow (\mathbb{w}, 2)-\mathbf{Cat}$$

is an isomorphism, and\textit{App} $\cong (\mathbb{w}, 2)$-\textit{Cat}.

\textit{Hint.} To see that $\alpha$ is interpolating, observe that for each $S \subseteq [0, \infty]$ and $u \in [0, \infty]$ with $S \neq \emptyset$ and $u = \inf S$, there is some $a \in \beta[0, \infty]$ with $S \in a$ and $\xi(a) = u$. 

\textit{Exercise 3.G} Find the category $\text{App}$ of all approach spaces $(X, \mathcal{T})$ with $\mathcal{T}$ a continuous $\mathbb{P}_+$-action on $X$ and an approach space $(X, \mathcal{T})$ with $\mathcal{T}$ a continuous $\mathbb{P}_+$-action on $X$ and a continuous $\mathbb{P}_+$-action on $X$.
4 Injective lax algebras as Eilenberg–Moore algebras

Our primary goal in this subsection is to show that certain injective objects in $\mathbb{T}$-$\text{Mon}$ are the Eilenberg–Moore algebras of the monad $\mathbb{T}$. The motivating example is given by continuous lattices, which are the injective objects of the category $\text{Top}_0$ of $T_0$-spaces; however, we do not assume any prior knowledge of these particular ordered structures and will derive the necessary concepts in 4.4. A consequence of the formal approach is a unified treatment for the description of $\text{Sup}$ as a category of injectives for $\text{Ord}$, $\text{Cnt}$ for $\text{Top}$, $\text{Dst}$ for $\text{Cls}$, $\text{Frm}$ for $\text{Cls}_{\text{fin}}$, and $\text{Sup}^{\text{P+}}$ for $\text{Met}$.

4.1 Eilenberg–Moore algebras as Kleisli monoids. The embedding $\text{Set}^\mathbb{T} \hookrightarrow (\mathbb{T}, \mathcal{V})$-$\text{Cat}$ of Proposition III.1.6.5 describes $\mathbb{T}$-algebras as $(\mathbb{T}, \mathcal{V})$-algebras when the lax extension of $\mathbb{T}$ is flat. Here, we show that a similar situation can occur when the extension is not flat.

By Exercise II.3.H, a morphism $\tau : P \to \mathbb{T}$ from the powerset monad $P$ to a monad $\mathbb{T} = (T, m, e)$ on $\text{Set}$ yields a functor $\text{Set}^\mathbb{T} : \text{Set} \to \text{Set}^P$, $(X, a) \mapsto (X, a \cdot \tau X)$ that commutes with the respective underlying-set functors. Via the isomorphism $\text{Set}^P \cong \text{Sup}$ (Example II.3.2.2(2)), every $P$-algebra structure $s : PX \to X$ represents a supremum with right adjoint $\downarrow_X := s^\downarrow : X \to PX$ the down-set map of $X$, that is, the separated order relation on $X$ (II.1.7). Hence, every $\mathbb{T}$-algebra morphism becomes a sup-map $f : (X, a \cdot \tau X) \to (Y, b \cdot \tau Y)$ that has a right adjoint

$$f^\downarrow : Y \to X$$

in $\text{Ord}$. In particular, a $\mathbb{T}$-algebra structure $a : TX \to X$ has a right adjoint $a^\downarrow : X \to TX$ which satisfies

$$a^\downarrow \circ a^\downarrow \leq a^\downarrow \quad \text{and} \quad e_X \leq a^\downarrow . \quad (4.1.i)$$

Indeed, the second inequality follows by adjunction from $a \cdot e_X = 1_X$; the first is a consequence of $a \cdot Ta = a \cdot m_X$, since this identity implies $m_X \leq a^\downarrow \cdot a \cdot Ta$ and therefore

$$m_X \cdot T(a^\downarrow) \cdot a^\downarrow \leq a^\downarrow \cdot a \cdot Ta \cdot T(a^\downarrow) \cdot a^\downarrow = a^\downarrow$$

($a \cdot a^\downarrow = 1_X$ because $1_X = a \cdot e_X \leq a \cdot a^\downarrow \leq 1_X$), that is, $(X, a^\downarrow)$ is a $\mathbb{T}$-monoid. If $f : (X, a) \to (Y, b)$ is a $\mathbb{T}$-algebra homomorphism, then $b \cdot Tf = f \cdot a$ yields

$$Tf \cdot a^\downarrow \leq b^\downarrow \cdot f$$

by adjunction, so $f : (X, a^\downarrow) \to (Y, b^\downarrow)$ is a morphism of $\mathbb{T}$-monoids. Hence, when $(\mathbb{T}, \tau)$ is power-enriched, there is a functor

$$\text{Set}^\mathbb{T} \to \mathbb{T}$-$\text{Mon}$, \quad (X, a) \mapsto (X, a^\downarrow)$$
that commutes with the respective underlying-set functors. In particular, \((TX, m_X^\perp)\) is a \(\mathbb{T}\)-monoid, and every \(\mathbb{T}\)-monoid structure \(\nu\) on \(X\) becomes a \(\mathbb{T}\)-monoid morphism \(\nu : (X, \nu) \to (TX, m_X^\perp)\), since \(\nu \circ \nu \leq \nu\) equivalently means
\[ T\nu \cdot \nu \leq m_X^\perp \cdot \nu . \]

We note that, unlike the functor \(\text{Set} \hookrightarrow (\mathbb{T}, \mathcal{V})\text{-Cat}\) of Proposition \(\text{III.1.6.5}\), the functor \(\text{Set}^\mathbb{T} \to \mathbb{T}\text{-Mon}\) fails to be full in general, as the following Example shows.

**4.1.1 Example.** For the powerset monad \(\mathbb{P}\), the functor \(\text{Set}^\mathbb{P} \to \mathbb{P}\text{-Mon}\) simply describes the forgetful functor \(\text{Set}^\mathbb{P} \cong \text{Sup} \to \text{Ord} \cong \mathbb{P}\text{-Mon}\) (Example \(\text{I.3.2}(2)\)).

**4.2 Monads on categories of Kleisli monoids.** In order to characterize injective objects of \(\mathbb{T}\text{-Mon}\), it is convenient to introduce a monad \(\mathbb{T}'\) derived from the original monad \(\mathbb{T}\) on \(\text{Set}\), as follows. Consider a morphism \(\alpha : S \to \mathbb{T}\) of power-enriched monads \(S = (S, n, d)\) and \(\mathbb{T} = (T, m, e)\). By composing the functor \(\text{Set}^\alpha : \text{Set}^S \to \text{Set}^\mathbb{T}\) (Exercise \(\text{II.3.H}\)) with \(G : \text{Set}^S \to \text{S-Mon}\) (4.1), we obtain a functor
\[ G^{\mathbb{T}'} : \text{Set}^\mathbb{T} \to \text{S-Mon} \]

that sends a \(\mathbb{T}\)-algebra \((X, a)\) to the \(\text{S}\)-monoid \((X, (a \cdot \alpha_X)^\perp)\) and leaves maps unchanged. This functor \(G^{\mathbb{T}'}\) is right adjoint. Rather than describing its left adjoint explicitly, we construct below the induced monad \(\mathbb{T}'\) on \(\text{S-Mon}\) via its Kleisli triple components \(T', e'_X\), and \((\_)^{\mathbb{T}'}\) (Proposition \(4.2.2\)). Theorem \(4.3.2\) then shows that the new monad has the same Eilenberg–Moore category as \(\mathbb{T}\), a fact that justifies \(a \ posteriori\) the notation for the functor \(G^{\mathbb{T}'}\).

In \(4.4\) we exploit the case where \(\alpha = \tau : \mathbb{P} \to \mathcal{F}\) is the principal filter monad morphism to identify the category \(\text{Set}^\mathcal{F}\) as the category \(\text{Cnt}\) of continuous lattices via the monadic functor \(\text{Set}^\mathcal{F} \to \text{Ord} \cong \mathbb{P}\text{-Mon}\). In \(4.6\) we show that when \(\alpha = 1\) : \(\mathbb{T} \to \mathbb{T}\) is the identity, the monadic functor \(\text{Set}^\mathcal{F} \to \mathbb{T}\text{-Mon}\) is Kock–Zöberlein; this property then facilitates our study of injective objects in \(4.6\).

**Construction.**

1. Given an \(\text{S}\)-monoid structure \(\mu : X \to SX\), the map \(\nu := \alpha_X \cdot \mu\) yields a \(\mathbb{T}\)-monoid structure on \(X\) (Proposition \(1.3.3\)). We define the set \(T'X\) of \(\nu^\mathbb{T}\)-invariants as the equalizer in \(\text{Set}\) of the pair \((\nu^\mathcal{F}, 1_{TX})\):
\[ T'X \xrightarrow{q_X} TX \xrightarrow{\nu^\mathcal{F}} TX . \]
The universal property of \( q_X \) implies the existence of a map \( p_X : T X \rightarrow T' X \) such that
\[
q_X \cdot p_X = \nu \quad \text{and} \quad p_X \cdot q_X = 1_{T' X}.
\]
Indeed, as \( \nu \cdot \nu = \nu \) (by (1.3.i)), there is a map \( p_X : T X \rightarrow T' X \) with \( q_X \cdot p_X = \nu \); therefore, we have \( q_X \cdot p_X = q_X = \nu \cdot q_X = \nu \cdot q_X = q_X \), so that \( p_X \cdot q_X = 1_{T' X} \) by unicity of the induced map \( T' X \rightarrow T' X \).

The set \( T' X \) can be equipped with the \( S \)-monoid structure \( \omega_X : T' X \rightarrow ST' X \) given by
\[
\omega_X := S p_X \cdot (m_X \cdot \alpha_{T X})^{\downarrow} \cdot q_X.
\]

Lemma 4.2.1 below shows that \( q_X : (T' X, \omega_X) \rightarrow (T X, (m_X \cdot \alpha_{T X})^{\downarrow}) \) is also an equalizer in \( S_{-}\text{Mon} \). This ensures that the maps \( e'_X \) and \( f'^T \) defined in the following points are morphisms of \( S \)-monoids.

(2) Since \( \nu \cdot \nu = \nu \), there exists a map \( e'_X : X \rightarrow T' X \) such that \( q_X \cdot e'_X = \nu \):

\[
\begin{array}{ccc}
X & \xrightarrow{e'_X} & T' X \\
\downarrow & & \downarrow q_X \\
T' X & \xrightarrow{\nu} & T X & \xrightarrow{1_{T X}} & T X
\end{array}
\]

This yields a morphism of \( S \)-monoids \( e'_X : (X, \mu) \rightarrow (T' X, \omega_X) \). Since \( q_X \cdot e'_X = \nu \) and \( p_X \cdot q_X = 1_{T' X} \), one can equivalently obtain \( e'_X \) as either
\[
e'_X = p_X \cdot \nu \quad \text{or} \quad e'_X = p_X \cdot e_X
\]
because \( p_X \cdot \nu = p_X \cdot \nu \cdot e_X = p_X \cdot q_X \cdot p_X \cdot e_X = p_X \cdot e_X \).

(3) If \( (Y, \mu_Y) \) is another \( S \)-monoid, and \( f : (Y, \mu_Y) \rightarrow (T' X, \omega_X) \) is an \( S \)-monoid morphism, then
\[
\nu \cdot (q_X \cdot f)^{\downarrow} = (\nu \cdot q_X \cdot f)^{\downarrow} = (q_X \cdot f)^{\downarrow}.
\]
Hence, there exists a unique map \( f'^T : T' Y \rightarrow T' X \) making the following diagram commute:

\[
\begin{array}{ccc}
T' Y & \xrightarrow{(q_X \cdot f)^{\downarrow} \cdot q_Y} & T' X \\
\downarrow f'^T & & \downarrow q_X \\
T' X & \xrightarrow{\nu} & T X & \xrightarrow{1_{T X}} & T X
\end{array}
\]

This yields a morphism of \( S \)-monoids \( f'^T : (T' Y, \omega_Y) \rightarrow (T' X, \omega_X) \) that can also be obtained directly as
\[
f'^T = p_X \cdot (q_X \cdot f)^{\downarrow} \cdot q_Y.
\]
4.2.1 Lemma. For a morphism \( \alpha : S \rightarrow \mathbb{T} \) of power-enriched monads \( S = (S, n, d) \) and \( \mathbb{T} = (T, m, e) \), the map
\[
q_X : (T'X, \omega_X) \rightarrow (TX, (m_X \cdot \alpha_{TX})^\dagger)
\]
(defined in the previous construction) is an equalizer in \( S\text{-Mon} \). As a consequence, \( e'_X : (X, \mu) \rightarrow (T'X, \omega_X) \) and \( f^\top : (T'Y, \omega_Y) \rightarrow (T'X, \omega_X) \) are morphisms of \( S\text{-monoids} \).

Proof. To verify that \( \omega_X : T'X \rightarrow ST'X \) is an \( S\text{-monoid} \) structure, observe that
\[
d^X = d^X \cdot p_X \cdot q_X = Sp_X \cdot d^X \cdot q_X \leq Sp_X \cdot (m_X \cdot \alpha_{TX})^\dagger \cdot q_X = \omega_X
\]
because \( (m_X \cdot \alpha_{TX}) \cdot d^X = 1_{TX} \), and
\[
\omega_X \cdot \omega_X = Sp_X \cdot n^X \cdot S(m_X \cdot \alpha_{TX})^\dagger \cdot Su^\top \cdot (m_X \cdot \alpha_{TX})^\dagger \cdot q_X \quad (q_X \cdot p_X = \nu^\top)
\]
\[
\leq Sp_X \cdot n^X \cdot (S(m_X \cdot \alpha_{TX}))^\dagger \cdot Su^\top \cdot (m_X \cdot \alpha_{TX})^\dagger \cdot q_X \quad (m_X \cdot \alpha_{TX} \text{ a retraction})
\]
\[
\leq Sp_X \cdot n^X \cdot (Sm_X \cdot S\alpha^TX)^\dagger \cdot (m_X \cdot \alpha_{TX})^\dagger \cdot \nu^\top \cdot q_X \quad (\nu^\top \text{ morphism in } S\text{-Mon})
\]
\[
= Sp_X \cdot n^X \cdot (m_X \cdot \alpha_{TX} \cdot n^X)^\dagger \cdot \nu^\top \cdot q_X \quad (\alpha \text{ monad morphism})
\]
\[
= Sp_X \cdot (m_X \cdot \alpha_{TX})^\dagger \cdot q_X = \omega_X
\]
\[
= Sp_X \cdot (m_X \cdot \alpha_{TX})^\dagger \cdot q_X = \omega_X
\]
One can reason similarly to obtain
\[
Sq_X \cdot \omega_X = Su^\top \cdot (m_X \cdot \alpha_{TX})^\dagger \cdot q_X \leq (m_X \cdot \alpha_{TX})^\dagger \cdot \nu^\top \cdot q_X = (m_X \cdot \alpha_{TX})^\dagger \cdot q_X
\]
so that \( q_X : (T'X, \omega_X) \rightarrow (TX, (m_X \cdot \alpha_{TX})^\dagger) \) is a morphism in \( S\text{-Mon} \). Suppose now that \( g : (Y, \mu_Y) \rightarrow (TX, (m_X \cdot \alpha_{TX})^\dagger) \) is a morphism in \( S\text{-Mon} \) satisfying \( \nu^\top \cdot g = g \). Since \( q_X : T'X \rightarrow TX \) is an equalizer of \( (\nu^\top, 1_{TX}) \) in \( \text{Set} \), there exists a unique map \( h : Y \rightarrow T'X \) with \( g = q_X \cdot h \); moreover,
\[
Sh \cdot \mu_Y = Sp_X \cdot Sg \cdot \mu_Y \leq Sp_X \cdot (m_X \cdot \alpha_{TX})^\dagger \cdot g = \omega_X \cdot h
\]
which shows that \( h : (Y, \mu_Y) \rightarrow (T'X, \omega_X) \) is a morphism in \( S\text{-Mon} \). As a consequence, \( q_X \) is an equalizer in \( S\text{-Mon} \), and \( e'_X, f^\top \) are the underlying maps of the corresponding unique \( S\text{-monoid} \) morphisms into \( (T'X, \omega_X) \).

4.2.2 Proposition. If \( \alpha : S \rightarrow \mathbb{T} \) is a morphism of power-enriched monads, then the construction described in (1)–(3) above defines a Kleisli triple \( (T', (-)^\top, e') \) on \( S\text{-Mon} \). Hence, \( \mathbb{T}' = (T', m', e') \) is a monad on \( S\text{-Mon} \) with multiplication \( m'_X = 1^T_{TX} = p_X \cdot q_X^\top \cdot q_T^\top \).

Proof. Lemma 4.2.1 ensures that the components \( T', (-)^\top, \) and \( e' \) are of the appropriate type to yield a Kleisli triple on \( S\text{-Mon} \). For an \( S\text{-monoid} \) morphism \( h : (X, \mu) \rightarrow (TY, (\alpha_{TY} \cdot m_Y)^\top) \), we observe that if \( \nu = \alpha_X \cdot \mu \), then
\[
h^\top \cdot \nu = h
The conditions II(3.7.i) are now easily verified by using this observation. The monad $T'$ forms the components of a natural transformation $\varnothing$.

Let 4.3.1 Lemma. We now show that the category of $T$-monad $\vdash d$ therefore describes the down-set monad $\varnothing$.

4.2.3 Examples.

(1) If $S = \mathcal{I} = \mathcal{P}$, then a $\mathcal{P}$-monoid is an ordered set $(X, \downarrow_X)$ (Example 1.3.2(2)). In this case, an element $A \in PX$ is a $\downarrow_X$-invariant precisely when $\downarrow_X(A) = A$, that is, when $A$ is down-closed:

$$\bigcup \{\downarrow_X x \mid x \in A\} = A.$$

The construction in 4.2 therefore describes the down-set monad $\mathcal{D}_n = (\mathcal{D}_n, \sqcup_{\mathcal{D}_n}, \downarrow)$ on $\text{Ord}$ (Example II 4.9.3).

(2) If $S = \mathcal{I} = \mathcal{F}$ is the filter monad, then an $\mathcal{F}$-monoid is a topological space $(X, \nu)$, where $\nu : X \to FX$ is the neighborhood filter map (1.1). A filter $a \in FX$ is $\mathcal{F}$-invariant if and only if $a$ is spanned by open sets of $X$:

$$A \in \nu^{-1}(a) \iff \nu^{-1}(A^\uparrow) \in a \iff \{x \in X \mid A \in \nu(x)\} \in a,$$

so $\nu^\uparrow(a) = a$ means that for every $A \in a$ the interior must also be in $a$. The open-filter monad on $\text{Top} \cong \mathcal{F}$-Mon, that is, the monad $\mathcal{F} = (\mathcal{F}', e', m')$ obtained via Proposition 4.2.2 can therefore be described as follows. The functor $\mathcal{F}' : \text{Top} \to \text{Top}$ sends a topological space $X$ to the set $\mathcal{F}'X$ of filters in $\mathcal{O}X$ equipped with its Scott topology; $\mathcal{F}'$ also sends a continuous map $f : X \to Y$ to the continuous map $\mathcal{F}'f : \mathcal{F}'X \to \mathcal{F}'Y$ given by $O \in \mathcal{F}'f(a) \iff f^{-1}(O) \in a$ for all $a \in \mathcal{F}'Y$. The unit $\nu = e_X' : X \to \mathcal{F}'X$ sends a point $x$ to its set of open neighborhoods, and the multiplication $m_X' : \mathcal{F}'\mathcal{F}'X \to \mathcal{F}'X$ is the restriction of the Kowalsky sum.

4.3 Eilenberg–Moore algebras over $\mathcal{S}$-Mon. For power-enriched monads $\mathcal{S}$ and $\mathcal{T}$, the monad $\mathcal{T}'$ on $\mathcal{S}$-Mon is derived from $\mathcal{T}$ via the morphism $\alpha : \mathcal{S} \to \mathcal{T}$. Using the notations of 4.2 we now show that the category of $\mathcal{T}'$-algebras is isomorphic to $\text{Set}^{\mathcal{I}}$.

4.3.1 Lemma. Let $U : \mathcal{S}$-Mon $\to \text{Set}$ denote the forgetful functor. The maps $p_X$ of 4.2 form the components of a natural transformation $p : TU \to UT'$ that defines a lifting of $U$ through $(G^\uparrow, G')$ (Exercise II 3.H).

Proof. An $\mathcal{S}$-monoid morphism $f : (Y, \mu_Y) \to (X, \mu)$ yields a $\mathcal{T}$-monoid morphism $f : (Y, \nu_Y) \to (TX, \nu)$ (where $\nu_Y = \alpha_Y \cdot \mu_Y$ and $\nu = \alpha_X \cdot \mu$), so that

$$\nu \cdot f = (\nu \cdot f)^\uparrow \cdot e_Y \leq (\nu \cdot f)^\uparrow \cdot \nu_Y = \nu^\uparrow \cdot T f \cdot \nu_Y \leq \nu^\uparrow \cdot \nu \cdot f = \nu \cdot f.$$
Therefore, \((\nu \cdot f)^\top \cdot \nu_Y = \nu \cdot f\), and using \(T'f = (e_X' \cdot f)^\top\) we obtain
\[
T'f \cdot \nu_Y = p_X \cdot (\nu \cdot f)^\top \cdot \nu_Y = p_X \cdot ((\nu \cdot f)^\top \cdot \nu_Y)^\top = p_X \cdot (\nu \cdot f)^\top = p_X \cdot \nu^\top \cdot Tf = p_X \cdot Tf,
\]
so that \(p : T\!U \to U T'\) is a natural transformation. Moreover, for \(m_X' = (1_{r'X})^\top = p_X \cdot (q_X)^\top \cdot q_{r'X}\),
\[
m_X' \cdot p_{r'X} \cdot Tp_X = p_X \cdot (q_X)^\top \cdot (\alpha_X' \cdot \omega_X)^\top \cdot Tp_X
\]
\[
= p_X \cdot (m_X \cdot T\nu^\top \cdot \alpha_X \cdot (m_X \cdot \alpha_X')^\top \cdot q_X)^\top \cdot Tp_X
\]
\[
= p_X \cdot (\nu^\top \cdot m_X \cdot \alpha_X \cdot (m_X \cdot \alpha_X')^\top \cdot q_X)^\top \cdot Tp_X
\]
\[
= p_X \cdot (\nu^\top \cdot q_X)^\top \cdot Tp_X
\]
\[
= p_X \cdot (q_X)^\top \cdot Tp_X
\]
\[
= p_X \cdot m_X
\]
Since \(e_X' = p_X \cdot e_X\) we conclude that \(r\) does indeed define a lifting of \(U\) through \((G^\top, G^\top)\). □

4.3.2 Theorem. If \(\alpha : \mathbb{S} \to \mathbb{T}\) is a morphism of power-enriched monads, then there is an isomorphism

\[
\text{Set}^\top \cong \mathbb{S}\text{-Mon}^\top
\]
of Eilenberg–Moore categories that commutes with the underlying-set functors.

Proof. Suppose first that \((X, a)\) is a \(\mathbb{T}\)-algebra. One obtains an \(\mathbb{S}\)-monoid \((X, \mu)\), with \(\mu = (a \cdot \alpha_X)^\top\), that can be equipped with the structure \(a' : (T'X, \omega_X) \to (X, \mu)\) defined by
\[
a' := a \cdot q_X.
\]
Since \(a : (TX, m_X) \to (X, a)\) is a \(\mathbb{T}\)-algebra morphism, its \(G^\top\)-image is an \(\mathbb{S}\)-monoid morphism, and therefore so is \(a'\). To see that \(a'\) satisfies the algebra conditions for the monad \(\mathbb{T}'\), we use the definition of \(e_X'\):
\[
a' \cdot e_X' = a \cdot q_X \cdot e_X' = a \cdot \alpha_X \cdot (a \cdot \alpha_X)^\top = 1_X.
\]
Suppose now that \(f, g : (Y, \mu_Y) \to (Y'X, \omega_X)\) are morphisms in \(\mathbb{S}\text{-Mon}\) satisfying \(a' \cdot f = a' \cdot g\), or equivalently, \(a \cdot q_X \cdot f = a \cdot q_X \cdot g\); since \(a\) is a \(\mathbb{T}\)-algebra structure, one has \(a \cdot (q_X \cdot f)^\top = a \cdot (q_X \cdot g)^\top\) (Exercise 3.3.D), so that
\[
a' \cdot f^\top = a \cdot q_X \cdot f^\top = a \cdot (q_X \cdot f)^\top \cdot q_Y = a \cdot (q_X \cdot g)^\top \cdot q_Y = a \cdot q_X \cdot g^\top = a' \cdot g^\top.
\]
Therefore, \(((X, \mu), a')\) is a \(\mathbb{T}'\)-algebra. A \(\mathbb{T}\)-algebra morphism \(f : (X, a) \to (Y, a_Y)\) yields a morphism \(f : (X, (a \cdot \alpha_X)^\top) \to (Y, (a_Y \cdot \alpha_Y)^\top)\) in \(\mathbb{S}\text{-Mon}\). To verify that \(a'_Y \cdot (e'_Y \cdot f)^\top = f \cdot a'\), we first observe
\[
a_Y \cdot (\alpha_Y \cdot (a_Y \cdot \alpha_Y)^\top \cdot f)^\top = a_Y \cdot m_X \cdot T(\alpha_Y \cdot (a_Y \cdot \alpha_Y)^\top)^\top \cdot T f
\]
\[
= a_Y \cdot T \alpha_Y \cdot T(\alpha_Y \cdot (a_Y \cdot \alpha_Y)^\top)^\top \cdot T f
\]
\[
= a_Y \cdot T f.
\]
Hence,
\[
    a_Y' \cdot (e_Y' \cdot f)^T = a_Y \cdot (q_Y \cdot e_Y' \cdot f)^T \cdot q_X
\]
\[
    = a_Y \cdot (\alpha_Y \cdot (a_Y \cdot \alpha_Y)^{-1} \cdot f)^T \cdot q_X
\]
\[
    = f \cdot a \cdot q_X
\]
\[
    = f \cdot a',
\]
which proves that \(f : ((X, \mu), a') \to ((Y, \mu_Y), a'_Y)\) is a morphism of \(\mathbb{T}'\)-algebras. Thus, the assignment of \(((X, (a \cdot \alpha_X)^{-1}), a \cdot q_X)\) to a \(\mathbb{T}\)-algebra \((X, a)\) yields a functor \(K : \text{Set}^{\mathbb{T}} \to \mathbb{S} \text{-Mon}^{\mathbb{T}}\) that commutes with the underlying-set functors.

The lifting \(p : TU \to UT'\) of \(U\) (see Lemma 4.3.1) yields a functor \(\tilde{U} : \mathbb{S} \text{-Mon}^{\mathbb{T}} \to \text{Set}^{\mathbb{T}}\) that sends a \(\mathbb{T}'\)-algebra \(((X, \mu), a')\) to \((X, a)\), where \(a : TX \to X\) is defined by
\[
a := a' \cdot p_X,
\]
and is invariant on maps.

Given a \(\mathbb{T}\)-algebra \((X, a)\), the structure of \(\tilde{U}K(X, a)\) is described by
\[
a \cdot q_X \cdot p_X = a \cdot m_X \cdot T(\alpha_X \cdot (a \cdot \alpha_X)^{-1}) = a \cdot T(a \cdot \alpha_X) \cdot T(a \cdot \alpha_X)^{-1} = a.
\]

To study the image of a \(\mathbb{T}'\)-algebra \(((X, \mu), a')\) via \(K\tilde{U}\), note first that \(a' : (T'X, \omega_X) \to (X, \mu)\) is a morphism in \(\mathbb{S} \text{-Mon}\). Thus, after setting \(\nu = \alpha_X \cdot \mu\) and observing that \((m_X \cdot \alpha_{TX}) \cdot S(\alpha_X \cdot \mu) = \nu^{\mathbb{T}} \cdot \alpha_X\), one obtains
\[
1_{SX} = S(a' \cdot p_X) \cdot S(\alpha_X \cdot \mu)
\]
\[
\leq S(a' \cdot p_X) \cdot (m_X \cdot \alpha_{TX})^{-1} \cdot \nu^{\mathbb{T}} \cdot \alpha_X
\]
\[
= S\alpha' \cdot \omega_X \cdot p_X \cdot \alpha_X
\]
\[
\leq \mu \cdot (a' \cdot p_X \cdot \alpha_X).
\]

This inequality, combined with \((a' \cdot p_X \cdot \alpha_X) \cdot \mu = 1_X\) yields
\[
\mu = (a' \cdot p_X \cdot \alpha_X)^{-1}.
\]

Hence, the image under \(K\tilde{U}\) of the \(\mathbb{T}'\)-algebra \(((X, \mu), a')\) is the \(\mathbb{T}'\)-algebra with underlying \(\mathbb{S}\)-monoid \(((X, (a' \cdot p_X \cdot \alpha_X)^{-1}) = (X, \mu)\) and structure
\[
a' \cdot p_X \cdot q_X = a'.
\]

One concludes that \(K\) and \(\tilde{U}\) are inverse of each other, and \(\text{Set}^{\mathbb{T}} \cong \mathbb{S} \text{-Mon}^{\mathbb{T}}\).  \(\square\)

In general, a \(\mathbb{T}\)-monoid \((X, \nu)\) on a category \(X\) is separated whenever \(\nu\) is a monomorphism. For a power-enriched monad \(\mathbb{T}\), this condition amounts to the requirement that the initial order induced on \(X\) by \(\nu : X \to TX\),
\[
x \leq y \iff \nu(x) \leq \nu(y)
\](4.3.i)
for all \(x, y \in X\), is separated. The full subcategory of \(\mathbb{T}\)-\text{Mon} whose objects are the separated Kleisli monoids is denoted by \(\mathbb{T}\text{-Mon}_{\text{sep}}\).

**4.3.3 Corollary.** Given a morphism \(\alpha : \mathbb{S} \rightarrow \mathbb{T}\) of power-enriched monads, the monad \(\mathbb{T}'\) of Proposition 4.2.2 restricts to a monad on \(\mathbb{S}\text{-Mon}_{\text{sep}}\), and there is an isomorphism

\[
\text{Set}^{\mathbb{T}'} \cong (\mathbb{S}\text{-Mon}_{\text{sep}})^{\mathbb{T}'}
\]

that commutes with the underlying-set functors.

**Proof.** The results of 4.2 leading up to Theorem 4.3.2 can be reproduced by replacing \(\mathbb{S}\)-\text{Mon} by \(\mathbb{S}\text{-Mon}_{\text{sep}}\). Indeed, the functor \(G^{\mathbb{T}'} : \text{Set}^{\mathbb{T}} \rightarrow \mathbb{S}\text{-Mon}\) factors through \(\mathbb{S}\text{-Mon}_{\text{sep}}\), and \((T'X, \omega_X)\) is separated for all \(\mathbb{S}\)-monoids \((X, \mu)\): setting \(\nu = \alpha_X \cdot \mu\), one has

\[
p_X \cdot (m_X \cdot \alpha_{TX}) \cdot S q_X \cdot \omega_X = p_X \cdot (m_X \cdot \alpha_{TX}) \cdot S \nu^T \cdot (m_X \cdot \alpha_{TX})^t \cdot q_X = p_X \cdot \nu^T \cdot (m_X \cdot \alpha_{TX}) \cdot (m_X \cdot \alpha_{TX})^t \cdot q_X = 1_{T'X},
\]

that is, \(\omega_X\) is a section.

\[\square\]

**4.3.4 Examples.**

(1) The initial order on a \(\mathbb{P}\)-monoid \((X, \downarrow)\) is given by

\[
x \leq y \iff \downarrow x \leq \downarrow y
\]

for all \(x, y \in X\). Hence, if \(\downarrow : X \rightarrow PX\) is a monomorphism, then \(\downarrow x = \downarrow y\) implies \(x = y\), so \(X\) is a separated ordered set, and we have \(\mathbb{P}\text{-Mon}_{\text{sep}} \cong \text{Ord}_{\text{sep}}\). Hence, with \(\mathbb{T} = \mathbb{S} = \mathbb{P}\), Theorem 4.3.2 and Corollary 4.3.3 state that the forgetful functors

\[
\text{Sup} \rightarrow \text{Ord} \quad \text{and} \quad \text{Sup} \rightarrow \text{Ord}_{\text{sep}}
\]

are strictly monadic (see Example 4.1.1).

(2) The initial order induced on a topological space, seen as an \(\mathbb{F}\)-monoid, by its neighborhood map \(\nu : X \rightarrow FX\) is its underlying order (see \[\text{II.1.9}\]). Hence, the isomorphism \(\mathbb{F}\text{-Mon} \cong \text{Top}\) identifies separated \(\mathbb{F}\)-monoids with \(T_0\)-spaces (Exercise \[\text{I.B}\]). With \(\text{Top}_0\) the corresponding full category of \(\text{Top}\), we therefore have \(\mathbb{F}\text{-Mon}_{\text{sep}} \cong \text{Top}_0\). With \(\mathbb{S} = \mathbb{F}\) or \(\mathbb{S} = \mathbb{P}\), we obtain strictly monadic functors

\[
\text{Set}^{\mathbb{F}} \rightarrow \text{Top}, \quad \text{Set}^{\mathbb{F}} \rightarrow \text{Top}_0, \quad \text{Set}^{\mathbb{F}} \rightarrow \text{Ord}, \quad \text{and} \quad \text{Set}^{\mathbb{F}} \rightarrow \text{Ord}_{\text{sep}},
\]

thanks to Theorem 4.3.2 and Corollary 4.3.3. The identification of \(\text{Set}^{\mathbb{F}}\) as a category of lattices requires more work and is considered in the next section.
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4.4 Continuous lattices. For a monad morphism $\alpha : P \to T$, the study of $T$-algebras can be facilitated by the study of the monad $T'$ on $\text{Ord} \cong P\text{-Mon}$, as a consequence Theorem [1.3.2].

This approach leads us here to the identification of continuous lattices as the Ini-injective objects of $\text{Top}$, for Ini the class of $U$-initial morphisms (with $U : \text{Top} \to \text{Set}$ forgetful).

A complete lattice $X$ is continuous if the restriction of the supremum map $\bigvee_X : \text{Dn}X \to X$ to the set $\text{Idl}X$ of ideals in $X$ has a left adjoint $\downarrow_X$:

$$\downarrow_X \dashv \bigvee_X : \text{Idl}X \to X$$

(compare with complete distributivity [II.1.11]). The category of continuous lattices and inf-maps which preserve up-directed suprema is denoted by $\text{Cnt}$.

Since our focus in this subsection is on convergence of filters rather than that of ideals, from now on we will concentrate on the dual notion of cocontinuous lattices.

A complete lattice $X$ is cocontinuous if the restriction of the infimum map $\bigwedge_X : \text{Up}X \to X$ to the set $\text{Fil}X$ of filters in $X$ has a right adjoint $\uparrow_X$:

$$\bigwedge_X \dashv \uparrow_X : X \to \text{Fil}X.$$ 

The order on $\text{Fil}X$ is induced by the order on $\text{Up}X$ and is therefore given by reverse inclusion, see [II.1.7] hence, one has

$$\bigwedge S \leq a \iff S \supseteq \uparrow a$$

for all $S \in \text{Fil}X$, $a \in X$, and

$$\uparrow a = \bigcap\{S \in \text{Fil}X \mid \bigwedge S \leq a\}$$

by Proposition [II.1.8.2]. Note also that $\bigwedge \uparrow a = a$ for all $a \in X$, that is, $\uparrow_X : X \to FX$ is a fully faithful embedding. Cocontinuous lattices with sup-maps that preserve down-directed infima form a 2-category which is 2-isomorphic to $\text{Cnt}^{\text{co}}$ (the isomorphism sends $f : X \to Y$ to $f^{\text{op}} : X^{\text{op}} \to Y^{\text{op}}$). Hence, in the sequel, we will write $\text{Cnt}^{\text{co}}$ to designate the category of cocontinuous lattices.

The function $\text{Fil}$ defined on ordered sets is the object part of a 2-functor $\text{Fil} : \text{Ord} \to \text{Ord}$ that is the restriction of the up-set 2-functor $\text{Up}$ to filters: for a map $f : X \to Y$, one has

$$\text{Fil} f(A) = \bigcup_{x \in A} \uparrow f(x)$$

for all filters $A \subseteq X$. The up-set map of an ordered set $X$ corestricts to a monotone map $\uparrow_X : X \to \text{Fil}X$, and union yields the infimum map $\bigwedge_{\text{Fil}X} : \text{Fil}\text{Fil}X \to \text{Fil}X$ to form the ordered-filter monad

$$\mathcal{F} = (\text{Fil}, \bigwedge_{\text{Fil}}, \uparrow)$$
4. INJECTIVE LAX ALGEBRAS AS EILENBERG–MOORE ALGEBRAS

There is a distributive law \( \lambda : \text{DnFil} \to \text{FilDn} \) of the down-set monad

\[
\text{Dn} = (\text{Dn}, \lor, \downarrow)
\]

(Example [I]4.9.3) over \( \text{Fil} \) whose components are given by

\[
\lambda_X(\chi) = \{ A \in \text{Dn}X \mid \forall B \in \chi \ (A \cap B \neq \emptyset) \}
\]

for all ordered sets \( X \) and \( \chi \in \text{DnFil}X \) (Exercise 4.A). The resulting monad is the down-set-filter monad

\[
\text{FilDn} = (\text{FilDn}, \land, \lor, \downarrow, \uparrow, \downarrow). 
\]

4.4.1 Proposition. The category \( \text{Cnt}^\text{co} \) is strictly monadic over \( \text{Ord} \). More precisely, there is a 2-isomorphism

\[
\text{Cnt}^\text{co} \cong \text{Ord}^\text{FilDn}
\]

which commutes with the forgetful functors to \( \text{Ord} \).

Proof. For a cocontinuous lattice \( X \), the diagram

\[
X \xrightarrow{\Lambda_X} \text{Fil}X \xleftarrow{\text{Fil} \lor} \text{FilDn}X
\]

motivates us to define a map \( a := \Lambda_X \cdot \text{Fil} \lor \). Let us verify that \( a \) yields the structure morphism of an \( \text{FilDn} \)-algebra. The unit condition \( a \cdot \text{Fil} \downarrow \cdot \uparrow = 1_X \) is immediate, so we only need to verify the multiplication condition

\[
a \cdot \text{FilDn}a = a \cdot \Lambda_{\text{FilDn}X} \cdot \text{Fil} \lor \text{FilDn}X:
\]

\[
= \Lambda_X \cdot \text{Fil} (a \cdot \lor \text{FilDn}X) \quad \text{(a left adjoint by (4.4.i))}
\]

\[
= \Lambda_X \cdot \text{Fil} \lor \cdot \Lambda_{\text{FilDn}X} \cdot \text{Fil} \lor \text{FilDn}X \quad \text{(} \Lambda_X \text{ preserves infima)}
\]

\[
= \Lambda_X \cdot \text{Fil} \lor \cdot \Lambda_{\text{FilDn}X} \cdot \text{Fil} \lor \text{FilDn}X \quad \text{(} \Lambda_{\text{Fil}} \text{ natural transformation.)}
\]

Consider now an \( \text{FilDn} \)-algebra \((X, a : \text{FilDn}X \to X)\). Since there is a monad morphism \( \uparrow \text{Dn} : \text{Dn} \to \text{FilDn} \), the ordered set \( X \) is a \( \text{Dn} \)-algebra, that is, a complete lattice with supremum given by \( \lor X = a \cdot \uparrow \text{Dn}X \). The \( \text{FilDn} \)-algebra morphism \( a \) is a sup-map and a retraction, so it has a fully faithful right adjoint \( a^\downarrow : X \to \text{FilDn}X \) and we have the adjunctions

\[
X \xleftarrow{a^\downarrow} \text{FilDn}X \xrightarrow{\text{Fil} \downarrow} \text{Fil}X.
\]

Since the components of the ordered-filter monad multiplication are \( \Lambda_{\text{Fil}X} \), we have in particular \( \Lambda_{\text{FilDn}X} \cdot \text{Fil} \uparrow \text{Dn}X = 1_{\text{FilDn}X} \). By naturality of \( \downarrow \) and the algebra multiplication condition,

\[
a \cdot \text{Fil} \downarrow X \cdot \text{Fil} (a \cdot \uparrow \text{Dn}X) = a \cdot \Lambda_{\text{FilDn}X} \cdot \text{Fil} \lor \text{FilDn}X \cdot \text{Fil} \downarrow \text{FilDn}X \cdot \text{Fil} \uparrow \text{Dn}X = a.
\]
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One deduces that \( a \cdot \operatorname{Fil} \downarrow X \) admits \( \operatorname{Fil}(a \cdot \uparrow_{Dn} X) \cdot a^{-} \) as a right adjoint. As the infimum operation on \( X \) (obtained via the monad morphism \( \operatorname{Fil} \downarrow: \operatorname{Fil} \rightarrow \operatorname{FilDn} \)) is precisely \( \wedge_X = a \cdot \operatorname{Fil} \downarrow X \), the ordered set \( X \) is a cocontinuous lattice (and \( a = \wedge_X \cdot \operatorname{Fil} \vee X \) by the previous displayed identity).

Finally, a cocontinuous lattice morphism \( f: X \rightarrow Y \) is also an \( \operatorname{FilDn} \)-algebra morphism, and a morphism \( f: (X, a) \rightarrow (Y, b) \) of \( \operatorname{FilDn} \)-algebras naturally preserves both suprema and down-directed infima because it is both a \( Dn \)-algebra and an \( \operatorname{Fil} \)-algebra morphism.

The monad \( \operatorname{FilDn} \) on \( \operatorname{Ord} \) is the monad \( F' \) on \( \mathbb{P}\)-\( \operatorname{Mon} \) obtained in 4.2 from the filter monad \( F \) on \( \operatorname{Set} \). Indeed, consider the principal filter natural transformation \( \tau: \mathbb{P} \rightarrow F \) and a \( \mathbb{P} \)-monoid \( (X, \downarrow_X) \) (Example 1.3.2(2)). The construction of 4.2 associates to this ordered set the topological space \( (X, \nu) \) whose neighborhood map is given at each \( x \in X \) by the principal filter of \( \downarrow_X x \in PX: \nu(x) = \uparrow_{PX}(\downarrow_X x) \).

Hence, \( (X, \nu) \) is the Alexandroff space associated with the ordered set \( X \) (II.5.10.5), and open sets are the down-closed sets. Thanks to Example 4.2.3(2), the set of \( \nu^F \)-invariant filters can be identified with the set of filters on \( Dn X \), so that \( F' = \operatorname{FilDn} \) is the down-set-filter monad on \( \operatorname{Ord} \). Proposition 4.4.1 and Theorem 4.3.2 now yield the isomorphism

\[ \operatorname{Cnt}^{co} \cong \operatorname{Set}^{\mathbb{F}}. \]

The functor \( G^{F'}: \operatorname{Set}^{\mathbb{F}} \rightarrow \mathbb{F}\)-\( \operatorname{Mon} \) of 4.2 sending a \( T \)-algebra \( (X, a) \) to the \( T \)-monoid \( (X, a^{-}) \) therefore describes the functor \( \operatorname{Cnt}^{co} \rightarrow \operatorname{Top} \) that equips a cocontinuous lattice with its \textit{Scott topology} and sends a filtered-inf-preserving sup-map to a continuous map (see also Exercise 4.D).

Moreover, separated \( \mathbb{F} \)-monoids are precisely \( \mathcal{T}0 \)-spaces and separated \( \mathbb{P} \)-monoids are separated ordered sets (Examples 4.3.4) so Corollary 4.3.3 tells us that the forgetful functor

\[ \operatorname{Cnt}^{co} \rightarrow \operatorname{Ord}_{\text{sep}} \]

is strictly monadic.

4.5 Kock–Zöblerlein monads on \( \mathbb{T}\)-\( \operatorname{Mon} \). The powerset monad \( \mathbb{P} \) induces the Kock–Zöblerlein monad \( \mathbb{D}n \) on \( \operatorname{Ord} \) (Example 4.2.3(1)), and we will see that \( \mathbb{F} \) yields such a monad on \( \operatorname{Top} \) (Example 4.2.3(2)). Before proving in Theorem 4.5.3 that an arbitrary power-enriched monad \( \mathbb{T} \) induces a Kock–Zöblerlein monad \( \mathbb{T}' \) on \( \mathbb{T}\)-\( \operatorname{Mon} \), as in the cases \( \mathbb{T} = \mathbb{P} \) or \( \mathbb{T} = \mathbb{F} \), we must first show that \( \mathbb{T}\)-\( \operatorname{Mon} \) is an ordered category.

4.5.1 Lemma. If \( \mathbb{T} = (T, m, e) \) is a power-enriched monad, the initial order (4.3.1) induced by the Kleisli monoid structures yields functors

\[ \mathbb{T}\)-\( \operatorname{Mon} \rightarrow \operatorname{Ord} \quad \text{and} \quad \mathbb{T}\)-\( \operatorname{Mon}_{\text{sep}} \rightarrow \operatorname{Ord}_{\text{sep}} \]

that commute with the underlying-set functors. As a consequence, \( \mathbb{T}\)-\( \operatorname{Mon} \) is an ordered and \( \mathbb{T}\)-\( \operatorname{Mon}_{\text{sep}} \) a separated ordered category.
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Proof. To see that morphisms in \(\mathbb{T}-\text{Mon}\) are monotone, consider \(f : (X, \nu) \to (Y, \mu)\) and suppose that \(x, y \in X\) are such that \(x \leq y\), or equivalently \(\nu(x) \leq \nu(y)\). Then

\[
e_Y \cdot f(x) = Tf \cdot e_X(x) \leq Tf \cdot \nu(x) \leq Tf \cdot \nu(y) \leq \mu \cdot f(y);
\]

by composing with \(\mu^\top\) on the left, one obtains \(\mu \cdot f(x) \leq \mu \cdot f(y)\), so that \(f(x) \leq f(y)\) by definition of the order on \(Y\). Hence, \(\mathbb{T}\)-monoid morphisms are monotone, and we have the claimed functors into \(\text{Ord}\) and \(\text{Ord}_{\text{sep}}\). Since hom-sets of \(\mathbb{T}-\text{Mon}\) are ordered pointwise, we have monotone maps

\[
f \cdot (-) : \mathbb{T}-\text{Mon}(Z, X) \to \mathbb{T}-\text{Mon}(Z, Y) \quad \text{and} \quad (-) \cdot g : \mathbb{T}-\text{Mon}(X, Y) \to \mathbb{T}-\text{Mon}(Z, Y)
\]

for all \(f \in \mathbb{T}-\text{Mon}(X, Y)\) and \(g \in \mathbb{T}-\text{Mon}(Z, X)\), and \(\mathbb{T}-\text{Mon}\) is a separated ordered category. \(\square\)

The initial order allows us to refine our analysis of the maps \(q_X\) and \(p_X\) defined in 4.2.

### 4.5.2 Lemma

Let \(\mathbb{T} = (T, m, e)\) be a power-enriched monad, \(\mathbb{T}'\) its induced monad on \(\mathbb{T}-\text{Mon}\) (with \(\mathbb{S} = \mathbb{T}\) and \(\alpha = 1_{\mathbb{T}}\) in 4.2), and \((X, \nu)\) a \(\mathbb{T}\)-monoid. Then, with respect to the initial order on \(T'X\) induced by \(\omega_X\), the section \(q_X : (T'X, \omega_X) \to (TX, m^\top_X)\) is an order-embedding and the retraction \(p_X : (TX, m^\top_X) \to (T'X, \omega_X)\) is a \(\mathbb{T}\)-monoid morphism. In fact, \(p_X \dashv q_X\).

Proof. By Lemmata 4.2.1 and 4.5.1 the maps \(q_X\) are monotone. Suppose that for a \(\mathbb{T}\)-monoid \((X, \nu)\) there are \(x, y \in TX\) with \(q_X(x) \leq q_X(y)\), or equivalently, such that \(m^\top_X \cdot q_X(x) \leq m^\top_X \cdot q_X(y)\). One can compose each side of this last inequality with the monotone map \(Tp_X\) on the left to obtain \(\omega_X(x) \leq \omega_X(y)\), that is, \(x \leq y\) by definition of the order on \(T'X\).

Since \(\mathbb{T}\) is power-enriched, \(e_X \leq \nu\) so \(1_{TX} = e_X^\top \leq \nu^\top = q_X \cdot p_X\) and

\[
Tp_X \cdot m^\top_X \leq Tp_X \cdot m^\top_X \cdot q_X \cdot p_X = \omega_X \cdot p_X.
\]

Hence, \(p_X\) is a \(\mathbb{T}\)-monoid morphism.

The last statement then follows from \(1_{TX} \leq \nu^\top = q_X \cdot p_X\) and \(p_X \cdot q_X = 1_{T'X}\). \(\square\)

### 4.5.3 Theorem

For a power-enriched monad \(\mathbb{T}\), the derived monad \(\mathbb{T}'\) on \(\mathbb{T}-\text{Mon}\) (or \(\mathbb{T}-\text{Mon}_{\text{sep}}\)) is of Kock–Zöberlein type.

Proof. Since \(\mathbb{T}\) is power-enriched, one has for \(\mathbb{T}\)-monoid morphisms \(f, g : (X, \nu) \to (Y, \mu)\)

\[
f \leq g \implies (e'_Y \cdot f)^\top = p_Y \cdot (q_Y \cdot e'_Y \cdot f)^\top \cdot q_X \leq p_Y \cdot (q_Y \cdot e'_Y \cdot g)^\top \cdot q_X = (e'_Y \cdot g)^\top,
\]

so \(T'\) is a 2-functor (since \((e'_Y \cdot f)^\top = T'f\)
To verify that \( m'_X \cdot e'_{T^X} \) for all \( \mathbb{T} \)-monoids \((X, \nu)\), it suffices to verify \( 1_{T^X} \leq e'_{T^X} \cdot m'_X \) (because \( m'_X \cdot e'_{T^X} = 1_{T^X} \) always holds). In view of this, we write

\[
q_{T^X} \leq Tp_X \cdot m_{X}^{-1} \cdot m_X \cdot Tq_X \cdot q_{T^X}
= Tp_X \cdot m_{X}^{-1} \cdot (q_X)\top \cdot q_{T^X}
= Tp_X \cdot m_{X}^{-1} \cdot (\nu^\top \cdot q_X)\top \cdot q_{T^X}
= Tp_X \cdot m_{X}^{-1} \cdot q_X \cdot p_X \cdot (q_X)\top \cdot q_{T^X}
= \omega_X \cdot m_{X}^{-1} = q_{T^X} \cdot e_{T^X} \cdot m_{X}^{-1}.
\]

Hence, \( 1_{T^X} \leq e'_{T^X} \cdot m'_X \) because \( q_{T^X} \) is an order-embedding (Lemma 4.5.2). The same proof obviously holds if \( \mathbb{T}-\text{Mon} \) is replaced by \( \mathbb{T}-\text{Mon}_{\text{sep}} \).

4.5.4 Examples.

1. If \( \mathbb{T} = \mathbb{P} \), Example 4.2.3(1) shows that \( \mathbb{P}' = \mathbb{D} \) is the down-set monad on \( \text{Ord} \). Theorem 4.5.3 states that this monad is Kock–Zöberlein and we recover Example II.4.9.3.

2. The open-filter monad of Example 4.2.3(2) is the monad \( \mathbb{F}' \) on \( \mathbb{F}-\text{Mon} \cong \text{Top} \) derived from the filter monad \( \mathbb{F} \), and is therefore Kock–Zöberlein.

4.6 Eilenberg–Moore algebras and injective Kleisli monoids. We are now ready to look into the identification of injective objects that motivated this Section 4.

Let \( U : \mathbb{T}-\text{Mon} \rightarrow \text{Set} \) be the forgetful functor associated to a power-enriched monad \( \mathbb{T} \). We denote by \( \text{Ini} U \) of all \( U \)-initial \( \mathbb{T} \)-monoid morphisms, and by \( \text{RegMono}(\mathbb{T}-\text{Mon}) \) of all \( U \)-initial \( \mathbb{T} \)-monoid morphisms whose underlying maps are monomorphisms (Exercise II.5.D).

In Theorem 4.6.3 below, we identify \( \mathcal{M} \)-injective Kleisli monoids when \( \mathcal{M} = \text{Ini} \) or \( \mathcal{M} = \text{RegMono} \) as \( \mathbb{T} \)-algebras (see Exercise II.5.M). If \( A \) is an ordered category, we denote by \( \mathcal{M}-\text{Inj}(A) \) the category of \( \mathcal{M} \)-injective \( A \)-objects with left adjoint \( A \)-morphisms.

When \( \mathbb{T} \) is power-enriched, every map \( Tf : (TX, m_X^{-1}) \rightarrow (TY, m_Y^{-1}) \). In fact, its right adjoint \( (Tf)^\top \) is a right adjoint \( (Tf)^\top : (TY, m_Y^{-1}) \rightarrow (TX, m_X^{-1}) \) in \( \mathbb{T}-\text{Mon} \). Indeed, \((-)^\top \) applied to \( Tf \cdot (Tf)^\top \leq 1_{TX} \) yields the inequality in

\[
Tf \cdot m_X \cdot T(Tf)^\top = m_Y \cdot TTf \cdot T(Tf)^\top \leq m_Y,
\]

so that \( (Tf)^\top \cdot m_Y \leq m_X^{-1} \cdot (Tf)^\top \) by adjunction.

For the associated monad \( \mathbb{T}' = (T', m', e') \) on \( \mathbb{T}-\text{Mon} \) (induced as in 4.2 by the identity monad morphism \( \alpha = 1_\mathbb{T} : \mathbb{T} \rightarrow \mathbb{T} \)), if \( f : (X, \nu) \rightarrow (Y, \mu) \) is a \( \mathbb{T} \)-monoid morphism, one can define the right adjoint \( \mathbb{T} \)-monoid morphism \( (Tf)^\top : (T'Y, \omega_Y) \rightarrow (T'X, \omega_X) \) by

\[
(Tf)^\top := p_X \cdot (Tf)^\top \cdot q_Y.
\]
Indeed, \((T^t f)^\downarrow\) is a composite of \(\top\)-monoid morphisms (Lemma 4.5.2) and one verifies that
\[
1_{TX} \leq (T^t f)^\downarrow \cdot T^t f \quad \text{and} \quad T^t f \cdot (T^t f)^\downarrow \leq 1_{TY}
\]
by using the identity \(T^t f = p_Y \cdot T f \cdot q_X\).

4.6.1 Lemma. If \(\top\) is a power-enriched monad, the \(U\)-initial \(\top\)-monoid structure on \(X\) induced by \(f : X \to U(Y, \mu)\) is
\[
\nu := (T f)^\downarrow \cdot \mu \cdot f .
\]
As a consequence, \(e_X' : (X, \nu) \to (TX, \omega_X)\) is \(U\)-initial.

Proof. The fact that \((T f)^\downarrow \cdot \mu \cdot f\) defines an initial \(\top\)-monoid morphism \(f : (X, \nu) \to (Y, \mu)\)
follows from straightforward verifications. The initial structure on \(X\) induced by \(e_X' : X \to U(TX, \omega_X)\) is therefore
\[
(T e_X')^\downarrow \cdot \omega_X \cdot e_X' = (T p_X \cdot T \nu)^\downarrow \cdot T p_X \cdot m_X^\downarrow \cdot q_X \cdot p_X \cdot e_X = (T \nu)^\downarrow \cdot m_X^\downarrow \cdot \nu .
\]
Furthermore, \((\nu^\top)^\downarrow = e_X^\top \cdot (\nu^\top)^\downarrow \leq \nu^\top \cdot (\nu^\top)^\downarrow \leq 1_{TX}\) so
\[
\nu \leq (\nu^\top)^\downarrow \cdot \nu^\top \cdot \nu = (\nu^\top)^\downarrow \cdot \nu \leq \nu ,
\]
that is, \((T \nu)^\downarrow \cdot m_X^\downarrow \cdot \nu = \nu\) so the last claim is verified.

4.6.2 Lemma. Let \(\top\) be a power-enriched monad. In the notations of 4.2 (with \(S = \top\) and \(\alpha = 1_{\top}\)), a morphism of \(\top\)-monoids \(f : (X, \nu) \to (Y, \mu)\) satisfies
\[
\nu = (T f)^\downarrow \cdot \mu \cdot f \iff e_X' = (T^t f)^\downarrow \cdot e_Y \cdot f .
\]

Proof. Suppose that \(\nu = (T f)^\downarrow \cdot \mu \cdot f\) holds. Then
\[
(T^t f)^\downarrow \cdot e_Y \cdot f = p_X \cdot (T f)^\downarrow \cdot q_Y \cdot e_Y \cdot f = p_X \cdot (T f)^\downarrow \cdot \mu \cdot f = p_X \cdot \nu = e_X' .
\]
Conversely, if \(e_X' = (T^t f)^\downarrow \cdot e_Y \cdot f\), then, recalling that \(T^t f \cdot p_X = p_Y \cdot T f\) by Lemma 4.3.1 and that \(q_X = p_X^\downarrow\) by Lemma 4.5.2, we can write
\[
\nu = q_X \cdot e_X' = p_X^\downarrow \cdot (T^t f)^\downarrow \cdot e_Y \cdot f = (p_Y \cdot T f)^\downarrow \cdot e_Y \cdot f = (T f)^\downarrow \cdot q_Y \cdot e_Y \cdot f = (T f)^\downarrow \cdot \mu \cdot f .
\]

4.6.3 Theorem. If \(\top\) is power-enriched, then there is an isomorphism of categories
\[
\text{Ini-Inj}(\top\text{-Mon}) \cong (\top\text{-Mon})^{\top}
\]
which commutes with the forgetful functors to \(\top\text{-Mon}\). In particular, the Ini-injective \(\top\)-monoids are precisely the \(\top'\)-algebras.
The morphism given a $U$-Conversely, any to $f$ that is, $b$ right adjoint $Suppose that $f(X, \nu) \to (T'X, \omega_X)$ is $U$-initial, so there is a $T$-initial morphism $a' : (T'X, \omega_X) \to (X, \nu)$ that extends $1_X : (X, \nu) \to (X, \nu)$ along $e'_X$:

$$\begin{align*}
(X, \nu) & \xrightarrow{e'_X} (T'X, \omega_X) \\
1_X & \downarrow \downarrow a' \\
(X, \nu) & \xrightarrow{(X, \nu)}
\end{align*}$$

(4.6.i)

that is, such that $a' \cdot e'_X = 1_X$. Moreover, by using that $T'$ is Kock–Zöberlein and $e'$ a natural transformation, we get

$$1_{T'X} = T'a' \cdot T'e'_X \leq T'a' \cdot e'_{T'X} = e'_X \cdot a',$$

so $a' \vdash e'_X$. Also, since $T'$ is a Kock–Zöberlein monad, $a' \cdot T'a' \simeq a' \cdot m'_X$ (that is, $a' \cdot T'a' \leq a' \cdot m'_X$ and $a' \cdot m'_X \leq a' \cdot T'a'$, see [14.4]; this equivalence is induced by the separated order on $T'X$, so it is actually an equality $a' \cdot T'a' = a' \cdot m'_X$. Hence, $(X, a')$ is a $T'$-algebra. Let us finally return to the definition of $a'$: by the adjunction $a' \vdash e'_X$, the morphism $a'$ is determined up to equivalence in $X$; since $a' \cdot p_X \cdot \nu = 1_X$ implies that the order on $X$ is separated, $a'$ is really uniquely determined.

Suppose that $f : (X, \nu) \to (Y, \mu)$ is a morphism between $U$-initial $\mathcal{T}$-monoids that has a right adjoint $f : (Y, \mu) \to (X, \nu)$ in $\mathcal{T}$-Mon. The previous construction yields $\mathcal{T}$-algebras $(X, a')$ and $(Y, b')$, respectively. One then has

$$b' \cdot T'f = b' \cdot p_Y \cdot Tf \cdot q_X \\
\leq b' \cdot p_Y \cdot Tf \cdot q_X \cdot e'_X \cdot a' \quad (a' \vdash e'_X) \\
\leq b' \cdot p_Y \cdot \mu \cdot f \cdot a' \quad (Tf \cdot \nu \leq \mu \cdot f) \\
= f \cdot a' \quad (b' \cdot p_Y \cdot \mu = b' \cdot e'_Y = 1_Y),$$

that is, $b' \cdot T'f \leq f \cdot a'$. In the same way, one obtains $a' \cdot T'(f^{-1}) \leq f^{-1} \cdot b'$, which is equivalent to $f \cdot a' \leq b' \cdot T'f$, and we can conclude that $f : (X, a') \to (Y, b')$ is a morphism of $\mathcal{T}$-algebras.

Conversely, any $\mathcal{T}$-algebra $((X, \nu), a')$ makes the diagram (4.6.1) above commute. Thus, given a $U$-initial morphism $j : (Y, \mu) \to (Z, \zeta)$, and a morphism $f : (Y, \mu) \to (X, \nu)$, one can define $\overline{f} := a' \cdot T'f \cdot (T'j)^{-1} \cdot e'_Z$:

$$\begin{align*}
(T'Y, \omega_X) & \xrightarrow{(T'j)^{-1}} (T'Z, \omega_Z) \xleftarrow{e'_Z} (Z, \zeta) \\
\downarrow & \downarrow j \\
(T'X, \omega_X) & \xrightarrow{a'} (X, \nu)
\end{align*}$$

The morphism $\overline{f} : (Z, \zeta) \to (X, \nu)$ does indeed extend $f$ along $j$, since by Lemmata [4.6.2] and [4.6.1] one has

$$\overline{f} \cdot j = a' \cdot T'f \cdot (T'j)^{-1} \cdot e'_Z \cdot j = a' \cdot T'f \cdot e'_Y = a' \cdot e'_X \cdot f = f.$$
This proves that \((X, \nu)\) is an Ini-injective object of \(\mathbb{T}\)-\textit{Mon}.

If \(f : ((X, \nu), a') \rightarrow ((Y, \mu), b')\) is a morphism of \(\mathbb{T}'\)-algebras, then the \(\mathbb{T}\)-monoid morphism \(f^\ddagger : (Y, \mu) \rightarrow (X, \nu)\) defined by

\[
f^\ddagger = a' \cdot p_X \cdot (Tf)^\ddagger \cdot \mu
\]

is right adjoint to \(f\). Indeed, one readily verifies the inequalities

\[
1_X \leq f^\ddagger \cdot f \quad \text{and} \quad f \cdot f^\ddagger \leq 1_Y .
\]

The passages from Ini-injective Kleisli monoids to Eilenberg–Moore algebras and back described above obviously define functors that commute with the forgetful functors to \(\mathbb{T}\)-\textit{Mon} and that are inverse to each other.

**4.6.4 Corollary.** If \(\mathbb{T}\) is power-enriched, then there is an isomorphism between the category of \(\text{RegMono-} \text{Ini-inj}\)-separated \(\mathbb{T}\)-monoids (with left adjoint morphisms) and the category of \(\mathbb{T}'\)-algebras (with \(\mathbb{T}'\) seen as a monad on \(\mathbb{T}\)-\textit{Mon} sep):

\[
\text{RegMono-} \text{Ini-inj}(\mathbb{T}\text{-}\text{Mon} \text{sep}) \cong (\mathbb{T}\text{-}\text{Mon} \text{sep})^{\mathbb{T}'}. 
\]

Moreover, the functors forming the isomorphism commute with the forgetful functors to \(\mathbb{T}\)-\textit{Mon}.

**Proof.** The proof that an Ini-injective object is an Eilenberg–Moore algebra in Theorem 4.6.3 relies on the fact that the structure morphism \(\nu : X \rightarrow TX\) of a Kleisli monoid \((X, \nu)\) belongs to the class of \(U\)-initial morphisms. By definition, the structure morphisms of \textit{separated} Kleisli monoids have monic underlying maps, so the same proof yields the stated isomorphism.

**4.6.5 Corollary.** Given an associative lax extension \(\hat{\mathbb{T}}\) of \(\mathbb{T}\) to \(\mathcal{V}\)-\textit{Rel}, there is an isomorphism

\[
\text{Ini-} \text{Inj}(\mathbb{T}\text{-}\text{Cat}) \cong (\mathbb{I}\text{-}\text{Mon})^{\mathbb{T}'}. 
\]

In particular, the Ini-injective \((\mathbb{T}, \mathcal{V})\)-categories are precisely the \(\mathbb{I}'\)-algebras (where \(\mathbb{I}\) is the discrete presheaf monad associated to \(\hat{\mathbb{T}}\)).

**Proof.** By Proposition 3.2.1, the discrete presheaf monad associated to \(\hat{\mathbb{T}}\) is power-enriched. The result then follows directly from the isomorphisms of Theorems 4.6.3 and 3.2.2.

The following direct consequences of Theorem 4.6.3 and its corollaries allow us in particular to identify the injective objects of \(\text{Ord}_\text{sep}\) and \(\text{Top}_0\).
4.6.6 Examples.

(1) Since $\text{Set}^P \cong \text{Sup}$ and $\mathbb{P}-\text{Mon} \cong \text{Ord}$, complete lattices are both the Ini-injective objects in $\text{Ord}$ and the RegMono-injective objects in $\text{Ord}_{\text{sep}}$ (in this case the, regular monomorphisms are the order-embeddings).

(2) Cocontinuous lattices can equivalently be seen as the Ini-injective objects in $\text{Top}$ or the RegMono-injective objects in $\text{Top}_0$ via the isomorphisms $\text{Set}^P \cong \text{Cnt}^\text{co}$, and $\mathbb{F}-\text{Mon} \cong \text{Top}$ or $\mathbb{F}-\text{Mon}_{\text{sep}} \cong \text{Top}_0$, see Exercise 4.B (here, regular monomorphisms are the topological embeddings).

(3) Completely distributive lattices are the Ini-injective objects of the category $\text{Cls}$ of closure spaces (Exercise 4.E), and frames are the Ini-injective objects of the category $\text{Cls}_{\text{fin}}$ of finitary closure spaces (Exercise 4.F).

(4) For a given quantale $\mathcal{V}$, the Ini-injective objects of $\mathcal{V}-\text{Cat}$ are the $\mathcal{V}$-actions in $\text{Sup}$ (Exercise 4.G).

(5) Quantales are the Ini-injective objects of the category $\text{MultiOrd} \cong (\mathbb{L}, 2)-\text{Cat}$ of multi-ordered sets and their morphisms (see $\text{V.1.4}$ and Exercise 4.H).

Exercises

4.A A distributive law for the down-set-filter monad. The down-set-filter monad $\mathbb{F} \mathbb{D} \mathbb{n}$ is a composite monad obtained from a distributive law of $\mathbb{D} \mathbb{n}$ over $\mathbb{F} \mathbb{N}$ (see 4.4). The monad $\mathbb{D} \mathbb{n}$ is of Kock–Zöberlein type and $\mathbb{F} \mathbb{N}$ is of dual Kock–Zöberlein type, but $\mathbb{F} \mathbb{D} \mathbb{n}$ is neither. Moreover, if $(X, a)$ is a $\mathbb{D} \mathbb{n}$-algebra, then the order on $X$ is necessarily separated, so the 2-isomorphisms of Example II.4.9.3 become

$$\text{Ord}^{\mathbb{D} \mathbb{n}} \cong \text{Sup} \quad \text{and} \quad \text{Ord}^{\mathbb{U} \mathbb{p}} \cong \text{Inf}.$$  

Hence, $\text{Ord}^{\mathbb{F} \mathbb{N}}$ is 2-isomorphic to the category of ordered sets in which every down-directed set has an infima, with maps preserving down-directed infima.

4.B $T_0$-spaces. Via the isomorphism $\mathbb{F}-\text{Mon} \cong \text{Top}$ of Proposition 1.1.1, the category $\mathbb{F}-\text{Mon}_{\text{sep}}$ of separated $\mathbb{F}$-monoids is isomorphic to the category $\text{Top}_0$ of $T_0$-spaces.

4.C Separated representable Kleisli monoids. Let $\mathbb{T} = (T, m, e)$ be a power-enriched monad with its Kleisli extension $\tilde{T}$ to $\text{Rel}$. It follows from III.5.1 that $\tilde{T}$ induces a monad $\mathbb{T} = (T, m, e)$ on $\text{Ord}$. Show:

(1) the functor $T : \text{Set} \to \text{Set}$ sends surjections to split epimorphisms;

(2) the full subcategory $(\text{Ord}^T)_{\text{sep}}$ of $\text{Ord}^T$ spanned by all separated ordered sets is reflective;

(3) the composite functor $(\text{Ord}^T)_{\text{sep}} \hookrightarrow \text{Ord}^T \xrightarrow{K} (\mathbb{T}, 2)-\text{Cat}$ (see III.5.3) is strictly monadic;
4. INJECTIVE LAX ALGEBRAS AS EILENBERG–MOORE ALGEBRAS

(4) there is an isomorphism \( \text{Set}^T \cong (\text{Ord}^T)_{\text{sep}} \) that commutes with the underlying-set functors;

(5) the functor \( \text{Set}^T \to \mathbb{T}\text{-Mon} \) of 4.2 factors as

\[
\text{Set}^T \cong (\text{Ord}^T)_{\text{sep}} \to \text{Ord}^T \to (\mathbb{T}, 2)-\text{Cat} \cong \mathbb{T}\text{-Mon}.
\]

4.D Open sets and continuity in the Scott topology. The open sets of the Scott topology on a cocontinuous lattice \( X \) are those down-sets \( O \in \text{Dn} \, X \) such that

\[
\forall A \in \text{Fil} \, X \left( \bigwedge A \in O \implies A \setminus O \neq \emptyset \right).
\]

If \( Y \) is another cocontinuous lattice, then a map \( f : X \to Y \) is continuous if and only if it preserves down-directed infima.

4.E Completely distributive lattices as Eilenberg–Moore algebras. The Eilenberg–Moore algebras of the up-set monad \( \mathbb{U} \) on \( \text{Set} \) is the category \( \text{Dst} \) of completely distributive lattices and maps that preserve all infima and suprema:

\[
\text{Set}^\mathbb{U} \cong \text{Dst}.
\]

The Ini-injective objects of the category \( \text{Cls} \) of closure spaces are precisely the completely distributive lattices (use Example 1.3.2(4)).

4.F Frames as Eilenberg–Moore algebras. The Eilenberg–Moore algebras of the finitary-up-set monad of Exercise 1.E is the category frames and frame homomorphisms:

\[
\text{Set}^{\mathbb{U}_{\text{fin}}} \cong \text{Frm}.
\]

The Ini-injective objects of the category \( \text{Cls}_{\text{fin}} \) of finitary closure spaces are precisely frames (use Example 1.E).

4.G Categories associated to the \( \mathcal{V} \)-powerset monad. The \( \mathcal{V} \)-powerset monad (Exercise III.1.D) is power-enriched, and

\[
\mathbb{P}_{\mathcal{V}}\text{-Mon} \cong \mathcal{V}\text{-Cat} \quad \text{and} \quad \text{Set}^{\mathbb{P}_{\mathcal{V}}} \cong \text{Sup}^{\mathcal{V}},
\]

(see Example II.4.3.1(3)). Hence, the Ini-injective objects of \( \mathcal{V}\text{-Cat} \) are the \( \mathcal{V} \)-actions in \( \text{Sup}. \)

4.H Quantales. There is a distributive law \( \delta : LP \to PL \) of the list monad \( \mathbb{L} \) over the powerset monad \( \mathbb{P} \) given by

\[
\delta(A_1, \ldots, A_n) = \{ (a_1, \ldots, a_n) \mid a_i \in A_i \}.
\]

The resulting composite monad \( \mathbb{P}_{\mathbb{L}} \) is power-enriched, and is induced by the monadic composition of the monadic forgetful functors \( \text{Qnt} \to \text{Mon} \) and \( \text{Mon} \to \text{Set} \) (see Exercise II.2.T and Example II.3.2.2(1)). Show that \( \mathbb{P}_{\mathbb{L}} \) is isomorphic to the discrete presheaf monad of the list monad, so that

\[
\mathbb{P}_{\mathbb{L}}\text{-Mon} \cong (\mathbb{L}, 2)-\text{Cat}
\]

(see Exercise 3.C).
5 Domains as lax algebras and Kleisli monoids

It is shown in 4.4 that the cocontinuous lattices are the Eilenberg–Moore algebras of the filter monad \( F \) on \( \text{Set} \), whereas in Corollary 1.5.4 we saw that the category \( (\mathcal{F}, 2)\text{-Cat} \) of lax algebras of the Kleisli extension of the filter monad is isomorphic to the category \( \text{Top} \) of all topological spaces. We saw in 4.4 that there is a monadic functor \( \text{Cnt}^{\text{co}} \cong \text{Set}^{\text{F}} \rightarrow \mathcal{F}\text{-Mon} \cong (\mathcal{F}, 2)\text{-Cat} \cong \text{Top} \) which associates to each cocontinuous lattice its Scott topology (see Exercise 4.D). We shall see that the lax algebra thus associated with each cocontinuous lattice is \textit{strict}, meaning that it satisfies the lax multiplicative Eilenberg–Moore \textit{strictly}. Further, we shall show in Theorem 5.4.1 that the strict lax algebras of \( F \) constitute a class of topological spaces that not only generalize the continuous lattices but in fact provide a close topological generalization of the broader class of \textit{continuous} dcpos (see 5.2). The spaces in question—which we call the \textit{observable realization spaces}—capture an essential domain-theoretic approximation property in topological terms, and the continuous dcpos occur among these spaces as precisely the \textit{sober} observable realization spaces. In 5.9 we will characterize those strict lax algebras that are associated to continuous lattices.

In Theorem 5.7.2 we shall also show that the observable realization spaces are Kleisli monoids of the \textit{ordered-filter monad} \( \mathcal{F}\mathbb{I} \) on \( \text{Ord} \).

5.1 Modules and adjunctions. Recall from II.2.2 that the category \( \text{Ord} \) of ordered sets and monotone maps is manifested within the category \( \text{Mod} \) of ordered sets and modules as their morphisms via functors

\[
(-)_*: \text{Ord} \rightarrow \text{Mod} \quad \text{and} \quad (-)^*: \text{Ord}^{\text{op}} \rightarrow \text{Mod}
\]

which send a monotone map \( f: X \rightarrow Y \) to the modules

\[
f_* = (\leq_Y) \cdot f : X \ni X \leftrightarrow Y \quad \text{and} \quad f^* = f^\circ (\leq_Y) : Y \ni X \leftrightarrow X,
\]

respectively, where \( f^\circ : Y \ni X \leftrightarrow X \) is the converse relation of \( f \) (given by \( y \ni f^\circ x \iff y = f(x) \)). Endowing the hom-sets \( \text{Ord}(X, Y) \) and \( \text{Mod}(X, Y) \) with the pointwise and inclusion orders, respectively, we find that for monotone maps \( f, g : X \rightarrow Y \),

\[
f_* \geq g_* \iff f \leq g \iff f^* \leq g^* ,
\]

so that we have order-embeddings

\[
(-)_*: \text{Ord}(X, Y)^{\text{op}} \rightarrow \text{Mod}(X, Y) \quad \text{and} \quad (-)^*: \text{Ord}(X, Y) \rightarrow \text{Mod}(Y, X).
\]

We recall (see IV.4) that for monotone maps \( f : X \rightarrow Y, g : Y \rightarrow X \), one has

\[
f \downarrow g \iff \forall x \in X, y \in Y \ (f(x) \leq y \iff x \leq g(y)) \iff f_* = g^* .
\]
As a quantaloid, \( \text{Mod} \) is in particular an ordered category and hence a 2-category. For any monotone map \( f : X \to Y \) we have

\[
1_X^\ast \leq f^* \cdot f_\ast \quad \text{and} \quad f_\ast \cdot f^* \leq 1_Y^\ast ,
\]

so that \( f_\ast \) is left adjoint to \( f^* \) in the 2-category \( \text{Mod} \); in symbols, \( f_\ast \dashv f^* \).

5.2 Cocontinuous ordered sets. Section 4.4 defines a cocontinuous lattice \( X \) as a complete lattice for which the infimum map \( \wedge_X : \text{Fil}X \to X \) has a right adjoint \( \uparrow_X \); if such a right adjoint exists, then it is given by

\[
x \in \uparrow a \iff \forall S \in \text{Fil}X \ (\wedge S \leq a \implies x \in S) . \tag{5.2.i}
\]

Considering an arbitrary ordered set \( X \) instead of a complete lattice, let us define a relation \( (\ll) : X \to X \) by

\[
a \ll x \iff \forall S \in \text{Fil}_A X \ (\wedge S \leq a \implies x \in S) ,
\]

where \( \text{Fil}_A X \) is the set of all filters \( S \) in \( X \) for which a meet \( \wedge S \) exists. Letting \( \uparrow x := \{ y \in X \mid x \ll y \} \) for each \( x \in X \), we say that the ordered set \( X \) is cocontinuous if

\[
\uparrow x \in \text{Fil}_A X \quad \text{and} \quad \wedge \uparrow x \cong x \tag{5.2.ii}
\]

for each \( x \in X \). For example, if \( X \) is a co-dcpo, that is, if \( X \) is separated and \( \text{Fil}_A X = \text{Fil}X \), then the infimum map \( \wedge_X : \text{Fil}X \to X \) has a right adjoint \( \uparrow_X \) if and only if \( X \) is cocontinuous; if this is the case then the right adjoint is given by \( \uparrow_X x = \uparrow x \) and we say that \( X \) is a cocontinuous co-dcpo. The dual notion, that of a continuous dcpo, is more often employed. The term dcpo was originally introduced as an acronym for directed-complete partial order.

For any ordered set \( X \), we may define the Scott topology on \( X \) as the collection of all Scott-open sets, that is, of all down-sets \( U \in \text{Dn}X \) such that

\[
\forall S \in \text{Fil}_A X \ (\wedge S \in U \implies \exists x \in S \cap U) .
\]

As in Exercise 4.D with regard to cocontinuous lattices, one may readily verify that a function \( f : X \to Y \) between ordered sets is continuous with respect to the Scott topologies on \( X \) and \( Y \) if and only if \( f \) preserves meets of down-directed subsets. Furthermore, the Scott topology on an ordered set \( X \) is order-compatible, in the sense that the underlying order associated to the Scott topology of an ordered set \( X \) coincides with the order relation possessed by \( X \). In a topological space \( X \) endowed with its underlying order, we say that a point \( x \) specializes a point \( y \) if \( x \leq y \).

The nature of the Scott topology is related to the preservation of down-directed meets by the neighborhood-filter map \( \nu : X \to FX \) of a topological space \( X \) endowed with its underlying order. For any down-directed subset \( S \subseteq X \) with a meet in \( X \), the image \( \nu(S) \)
is down-directed in $FX$ by monotonicity, and directed meets in $FX$ are given by union, so to say that $\nu$ preserves the meet $\wedge S$ means that

$$\nu(\wedge S) = \bigcup_{z \in S} \nu(z).$$

5.2.1 Definition. Let $X$ be a topological space, endowed with its underlying order, and let $S \subseteq X$ be a down-directed subset of $X$ with a meet $\wedge S$ in $X$. We say that $\wedge S$ is a topological meet of $S$ in $X$ if the neighborhood-filter map $\nu : X \to FX$ preserves this meet.

Given an ordered set $X$, the Scott topology on $X$ is the largest order-compatible topology whose neighborhood filter map $\nu$ preserves all directed meets, as follows:

5.2.2 Proposition. Let $X$ be a topological space endowed with its underlying order. The following are equivalent:

(i) $\mathcal{O}X$ is coarser than the Scott topology on $X$; that is, every open set is Scott-open;

(ii) every directed meet in $X$ is a topological meet.

Proof. This is Exercise 5.A.

We shall also require the following Lemma.

5.2.3 Lemma. Let $X$ be a cocontinuous ordered set. Then for any $y \in X$, the set

$$\downarrow y := \{ x \in X \mid x \ll y \}$$

is Scott-open.

Proof. Suppose that $S \in \text{Fil}_X$ and $\wedge S \in \downarrow y$. Observe that

$$\wedge S = \bigwedge \{ \uparrow x \mid x \in S \} = \bigwedge (\bigcup \{ \uparrow x \mid x \in S \}).$$

Since $\bigcup \{ \uparrow x \mid x \in S \}$ is the union of a down-directed subset of $\text{Fil}X$, it is an element of $\text{Fil}X$. The result follows.

One can provide an alternate characterization of the property of cocontinuity of an ordered set with reference to the Scott topology. For any topological space $X$ with its underlying order and with neighborhood-filter map $\nu : X \to FX$, we may define a relation $(\ll) : X \leftrightarrow X$ by

$$x \ll y \iff \downarrow y \in \nu(x),$$

so in particular the set

$$\downarrow y := \{ x \in X \mid x \ll y \}$$

is open for all $y \in X$. Since $\downarrow y$ is the set of points that specialize $y$, the statement that $x \ll y$ for points $x, y \in X$ is in general stronger than the statement that $x$ specializes $y$,
and we say that $x$ observably specializes $y$ if $x ≺ y$. We refer to $(≺)$ as the observable specialization relation associated to $X$.

In the case that $X$ is a cocontinuous ordered set under the Scott topology, we find that $(≺) = (∈)$, as follows. For an ordered set $X$ and a relation $r : X ↦ X$, we say that $r$ is approximating if for each $x \in X$, $r(x)$ is a filter in $X$ and $x$ is a meet of $r(x)$. For example, $X$ is cocontinuous if and only if $(∈)$ is approximating.

5.2.4 Lemma. For an ordered set $X$ endowed with the Scott topology, if either $(∈)$ or $(≺)$ is approximating, then $(≺) = (∈)$.

Proof. See Exercise 5.B

5.2.5 Proposition. For an ordered set $X$ endowed with the Scott topology, the following are equivalent:

(i) $X$ is cocontinuous.

(ii) the relation $(≺)$ is approximating;

(iii) each $x \in X$ is a topological meet of $\uparrow x := \{y \in X \mid x ≺ y\}$.

Proof. This is Exercise 5.C

5.3 Observable realization spaces. The third characterization of continuity in ordered sets in Proposition 5.2.5 affords an interesting generalization to arbitrary topological spaces:

5.3.1 Proposition. Let $X$ be a topological space endowed with its underlying order. There is an embedding of ordered sets $ι : \text{Fil}X → FX$ given by $S ↦ \uparrow_{PX}\downarrow y \mid y \in S\}$, and the following are equivalent:

(i) each $x \in X$ is a topological meet of $\uparrow x$;

(ii) there is a map $n : X → \text{Fil}X$ such that the neighborhood-filter map $ν : X → FX$ factors as $X ↦ n \mapsto \downarrow_{PX}\downarrow y \mid y \in S\}$;

(iii) there is a module $c : \text{Fil}X ↦ X$ such that the convergence relation $a : FX ↦ X$ of $X$ factors as $FX ↦ c \mapsto \text{Fil}X ↦ c \mapsto X$;

(iv) for all $U \in OX$ and $x \in U$, there exists $y \in U$ with $x ≺ y$. 
If these conditions hold then the map \( n \) of (ii) and the relation \( c \) of (iii) are uniquely determined as

\[
n = \uparrow : X \rightarrow \Fil X \quad \text{and} \quad c = a \cdot \iota = \uparrow^* : \Fil X \rightleftarrows X
\]

with \( S \overset{c}{\longrightarrow} x \iff \iota(S) \supseteq \nu(x) \iff S \supseteq \uparrow x \), and \( \uparrow : X \rightarrow \Fil X \) is an order-embedding.

**Proof.** It is readily verified that \( \iota \) as given is well-defined, injective, monotone, and fully faithful.

Suppose (iv) holds. Then for any \( x \in X \) we have that

\[
\bigcup_{y \in \uparrow x} \nu(y) \supseteq \nu(x)
\]

and since \( \uparrow x \subseteq \uparrow x \), this inclusion is in fact an equality, from which it follows that \( x \) is a meet of \( \uparrow x \). Further, \( \uparrow x \) is down-directed, since if \( y, z \in \uparrow x \) then the sets

\[
\uparrow y = \{ t \in X \mid t \prec y \} \quad \text{and} \quad \uparrow z = \{ t \in X \mid t \prec z \}
\]

are open neighborhoods of \( x \) and hence by (iv) there is \( u \in \uparrow x \) with \( u \in \uparrow y \cap \uparrow z \subseteq \uparrow y \cap \uparrow z \) as needed. Thus, \( x \) is a topological meet of \( \uparrow x \), showing that (i) holds.

Suppose (i) holds. Then for any \( x \in X \)

\[
\nu(x) = \bigcup_{y \in \uparrow x} \nu(y)
\]

and hence (iv) holds. We find also that (ii) holds, as follows. The preceding equation implies that \( \nu(x) \subseteq \iota(\uparrow x) \) since open sets are down-closed. This inclusion may be replaced by an equality, since for any \( y \in \uparrow x \) we have that \( \nu(x) \ni \downarrow y \), so we find that \( \nu \) factors through the map \( \uparrow : X \rightarrow \Fil X x \mapsto \uparrow x \) as \( \nu = \iota \cdot \uparrow \).

If there is a map \( n \) as in (ii), then for any \( x \in X \) we find that

\[
n(x) \ni y \iff \iota(n(x)) \ni \downarrow y \iff \nu(x) \ni \downarrow y \iff x \prec y
\]

for each \( y \in X \), whence \( n(x) = \uparrow x \) and we find also that (iv) holds. Further, since \( \nu \) and \( \iota \) are monotone and fully faithful and \( \iota \cdot n = \nu \), it follows that \( n \) is monotone and fully faithful. Letting \( c := n^* \), we have that

\[
a = \nu^* = (\iota \cdot n)^* = n^* \cdot \iota^* = c \cdot \iota^*,
\]

so (iii) holds.

Suppose we have a relation \( c \) as in (iii). Since \( \iota \) is fully faithful we have that \( \iota^* \cdot \iota_* = \1_{\Fil X} \)

and hence

\[
a \cdot \iota = a \cdot \iota_* = c \cdot \iota^* \cdot \iota_* = c.
\]

Further,

\[
1_X \leq 1^* \leq \nu^* \cdot \nu_* = a \cdot \nu_* = c \cdot \iota^* \cdot \nu_* = a \cdot \iota_* \cdot \iota^* \cdot \nu_* = \nu^* \cdot \iota_* \cdot \iota^* \cdot \nu_* = \nu^* \cdot \iota \circ \nu_*,
\]

so for each \( x \in X \) there is \( S \in \Fil X \) with \( \nu(x) \supseteq \iota(S) \supseteq \nu(x) \), whence \( \nu(x) = \iota(S) \), and one verifies that \( S = \uparrow x \) and hence \( \nu(x) = \iota(\uparrow x) \). Therefore, (ii) holds with \( n := \uparrow \). \qed
5.3.2 Definition. For a topological space with convergence relation \( a : FX \leftrightarrow X \), we find that the associated relation \( a \cdot \iota : \text{Fil}X \leftrightarrow X \) of the preceding proposition is such that \( S \stackrel{a\iota}{\rightarrow} x \) if and only if \( S \) enters every open neighborhood of \( x \), and if this is the case then we say that \( S \) \textit{observably realizes} \( x \). Note that \( S \) observably realizes \( x \) if and only if \( x \) is in the closure of \( S \).

If the equivalent conditions of Proposition 5.3.1 hold for a topological space \( X \), then we say that \( X \) is an \textit{observable realization space}.

5.3.3 Example. Every cocontinuous ordered set is an observable realization space when endowed with the Scott topology. In fact, we shall show in 5.8 that the cocontinuous co-dcpos under the Scott topology are precisely the \textit{sober} observable realization spaces. For a cocontinuous co-dcpo, the map \( \downarrow : X \rightarrow \text{Fil}X \) is given by \( x \mapsto \downarrow x \), and the \textit{observable realization relation} \( c = a \cdot \iota : \text{Fil}X \leftrightarrow X \) is given by
\[
S \stackrel{a\iota}{\rightarrow} x \iff \bigwedge S \leq x ;
\]
the associated filter convergence relation \( a = c \cdot \iota^* \) is called the relation of \textit{Scott convergence}.

Thus, when we express the domain-theoretic \textit{approximation property} of continuity in topological rather than order-theoretic terms, relative to an arbitrary topology, we obtain a characterization of the observable realization spaces (Proposition 5.3.1(i)). Moreover, we find that these spaces can also be characterized via condition 5.3.1(iii) as those in which filter convergence reduces to a relation of convergence of directed sets that captures the essential topological character of the notion of \textit{convergence by directed meets} supported by cocontinuous co-dcpos. Thus, these spaces provide our general notion of a \textit{domain}, such that in 5.8 we shall see that the continuous dcpos are those domains that are \textit{sober}.

The following Lemma entails in particular that every observable realization space carries a topology finer than the Scott topology on its specialization order (that is, a topology in which every Scott-open subset is open). The proof is straightforward.

5.3.4 Lemma. \textit{Suppose} \( X \) \textit{is a topological space with its underlying order such that} \((\prec)\) \textit{is approximating. Then} \( OX \) \textit{is finer than the Scott topology on} \( X \); \textit{that is, every Scott-open subset is open}.

5.4 Observable realization spaces as lax algebras. We now observe that the observable realization spaces may be characterized among arbitrary topological spaces as those whose corresponding \((\mathcal{F}, 2)\)-category (with respect to the Kleisli extension \( \mathcal{F} \), see 1.4) satisfies the lax Eilenberg–Moore multiplicative law up to equality rather than simply an inequality, as follows:

5.4.1 Theorem. \textit{Let} \( X \) \textit{be a topological space with associated} \((\mathcal{F}, 2)\)-\textit{category} \((X, a)\) \textit{(where} \( a : FX \leftrightarrow X \) \textit{is the filter convergence relation associated to} \( X \)). \textit{Then}
\[
a \cdot m_X = a \cdot \mathcal{F}a \iff X \textit{is an observable realization space}.
\]
Thus, the isomorphism \((F, 2)\text{-Cat} \cong \text{Top}\) of Corollary 1.5.4 restricts to an isomorphism \((F, 2)\text{-Cat} = \cong \text{ObsReSp}\) between the full subcategory \((F, 2)\text{-Cat} = \cong \text{ObsReSp}\) of \(F\text{-Cat}\), consisting of those lax algebras of \(F\) that satisfy the multiplicative law up to equality, and the full subcategory \(\text{ObsReSp}\) of \(\text{Top}\) with objects all observable realization spaces.

Proof. Since \(F\) is a power-enriched monad, we know by Proposition 1.2.1 that \(m_X : FFX \to FX\) is a sup-map and hence has a right adjoint \(m_X^\perp : FX \to FFX\) given by

\[
m_X^\perp(x) = \uparrow_{PFX} \{ A^F \mid A \in x \},
\]

for all \(x \in FX\) (where \(A^F = \{ y \in FX \mid A \in y \}\)). Also, with reference to (1.4.i), there is a monotone map \(a^\tau : FX \to FFX\), given by \(a^\tau(y) = \uparrow_{PFX} \{ A^\circ(A) \mid A \in y \}\), such that \(\tilde{F}a = (a^\tau)^*\). Note also that \(a = \nu^*\), where \(\nu : X \to FX\) is the neighborhood filter map of \(X\). Letting \(O_X(x)\) denote the set of open neighborhoods of a point \(x \in X\), we may reason as follows:

\[
a \cdot m_X = a \cdot \tilde{F}a \quad \iff \quad a \cdot m_X \leq a \cdot \tilde{F}a \quad ((X, a) \in (F, 2)\text{-Cat})
\]

\[
\iff \quad a \cdot (m_X)_* \leq a \cdot \tilde{F}a
\]

\[
\iff \quad \nu^* \cdot (m_X)_* \leq \nu^* \cdot \tilde{F}a
\]

\[
\iff \quad \nu^* \cdot (m_X^\perp)_* \leq \nu^* \cdot \tilde{F}a \quad (m_X \perp m_X^\perp)
\]

\[
\iff \quad \nu^* - (m_X^\perp)_* \leq (a^\tau)^*
\]

\[
\iff \quad (m_X^\perp)_* \nu \leq (a^\tau) \cdot \nu
\]

\[
\iff \quad \forall x \in X \left( \uparrow \{ A^F \mid A \in \nu(x) \} \supseteq \uparrow \{ A^\circ(A) \mid A \in \nu(x) \} \right)
\]

\[
\iff \quad \forall x \in X \left( \uparrow \{ V^F \mid V \in O_X(x) \} \supseteq \uparrow \{ A^\circ(U) \mid U \in O_X(x) \} \right)
\]

\[
\iff \quad \forall x \in X, U \in O_X(x) \left( \exists V \in O_X(x) \left( V^F \subseteq A^\circ(U) \right) \right)
\]

This last condition is equivalent to condition (iv) of Proposition 5.3.1 as follows. Consider any \(x \in X\) and \(U \in O_X(x)\). For any \(V \in O_X(x)\), we have that

\[
V^F \subseteq A^\circ(U) \iff \uparrow \{ V \} \in A^\circ(U)
\]

and hence moreover

\[
\exists V \in O_X(x) \left( V^F \subseteq A^\circ(U) \right) \iff \exists V \in O_X(x), y \in U \left( \uparrow \{ V \} \xrightarrow{a} y \right)
\]

\[
\iff \exists y \in U \left( x < y \right),
\]
with the last equivalence holding as follows. Firstly, if \( V \in O_X(x), y \in U \), and \( \{V\} \xrightarrow{a} y \) then any \( z \in V \) specializes \( y \) since \( \nu(z) \supseteq \{V\} \supseteq \nu(y) \), so that \( x \in V \subseteq \downarrow y \) and hence \( x \prec y \). Conversely, if \( x \prec y \) then there is \( V \in O_X \) with \( x \in V \subseteq \downarrow y \), so any open neighborhood \( U \) of \( y \), \( V \subseteq \downarrow y \subseteq U \), whence we have that \( \{V\} \supseteq \nu(y) \); equivalently, \( \{V\} \xrightarrow{a} y \).

5.4.2 Remark. Related to the characterization of the observable realization spaces given in Theorem 5.4.1, we already mentioned in Remark III.5.6.5 that \((\mathcal{B}, \mathcal{2})\text{-Cat}_{\mathcal{B}}\), taken with respect to the Barr extension, is isomorphic to the full subcategory of \( \text{Top} \) consisting of exponentiable spaces (see Proposition III.5.6.6 and Theorem III.5.8.5). In fact, if the last condition in the first chain of equivalences in the proof of Theorem 5.4.1 is modified by substituting the set \( V^\mathcal{B} = \{x \in V^\mathcal{F} \mid x \text{ is an ultrafilter}\} \) in place of \( V^\mathcal{F} \), we obtain the following result.

5.4.3 Corollary. Every observable realization space is exponentiable in \( \text{Top} \).

5.5 Observable specialization systems. We now show that observable realization spaces may be axiomatized quite directly as a set equipped with a binary relation \((\prec)\). We then find that this axiomatization in turn leads to a description of these spaces as Kleisli monoids.

5.5.1 Definition. Let \( X \) be a set and \((\prec) : X \rightarrow X \) a relation. We say that \((X, (\prec))\) is an observable specialization system if the following statements hold for all \( x, y, z \in X \):

1. \( x \prec y \prec z \implies x \prec z \);
2. \( \exists u \in X \ (x \prec u) \);
3. \( x \prec y, z \implies (\exists u : x \prec u \prec y, z) \);
4. \( x \prec y \& (\forall u \ (z \prec u \implies y \prec u)) \implies x \prec z \).

If (1)–(3) hold, then we say that \((X, \prec)\) is an abstract basis.

The relation \((\prec)\) induces an order on \( X \), given by

\[
x \leq y \iff \downarrow x \supseteq \downarrow y \tag{5.5.1}
\]

for \( x, y \in X \), where \( \downarrow x := \{y \in X \mid x \prec y\} \). With respect to this order, note that (4) requires exactly that \((\prec) : X \rightarrow X \) is a module, since one always has \((\prec) \cdot (\leq) \subseteq (\prec)\).

The category \( \text{ObsSys} \) has objects all observable specialization systems and morphisms all maps \( f : (X, \prec) \rightarrow (Y, \prec) \) such that

\[
\forall x \in X \ y \in Y \ (f(x) \prec y \implies \exists u \in X \ (x \prec u \& f(\downarrow u) \subseteq \downarrow y)),
\]

where \( \downarrow y := \{z \in Y \mid z \prec y\} \).
5.5.2 Theorem. There is an isomorphism

\[ \text{ObsReSp} \cong \text{ObsSys} \]

which commutes with the underlying-set functors and sends an observable realization space \( X \) to the pair \((X, \prec)\) consisting of the underlying set and observable specialization relation \((\prec)\) of \( X \).

Quite generally, if \((X, \prec)\) is an abstract basis, then \( \mathcal{B} := \{ \uparrow y \mid y \in X \} \) is a base for a topology on \( X \), under which \( X \) is an observable realization space, and if \((X, \prec)\) is moreover an observable specialization system, then \((\prec)\) coincides with the observable specialization relation of this observable realization space.

Proof. If \( X \) is an observable realization space, then by Proposition 5.3.1, \( \uparrow : X \rightarrow \text{Fil} X \) is an order-embedding, so the underlying order \((\leq_X)\) is the order induced by \((\prec_X, 5.5.1)\), and it is an easy exercise to verify that (1)–(4) hold for \((\prec_X)\).

Next suppose \((X, \prec)\) is an abstract basis. It is readily verified that \( \mathcal{B} \) as given is a base for a topology on \( X \) (meaning that for each \( x \in X \), the set \( \{ B \in \mathcal{B} \mid x \in B \} \) is either empty or down-directed) and that the order induced by \((\prec)\) \((5.5.1)\) coincides with the associated underlying order \((\leq_X)\) of the resulting topological space \( X \). It then follows by (1) that \((\prec) \leq (\leq_X)\), and in turn that \((\prec) \leq (\prec_X)\) using (3). We may use this and (3) to deduce by means of Proposition 5.3.1 (iv) that \( X \) is an observable realization space.

Supposing moreover that \((X, \prec)\) also satisfies (4), we now show that \((\prec_X) \leq (\prec)\) as well. If \( x \prec_X y \) then there is some \( u \) with \( x \in \uparrow u \subseteq \downarrow_X y \), and by (3) there is some \( v \) with \( x \prec v \prec u \), whence \( x \prec v \leq_X y \), so by (4), \( x \prec y \).

Noting also that for an observable realization space \( X \) the base \( \mathcal{B} \) associated to \((X, \prec_X)\) is a base for \( X \), the result follows. \( \square \)

Thus, while the topological concept of a domain embodied by the observable realization spaces is not the usual order-theoretic one, in which the specialization order is primitive and the topology is derivative, the observable realization spaces are nevertheless determined by a transitive relation, but one that is not necessarily reflexive (nor irreflexive). For these spaces, observable specialization, rather than specialization, provides the primitive notion.

5.6 Ordered abstract bases and round filters. For an ordered set \( X \), recall that a module \((\prec) : X \leftrightarrow X\) is an auxiliary relation on \( X \) if \((\prec) \leq (\leq_X)\) (see also Example 2.5.5(6)). Observe that any auxiliary relation \((\prec)\) on \( X \) is necessarily transitive, since \((\prec) \cdot (\prec) \leq (\prec) \cdot (\leq_X) = (\prec)\).

5.6.1 Definition. For an auxiliary relation \((\prec)\) on an ordered set \( X \), we say that \((X, \prec)\) is an ordered abstract basis if the underlying set \( X \) and the relation \((\prec)\) constitute an abstract basis.
Ordered abstract bases provide a mild generalization of observable specialization systems, as follows:

**5.6.2 Proposition.** Let $X$ be a set and $(\prec) : X \leftrightarrow X$ a relation. A pair $(X, \prec)$ is an observable specialization system if and only if $(X, \prec)$ is an ordered abstract basis when $X$ is endowed with the order induced by $(\prec)$ (see (5.5.i)).

The following alternative axioms lead us to a description of ordered abstract bases and observable specialization systems as Kleisli monoids in 5.7:

**5.6.3 Proposition.** For an ordered set $X$ and a relation $(\prec) : X \leftrightarrow X$, the pair $(X, \prec)$ is an ordered abstract basis if and only if the following statements hold, where $\downarrow x := \{ y \in X \mid x \prec y \}$ for each $x \in X$:

1. $(\prec)$ is an auxiliary relation on $X$;
2. For each $x \in X$, $\downarrow x$ is down-directed;
3. For any $x, z \in X$, if $x \prec z$ then there is some $y \in X$ with $x \prec y \prec z$.

**5.6.4 Definition.** Given an ordered set $X$ equipped with an auxiliary relation $(\prec)$, we say that a filter $S$ in $X$ is a round filter in $(X, \prec)$ if for any $y \in S$ there is some $x \in S$ with $x \prec y$. Observe that $(X, \prec)$ is an ordered abstract basis if and only if $\downarrow x$ is a round filter for each $x \in X$.

**5.7 Domains as Kleisli monoids of the ordered-filter monad.** For an ordered set $X$, modules $(\prec) : X \leftrightarrow X$ correspond bijectively to monotone maps $n : X \rightarrow \text{Up} X$ via

$$x \prec y \iff n(x) \ni y,$$

and we have $n(x) = \downarrow x$. In the case that $(X, \prec)$ is an ordered abstract basis we have that each $n(x)$ is a round filter, so we obtain a map $n : X \rightarrow \text{Fil} X$. In fact, we may employ this correspondence to obtain an axiomatization of observable realization spaces and ordered abstract bases as Kleisli monoids of the ordered-filter monad $\mathbb{F} = (\text{Fil}, \uparrow, \cup)$ (see 4.4).

**5.7.1 Lemma.** For a monotone map $n : X \rightarrow \text{Up} X$ with corresponding module $(\prec) : X \leftrightarrow X$, the following are equivalent:

1. $n : X \rightarrow \text{Up} X$ restricts to an $\mathbb{F}$-monoid structure $n' : X \rightarrow \text{Fil} X$;
2. $(X, \prec)$ is an ordered abstract basis.

**Proof.** (i) is equivalent to the statement that $n$ restricts to a (monotone) map $n' : X \rightarrow \text{Fil} X$ with

$$\uparrow_X x \leq n' \quad \text{and} \quad \cup \cdot \text{Fil} n' \cdot n' \leq n'.$$
The first inequality requires that $\uparrow_X \supseteq \updownarrow x$ for all $x \in X$, or equivalently, that $(\prec) \leq (\leq_X)$; since $(\prec)$ is a module, this is equivalent to the requirement that $(\prec)$ be an auxiliary relation on $X$. Next observe that

$$y \in (\bigcup \cdot \text{Fil} n' \cdot n')(x) \iff \uparrow_X y \in (\text{Fil} n' \cdot n')(x) \iff \exists u \in n(x) \ (\uparrow_X y \subseteq n(u)) \iff \exists u \ (u \in n(x) \land y \in n(u))$$

for all $x, y \in X$, so that the second inequation requires that

$$\forall x, y \ (x \prec y \implies \exists u \ (x \prec u \prec y)).$$

\[5.7.2 \text{ Theorem.} \quad \text{Let } X \text{ be an ordered set. There is a bijection between } \text{Fil}-\text{monoids } (X, n : X \to \text{Fil} X) \text{ and ordered abstract bases } (X, \prec), \text{ wherein we associate to each Kleisli monoid } (X, n) \text{ the ordered abstract basis whose auxiliary relation } (\prec) \text{ is the module associated to the composite } X \overset{n}{\to} \text{Fil} X \overset{\uparrow}{\to} \text{Up} X \text{ under the bijection } \text{Ord}(X, \text{Up} X) \cong \text{Mod}(X, X) \text{ given above.}$$

Consequently, observable specialization systems $(X, \prec)$ correspond bijectively to Kleisli monoids $(X, n)$ in which $n : X \to \text{Fil} X$ is fully faithful, and hence there are isomorphisms

$$\text{ObsReSp} \cong \text{ObsSys} \cong \text{Fil}-\text{Mon}_{\text{ff}}$$

which commute with the underlying-set functors, where $\text{Fil}-\text{Mon}_{\text{ff}}$ is the full subcategory of $\text{Fil}-\text{Mon}$ consisting of those Kleisli monoids $(X, n)$ with $n$ fully faithful.

\textbf{Proof.} The preceding lemma yields the bijection between Kleisli monoids and ordered abstract bases, and an ordered abstract basis $(X, \prec)$ carries the induced order if and only if its mate $n : X \to \text{Fil} X$ is fully faithful.

Consider observable realization spaces $X, Y$ with associated Kleisli monoid structures $n_X, n_Y$ and a function $f : X \to Y$. If $f$ is continuous, then $f$ is monotone, so we may assume that $f$ is monotone. Then one may employ the characterization of morphisms given in Theorem 5.5.2 to verify that $f$ is continuous if and only if

$$\forall x \in X, y \in Y \ (f(x) \prec y \implies \exists u \in X \ (x \prec u \land f(u) \prec y),$$

using the fact that $(\prec_X)$ is interpolating and $(\prec_X) \leq (\leq_X)$. On the other hand, $f : (X, n_X) \to (Y, n_Y)$ is a morphism of Kleisli monoids if and only if $\text{Fil} f \cdot n_X \leq n_Y \cdot f$, equivalently, if and only if

$$\forall x \in X, y \in Y \ (f(x) \prec y \implies \exists v \ (x \prec v \land f(v) \leq y)).$$

This is entailed by continuity, since $(\prec_Y) \leq (\leq_Y)$, and entails continuity, since if $f(x) \prec y$ then we may interpolate to obtain $y'$ with $f(x) \prec y' \prec y$. \[\square\]
5.8 Continuous dcpos as sober domains. Herein we shall show that the continuous dcpos under the Scott topology are precisely the sober observable realization spaces. Moreover, we find that the observable realization spaces are closed under sobrification and hence that the continuous dcpos occur as a reflective subcategory of \( \text{ObsReSp} \). The sobrification of an observable realization space \( X \) may be formed by topologizing the set of round filters in \( X \).

For each point \( x \) in a topological space \( X \), the open neighborhood filter \( \mathcal{O}_X(x) = \mathcal{O}_X \cap \mathcal{N}_x \) of \( x \) is a filter in the frame \( \mathcal{O}_X \). The filter \( p = \mathcal{O}_X(x) \) is completely prime, meaning that for any family \( (U_i \in \mathcal{O}_X)_{i \in I} \),

\[
p \ni \bigcup_{i \in I} U_i \iff \exists i \in I (p \ni U_i) . \tag{5.8.i}
\]

Quite generally, given any frame \( O \), we refer to the completely prime filters in \( O \) as the points of \( O \), and we denote the set of all such points by \( \text{pt} \, O \). The idea is that we may liken the elements \( u \in O \) to open sets in a topological space and the points \( p \in \text{pt} \, O \) to points in a topological space, interpreting statements of the form \( p \ni u \) as statements of membership of a point in an open set. We can make this concrete by defining \( u^\square := \{ p \in \text{pt} \, O \mid p \ni u \} \) for each \( u \in O \) and observing that there is a resulting frame homomorphism

\[
O \to P \text{pt} \, O , \quad u \mapsto u^\square
\]

whose image is thus a topology on \( \text{pt} \, O \), so that the points of \( O \) are in fact the points of a topological space \( \text{pt} \, O \) which we call the spatialization of \( O \). However, the frame \( \text{pt} \, O \) need not be isomorphic to \( O \), and in fact the discipline of locale theory pursues a formulation of general topology in which frames, rather than the usual topological spaces, are taken as the primitive objects of study (see e.g. [Johnstone 1982]).

In the case that \( O \) is the frame \( \mathcal{O}_X \) of open sets of a topological space \( X \), the given frame homomorphism \( \mathcal{O}_X \to \text{pt} \, \mathcal{O}_X \) has a retraction given by \( U \mapsto \{ x \in X \mid \mathcal{O}_X(x) \in U \} \) and hence is actually an isomorphism of frames \( \mathcal{O}_X \cong \text{pt} \, \mathcal{O}_X \). The original space \( X \) is manifested within the spatialization \( \text{pt} \, \mathcal{O}_X \) via an initial continuous map \( \mathcal{O}_X : X \to \text{pt} \, \mathcal{O}_X \)—the open neighborhood filter map—and this map is an embedding if and only if \( X \) is T0. If this map is a bijection then, since it is initial, it is a homeomorphism (that is, an isomorphism in \( \text{Top} \)) \( X \cong \text{pt} \, \mathcal{O}_X \) and we say that \( X \) is sober, in which case the points of \( X \) may be viewed as order-theoretic features inherent in the frame \( \mathcal{O}_X \) of opens. Regardless, the isomorphism of frames \( \mathcal{O}_X \cong \text{pt} \, \mathcal{O}_X \) induces an evident bijection \( \text{pt} \, \mathcal{O}_X \cong \text{pt} \, \mathcal{O}_X \), and in fact this bijection coincides with the open neighborhood filter map of the space \( \text{pt} \, \mathcal{O}_X \), which is thus always sober and is called the sobrification of \( X \).

For any frame \( O \), there is a bijection \( \text{pt} \, O \cong \text{Frm}(O, 2) \) given by associating to a completely prime filter \( p \in \text{pt} \, O \) its characteristic function. Thus, we may alternatively construct the spatialization of \( O \) by taking \( \text{pt} \, O := \text{Frm}(O, 2) \) under the topology given by analogy with the above. In fact, it is trivially verified that the contravariant hom-functor \( \text{Frm}(-, 2) : \text{Frm}^{\text{op}} \to \text{Set} \) sends each frame homomorphism \( h : O \to N \) to a continuous map \( \text{pt} \, h : \text{pt} \, N \to \text{pt} \, O \) and hence factors through a functor \( \text{pt} : \text{Frm}^{\text{op}} \to \text{Top} \). Composing this functor with
the functor \( \mathcal{O} : \text{Top} \rightarrow \text{Frm}^{\text{op}} \) that sends a continuous map \( f : X \rightarrow Y \) to the frame homomorphism \( \mathcal{O}Y \rightarrow \mathcal{O}X \) given by inverse image, we obtain a covariant functor

\[
\text{Top} \xrightarrow{\mathcal{O}} \text{Frm}^{\text{op}} \xrightarrow{\text{pt}} \text{Top}
\]

which sends a space \( X \) to its sobrification. This functor factors through the full subcategory \( \text{Sob} \) of \( \text{Top} \) consisting of sober spaces as \( \text{Top} \xrightarrow{R} \text{Sob} \xrightarrow{J} \text{Top} \), and the maps \( \mathcal{O}X : X \rightarrow \text{pt} \mathcal{O}X \), which are isomorphisms for precisely the objects of \( \text{Sob} \), serve as the components of a natural transformation \( \rho : 1_{\text{Top}} \rightarrow JR \). One may verify immediately that \( \rho JR = JR \rho \) and then use this fact and the naturality of \( \rho \) to verify that \( \rho \) is the unit of an adjunction \( R \dashv J \), so that \( \text{Sob} \) is a reflective subcategory of \( \text{Top} \).

For observable realization spaces, we have the following:

**5.8.1 Proposition.** Let \( X \) be an observable realization space. Then the sobrification \( \text{pt} \mathcal{O}X \) of \( X \) is an observable realization space and is homeomorphic to the space \( \text{Fil} \circ X \) of all round filters in \( X \) under a topology consisting of the sets \( U^\circ = \{ R \in \text{Fil}_o X \mid \exists y (R \ni y \in U) \} \) with \( U \in \mathcal{O}X \), and there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{O}X} & \text{pt} \mathcal{O}X \\
\downarrow & & \downarrow \\
\text{Fil}_o X & \cong & \text{pt} \mathcal{O}X
\end{array}
\]

in \( \text{Top} \) whose bottom row is an isomorphism.

**Proof.** It is straightforward to verify that the map \( \mathcal{O}X \rightarrow \text{P} \text{Fil}_o X \) given by \( U \mapsto U^\circ \) is a frame homomorphism, so the image of this map is a topology \( \mathcal{O} \text{Fil}_o X \) on \( \text{Fil}_o X \). Moreover, since \( X \) is an observable realization space we have that for each \( U \in \mathcal{O}X \), \( x \in X \),

\[
x \in U \iff \exists y (x < y \in U) \iff \uparrow x \in U^\circ ,
\]

so the given frame homomorphism \( \mathcal{O}X \rightarrow \mathcal{O} \text{Fil}_o X \) has a section \( \uparrow \) and hence is an isomorphism of frames \( \mathcal{O}X \cong \mathcal{O} \text{Fil}_o X \). Thus, there is a homeomorphism \( \text{pt} \mathcal{O}X \cong \text{pt} \mathcal{O} \text{Fil}_o X \), wherein each completely prime filter \( q \in \text{pt} \mathcal{O} \text{Fil}_o X \) is the image \( q = \{ U^\circ \mid U \in p \} \) of a unique \( p \in \text{pt} \mathcal{O}X \). It is readily verified that \( R = \{ y \in X \mid p \ni \uparrow y \} \) is a round filter, and then for any \( U \in \mathcal{O}X \) we find that

\[
p \ni U \iff p \ni \bigcup_{y \in U} \uparrow y \iff \exists y \in U (p \ni \uparrow y) \iff R \ni U^\circ ,
\]

whence \( q = \mathcal{O} \text{Fil}_o X(R) \). This shows that the initial continuous map \( \mathcal{O} \text{Fil}_o X : \text{Fil}_o X \rightarrow \text{pt} \mathcal{O} \text{Fil}_o X \) is surjective, and we shall show below that this map is injective and hence a homeomorphism, so we obtain a composite homeomorphism \( \text{Fil}_o X \cong \text{pt} \mathcal{O} \text{Fil}_o X \cong \text{pt} \mathcal{O}X \) which, by (5.8.1), commutes with the maps \( \uparrow, \mathcal{O}X \) as in the above diagram.

Indeed, \( \text{Fil}_o X \) is T0, since its underlying order \( (\leq) \) is the reverse inclusion order, as follows. We have that

\[
R \in (\uparrow y)^\circ \iff \exists u \in X (R \ni u < y) \iff R \ni y
\]
for any \( R \in \text{Fil}_c X, y \in X \). Thus, for \( R, S \in \text{Fil}_c X \), one readily verifies the following implications,

\[
R \leq S \implies \forall y \in X \ (R \in (\downarrow y)^\circ \iff S \in (\downarrow y)^\circ) \iff R \supseteq S \implies R \leq S ,
\]

thus obtaining the needed conclusion.

Lastly, we show that \( \text{Fil}_c X \) is an observable realization space via Proposition 5.3.1(iv). Suppose \( R \in U^\circ \). Then

\[
R \in U^\circ = (\bigcup_{y \in U} \downarrow y)^\circ = \bigcup_{y \in U} (\downarrow y)^\circ
\]

and hence there is \( y \in U \) with \( R \in (\downarrow y)^\circ \). We know that \( \downarrow y \in \text{Fil}_c X \), and each \( S \in (\downarrow y)^\circ \) specializes \( \downarrow y \) since \( S \ni y \) and hence \( S \supseteq \downarrow y \). Thus, we have that \( R < \downarrow y \in U^\circ \).  

Before characterizing the sober observable realization spaces as precisely the continuous dcpo's under the Scott topology, we note the following well-known result concerning sober spaces:

5.8.2 Proposition. Let \( X \) be a sober space. Then, under the underlying order, \( X \) is a co-dcpo, and \( O_X \) is coarser than the Scott topology on \( X \) (that is, every open set is Scott-open).

Proof. It is straightforward to verify that the union of a directed subset of \( \text{pt } O_X \) is a completely prime filter on \( O_X \) and hence that directed meets in \( \text{pt } O_X \) are given by union (as is the case in \( T X \)), so that these directed meets are moreover preserved by the monotone map \( \uparrow_{PTX} : \text{pt } O_X \rightarrow T X \). Further, the top row of the following commutative diagram is an Ord-isomorphism

\[
\begin{array}{ccc}
X & \xrightarrow{\cong} & \text{pt } O_X \\
\downarrow \nu & & \downarrow \uparrow_{PTX} \\
TX & &
\end{array}
\]

and hence \( X \) is a separated ordered set with all directed meets such that these meets are preserved by \( \nu \). Therefore, \( X \) is a co-dcpo, and by Proposition 5.2.2 \( O_X \) is coarser than the Scott topology.  

In observable realization spaces, these conditions are also sufficient for sobriety, and we obtain the following result:

5.8.3 Theorem. For any topological space \( X \) endowed with the underlying order, the following are equivalent:

(i) \( X \) is a sober observable realization space;

(ii) \( X \) is a cocontinuous co-dcpo and carries the Scott topology.
Proof. First suppose (i). Then by the preceding Proposition, $X$ is a co-dcpo and $\mathcal{O}X$ is coarser than the Scott topology, so by Lemma 5.3.4, $\mathcal{O}X$ coincides with the Scott topology. Thus, since $X$ is an observable realization space, the co-dcpo $X$ is cocontinuous, by Corollary 5.2.5 and Proposition 5.3.1.

Next, suppose (ii). Since $X$ is a cocontinuous co-dcpo carrying the Scott topology, $X$ is an observable realization space. $X$ is T0, since it carries a separated underlying order, so to show that $X$ is sober it suffices by Proposition 5.8.1 to show that the map $\mathcal{O}X(\bigwedge R) = \bigcup_{y \in R} \mathcal{O}(y)$ is surjective, so let $R \in \text{Fil}_o X$. Since $X$ is a co-dcpo and carries the Scott topology, by Proposition 5.2.2 there is a topological meet $\bigwedge R$ of $R$ in $X$, so that $\mathcal{O}(\bigwedge R) = \bigcup_{y \in R} \mathcal{O}(y)$.

Thus, for any $z \in X$,

$$\bigwedge R \prec z \iff \mathcal{O}(\bigwedge R) \ni \downarrow z \iff \exists y (R \ni y \ni \downarrow z) \iff R \ni z,$$

since $R$ is a round filter, so $\downarrow \bigwedge R = R$.

Letting $\text{CntDcpo}^{\text{co}}$ be the category of cocontinuous co-dcpos and directed-meet-preserving maps, we obtain the following:

5.8.4 Corollary. There is a reflective full embedding

$$\text{CntDcpo}^{\text{co}} \hookrightarrow \text{ObsReSp}$$

which commutes with the underlying-set functors and endows a cocontinuous co-dcpo with its Scott topology and whose image is the full subcategory of $\text{ObsReSp}$ consisting of sober observable realization spaces. A left adjoint to this embedding may be taken as a restriction of the composite

$$\text{Top}^{\text{pt}} \circ \text{Top} \rightarrow \text{Ord},$$

of the sobrification functor $\text{pt}\mathcal{O}$ and the concrete functor $S$, which sends a topological space to its underlying order.

Proof. By the preceding Theorem and the remarks following the definition of the Scott topology above, there is an embedding $\text{CntDcpo}^{\text{co}} \hookrightarrow \text{ObsReSp}$ as described, and this embedding restricts to an isomorphism $\text{CntDcpo}^{\text{co}} \cong \text{Sob} \cap \text{ObsReSp}$ with inverse a restriction of $S$. As discussed above, the embedding $\text{Sob} \hookrightarrow \text{Top}$ has a left adjoint $R : \text{Top} \rightarrow \text{Sob}$ which is a restriction of $\text{pt}\mathcal{O}$, and since $\text{ObsReSp}$ is closed under sobrification by Proposition 5.8.1, $R$ restricts to a left adjoint of the inclusion $\text{Sob} \cap \text{ObsReSp} \hookrightarrow \text{ObsReSp}$, thus yielding the needed reflector.

5.9 Cocontinuous lattices among lax algebras. A cocontinuous lattice is a complete lattice that is cocontinuous (see 5.2). In 4.4 it was shown that the category of $\text{Cnt}^{\text{co}}$ of cocontinuous lattices and directed-meet-preserving sup-maps is isomorphic to the category $\text{Set}^{\mathcal{E}}$ of Eilenberg–Moore algebras of the filter monad $\mathcal{E}$. By Theorem 5.8.3 when we endow a cocontinuous lattice with its Scott topology, we obtain a (sober) observable realization space.
The resulting functor $\text{Cnt}^{\text{co}} \to \text{ObsReSp}$ is faithful, but not full, and given the isomorphism $\text{ObsReSp} \cong (\mathbb{F}, 2)\text{-Cat}_=$ of Theorem 5.4.1, we may form an evident commutative diagram

$$
\begin{array}{ccc}
\text{Cnt}^{\text{co}} & \to & \text{ObsReSp} \\
\cong & \downarrow & \cong \\
\text{Set}^\mathbb{F} & \to & (\mathbb{F}, 2)\text{-Cat}_= \\
\end{array}
$$

The functors in the leftmost square allow us to consider cocontinuous lattices as spaces, $\mathbb{F}$-algebras, and strict lax algebras of $\mathbb{F}$, as follows.

5.9.1 Proposition. Let $X$ be a cocontinuous lattice.

1. The relation $a : FX \to X$ of Scott convergence of filters (see Example 5.3.3) is determined by a map $l : FX \to X$, $\chi \mapsto \bigwedge \{z \in X \mid \downarrow z \in \chi\} = \bigwedge_{A \in \chi} \vee A$, in the sense that $a = l_* = (\leq_X) \cdot l$.

2. By (1), the neighborhood filter map $\nu : X \to FX$ is related to $l$ via $l_* = a = \nu^*$ and hence $l \dashv \nu$.

3. The isomorphism $\text{Cnt}^{\text{co}} \cong \text{Set}^\mathbb{F}$ of 4.4 associates to $X$ the $\mathbb{F}$-algebra $(X, l)$.

4. The given functor $\text{Cnt}^{\text{co}} \to \text{ObsReSp}$ endows the cocontinuous lattice $X$ with its Scott topology, and the isomorphism $\text{ObsReSp} \cong (\mathbb{F}, 2)\text{-Cat}_=$ of Theorem 5.4.1 associates to this space the strict lax algebra $(X, l_* : FX \to X)$.

Proof. The verification is straightforward. \qed

We now characterize the image of the given composite $\text{Cnt}^{\text{co}} \cong \text{Set}^\mathbb{F} \to (\mathbb{F}, 2)\text{-Cat}_=$, thus characterizing the cocontinuous lattices among strict lax algebras.

5.9.2 Theorem. Among lax algebras $(X, a : FX \to X)$ of $\mathbb{F}$, those associated to cocontinuous lattices in the above commutative diagram are precisely those that are

1. strict,

2. map-determined, in the sense that $a = l_*$ for some monotone map $l : FX \to X$, and

3. $T_0$, in the sense that $(\leq_X)$ is separated,

when we endow $X$ with the underlying order $(\leq_X) = a \cdot e_X$.

Thus, the category of cocontinuous lattices and directed-meet-preserving maps is evidently isomorphic to the full subcategory of $(\mathbb{F}, 2)\text{-Cat}_=$ whose objects satisfy (2) and (3).
Proof. By the above, it is clear that if \( X \) is a cocontinuous lattice, then the associated lax algebra satisfies all three conditions. Conversely, consider an arbitrary \((X,a) \in \text{ob}(\mathcal{F}, \mathbf{2})\)-Cat satisfying (1)–(3). Since \((X,a)\) is strict, the associated topological space is an observable realization space by Theorem 5.4.1. Therefore, by Proposition 5.2.2, Lemma 5.3.4, and Proposition 5.2.5, it suffices to show that \( X \) is a complete lattice and that the neighborhood-filter map \( \nu : X \to FX \) preserves directed meets.

Since \( a \) is the convergence relation of the associated space, we have that \( l^* = a = \nu^* \). Thus,

\[
FX \xrightarrow{\nu} X,
\]

so since \( FX \) is a complete lattice and \( \nu : X \to FX \) is a fully faithful functor, it follows that \( X \) is a complete lattice and that \( \nu \) preserves all meets.

5.9.3 Remark. Condition 5.9.2(2) means

\[
xax \iff l(x) \leq x
\]

for all \( x \in FX, x \in X \). In other words, \( l \) chooses for every \( x \in FX \) a smallest convergence point which, in conjunction with (3), is uniquely determined. Conversely, if every filter \( \chi \) has a least convergence point \( l(\chi) \), we obtain a map \( l : FX \to X \) which is automatically monotone and and satisfies \( a = l_* \).

Exercises

5.A Topological meets and the Scott topology. For a topological space \( X \) endowed with its underlying order, the statement that every open subset of \( X \) is Scott-open is equivalent to the statement that every directed meet in \( X \) is a topological meet (Proposition 5.2.2).

5.B Comparing \( \ll \) and \( \prec \). Let \( X \) be an ordered set endowed with the Scott topology. If either \( \ll \) or \( \prec \) is approximating, then \( \ll = \prec \) (see Lemma 5.2.4).

Hint. Note that we have \( \prec \leq \ll \) in general. Employ Lemma 5.2.3.

5.C Characterizations of cocontinuity. For an ordered set \( X \) endowed with the Scott topology, the following are equivalent:

(i) \( X \) is cocontinuous;

(ii) the relation \((\prec)\) is approximating;

(iii) each \( x \in X \) is a topological meet of \( \downarrow x := \{ y \in X \mid x \prec y \} \).

(See Proposition 5.2.5.)

5.D Closure properties of the class of observable realization spaces. Show that \( \text{ObsReSp} \) is closed in \( \text{Top} \) under finite products, arbitrary coproducts, and retracts.
Notes on Chapter IV

The provision of a satisfactory notion of convergence has been one of the major concerns of topology from its very beginnings. In his pioneering work [Fréchet, 1905, 1906], Fréchet not only introduced metric spaces (the designation is due to Hausdorff) but also defined sequential convergence in an abstract way. However, his concept of convergence turned out to be insufficient as it lacks in particular the ability to deal with iterated limits. A more general type of convergence based on directed sets rather than sequences was introduced by Moore and his student Smith [Moore, 1915; Moore and Smith, 1922], and in 1937 Birkhoff characterized T1-topological spaces in terms of this net convergence (the term net was coined by Kelley [1950]). At a first glance, the notion of net resembles closely the notion of sequence; however, already the definition of subnet turns out to be more sophisticated than the corresponding notion of subsequence. Birkhoff also showed that net convergence can be equivalently substituted by a notion of convergence based on certain systems of sets that he had introduced earlier [Birkhoff, 1935], called filter bases in today’s language. Approximately at the same time, Cartan [1937a, b] introduced the notion of filter convergence, and this idea became central in Bourbaki’s treatment of topology [Bourbaki, 1942]. The notions of filter and ultrafilter and their predecessors per se had emerged earlier in various mathematical contexts; in fact, the ideale Verdichtungsstelle mentioned by Riesz [1908] may be considered a precursor of the notion of ultrafilter, and the Polish School of measure theorists and functional analysts had studied the dual concept of ideal, with Tarski [1930] in effect proving that every filter is contained in an ultrafilter.

Grimeisen [1960, 1961] characterized topological spaces among pretopological spaces as precisely those which have the “iterated limit property”. Later on, Cook and Fischer [1967] presented four axioms that characterize topological spaces in terms of filter convergence. Barr [1970] extended Manes’ equational presentation of compact Hausdorff spaces (see [Manes, 1969]) to characterize topological spaces in terms of two simple inequalities for an ultrafilter convergence relation that are amenable to algebraic manipulation. The fact that Barr’s two ultrafilter convergence axioms may be taken verbatim as filter convergence axioms to already fully characterize topological spaces appeared only in Seal [2005]; a slightly weaker version of this result had been established earlier in Pisani [1999].

The original definition of a topological space given by Hausdorff [1914] used neighborhood systems as the primary structure. (His axioms included the “Hausdorff” separation axiom which, however, is independent of his other axioms.) The fact that this definition is equivalent to that of a monoid in the Kleisli category of the filter monad on $\mathbf{Set}$ (Proposition 1.1.1) was first observed in Gähler [1992]. Therein, Gähler introduced the notion of a preordered monad that is at the origin of our power-enriched monads 1.2 and their associated category of monadic topologies that, similarly to Kleisli monoids 1.3, are defined as monoids in the Kleisli category of the given monad. Höhle’s notion of a topological space object (see [Höhle, 2001]) uses the same idea in a framework different from the one used in this book, but his work targeted many of the categories also of interest here. The presentation of closure spaces and the up-set monad as in Example 1.3.2(4) appeared in Seal [2009] which also defined the Kleisli and strata extensions of a power-enriched monad as described in 1.4 and 2.4. The strata extensions of a general lax monad first appeared in Clementino and Hofmann [2004b]. Initial extensions of a functor and of a monad 2.1 were introduced in Schubert and Seal [2008] and Colebunders, Lowen, and Rosiers [2011], respectively. The Kleisli towers in 2.5 are a particular case of the tower extension construction over a topological functor from Zhang [2000], where Zhang generalized the presentation of approach spaces as towers [Lowen, 1989].

The construction of the discrete presheaf monad (as a power-enriched monad) from an arbitrary monad with an associative lax extension to $\mathbf{V}$-$\mathbf{Rel}$ as given in Section 3.2.4—which enables the surprising result that $(\mathcal{T}, \mathcal{V})$-categories may always be considered as relational algebras (see Corollary 3.2.3)—was motivated by the presentation of approach spaces as relational algebras in Lowen and Vroegrijk [2008]. The alternative description of the discrete presheaf monad in $\mathbf{3.3}$ is similar to that one of the monad $\pi$ in Colebunders, Lowen, and Rosiers [2011].
CHAPTER IV. KLEISLI MONOIDS

The identification of Eilenberg–Moore algebras for the filter monad on $\text{Set}$ or $\text{Top}_0$ as continuous lattices is due to Day [1975]. In 1981, Wyler noted that $\text{Cnt}$ is also monadic over $\text{Sup}$ and $\text{CompHaus}$. The fact that completely distributive lattices are the Eilenberg–Moore algebras for the up-set monad on $\text{Set}$ (Exercise 4.E) was noted in Pedicchio and Wood [1999]; the corresponding observation for frames (Exercise 4.F) over $\text{Set}$ can be found in Johnstone [1982]. The description of $\mathcal{P}_\mathcal{V}$-algebras as $\mathcal{V}$-actions in $\text{Sup}$ (Exercise 4.C) comes from Pedicchio and Tholen [1989] (following the more technical description of Machner [1985]); Pedicchio and Tholen [1989] also remarked that $\text{Set}^\mathcal{V}$ is monadic over $\text{Ord}$ and $\mathcal{V}$-$\text{Cat}$. Further developments in the context of categories enriched in a quantaloid can be found in Stubbe [2007]. The description of the distributive law of the list monad over the powerset monad and its category of Eilenberg–Moore algebras (Exercise 4.H) appeared in Manes and Mulry [2007]. Finally, the distributive law of Exercise 4.A can be found in Seal [2006] and uses the distributive law of $\text{Dn}$ over $\text{Up}$ described in Marmolejo et al. [2002] that, in turn, finds its source in a distributive law for the frame monad over $\text{Ord}$ mentioned by Linton at a meeting in 1984. The proof that the RegMono-injective $T_0$-spaces are the continuous lattices endowed with their Scott topology (Example 4.6.6(2)) goes back to Scott [1972], but the treatment we give of the subject in Section 4.6—and in particular the proof of Theorem 4.6.3—is more directly inspired by Escardó [1998]. The power-enriched treatment was developed in Seal [2010] and [Seal, 2011].

Section 5 is based largely on Lucyshyn-Wright [2009, 2011], in which Theorems 5.4.1 and 5.9.2 were first proved. The proof of Lemma 5.2.3 is derived from the proof (for dpos) that was contributed by Escardó to Gierz et al. [2003]. The given relation of Scott convergence was studied (with respect to the dual order) by Erné [1981, 1982] (under the name of $s_3$-convergence), considered there among two other generalizations to arbitrary preorders of Scott’s original notion of convergence in continuous lattices Scott [1972].

The class of spaces that we call the observable realization spaces appeared in papers by Banaschewski [1977, 1980, 1981, 1991, 1999]; Erné and Wilke [1983]; Ershov [1993]; Hoffmann [1979, 1981]; Lawson [1997, 1979]; but were often identified via quite different and less elementary characterizations. These spaces were called C-spaces by Erné. The basic role of these spaces in domain theory was noted by Ershov [1993], who had earlier developed domain theory in topological terms. The domain-theoretic relevance of C-spaces was also underscored in Lucyshyn-Wright [2009, 2011]. Lemma 5.3.4 is part of Exercise II.34 of Gierz et al. [2003], contributed by Escardó and Heckmann.

The axioms for an observable specialization system given in Definition 5.5.1 and the main result of Theorem 5.5.2 were given by Hoffmann [1981] (building upon work in Hoffmann [1979]), whereas the axioms (1)–(3) defining an abstract basis were given by Smyth earlier [1977/78] as a way of generating continuous dpos. Each abstract basis gives rise to an associated continuous dcpo, the set of round ideals of the basis (see for example Abramsky and Jung [1994]). Herein, following Lucyshyn-Wright [2009, 2011], we regard those abstract bases satisfying (4) as domains in their own right. These domains—the observable realization spaces—need not be sober, and the space of round ideals (or, rather, round filters) associated to such a space is in fact the sobrification of that space and is a continuous dcpo under the Scott topology (see 5.8). With regard to ordered abstract bases (see Definition 5.6.1), note that auxiliary relations with the given properties were studied by Hoffmann [1981].

Looking at \((\mathcal{T}, \mathcal{V})\)-categories as geometric objects, in this chapter we explore fundamental topological properties like compactness and Hausdorff separation, along with low-separation properties, regularity, normality, and extremal disconnectedness. We do so first taking a more traditional object-oriented view, before highlighting the central role of proper and open maps in \((\mathcal{T}, \mathcal{V})\text{-Cat}\). Each of these classes give a “topology” on \((\mathcal{T}, \mathcal{V})\text{-Cat}\) and, along with other classes, allows us to investigate relativized topological properties in an axiomatic categorical setting. We also explore the categorical notion of connected object in \((\mathcal{T}, \mathcal{V})\text{-Cat}\).

1 Hausdorff separation and compactness

Topological spaces have been described as lax algebras via ultrafilter convergence in III.2.2, that is, as sets \(X\) equipped with a relation \(a : \beta X \rightarrow X\) satisfying the two requirements of a \((\beta, 2)\)-category. The relation \(a\) will actually be a map if, when we write \(x \xrightarrow{\chi} x\) instead of \(x a x\),

\[
\forall x, y \in X, z \in TX (z \rightarrow x \& z \rightarrow y \implies x = y), \text{ that is, } a \cdot a^\circ \leq 1_X,
\]

and

\[
\forall \chi \in TX \exists x \in X (\chi \rightarrow x), \text{ that is, } 1_{TX} \leq a^\circ \cdot a.
\]

The first property identifies \(X\) as a Hausdorff space, and the second one as a compact space (see III.2.3). This observation leads us to consider the following notions in the general context of a monad \(\mathcal{T} = (T, m, e)\) with a lax extension \(\hat{T}\) to \(\mathcal{V}\text{-Rel}\), where \(\mathcal{V} = (\mathcal{V}, \otimes, k)\) is a quantale.

1.1 Basic definitions and properties. As we are considering topologically-inspired properties of \((\mathcal{T}, \mathcal{V})\)-categories, we will most often refer to the objects of \((\mathcal{T}, \mathcal{V})\text{-Cat}\) as \((\mathcal{T}, \mathcal{V})\text{-spaces}\), and to its morphisms as \((\mathcal{T}, \mathcal{V})\text{-continuous maps}\). Objects and morphisms of \(\mathcal{V}\text{-Cat}\) are simply called \(\mathcal{V}\text{-spaces}\) and \(\mathcal{V}\text{-continuous maps}\), respectively.

1.1.1 Definition. A \((\mathcal{T}, \mathcal{V})\)-space \((X, a)\) is Hausdorff if

\[
a \cdot a^\circ \leq 1_X,
\]
and it is compact if
\[ 1_{TX} \leq a^\circ \cdot a . \]

The resulting full subcategories of \((\mathbb{T}, \mathcal{V})\text{-Cat}\) are denoted by \((\mathbb{T}, \mathcal{V})\text{-Cat}_{\text{Haus}}\) and \((\mathbb{T}, \mathcal{V})\text{-Cat}_{\text{Comp}}\), respectively, and their intersection by \((\mathbb{T}, \mathcal{V})\text{-Cat}_{\text{CompHaus}}\).

1.1.2 Proposition. Let \((X, a)\) be a \((\mathbb{T}, \mathcal{V})\)-space.

(1) \((X, a)\) is Hausdorff if and only if for all \(x, y \in X\) and \(z \in TX\),
\[ \perp < a(z, x) \otimes a(z, y) \implies x = y \quad \text{and} \quad a(z, x) \otimes a(z, x) \leq k , \]
where the latter condition holds trivially when \(\mathcal{V}\) is integral.

(2) \((X, a)\) is compact if and only if for all \(x \in TX\),
\[ k \leq \bigvee_{z \in X} a(x, z) \otimes a(x, z) . \quad (1.1.i) \]

Proof. The assertions follow immediately from
\[
(a \cdot a^\circ)(x, y) = \bigvee_{z \in TX} a(z, x) \otimes a(z, y) , \quad \text{and} \quad (a^\circ \cdot a)(x, y) = \bigvee_{z \in X} a(x, z) \otimes a(y, z)
\]
for all \(x, y \in X, \, x, y \in TX\).

1.1.3 Remark. If the quantale \(\mathcal{V}\) is superior, that is, if
\[ k \leq \bigvee_{i \in I} u_i \otimes u_i \iff k \leq \bigvee_{i \in I} u_i \quad (1.1.ii) \]
for all families \(u_i \in \mathcal{V}, \, i \in I\), then the compactness condition \((1.1.i)\) simplifies to
\[ k \leq \bigvee_{z \in X} a(x, z) , \quad (1.1.iii) \]
for all \(x \in TX\). Note that for \(\mathcal{V}\) integral (so that \(k = \top\)) one has \(u \otimes u \leq u \otimes k = u\), so that the implication \(\implies\) of \((1.1.ii)\) holds trivially. If \(\mathcal{V}\) is a frame with \(\otimes = \wedge\), or when \(\mathcal{V} = P_+\), the implication \(\iff\) of \((1.1.ii)\) also holds, so that in this case the simplified compactness condition \((1.1.iii)\) applies.

An example of a non-superior quantale is given below in Remark 1.4.3(1).
1. HAUSDORFF SEPARATION AND COMPACTNESS

1.1.4 Examples.

(1) Let $\mathbb{I} = \downarrow$ be the identity monad (identically extended to $\mathcal{V}$-$\text{Rel}$). Then, for $x \neq y$ in a Hausdorff $(\mathbb{I}, \mathcal{V})$-space $(X, a)$, we must have $a(x, y) = \bot$. Consequently, if $k = \mathbb{I}$ in $\mathcal{V}$, necessarily $a = 1_X$, that is, $(X, a)$ is discrete. However, quite trivially, $(X, a)$ is always compact. Briefly, if $\mathcal{V}$ is integral, then $\mathcal{V}$-$\text{Cat}_{\text{Comp}} = \mathcal{V}$-$\text{Cat}$, while $\mathcal{V}$-$\text{Cat}_{\text{Haus}}$ is the full coreflective subcategory of discrete $\mathcal{V}$-categories in $\mathcal{V}$-$\text{Cat}$.

(2) In $\text{Top} \cong (\beta, 2)$-$\text{Cat}$ (see III.2.2), the Hausdorff separation property and the compactness property of 1.1.1 are equivalent to the usual notions that are expressed in terms of open sets, provided that the Axiom of Choice is granted: a topological space $X$ is Hausdorff if and only if for any distinct points $x$, $y$ one finds disjoint open subsets $U \ni x$, $V \ni y$, and $X$ is compact if and only if any open cover of $X$ (that is, any set $\mathcal{U}$ of open subsets with $\bigcup \mathcal{U} = X$) contains a finite subcover (see Propositions III.2.3.1 and III.2.3.2). Briefly,

$$((\beta, 2))$-\text{Cat}_{\text{Haus}} \cong \text{Haus}, (\beta, 2)$-\text{Cat}_{\text{Comp}} \cong \text{Comp} .$$

(3) In $\text{Top} \cong (\mathcal{F}, 2)$-$\text{Cat}$ (see Corollary IV.1.5.4), since the filter $PX$ on $X$ converges to every point in $X$, a Hausdorff $(\mathcal{F}, 2)$-space can have only at most one point: $(\mathcal{F}, 2)$-$\text{Cat}_{\text{Haus}}$ is equivalent to $\{\emptyset, 1\}$. Considering the filter $\{X\}$, one sees that a compact $(\mathcal{F}, 2)$-space $X$ must contain a point whose only neighborhood is $X$. Since every other filter on $X$ converges to that point as well, this property characterizes compactness in $(\mathcal{F}, 2)$-$\text{Cat}$. Spaces with this property are called supercompact since they may equivalently be described by the property that every open cover of $X$ contains at most one open subset of $X$ (see Exercise 1.A).

(4) If we replace the filter monad by the submonad $\mathcal{F}_p = (F_p, m, e)$, where $F_p X = FX \setminus \{PX\}$ is the set of proper filters on $X$, equipped with its Kleisli extension $\hat{\mathcal{F}}_p$, then the Hausdorff property assumes its usual meaning: $(\mathcal{F}_p, 2)$-$\text{Cat}_{\text{Haus}} \cong \text{Haus}$. However, $(\mathcal{F}_p, 2)$-$\text{Cat}_{\text{Comp}} = (\mathcal{F}_p, 2)$-$\text{Cat}_{\text{Comp}} \setminus \{\emptyset\}$. Consequently, compact Hausdorff objects in $(\mathcal{F}_p, 2)$-$\text{Cat}$ can have at most one point (see Exercise 1.A).

(5) An object $(X, a)$ in $(\beta, \mathcal{P}_+)$-$\text{Cat} \cong \text{App}$ (see III.2.4.5) is Hausdorff precisely when $a(z, x) < \infty \& a(z, y) < \infty$ implies $x = y$, for all $x, y \in X$, $z \in \beta X$. Obviously, $(X, a)$ is Hausdorff if and only if its pseudotopological modification $(X, oa)$ of $(X, a)$ is Hausdorff. (Here $o : [0, \infty]^\text{op} \to 2 = (\bot, \top)$ is the “optimist’s map” with $(o(v) = \top \iff v < \infty)$ which induces a functor

$$(\beta, \mathcal{P}_+)$-\text{Cat} \cong \text{App} \to (\beta, 2)$-\text{Gph} \cong \text{PsTop} ,$$

see Exercise III.1.1 and Examples III.4.1.3 a pseudotopological space $(X, \rightarrow)$ is called Hausdorff if $z \rightarrow x \& z \rightarrow y$ implies $x = y$, for all $x, y \in X$, $\chi \in \beta X$.) The approach space $(X, a)$ is compact if and only if

$$\inf_{x \in X} a(\chi_x, x) = 0$$
for all $x \in \beta X$, a property called $0$-compact in approach theory (see [Lowen, 1988]).

1.1.5 Proposition.

(1) $(\mathbb{T}, \mathcal{V})$-$\text{Cat}_{\text{Haus}}$ is closed under non-empty mono-sources in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$; it is closed under all mono-sources and, therefore, it is strongly epireflective in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$ if $\mathcal{V}$ is integral or $T_1 \cong 1$.

(2) $(\mathbb{T}, \mathcal{V})$-$\text{Cat}_{\text{Comp}}$ is closed under those sinks $g_i : (X_i, a_i) \to (Y, b)$ in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$ for which $(Tg_i)_{i \in I}$ is epic in $\text{Set}$.

Proof.

(1) For a mono-source $(f_i : (X, a) \to (Y_i, b_i))_{i \in I}$ with $I \neq \emptyset$ in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$, one has $1_X = \bigwedge_{i \in I} f_i^* \cdot f_i$ (see Proposition III.1.2.2). Hence, if all $(Y_i, b_i)$ are Hausdorff, from $a \cdot a^* \leq (f_i^* \cdot b_i \cdot Tf_i) \cdot ((Tf_i)^* \cdot b_i^* \cdot f_i) \leq f_i^* \cdot b_i \cdot f_i \leq f_i^* \cdot f_i$ one obtains $a \cdot a^* \leq 1_X$, as desired. If $I = \emptyset$ then $|X| \leq 1$ and the Hausdorff criterion 1.1.2(1) is trivially satisfied when $k = \top$ or $T1 \cong 1$. The additional claim follows with Proposition III.5.10.1.

(2) Similarly, the hypothesis on the sink $(g_i)_{i \in I}$ guarantees $1_{TY} = \bigvee_{i \in I} Tg_i \cdot (Tg_i)^*$. Consequently, with all $(X_i, a_i)$ compact, one obtains

$$b^* \cdot b \geq (Tg_i \cdot a_i^* \cdot g_i^*) \cdot (g_i \cdot a_i \cdot (Tg_i)^*) \geq Tg_i \cdot a_i^* \cdot a_i \cdot (Tg_i)^* \geq Tg_i \cdot (Tg_i)^*$$

and, hence, $b^* \cdot b \geq 1_{TY}$.

1.1.6 Corollary.

(1) For a surjective $(\mathbb{T}, \mathcal{V})$-continuous map $g : (X, a) \to (Y, b)$ between $(\mathbb{T}, \mathcal{V})$-spaces, if $(X, a)$ is compact, then $(Y, b)$ is compact.

(2) If the $\text{Set}$-functor $T$ preserves coproducts, $(\mathbb{T}, \mathcal{V})$-$\text{Cat}_{\text{Comp}}$ is closed under epi-sinks in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$. The same statement holds for finite coproducts with closure under finite epi-sinks.

Proof.

(1) $g$ is a split epimorphism in $\text{Set}$ and is preserved by $T$.

(2) Decompose $(Tg_i)_{i \in I}$ as

$$TX_i \longrightarrow \bigsqcup_i TX_i \xrightarrow{d} T\bigsqcup_i X_i \xrightarrow{Tg} TY.$$

Here the induced morphism $g$ is epic and therefore preserved by $T$, and the canonical comparison morphism $d$ is bijective by hypothesis. It actually suffices to know that $d$ is surjective to be able to conclude that $(Tg_i)_{i \in I}$ is an epimorphism and to apply Proposition III.1.5.
1. HAUSDORFF SEPARATION AND COMPACTNESS

1.1.7 Example. Compact spaces are closed under finite epi-sinks in \( \text{Top} \cong (\beta, 2)-\text{Cat} \) (since \( \beta : \text{Set} \to \text{Set} \) preserves finite coproducts), but not under countable epi-sinks (consider \((n : 1 \to \mathbb{N})_{n \in \mathbb{N}} \) with \( \mathbb{N} \) discrete). Likewise, 0-compact spaces are closed under finite epi-sinks in \( \text{App} \cong (\beta, \mathcal{P}_+)-\text{Cat} \), but not under countable epi-sinks.

Finally, Hausdorffness and compactness are preserved by algebraic functors (see III.3.4).

1.1.8 Proposition. Let \( \alpha : (S, \hat{S}) \to (T, \hat{T}) \) be a morphism of lax extensions. Then the associated algebraic functor \( A_\alpha : (T, \mathcal{V})-\text{Cat} \to (S, \mathcal{V})-\text{Cat} \) preserves Hausdorffness and compactness.

Proof. Preservation of Hausdorffness follows from \( (a \cdot \alpha_X) \cdot (a \cdot \alpha_X) = a \cdot \alpha_X \cdot \alpha_X \cdot a \leq a \cdot a \leq 1_X \), and preservation of compactness from \( (a \cdot \alpha_X) \cdot (a \cdot \alpha_X) = a_X \cdot a_X \cdot a_X \geq a_X \cdot a_X \geq 1_{TX} \).

1.2 Tychonoff’s Theorem, Čech–Stone compactification. For \( T \) and \( V \) as in 1.1, a \((T, V)\)-space \((X, a)\) is by definition compact Hausdorff when \( a : TX \to X \) is a map in the ordered category \( \mathcal{V}-\text{Rel} \). By Proposition III.1.2.1, \( a \) is then actually a \( \text{Set} \)-map \( TX \to X \), provided that \( V \) is integral and lean. In that case, the \( \mathcal{V} \)-relational inequalities between \( \text{Set} \)-maps must actually be equalities; likewise the \((T, V)\)-continuity condition \( f \cdot a \leq b \cdot Tf \) for \( f : (X, a) \to (Y, b) \) must actually be an equality if \((X, a), (Y, b)\) are both compact Hausdorff. This proves the following result.

1.2.1 Proposition. Let \( V \) be integral and lean. Then \((T, V)\)-\text{Cat}_{\text{CompHaus}} is precisely the full subcategory of \( \text{Set}^T \) containing those \( T \)-algebras \((X, a)\) with \( a \cdot \hat{T}a = a \cdot m_X \). In particular, if \( \hat{T} \) is flat, then

\[ (T, V)\)-\text{Cat}_{\text{CompHaus}} = \text{Set}^T. \]

1.2.2 Examples.

(1) The category of 0-compact approach spaces with Hausdorff pseudotopological modification (see Example 1.1.4(5)) is isomorphic to the category of compact Hausdorff spaces:

\[ (\beta, \mathcal{P}_+)-\text{Cat}_{\text{CompHaus}} \cong (\beta, 2)-\text{Cat}_{\text{CompHaus}} \cong \text{Set}\beta \cong \text{CompHaus}. \]

(2) Example 1.1.4(3) shows that flatness of \( \hat{T} \) is essential for \((T, V)\)-\text{Cat}_{\text{CompHaus}} = \text{Set}^T, while \text{Set}^T is the category of cocontinuous lattices (see IV.4.4), \((\mathbb{F}, 2)\)-\text{Cat}_{\text{CompHaus}} contains just singletons.

Proposition 1.2.1 entails a Tychonoff Theorem for compact Hausdorff objects in \((T, V)\)-\text{Cat}, with the needed hypotheses on \( T \) and \( V \) granted, as follows. With \((X_i, a_i)\) in \( \text{Set}^T \) for all \( i \in I \),
the product \( X = \prod_{i \in I} X_i \) in \( \text{Set}^T \) carries the structure \( a : TX \to X \) with \( \pi_i \cdot a = a_i \cdot T \pi_i \) (where \( \pi_i \) are the product projections for all \( i \in I \)), and using Proposition \[ \text{III.1.2.2} \] one obtains that \( a \) coincides with the product structure formed in \( (\mathbb{T}, \mathcal{V})\)-\text{Cat} (see Proposition \[ \text{III.3.1} \]):

\[
a = (\Lambda_{i \in I} \pi_i^0 \cdot \pi_i) \cdot a = \Lambda_{i \in I} \pi_i^0 \cdot \pi_i \cdot a = \Lambda_{i \in I} \pi_i^0 \cdot a_i \cdot T \pi_i .
\]

With the help of the Adjoint Functor Theorem \[ \text{II.2.12.1} \] we can now go one step further and guarantee the existence of a Čech–Stone compactification for \( (\mathbb{T}, \mathcal{V})\)-spaces.

**1.2.3 Theorem.** Let \( \mathcal{V} \) be integral and lean, and let \( \hat{T} \) be flat. Then \( (\mathbb{T}, \mathcal{V})\)-\text{Cat}_{\text{Comp Haus}} \) is reflective in \( (\mathbb{T}, \mathcal{V})\)-\text{Cat}_{\text{Haus}} \) which, in turn, is strongly epireflective in \( (\mathbb{T}, \mathcal{V})\)-\text{Cat}.

**Proof.** The embedding \( \text{Set}^T \to (\mathbb{T}, \mathcal{V})\)-\text{Cat} preserves not only small products, but also equalizers. Indeed, the \( \mathbb{T} \)-algebra structure of the equalizer \( j : E \to X \) of \( f, g : (X, a) \to (Y, b) \) in \( \text{Set}^T \) is the map \( a_0 : TE \to E \) with \( j \cdot a_0 = a \cdot Tj \); hence, \( a_0 = j^0 \cdot a \cdot Tj \), which is the structure of the equalizer formed in \( (\mathbb{T}, \mathcal{V})\)-\text{Cat}. Consequently, \( \text{Set}^T \) is closed under small limits in \( (\mathbb{T}, \mathcal{V})\)-\text{Cat} and therefore in \( (\mathbb{T}, \mathcal{V})\)-\text{Cat}_{\text{Haus}} \) (which is strongly epireflective in \( (\mathbb{T}, \mathcal{V})\)-\text{Cat}), see Proposition \[ \text{I.1.5} \].

In order to build a solution set in \( \text{Set}^T = (\mathbb{T}, \mathcal{V})\)-\text{Cat}_{\text{Comp Haus}} \) for \( (X, a) \) in \( (\mathbb{T}, \mathcal{V})\)-\text{Cat}_{\text{Haus}} \), consider any \( (\mathbb{T}, \mathcal{V}) \)-continuous \( f : (X, a) \to (Y, b) \) in \( \text{Set}^T \) and form the least \( \mathbb{T} \)-subalgebra \( \langle M \rangle \) of \( (Y, b) \) containing \( M = f(X) \), which may be constructed as the image \( h(TM) \) of the \( \mathbb{T} \)-homomorphism \( h : TM, m_M \to (Y, b) \) with \( h \cdot e_M = (M \to Y) \). Hence, \( f \) factors as \( (X \to \langle M \rangle \to Y) \), where the cardinality of \( \langle M \rangle \) cannot exceed the cardinality of \( TX \). Hence a solution set for \( (X, a) \) can be given by a representative system of non-isomorphic \( \mathbb{T} \)-algebras \( (Y, b) \) whose cardinalities do not exceed that of \( TX \).

Theorem \[ \text{1.2.3} \] relies on the algebraic realization of \( (\mathbb{T}, \mathcal{V})\)-\text{Cat}_{\text{Comp Haus}} \) as \( \text{Set}^T \). The theme of Hausdorff-separation and compactness will be taken up again in Sections \[ \text{3} \] and \[ \text{4} \] through an algebraic approach to proper maps.

**1.3 Compactness for Kleisli-extended monads.** The example \( \text{Top} \cong (\mathbb{F}, 2)\)-\text{Cat} \( \cong (\mathbb{F}, 2)\)-\text{Cat} shows that the notion of compactness may change dramatically when changing the parameters \( \mathbb{T} \) or \( \mathcal{V} \), see Examples \[ \text{I.1.4.2} \) and \( \text{I.1.4.3} \). Here, we wish to explore this change when \( (\mathbb{T}, \mathcal{V})\)-\text{Cat} is presented relationally as \( (\mathbb{T} \cap \mathcal{V}, 2)\)-\text{Cat}, as in Corollary \[ \text{IV.3.2.3} \].

With that goal in mind, let us first look at relational \( \mathbb{T} \)-algebras in general. Independently of the lax extension of \( \mathbb{T} \) to \( \text{Rel} \), for \( (X, a) \) in \( (\mathbb{T}, 2)\)-\text{Cat} one has (writing \( \chi \to x \) instead of \( a(\chi, x) = \mathbb{T} \)):

\[
\begin{align*}
(X, \to) \text{ is Hausdorff} & \iff \forall x, y \in X, z \in TX (z \to x \& z \to y \Rightarrow x = y) , \\
(X, \to) \text{ is compact} & \iff \forall \chi \in TX \exists z \in X (\chi \to z).
\end{align*}
\]

In the case where \( \mathbb{T} = (T, m, e) \) is power-enriched via \( \tau : \mathbb{P} \to \mathbb{T} \), so that \( \tau : (\mathbb{P}, \bar{P}) \to (\mathbb{T}, \bar{T}) \) is a morphism of the respective Kleisli extensions, the algebraic functor

\[
A_{\tau} : (\mathbb{T}, 2)\text{-Cat} \longrightarrow (\mathbb{P}, 2)\text{-Cat} \cong \text{Ord}
\]
simply equips \((X, \rightarrow)\) with the induced order considered in [II 3.3]

\[ x \leq y \iff \tau(\{x\}) \rightarrow y \iff e_X(x) \rightarrow y. \]

1.3.1 Proposition. For a monad \(\mathbb{T}\) power-enriched by \(\tau : \mathbb{P} \rightarrow \mathbb{T}\) and \((X, \rightarrow)\) in \((\mathbb{T}, 2, \mathbb{T})\)-Cat, one has:

1. \((X, \rightarrow)\) is Hausdorff if and only if \(|X| \leq 1.\)

2. If \((X, \rightarrow)\) is compact, \(X\) has a largest element, with the converse statement holding when \(\tau(X)\) is the largest element in \(TX\).

Proof.

(1) The complete lattice \(TX\) has a least element \(p\). By Proposition [III 3.3.6] and Remark [IV 1.4.4] one has

\[(p \leq \chi \& \chi \rightarrow x) \implies (p \rightarrow x)\]

for all \(x \in X, \chi \in TX\). Since \(e_X(x) \rightarrow x\), one has \(p \rightarrow x\) for all \(x \in X\), and (1) follows.

(2) For \((X, \rightarrow)\) compact there is \(x_0\) with \(\tau(X) \rightarrow x_0\). Since \(\tau(X) = \bigvee_{x \in X} e_X(x)\) (see Exercise [IV 1.1.B]) one obtains \(e_X(x) \rightarrow x_0\) and, hence, \(x \leq x_0\) for all \(x \in X\). Conversely, assuming \(x_0\) a largest element in \(X\), with Exercise [IV 1.1.H] one concludes \(\tau(X) \rightarrow x_0\) which, with \(\tau(X)\) the largest element in \(TX\), gives \(\chi \rightarrow x_0\) for all \(\chi \in TX\).

For \(\mathbb{T} = \mathbb{F}\) the filter monad, the least element \(p\) of \(FX\) is the filter \(PX\), while the largest element \(\tau(X)\) is the filter \(\{X\}\). Compactness in \((\mathbb{F}, 2)\)-Cat (that is, supercompactness, see [1.1.4(3)]) may seem like a rare property at first sight. However, the following Proposition shows that in general there is a rich supply of such spaces.

Recall from [IV 4.1] and Theorem [IV 1.5.3] that for a power-enriched monad \(\mathbb{T}\) one has the functors

\[\text{Set}^\mathbb{T} \rightarrow \mathbb{T}\text{-Mon} \rightarrow (\mathbb{T}, 2)\text{-Cat}\]

\[(X, \alpha) \mapsto (X, \alpha^\perp) = (X, \nu) \mapsto (X, \rightarrow),\]

where \(\alpha^\perp\) is right adjoint to the sup-map \(\alpha : TX \rightarrow X\), and

\[\chi \rightarrow x \iff \chi \leq \nu(x) \iff \chi \leq \alpha^\perp(x) \iff \alpha(\chi) \leq x \quad (1.3.i)\]

for all \(x \in X, \chi \in TX\). In particular, setting \(x = \alpha(\chi)\) one obtains the following result.

1.3.2 Proposition. For a power-enriched monad \(\mathbb{T}\), every \(\mathbb{T}\)-algebra is compact when considered as a \((\mathbb{T}, 2)\)-space via (1.3.1). In fact, for \((X, \alpha) \in \text{Set}^\mathbb{T}\) and every \(\chi \in TX\) there is a least point \(x \in X\) with \(\chi \rightarrow x\).
1.3.3 Corollary. Every continuous lattice endowed with its Scott topology is supercompact.

Let us now consider the discrete presheaf monad

$$
\Pi = \Pi(T, \mathcal{V}, \hat{T})
$$

induced by a monad $T$ and an associative lax extension $\hat{T}$ to $\mathcal{V}$-$\text{Rel}$ and apply Proposition 1.3.2 to $\Pi$ with its Kleisli extension (see Corollary IV.3.2.3):

$$
\text{Set}^{\Pi} \longrightarrow \Pi\text{-Mon} \longrightarrow (\Pi, 2)\text{-Cat} \cong (T, \mathcal{V})\text{-Cat}
$$

1.3.4 Corollary. For a monad $T$ with an associative lax extension $\hat{T}$ to $\mathcal{V}$-$\text{Rel}$, every $\Pi(T, \mathcal{V})$-algebra is compact as an object of $(\Pi, 2)$-$\text{Cat}$.

1.3.5 Example. Consider $T = \beta$ with its Barr extension to $\mathcal{V} = P_+ = ([0, \infty]^{op}, +, 0)$, which carries a $\Pi(T, \mathcal{V})$-algebra structure

$$
\alpha : \Pi[0, \infty] = (T, \mathcal{V})\text{-URel}(1, V) \rightarrow (T, \mathcal{V})\text{-URel}(1, 1) \cong [0, \infty]
$$

$$
\varphi \mapsto \varphi \circ \iota = \inf_{x \in [0, \infty]} \varphi(x) + \xi(x),
$$

where $\iota : 1 \rightarrow V$ is given by the identity map on $[0, \infty]$ and $\xi(x) = \sup_{A \in \chi} \inf_{u \in A} u$ for all $\chi \in \beta[0, \infty]$; see IV.3.3. (Note that the convergence map $\xi : \beta[0, \infty] \rightarrow [0, \infty]$ provides $[0, \infty]$ with the standard topology of $[0, \infty]$.) By Corollary 1.3.4, the space $([0, \infty], \rightarrow)$ becomes compact in $(\Pi, 2)$-$\text{Cat} \cong \text{App}$.

Finally, let us examine compactness in $(\Pi(T, \mathcal{V}), 2)$-$\text{Cat}$ beyond spaces that arise from $\Pi$-algebras. Recall from IV.3.2 the isomorphisms

$$
(\Pi, \mathcal{V})\text{-Cat} \xrightarrow{\cong} \Pi\text{-Mon} \xrightarrow{\cong} (\Pi, 2)\text{-Cat}
$$

$$(X, a) \longrightarrow (X, a^\flat) \longrightarrow (X, \rightarrow),$$

where $a^\flat : X \rightarrow \Pi X$ is given by $a^\flat(x) = x^\flat \cdot a = a(-, x)$, and

$$
\varphi \rightarrow x \iff \varphi \leq a(-, x)
$$

for all $\varphi \in \Pi X$.

1.3.6 Proposition. For a monad $T$ with a flat associative lax extension to $\mathcal{V}$-$\text{Rel}$ and a $(T, \mathcal{V})$-space $(X, a)$, each of the following statements implies the next:

(i) $(X, \rightarrow)$ is compact in $(\Pi(T, \mathcal{V}), 2)$-$\text{Cat}$;
(ii) \( \forall \chi \in TX \exists x \in X \ (a(\chi, x) \geq k) \);

(iii) \((X, a)\) is compact in \((\mathcal{T}, \mathcal{V})\)-Cat.

**Proof.** For (i) \( \implies \) (ii), we observe that compactness of \((X, \rightarrow)\) means

\[ \forall \varphi \in \Pi X \exists x \in X \ (\varphi \rightarrow x) . \]

Hence, given \( x \in TX \) we may exploit this property for \( \varphi = Y_X(\chi) \), with \( Y : \mathcal{T} \rightarrow \Pi \) as in Proposition \[\text{IV.3.2.5}\]. With \( \varphi \rightarrow x \in X \) we then have

\[ k = \varphi(\chi) \leq a(\chi, x) , \]

as desired. (ii) \( \implies \) (iii) is immediate. \( \square \)

1.3.7 Remark. For \( T = \mathbb{I} \) the identity monad identically extended to \( \mathcal{V}\) Rel, \( \Pi(\mathbb{I}, \mathcal{V}) = \mathbb{P}_\mathcal{V} \) is the \( \mathcal{V}\)-powerset monad and \( (\mathbb{P}, 2)\)-Cat \( \cong \mathcal{V}\) Cat; see Corollary \[\text{IV.3.2.3}\].

For \( \mathcal{V} = 2 \), so that \( \mathcal{V}\) Cat \( \cong \text{Ord} \) and \( \varphi \in \mathbb{P}_2 X \) corresponds to \( A \subseteq X \), one has

\[ A \rightarrow x \iff A \subseteq \downarrow x , \]

so that compactness of \((X, \leq)\) in \((\mathbb{P}, 2)\)-Cat means existence of a largest element in \( X \) (as already confirmed in Proposition \[\text{I.3.1(2)}\]). When \( \mathcal{V} = \mathbb{P}_+ \), so that \( \mathcal{V}\) Cat \( \cong \text{Met} \), for \( \varphi \in \Pi X \) one has

\[ \varphi \rightarrow x \iff \varphi \geq a(\cdot, x) \]

for a metric space \((X, a)\) considered as a \((\mathbb{P}, 2)\)-space. With \( \varphi \) constantly 0 one sees:

\[ (X, \rightarrow) \text{ compact} \iff \exists x \in X \forall y \in X \ (a(y, x) = 0) \iff (X, \leq) \text{ compact}, \]

where \( \leq \) is the underlying order of \((X, a)\), making \((X, \leq)\) an object of \((\mathbb{P}(\mathbb{I}, 2), 2)\)-Cat.

This invariance property of compactness is now shown to hold in full generality, not just for \( T = \mathbb{I} \).

As in Theorem \[\text{IV.3.4.4}\] let \( \varphi : (V, \otimes, k) \rightarrow (W, \otimes, l) \) be a homomorphism of integral quantales with a right adjoint \( \psi \) (in \( \text{Ord} \)); furthermore, for a monad \( \mathbb{T} \) we assume \( \varphi \) to be strictly compatible with the associative lax extensions \( \hat{T}, \tilde{T} \) to \( \mathcal{V}\) Rel, \( \mathcal{W}\) Rel, respectively, so that

\[ \tilde{T}(\varphi r) = \varphi(\hat{T} r) \]

for all \( \mathcal{V}\)-relations \( r \). With \( \Pi_\mathcal{V} = \Pi(\mathbb{T}, \mathcal{V}), \Pi_\mathcal{W} = \Pi(\mathbb{T}, \mathcal{W}) \) there is then a morphism

\[ \varphi^T = \Pi(\mathbb{T}, \varphi) : \Pi_\mathcal{V} \rightarrow \Pi_\mathcal{W} \]
of power-enriched monads whose algebraic functor commutes with the change-of-base functor induced by $ψ$:

$$(\mathbb{T}, W)\text{-Cat} \xrightarrow{B_ψ} (\prod_W 2)\text{-Cat} \quad (X, a) \xmapsto{\lambda} (X, \rightarrow)$$

$$(\mathbb{T}, V)\text{-Cat} \xrightarrow{A_ψ} (\prod_V 2)\text{-Cat} \quad (X, ψa) \xmapsto{\lambda} (X, \rightsquigarrow) ,$$

where

$$\rho \rightarrow x \iff \rho \leq a(−, x) ,$$

$$\theta \rightsquigarrow x \iff \theta \leq ψa(−, x) \iff \varphi θ ≤ a(−, x) ,$$

for all $\rho \in \Pi_WX, \theta \in \Pi_VX, x \in X$.

1.3.8 Theorem. For $(X, a) \in (\mathbb{T}, W)\text{-Cat},$

$$(X, \rightarrow) \text{ compact in } (\prod_W 2)\text{-Cat} \iff (X, \rightsquigarrow) \text{ compact in } (\prod_V 2)\text{-Cat} .$$

Proof. “ ⇒ ” follows from Proposition 1.1.8. For “ ⇐ ”, consider $θ : TX \rightarrow 1$ with $θ(χ) = \top$ for all $χ \in TX$. Since

$$\theta \cdot \hat{1}_X ≥ θ \quad \text{and} \quad e^o_1 \cdot \hat{θ} \cdot m^o_X ≥ e^o_1 \cdot \hat{θ} \cdot e_{TX} ≥ θ ,$$

$θ$ is unitary when $k = \top$. Compactness of $(X, \rightsquigarrow)$ gives $x_0 \in X$ with $θ \rightarrow x_0$, so that $θ ≤ ψa(−, x_0)$ implies $ϕθ ≤ a(−, x_0)$, with $ϕθ$ the top element in $Π_WX$. Consequently, $ρ \rightarrow x_0$ for all $ρ \in Π_WX$. $\square$

1.3.9 Corollary. An approach space is compact as an object of $(\prod(β, P_+), 2)\text{-Cat}$ if and only if its topological coreflection is supercompact.

Proof. Apply Theorem 1.3.8 with $ϕ = υ : 2 \rightarrow P_+$ and $ψ = p$ the “pessimist’s map” (see Exercise II.1.1). $\square$

1.4 Examples involving monoids. For a monoid $(H, µ, η)$ we consider the monad

$$\mathbb{H} = (H \times (−), m, e)$$

on $\text{Set}$, with $m_X = µ \times 1_X : H \times H \times X \rightarrow H \times X$ and $e_X = (η, 1_X) : X \rightarrow H \times X$ (see Exercise II.3.B). The Barr extension $\overline{H} : \text{Rel} \rightarrow \text{Rel}$ is a lax extension of the monad $\mathbb{H}$, and it is given explicitly by

$$(α, x) \overline{H} (β, y) \iff α = β & x r y ,$$
for \( r : X \rightarrow Y, x \in X, y \in Y, \) and \( \alpha, \beta \in H. \) For a relation \( a : H \times X \rightarrow X, \) when we write \( x \xrightarrow{\alpha} y \) instead of \( (\alpha, x) a y \) (that is, “\( x \) is related to \( y \) with weight \( \alpha \)”), an \((H, 2)\)-category \((X, a)\) is characterized by the conditions

\[
x \xrightarrow{\eta} x \quad \text{and} \quad (x \xrightarrow{\alpha} y \& y \xrightarrow{\beta} z \implies x \xrightarrow{\beta \cdot \alpha} z),
\]

where \( \beta \cdot \alpha = \mu(\beta, \alpha) \) in \( H; \) a morphism \( f : (X, a) \rightarrow (Y, b) \) must satisfy

\[
x \xrightarrow{\alpha} y \implies f(x) \xrightarrow{\alpha} f(y)
\]
for all \( x, y \in X, \alpha \in H. \) For an \((H, 2)\)-category \((X, a)\), one has:

\[
(X, a) \text{ is compact } \iff \forall x \in X, \alpha \in H \exists y \in Y (x \xrightarrow{\alpha} y), \\
(X, a) \text{ is Hausdorff } \iff \forall x, y, z \in X, \alpha \in H (x \xrightarrow{\alpha} y \& x \xrightarrow{\alpha} z \implies y = z),
\]

and the conjunction of both properties makes \( a : H \times X \rightarrow X \) an action of \( H \) on \( X; \)
\( x \xrightarrow{\alpha} y \) means \( y = \alpha \cdot x \) when we write the action multiplicatively. These observations illustrate the assertions of Proposition'] and Theorem [1.2.3]

1.4.1 Corollary. The category of compact Hausdorff \((H, 2)\)-spaces and \((H, 2)\)-continuous maps is isomorphic to the category of \( H \)-actions and equivariant maps:

\[
\text{CompHaus} \cong \text{Set}^H
\]

It is reflective in \((H, 2)\)-Cat which, in turn, is strongly epireflective in \((H, 2)\)-Cat.

Our arrow notation for the structure of an \((H, 2)\)-category \((X, a)\) emphasizes that \( X \) is actually the object set of a small category, denoted again by \( X \), with hom-sets

\[
X(x, y) = \{(x, \alpha, y) \mid \alpha \in H, x \xrightarrow{\alpha} y \text{ in } (X, a)\};
\]

moreover, this category comes with a faithful functor

\[
p : X \rightarrow H, \quad (x, \alpha, y) \mapsto \alpha,
\]

with \( H \) considered as a one-object category. (Hence, \( p \) identifies the objects while leaving the morphisms intact.) Considering now more generally small categories \( X \) which come equipped with an \( H \)-valued norm, that is, a functor \( n : X \rightarrow H, \) we see that there is a full embedding

\[
E : (H, 2)\text{-Cat} \rightarrow \text{Cat}/H, \quad (X, a) \mapsto (X, p).
\]

1.4.2 Proposition. The functor \( E \) is reflective and identifies \((H, 2)\)-categories as those small categories over \( H \) whose norm is faithful.
**Proof.** The reflection of an object \((X, n)\) in \(\mathbf{Cat}/H\) into \((\mathfrak{H}, 2)\)-\(\mathbf{Cat}\) may be formed by \(X := \text{ob} X\) with \((\mathfrak{H}, 2)\)-structure

\[
x \xrightarrow{\alpha} y \iff \exists \varphi \in X(x, y) \ (n(\varphi) = \alpha)
\]

which makes \(n : (X, n) \to (X, p)\) a morphism in \(\mathbf{Cat}/H\) and, in fact, the reflection morphism, as one easily verifies. Furthermore, \(n\) as a morphism in \(\mathbf{Cat}/H\) becomes an isomorphism if and only if the functor \(n : X \to H\) is faithful. Consequently, \((\mathfrak{H}, 2)\)-\(\mathbf{Cat}\) is equivalent to the full subcategory of \(\mathbf{Cat}/H\) whose objects have a faithful norm. 

---

**1.4.3 Remarks.**

(1) The category \((\mathfrak{H}, 2)\)-\(\mathbf{Cat}\) may also be presented as \(\mathcal{V}\)-\(\mathbf{Cat}\) with \(\mathcal{V}\) the powerset of \(H\), ordered by inclusion, and equipped with the tensor product \(\otimes\) defined by

\[
A \otimes B = \{\alpha \cdot \beta \mid \alpha \in A, \beta \in B\},
\]

that preserves unions (see Exercise [III.1.M]). Then \(\{\eta\}\) is the tensor unit, but not the top element of \(\mathcal{V}\), hence \(\mathcal{V}\) is not integral unless \(H = 1\). Moreover, if, for instance, \(H\) is a non-trivial group, \(\mathcal{V}\) is not superior: let \(A = \{\alpha, \alpha^{-1}\}\), with \(\alpha \neq \eta\); then \(\{\eta\} \not\subseteq A\), although \(\{\eta\} \subseteq A \otimes A\).

It is easy to check that if, to each \(\mathcal{V}\)-space \((X, a)\), we assign the \((\mathfrak{H}, 2)\)-space \((X, \tilde{a})\), with \(\tilde{a}(\alpha, x, y) = \top\) if and only if \(\alpha \in a(x, y)\), every \(\mathcal{V}\)-functor \(f : (X, a) \to (Y, b)\) is an \((\mathfrak{H}, 2)\)-functor \(f : (X, \tilde{a}) \to (Y, \tilde{b})\), and in fact this correspondence defines an isomorphism \(\mathcal{V}\)-\(\mathbf{Cat} \to (\mathfrak{H}, 2)\)-\(\mathbf{Cat}\) (see also Example [III.3.2.2]). We note, however, that Hausdorff and compact \(\mathcal{V}\)-spaces do not coincide with Hausdorff and compact \((\mathfrak{H}, 2)\)-spaces.

(2) In the trivial case \(H = 1\), we have \(\mathfrak{H} = \emptyset\), so that \((\mathfrak{H}, 2)\)-\(\mathbf{Cat}\) reproduces \(2\)-\(\mathbf{Cat} \cong \mathbf{Ord}\), also identified as the full subcategory of \(\mathbf{Cat}\) given by small categories \(X\) for which \(X \to 1\) is faithful.

We now turn to the list monad \(\mathbb{L} = (L, m, e)\) induced by the right adjoint functor \(\mathbf{Mon} \to \mathbf{Set}\) (see Example [II.3.1.2]). The Barr extension \(\mathcal{T} : \mathbf{Rel} \to \mathbf{Rel}\) of the list monad is given by

\[
(x_1, \ldots, x_n) \mathcal{T} r (y_1, \ldots, y_m) \iff n = m \& x_1 r y_1 \& \ldots \& x_n r y_n
\]

for all \((x_1, \ldots, x_n) \in LX\), \((y_1, \ldots, y_m) \in LY\), and relations \(r : X \to Y\); it is obviously an associative flat lax extension of \(\mathbb{L}\) (by Corollary [III.1.2.2] and Exercise [III.1.Q]). An \((\mathbb{L}, 2)\)-category is a multi-ordered set, that is, a set \(X\) with a relation \(a : LX \to X\) that, when we write

\[
(x_1, \ldots, x_n) \vdash y \text{ for } (x_1, \ldots, x_n) a y,
\]

must satisfy \((x) \vdash x\) and the transitivity condition

\[
(x_1^1, \ldots, x_{n_1}^1) \vdash y_1, \ldots, (x_1^m, \ldots, x_{n_m}^m) \vdash y_m \& (y_1, \ldots, y_m) \vdash z \implies (x_1^1, \ldots, x_{n_m}^m) \vdash z.
\]
For an $\langle L, 2 \rangle$-continuous map $f : (X, a) \to (Y, b)$ one must have the monotonicity condition
\[(x_1, \ldots, x_n) \vdash y \implies (f(x_1), \ldots, f(x_n)) \vdash f(y) .\]

An $\langle L, 2 \rangle$-space $(X, a)$ is compact precisely when for all integer $n \geq 0$,
\[\forall x_1, \ldots, x_n \in X \exists z \in X \ ((x_1, \ldots, x_n) \vdash z) ,\]
and $(X, a)$ is Hausdorff if and only if for all $n \geq 0$,
\[\forall x_1, \ldots, x_n, y, z \in X \ ((x_1, \ldots, x_n) \vdash y \& (x_1, \ldots, x_n) \vdash z \implies y = z) .\]

In particular, compactness forces $X \neq \emptyset$ (from the case $n = 0$). The conjunction of both properties makes $(x_1, \ldots, x_n) \vdash y$ the $n$-ary extension of a binary monoid operation on $X$.

1.4.4 Corollary. The category of compact Hausdorff $\langle L, 2 \rangle$-spaces and $\langle L, 2 \rangle$-continuous maps is isomorphic to the category of monoids and their homomorphisms:
\[\langle L, 2 \rangle\text{-Cat}_{\text{CompHaus}} \cong \text{Mon} .\]

It is reflective in $\langle L, 2 \rangle\text{-Cat}_{\text{Haus}}$ which, in turn, is strongly epireflective in $\langle L, 2 \rangle\text{-Cat}$.

Proof. The result is a direct consequence of Proposition 1.2.1 and Theorem 1.2.3. \qed

Exercises

1.A Supercompact topological spaces. Show that the compact objects in $\langle \mathbb{F}_p, 2 \rangle\text{-Cat}$ are precisely the supercompact topological spaces (see 1.1.4(4)). Furthermore,
\[\langle \mathbb{F}_p, 2 \rangle\text{-Cat}_{\text{Haus}} \cong \text{Haus} \quad \text{and} \quad \langle \mathbb{F}, 2 \rangle\text{-Cat}_{\text{CompHaus}} \cong \{X \in \text{Top} \mid |X| \leq 1 \} .\]

1.B Algebraic functors induced by monoid homomorphisms.

(1) For a homomorphism $\alpha : H \to K$ of monoids one obtains a morphism $\alpha : \mathbb{H} \to \mathbb{K}$ of monads on $\text{Set}$ which, in turn, induces an algebraic functor $A_\alpha : (\mathbb{K}, 2)\text{-Cat} \to (\mathbb{H}, 2)\text{-Cat}$ (see Exercise II.3.B and 1.4).

(2) Considering that the monad associated to the trivial monoid $1$ is the identity monad $\mathbb{1}$ on $\text{Set}$, show that for every monoid $H$ there are algebraic functors $A : (\mathbb{H}, 2)\text{-Cat} \to \text{Ord}$ and $P : \text{Ord} \to (\mathbb{H}, 2)\text{-Cat}$ with $AP = 1_{\text{Ord}}$.

1.C Induced algebraic functors that are not adjoint. Let $H = \langle \{0, 1\}, +, 0 \rangle$ and $K = \langle \{0, 1\}, \cdot, 1 \rangle$ be the monoids that make up the ring $\mathbb{Z}/2\mathbb{Z}$.

(1) Show that $\langle \mathbb{H}, 2 \rangle\text{-Cat}$ is isomorphic to the category of ordered sets $(X, \leq)$ that come with an additional relation $\prec$ satisfying
\[(u \leq x \prec y \leq z \implies u \prec z) \quad \text{and} \quad (x \prec y \prec z \implies x \leq z) .\]
(2) Show that $(\mathbb{K}, 2)$-$\text{Cat}$ may be described as the category of ordered sets $(X, \leq)$ that come with an additional relation $\prec$ satisfying

$$u \leq x \prec y \prec z \leq w \implies u \prec w.$$ 

(3) Identify the compact and Hausdorff objects in $(\mathbb{H}, 2)$-$\text{Cat}$ and $(\mathbb{K}, 2)$-$\text{Cat}$.

(4) Describe the functors $A$ and $P$ of 1.B for both $H$ and $K$ and show that in both cases $A$ and $P$ fail to be adjoint to each other.

1.D Superior but not integral. The powerset of the multiplicative monoid $\{0, 1\}$ yields a 4-element commutative and superior, but non-integral, quantale.

1.E Characterizing superior completely distributive quantales. If the $\otimes$-neutral element $k$ of $V$ satisfies $k = \bigvee_{u \ll k} u$, then the following conditions are equivalent:

(i) $\forall u_i \in V (i \in I) (k \leq \bigvee_{i \in I} u_i \implies k \leq \bigvee_{i \in I} (u_i \otimes u_i));$

(ii) $k \leq \bigvee_{u \ll k} u \otimes u.$

Conclude that an integral completely distributive quantale is superior if and only if (ii) holds.
2. Low separation, regularity and normality

We continue to work with a monad $T = (T, m, e)$ on $\mathbf{Set}$, laxly extended by $\hat{T}$ to $\mathbf{V}$-$\mathbf{Rel}$, for a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$.

2.1 Order separation. Every $(T, \mathcal{V})$-category $(X,a)$ comes with an underlying order, defined by

$$x \leq y \iff k \leq a(e_X(x), y)$$

for all $x, y \in X$ (see Proposition III.3.3.1). We recall that $(X, a)$ separated if its underlying order is separated; that is, if for all $x, y \in X$

$$x \leq y \land y \leq x \implies x = y .$$

Hence, for $T = \mathbb{I}$ and $\mathcal{V} = 2$ or $P_+$, this notion returns the terminology used for ordered sets and metric spaces (see III.1.3 and Examples III.1.3.1). For topological spaces, whether considered as objects of $(\mathbb{B}, 2)$-$\mathbf{Cat}$ or $(\hat{T}, 2)$-$\mathbf{Cat}$, separation means $T\emptyset$-separation:

$$\hat{x} \to y \land y \to x \implies x = y ;$$

equivalently, the map $\nu : X \to F_X$ which assigns to every point its neighborhood filter is injective. An approach space $(X, a)$, as an object of $(\mathbb{B}, P_+)$-$\mathbf{Cat}$, is separated if and only if for all $x, y \in X$

$$a(\hat{x}, y) = 0 = a(\hat{y}, x) \implies x = y ;$$

equivalently, in terms of its approach distance $\delta$, if

$$\delta(x, \{y\}) = 0 = \delta(y, \{x\}) \implies x = y .$$

2.1.1 Proposition. Let $(X, a)$ be a $(T, \mathcal{V})$-space.

(1) If $(X, a)$ is Hausdorff, then $(X, a)$ is separated.

(2) If $(X, a)$ is separated, any $(T, \mathcal{V})$-continuous map $f : (2, T) \to (X, a)$ from a two-element indiscrete $(T, \mathcal{V})$-space is constant, and this property is equivalent to $(X, a)$ being separated if $a(a, i) = T$ for all $a \in T2 \setminus \{e_2(i)\}$, $i \in 2 = \{0, 1\}$.

(3) The full subcategory $(T, \mathcal{V})$-$\mathbf{Cat}_{\text{sep}}$ of separated $(T, \mathcal{V})$-spaces is closed under monosources in $(T, \mathcal{V})$-$\mathbf{Cat}$.

Proof.

(1) If $(X, a)$ is Hausdorff, already $k \leq a(e_X(x), y)$ implies

$$\bot < k = k \otimes k \leq a(e_X(x), x) \otimes a(e_X(x), y)$$

and then $x = y$. 
(2) For a \((\mathbb{T}, \mathcal{V})\)-functor \(f : (\mathbb{T}, \top) \to (X, a)\) one has
\[ k \leq \top \leq a(Tf(e_2(0)), f(1)) = a(e_X(f(0)), f(1)) \]
and, likewise, \(k \leq a(e_X(f(1)), f(0))\), hence \(f(0) = f(1)\) if \((X, a)\) is separated. Conversely, if \(x \leq y\) and \(y \leq x\) in \((X, a)\), the additional hypothesis makes \(f : (0 \mapsto x, 1 \mapsto y)\) \((\mathbb{T}, \mathcal{V})\)-continuous, hence constant.

(3) This is a straightforward exercise. 

Closure under mono-sources makes \((\mathbb{T}, \mathcal{V})\)-\text{Cat} strongly epireflective in \((\mathbb{T}, \mathcal{V})\)-\text{Cat}. We give an easy \textit{ad-hoc} description of the reflector, modeled after the situation in \text{Top}. For a topological space \(X\), the quotient topology on the \(T_0\)-reflection \(X/\sim\) with \(\{x\} = \{y\}\) has the special property that it makes the projection \(p : X \to X/\sim\) not only \(O\)-final but also \(O\)-initial, with respect to \(O : \text{Top} \to \text{Set}\). This property is crucial for establishing the reflection for \((\mathbb{T}, \mathcal{V})\)-spaces.

\(\circ\) 2.1.2 Theorem. For a \((\mathbb{T}, \mathcal{V})\)-space \((X, a)\), the projection \(p : X \to X/\sim\) of the separated-order reflection, given by \(x \sim y \iff x \leq y \& y \leq x\), serves also as a reflection into \((\mathbb{T}, \mathcal{V})\)-\text{Cat}_{\text{sep}} when \(X/\sim\) is endowed with the \((\mathbb{T}, \mathcal{V})\)-category structure \(\tilde{a} = p \cdot a \cdot (Tp)^\circ\), making \(p\) both \(O\)-final and \(O\)-initial with respect to the functor \(O : (\mathbb{T}, \mathcal{V})\)-\text{Cat} \to \text{Set}.

\textit{Proof.} By definition of \(\sim\), one has \(p^\circ \cdot p \leq a \cdot e_X\), which implies
\[ \tilde{a} \cdot e_{X/\sim} = p \cdot a \cdot (Tp)^\circ \cdot e_{X/\sim} \]
\[ \geq p \cdot a \cdot e_X \cdot p^\circ \]
\[ \geq p \cdot p^\circ = 1_{X/\sim} \]
as well as
\[ p^\circ \cdot \tilde{a} \cdotTp = p^\circ \cdot p \cdot a \cdot (Tp)^\circ \cdotTp \]
\[ \leq a \cdot e_X \cdot a \cdot \hat{T}(a \cdot e_X) \]
\[ \leq a \cdot e_X \cdot a \]
\[ \leq a \cdot \hat{T}a \cdot e_{TX} \]
\[ = a \cdot \tilde{a} \cdot Tp \cdot (Tp)^\circ \cdot e_{X/\sim} \]
\[ \leq p \cdot a \cdot \hat{T}(p^\circ \cdot \tilde{a} \cdot Tp) \cdot m_{\sim} \cdot (Tp)^\circ \]
\[ \leq p \cdot a \cdot \hat{T}a \cdot m_{\sim} \cdot (Tp)^\circ \]
\[ = p \cdot a \cdot (Tp)^\circ = \tilde{a} \cdot \tilde{m}_{X/\sim} \]
(Here we used the fact that $Tp$ and $TTp$ are surjective since $p$ is surjective, the Axiom of Choice granted.) Consequently, $(X/\sim, \tilde{a})$ is a $(\mathbb{T}, \mathcal{V})$-space, and $p$ is both $O$-initial and $O$-final. Quite trivially, $(X/\sim, \tilde{a})$ is separated since $p(x) \leq p(y)$ implies
\[
k \leq \tilde{a}(e_{X/\sim}(p(x), p(y))) \\
\leq (p^o \cdot \tilde{a} \cdot e_{X/\sim} \cdot p)(x, y) \\
\leq (p^o \cdot \tilde{a} \cdot Tp \cdot e_{X})(x, y) \\
= a(e_{X}(x), y),
\]
so $x \leq y$.

It now suffices to show that for any $(\mathbb{T}, \mathcal{V})$-continuous map $f : (X, a) \to (Y, b)$ with $(Y, b)$ separated one has $(x \sim y \implies f(x) = f(y))$, since the induced map $\tilde{f} : X/\sim \to Y$ with $\tilde{f} \cdot p = f$ is $(\mathbb{T}, \mathcal{V})$-continuous by $O$-finality of $p$. Moreover, $p(x) = p(y)$ implies
\[
k \leq a(e_{X}(x), y) \leq b(Tf \cdot e_{X}(x), f(y)) = b(e_{Y}(f(x)), f(y))
\]
and, likewise, $k \leq b(e_{Y}(f(x)), f(y))$, so $f(x) = f(y)$ by separatedness of $(Y, b)$. \qed

2.1.3 Corollary. The separated-order reflection of a $\mathcal{V}$-category carries its reflection into $\mathcal{V}$-$\text{Cat}_{\text{sep}}$; that is, the solid-arrow pullback diagram
\[
\begin{array}{ccc}
\mathcal{V}$-$\text{Cat}$ & \longrightarrow & \text{Ord}_{\text{sep}} \\
\uparrow & & \uparrow \\
\mathcal{V}$-$\text{Cat}_{\text{sep}} & \longrightarrow & \text{Ord}
\end{array}
\]
in $\text{CAT}$ also commutes with its dotted-arrow left adjoints.

2.2 Between order separation and Hausdorff separation. For a $(\mathbb{T}, \mathcal{V})$-space $(X, a)$, we now consider an array of low separation and symmetry conditions, with the terminology borrowed from the role model $\text{Top} \cong (\mathbb{O}, 2)$-$\text{Cat}$, as follows:

(T0) $(a \cdot e_{X}) \wedge (a \cdot e_{X})^o \leq 1_{X}$; \hspace{1cm} (R0) $(a \cdot e_{X})^o \leq a \cdot e_{X}$;

(T1) $a \cdot e_{X} \leq 1_{X}$; \hspace{1cm} (R1) $a \cdot a^o \leq a \cdot e_{X}$.

The following scheme shows how these conditions are related with the Hausdorff separation condition $a \cdot a^o \leq 1_{X}$ and with separatedness of $(X, a)$:

2.2.1 Proposition. The following implication hold for a $(\mathbb{T}, \mathcal{V})$-space $(X, a)$:

\[
\begin{array}{ccc}
\text{Hausdorff} & \iff & T1 \& R1 \\
\downarrow & & \downarrow \\
T1 & \iff & T0 \& R0 \\
\downarrow & & \downarrow \\
& & \text{separated.}
\end{array}
\]
Proof. Hausdorff $\implies$ T1 & R1: Since $e_X \leq a^\circ$, one has $a \cdot e_X \leq a \cdot a^\circ \leq 1_X \leq a \cdot e_X$.

T1 & R1 $\implies$ Hausdorff: One has $a \cdot a^\circ \leq a \cdot e_X \leq 1_X$.

T1 $\implies$ T0 & R0: One has $(a \cdot e_X) \land (a \cdot e_X)^\circ \leq a \cdot e_X \leq 1_X$ and $(a \cdot e_X)^\circ \leq 1_X \leq a \cdot e_X$.

T0 & R0 $\implies$ T1: One has $a \cdot e_X = (a \cdot e_X)^{\circ\circ} \leq (a \cdot e_X)^\circ$, so $a \cdot e_X = (a \cdot e_X) \land (a \cdot e_X)^\circ \leq 1_X$.

R1 $\implies$ R0: One has $(a \cdot e_X)^\circ = e_X^\circ \cdot a^\circ \leq a \cdot a^\circ \leq a \cdot e_X$.

T0 $\implies$ separated: If $k \leq a(e_X(x), y) \land a(e_X(y), x)$, then $\bot \leq k \leq ((a \cdot e_X) \land (a \cdot e_X)^\circ)(x, y) \leq 1_X(x, y)$, hence $x = y$. 

$\square$

2.2.2 Examples.

(1) In $\text{Ord} \cong 2$-$\text{Cat}$, T0-separation coincides with separation (see II.1.3), while R1 = R0 means symmetry, that is, the order is an equivalence relation. Also in $\text{Met} \cong \mathcal{P}_+ \text{-Cat}$, R1 = R0 assumes the usual meaning of symmetry: $a(x, y) = a(y, x)$ for all points in the metric space $(X, a)$. Like Hausdorffness, T1 means being discrete. When $(X, a)$ is symmetric, even T0 means being discrete and is therefore considerably stronger than being order-separated. But a two-point space $X = \{u, v\}$ with $a(u, v) = \infty$ and all other distances 0 is T0 but not T1.

(2) In $\text{Top} \cong (\beta, 2)$-$\text{Cat}$, or in $\text{Top} \cong (\mathcal{F}, 2)$-$\text{Cat}$, T0 means order-separated and T1 means that singleton sets are closed, that is, these conditions assume their usual meanings. Likewise for R0 (if $y \in \{x\}$, then $x \in \{y\}$) and R1 (if $U \cap W \neq \emptyset$ for all neighborhoods $U$ of $x$ and $W$ of $y$, then $\{x\} = \{y\}$). In $\text{App} \cong (\beta, \mathcal{P}_+) \text{-Cat}$ one has the following straightforward characterizations:

\[ (X, a) \text{ is T0 } \iff \forall x, y \in X \ (a(\hat{x}, y) < \infty \land a(\hat{y}, x) < \infty \implies x = y) ; \]

\[ (X, a) \text{ is T1 } \iff \forall x, y \in X \ (a(\hat{x}, y) < \infty \implies x = y) ; \]

\[ (X, a) \text{ is R0 } \iff \forall x, y \in X \ (a(\hat{y}, x) = a(\hat{x}, y)) ; \]

\[ (X, a) \text{ is R1 } \iff \forall x, y \in X \forall z \in \beta X (\delta(y, \{x\}) \leq a(z, x) + a(z, y)) , \]

where $\delta$ is the approach distance of $(X, a)$.

2.2.3 Proposition.

(1) Let $\mathcal{V}$ be an integral quantale. The T0 and T1 properties are closed under mono-sources in $(\mathbb{T}, \mathcal{V}) \text{-Cat}$. Hence, the corresponding full subcategories are strongly epireflective in $(\mathbb{T}, \mathcal{V}) \text{-Cat}$.

(2) The R0 and R1 properties are closed under $O$-initial sources in $(\mathbb{T}, \mathcal{V}) \text{-Cat}$, where $O : (\mathbb{T}, \mathcal{V}) \text{-Cat} \to \text{Set}$. Hence, the corresponding full subcategories are both mono- and epireflective in $(\mathbb{T}, \mathcal{V}) \text{-Cat}$.
Proof. Let $f_i : (X, a) \to (Y_i, b_i)$ ($i \in I$) be a source in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$ that is assumed to be monic when all $(Y_i, b_i)$ are assumed to be T0 or T1, and O-initial when all $(Y_i, b_i)$ are assumed to be R0 or R1.

Suppose that all $(Y_i, b_i)$ are T0; then

$$a \cdot e_X \land (a \cdot e_X) \leq \wedge_{i,j \in I}(f_i \circ b_i \cdot T f_i \cdot e_X) \land (f_j \circ b_j \cdot T f_j \cdot e_X)$$

$$\leq \wedge_{i \in I}(f_i \circ b_i \cdot e_{Y_i} \cdot f_i) \land (f_i \circ e_{Y_i} \cdot b_i \cdot f_i)$$

$$\leq \wedge_{i \in I} f_i \circ (b_i \cdot e_{Y_i} \cdot (b_i \cdot e_{Y_i}) \cdot f_i)$$

$$\leq \wedge_{i \in I} f_i \circ f_i = 1_X$$

$((Y_i, b_i)$ are T0).

The proof for T1 is similar.

If all $(Y_i, b_i)$ are R0, then

$$(a \cdot e_X) \leq \wedge_{i \in I}(f_i \circ b_i \cdot T f_i \cdot e_X)$$

$$= \wedge_{i \in I} f_i \circ (b_i \cdot e_{Y_i}) \cdot f_i$$

$$\leq \wedge_{i \in I} f_i \circ (b_i \cdot e_{Y_i} \cdot f_i)$$

$$= (\wedge_{i \in I} f_i \circ b_i \cdot T f_i) \cdot e_X = a \cdot e_X .$$

Finally, if all $(Y_i, b_i)$ are R1, then

$$a \cdot a^0 \leq \wedge_{i \in I}(f_i \circ b_i \cdot T f_i) \land \wedge_{j \in I}(f_j \circ b_j \cdot T f_j)^0$$

$$\leq \wedge_{i \in I} f_i \circ b_i \cdot T f_i \cdot (T f_i)^0 \cdot b_i \cdot f_i$$

$$\leq \wedge_{i \in I} f_i \circ b_i \cdot b_i \cdot f_i$$

$$= \wedge_{i \in I} f_i \circ b_i \cdot e_{Y_i} \cdot f_i$$

$((Y_i, b_i)$ are R1)

$$= (\wedge_{i \in I} f_i \circ b_i \cdot T f_i) \cdot e_X = a \cdot e_X .$$

2.3 Regular spaces. Throughout this subsection we assume that

- $\mathcal{V}$ is commutative.

In order to formulate regularity and normality for $(\mathbb{T}, \mathcal{V})$-spaces we will make essential use of the $\mathcal{V}$-relation (see Theorem III.5.3.5)

$$(TX \overset{a}{\longrightarrow} TX) := (TX \overset{m_X^0}{\longrightarrow} TX \overset{\hat{a}}{\longrightarrow} TX) .$$

For every $(\mathbb{T}, \mathcal{V})$-space $(X, a)$, the $\mathcal{V}$-relation $\hat{a} = \hat{T} a \cdot m_X^0$ is a $\mathcal{V}$-graph structure on $TX$:

$$1_{TX} \leq \hat{T} 1_X = \hat{T}(e_X^0) \cdot m_X^0 \leq \hat{T} a \cdot m_X^0 .$$
Therefore, both \((X, a \cdot \hat{a})\) and \((X, a \cdot \hat{a}^\circ)\) are \((\mathbb{T}, \mathcal{V})\)-graphs. While the inequality \(a \cdot \hat{a} \leq a\) is exactly the transitivity condition for \(a\), the condition \(a \cdot \hat{a}^\circ \leq a\) encodes an interesting separation property of \(a\).

2.3.1 Definition. A \((\mathbb{T}, \mathcal{V})\)-space \((X, a)\) is regular if \(a \cdot \hat{a}^\circ \leq a\), that is \(a \cdot m_X \cdot (\hat{T}a)^\circ \leq a\), or, in pointwise form,

\[
\hat{T}a(\mathcal{Y}, \chi) \otimes a(m_X(\mathcal{Y}), z) \leq a(\chi, z),
\]

for all \(\mathcal{Y} \in \mathcal{T}^2X, \chi \in TX\) and \(z \in X\).

We denote the resulting full subcategory of \((\mathbb{T}, \mathcal{V})\)-\text{Cat} by \((\mathbb{T}, \mathcal{V})\)-\text{Cat}_{reg}.

2.3.2 Examples.

(1) If \(\mathbb{T} = I\) is the identity monad (identically extended to \(\mathcal{V}\)-Rel), then \(a \cdot \hat{a}^\circ = a \cdot a^\circ\), so that \(a \cdot a^\circ \leq a\) if and only if \(a = a^\circ\). Indeed, if \(a = a^\circ\) then \(a \cdot a^\circ \leq a\) follows from transitivity. Conversely, if \(a \cdot a^\circ \leq a\), then \(a^\circ \leq a \cdot a^\circ \leq a\), which for \(\mathcal{V}\)-spaces regularity means symmetry.

(2) A topological space \(X\) considered as a \((\beta, 2)\)-space \((X, a)\) is regular if and only if it is regular in the usual sense, that is: if for \(x \in X\) and \(A \subseteq X\) closed with \(x \not\in A\), there exist open sets \(U, W \subseteq X\) with \(x \in U, A \subseteq W\) and \(U \cap W = \emptyset\). To prove this claim we first recall that, as observed in Examples III.5.3.7, for \(x, y \in \beta X\), with \((\leq) = \hat{a}\),

\[
y \leq x \iff \forall A \subseteq X, A \text{ closed}(A \in y \implies A \in x) \\
\iff \forall A \subseteq X, A \text{ open}(A \in x \implies A \in y).
\]

Furthermore, the condition \(a \cdot \hat{a}^\circ \leq a\) means that

\[
y \leq x \land y \rightarrow x \implies x \rightarrow x,
\]

for all \(\chi, y \in \beta X\) and \(x \in X\). Therefore, if we write \(y \rightarrow \chi\) whenever \(y \leq \chi\), we may depict this property as follows:

\[
\xymatrix{ y \ar@{..>}[r] & \chi \ar[r] & x \implies x }.
\]

If \(X\) is regular in the usual topological sense, then the closed neighborhoods of \(x \in X\) form a neighborhood base for \(x\). Therefore, if \(y \rightarrow x\), that is, if \(y\) contains the neighborhoods of \(x\), and every closed subset \(A\) of \(X\) belonging to \(y\) also belongs to \(x\), then clearly \(x \rightarrow x\). If \(X\) is not regular, that is, if there is \(x \in X\) with a neighborhood \(W\) that does not contain any closed neighborhood of \(x\), consider an ultrafilter \(\chi\) containing all the closed neighborhoods of \(x\) and \(X \setminus W\), and an ultrafilter \(y\) containing all the neighborhoods of \(x\). Then \(y \leq \chi\) and \(y \rightarrow y\), therefore \(a \cdot \hat{a}^\circ(\chi, x) = \top\). However, by construction, one does not have \(\chi \rightarrow x\), so \(a \cdot \hat{a} \leq a\) fails.
(3) Let $\mathbb{I} = \mathcal{H}$ as in 1.4. For an $(\mathcal{H}, 2)$-space $(X, a)$, the order $(\preceq) = \hat{a}$ is defined by

$$((\beta, y), (\alpha, x)) \preceq \iff H \cdot m^\mathcal{H}((\beta, y), (\alpha, x)) = \top$$

$$\iff \exists \gamma \in H \ (\beta = \alpha \cdot \gamma \& \ y \xrightarrow{\gamma} x),$$

for $(\alpha, x), (\beta, y) \in H \times X$. Hence $(X, a)$ is regular if and only if

$$x \xrightarrow{\alpha} z \iff x \xrightarrow{\beta} z$$

for all $\alpha, \beta \in H$ and all $x, y, z \in X$.

(4) Let $\mathbb{I}$ be the list monad $L$ as in 1.4. If $(X, a)$ is a multi-ordered set, then, for $(\preceq) = \hat{a}$ and $(x_1, \ldots, x_n), (y_1, \ldots, y_m) \in LX$, one has $(y_1, \ldots, y_m) \preceq (x_1, \ldots, x_n)$ if and only if there is an $n$-partition of $m$, that is, there exist $1 \leq m_1 < m_2 < \ldots < m_n = m$ such that

$$(y_1, \ldots, y_{m_1}) \mid x_1, \ldots, (y_{m_{n-1}+1}, \ldots, y_m) \mid x_n.$$ 

Hence, $(X, a)$ is regular if and only if for all $(x_1, \ldots, x_n), (y_1, \ldots, y_m) \in LX, z \in X$, one has

$$(y_1, \ldots, y_m) \preceq (x_1, \ldots, x_n) \& (y_1, \ldots, y_m) \mid z \iff (x_1, \ldots, x_n) \mid z.$$

(5) Let $\mathbb{I} = \beta$ be the ultrafilter monad and $\mathcal{V} = \mathcal{P}_+$. An approach space, considered as a $(\beta, \mathcal{P}_+)$-space $(X, a)$, is regular if and only if for any $x, y \in \beta X$ and $x \in X$,

$$a(x, x) \leq \hat{a}(y, x) + a(y, x),$$

where $\hat{a}(y, x) = \inf\{u \in [0, \infty] \mid \forall A \in y \ (A^{(u)} \in x)\}$ (see Examples III.5.3.7). Analogously to the characterization for topological spaces one can write

$$x \xrightarrow{u+v} x \iff x \leq x \xrightarrow{u+v} x.$$  

2.3.3 Proposition. For a $(\beta, \mathcal{P}_+)$-space $(X, a)$ with approach distance $\delta$, the following conditions are equivalent:

(i) $(X, a)$ is regular;

(ii) for every filter $f$ on $X, u \in [0, \infty]$ and every $B \subseteq X$ with $B \cap F^{(u)} \neq \emptyset$ for all $F \in f$,

$$\delta(x, B) \leq \sup_{A \in \mathcal{A}} \delta(x, A) + u,$$

with $\mathcal{A} = \{A \subseteq X \mid \forall F \in f \ (A \cap F \neq \emptyset)\};$
(iii) for every ultrafilter $y$ on $X$, $u \in [0, \infty]$ and every $B \subseteq X$ with $B \cap A^{(u)} \neq \emptyset$ for all $A \in y$,
\[ \delta(x, B) \leq a(y, x) + u. \]

Proof. (i) $\implies$ (ii): Let $f$ be a filter on $X$, $u \in [0, \infty]$ and $B \subseteq X$ with $B \cap F^{(u)} \neq \emptyset$ for every $F \in f$. Let $\chi$ be an ultrafilter on $X$ such that $B \in \chi$ and $F^{(u)} \in \chi$ whenever $F \in f$. Since $\{W \subseteq X \mid W^{(u)} \notin \chi\}$ is an ideal disjoint from $f$, by Corollary III.11.13.5 there exists an ultrafilter $y$ such that $f \subseteq y$ and $W^{(u)} \in y$ whenever $W \in \chi$. Then, for every $x \in X$,
\[ \delta(x, B) \leq a(\chi, x) \leq \hat{a}(y, \chi) + a(y, x) \leq u + a(y, x) \leq u + \sup_{A \in A} \delta(x, A). \]

(ii) $\implies$ (iii) is straightforward. For (iii) $\implies$ (i), consider $\chi, y \in \beta X, x \in X$ and $u \in [0, \infty]$ such that $A^{(u)} \in \chi$ whenever $A \in y$; for every $B \in \chi$, one then has $\delta(x, B) = a(y, x) + u$ and therefore
\[ a(\chi, x) = \sup_{B \in \chi} \delta(x, B) \leq a(y, x) + \inf\{u \mid \forall A \in y (A^{(u)} \in \chi)\} = a(y, x) + \hat{a}(y, \chi). \]

Here are some expected general assertions about regularity.

2.3.4 Proposition.

(1) If $\mathcal{V}$ is lean and integral and $\hat{T}$ is flat, then every compact Hausdorff $(\mathbb{T}, \mathcal{V})$-space is regular.

(2) $(\mathbb{T}, \mathcal{V})$-$\text{Cat}_{\text{reg}}$ is closed under $O$-initial sources (where $O : (\mathbb{T}, \mathcal{V})$-$\text{Cat} \to \text{Set}$ is the forgetful functor), hence both epi- and mono-reflective in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$. 

Proof. 

(1) Let $(X, a)$ be a compact Hausdorff $(\mathbb{T}, \mathcal{V})$-space. If $\mathcal{V}$ is lean and integral, $a$ is a map and $a \cdot T a = a \cdot m_X$. Since $\hat{T}$ is flat,
\[ a \cdot \hat{a} = a \cdot m_X \cdot (T a)^{\circ} = a \cdot T a \cdot (T a)^{\circ} \leq a \cdot T (a \cdot a^{\circ}) \leq a. \]

(2) Let $f_i : (X, a) \to (Y_i, b_i)$ be an $O$-initial source, that is $a = \wedge_{i \in I} f_i^{\circ} \cdot b_i \cdot T f_i$, with $(Y_i, b_i)$ regular for every $i \in I$. Then, for every $i \in I$,
\[
\begin{align*}
    a \cdot m_X \cdot (\hat{T} a)^{\circ} & \leq f_i^{\circ} \cdot b_i \cdot T f_i \cdot m_X \cdot (\hat{T} (f_i^{\circ} \cdot b_i \cdot T f_i))^{\circ} \\
    & = f_i^{\circ} \cdot b_i \cdot T f_i \cdot m_X \cdot ((T f_i)^{\circ} \cdot \hat{T} b_i \cdot TT f_i)^{\circ} \\
    & = f_i^{\circ} \cdot b_i \cdot T f_i \cdot m_X \cdot ((T T f_i)^{\circ} \cdot (\hat{T} b_i)^{\circ} \cdot T f_i) \\
    & = f_i^{\circ} \cdot b_i \cdot m_{Y_i} \cdot TT f_i \cdot ((T T f_i)^{\circ} \cdot (\hat{T} b_i)^{\circ} \cdot T f_i) \\
    & \leq f_i^{\circ} \cdot b_i \cdot m_{Y_i} \cdot (\hat{T} b_i)^{\circ} \cdot T f_i \\
    & \leq f_i^{\circ} \cdot b_i \cdot T f_i 
\end{align*}
\]
and therefore $a \cdot m_X \cdot (\hat{T} a)^{\circ} \leq \wedge_{i \in I} f_i^{\circ} \cdot b_i \cdot T f_i = a$. 

\[ \square \]
2. LOW SEPARATION, REGULARITY AND NORMALITY

2.3.5 Remarks.

(1) That a regular \((\mathbb{T}, \mathcal{V})\)-space does not need to be Hausdorff (or even separated) can be seen already at the level of \(\mathcal{V}\)-spaces: while Hausdorffness means discreteness, regularity means symmetry.

(2) As shown in Theorem III.5.3.5, if \(\mathcal{T}\) is associative, for a \((\mathbb{T}, \mathcal{V})\)-space \((X, a)\), \(\hat{a}\) is not only reflexive but also transitive, that is, \((TX, \hat{a})\) is a \(\mathcal{V}\)-space. Moreover, the \((\mathbb{T}, \mathcal{V})\)-space \((X, a)\) is regular whenever the \(\mathcal{V}\)-space \((TX, \hat{a})\) is regular. Indeed, if \(\hat{a} = \hat{a} \circ \cdot \hat{a}\), then using the equality \(a = a \cdot \hat{a} \circ \cdot \hat{a} \leq a \cdot \hat{a} = a\).

The converse statement is not true in general (see Exercise 2.F).

(3) A necessary condition for the \((\mathbb{T}, \mathcal{V})\)-space \((X, a)\) to be regular is the regularity of the \(\mathcal{V}\)-space \((X, a \cdot e_X)\), provided that the lax extension \(\mathcal{T}\) of \(T\) is symmetric, that is \(\mathcal{T} r = \mathcal{T} (\mathcal{T} T r)\) for all \(r\) in \(\mathcal{V} \text{-Rel}\). The next result generalizes this remark.

2.3.6 Proposition. Let \(\alpha : (S, \mathcal{S}) \to (\mathbb{T}, \mathcal{T})\) be a morphism of symmetric lax extensions. Then the algebraic functor \(A_{\alpha} : (\mathbb{T}, \mathcal{V})\text{-Cat} \to (S, \mathcal{V})\text{-Cat}\) preserves regularity.

Proof. For \((X, a)\) regular in \((\mathbb{T}, \mathcal{V})\text{-Cat}\) one has (with \(S = (S, n, d)\):

\[
a \cdot \alpha_X \cdot \hat{a} \circ \cdot \alpha_X = a \cdot \alpha_X \cdot n_X \cdot (S \alpha_X) \circ \cdot (\mathcal{S} a) \circ \\
= a \cdot m_X \cdot T \alpha_X \cdot \alpha_{S X} \cdot (S \alpha_X) \circ \cdot \hat{S} (a) \circ \quad (\alpha \text{ monad morphism}) \\
\leq a \cdot m_X \cdot T \alpha_X \cdot (T \alpha_X) \circ \cdot \alpha_{T X} \cdot \hat{S} (a) \circ \quad (T \alpha \cdot \alpha S = \alpha T \cdot S \alpha) \\
\leq a \cdot m_X \cdot \hat{T} (a) \circ \cdot \alpha_X \quad (\alpha \text{ lax extension morphism}) \\
\leq a \cdot \hat{a} \circ \cdot \alpha_X \\
\leq a \cdot \alpha_X.
\]

For \((\mathbb{T}, \mathcal{V}) = (\beta, \mathcal{P}_+)\) one concludes in particular that the underlying metric of a regular approach space is symmetric: see 2.3.2(1).

2.4 Normal and extremally disconnected spaces. Throughout this subsection we assume that

- \(\mathcal{V}\) is commutative;
- \(\mathcal{T}\) is associative (see 2.3.5(2)).

Recall that a topological space is normal if for all \(A, B\) closed with \(A \cap B = \emptyset\), there are open sets \(U\) and \(W\) such that \(A \subseteq U\), \(B \subseteq W\) and \(U \cap W = \emptyset\). It was already shown in
Proposition III.5.6.2 that normality for a \((\beta, 2)\)-space \((X, a)\) can be expressed by using the ordered set \((\beta X, \hat{a})\):

2.4.1 Proposition. For a topological space \(X\) presented as a \((\beta, 2)\)-space \((X, a)\), the following conditions are equivalent:

(i) \(X\) is a normal topological space.

(ii) \(\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a}\).

The proposition leads us to the following definition.

2.4.2 Definition. A \((T, V)\)-space \((X, a)\) is normal if

\[
\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a},
\]

that is,

\[
\hat{a}(x, \chi) \otimes \hat{a}(z, y) \leq \bigvee_{w \in TX} (\hat{a}(\chi, w) \otimes \hat{a}(y, w)),
\]

for all \(x, y, z \in TX\).

2.4.3 Proposition. For a \((T, V)\)-space \((X, a)\), the following conditions are equivalent:

(i) \((X, a)\) is normal;

(ii) \((TX, \hat{a})\) is a normal \(V\)-space;

(iii) \((TX, \hat{a}^\circ \cdot \hat{a})\) is a \(V\)-space.

Proof. For (i) \(\iff\) (ii), we just notice that normality for \((TX, \hat{a})\) means \(\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a}\).

(ii) \(\implies\) (iii): The structure \(\hat{a}^\circ \cdot \hat{a}\) is obviously reflexive. If \(\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a}\), then

\[
\hat{a}^\circ \cdot \hat{a} \cdot \hat{a}^\circ \cdot \hat{a} \leq \hat{a}^\circ \cdot \hat{a}^\circ \cdot \hat{a} \cdot \hat{a} \leq \hat{a}^\circ \cdot \hat{a}.
\]

(iii) \(\implies\) (i): Transitivity of \(\hat{a}^\circ \cdot \hat{a}\) and reflexivity of \(\hat{a}\) and \(\hat{a}^\circ\) give

\[
\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a} \cdot \hat{a}^\circ \cdot \hat{a} \leq \hat{a}^\circ \cdot \hat{a},
\]

that is, the \((T, V)\)-space \((X, a)\) is normal.

Reversing the inequality (2.4.i) has also an interesting topological meaning. It leads us to consider extremally disconnected objects. Recall that a topological space \(X\) is extremally disconnected if the closure of every open set in \(X\) is open.

2.4.4 Proposition. For a topological space \(X\) presented as a \((\beta, 2)\)-space \((X, a)\), the following conditions are equivalent:

(i) \(X\) is extremally disconnected;
(ii) for all open subsets $U, W$ of $X$, if $U \cap W = \emptyset$ then $\overline{U} \cap \overline{W} = \emptyset$;

(iii) $\hat{a} \circ \hat{a} \leq \hat{a} \cdot \hat{a}$.

Proof. (i) $\implies$ (ii): Let $U, W$ be open subsets of $X$. If $U \cap W = \emptyset$, then $\overline{U} \cap \overline{W} = \emptyset$, and, since $U$ is open, $U \setminus W = \emptyset$.

(ii) $\implies$ (i): Let $U$ be an open subset of $X$ and $W = X \setminus \overline{U}$. Then $U \cap W = \emptyset$ and therefore $\overline{U} \cap \overline{W} = \emptyset$. This implies that $U$ is open since $U \subseteq X \setminus W \subseteq X \setminus W = U$.

To show (ii) $\iff$ (iii), first we point out that, for any ultrafilters $\chi, y$ on $X$ and with $\hat{a} = (\leq)$,

$$\hat{a} \circ \hat{a}(\chi, y) = \top \iff \exists z \in \beta X \ (\chi \leq z \text{ and } y \leq z)$$
$$\iff \exists z \in \beta X \ (\forall B \subseteq X, B \text{ closed } B \in \chi \cup y \implies B \in z)$$

$$\hat{a} \cdot \hat{a}(\chi, y) = \top \iff \exists w \in \beta X \ (w \leq \chi \text{ and } w \leq y)$$
$$\iff \exists w \in \beta X \ (\forall A \subseteq X, A \text{ open } A \in \chi \cup y \implies A \in w)$$.

Assuming (ii), let $\chi, y, z$ be ultrafilters on $X$ with $\chi \leq z$ and $y \leq z$. For any open subsets $U, W$ of $X$, if $U \in \chi$ and $W \in y$, then $\overline{U}, \overline{W} \in z$ and therefore $\overline{U} \cap \overline{W} \neq \emptyset$, which implies with (ii) that $U \cap W \neq \emptyset$. The filter base

$$\{U \cap W \mid U, W \text{ open subsets of } X, U \in \chi, W \in y\}$$

is contained in an ultrafilter $w$. By construction, $w \leq \chi$ and $w \leq y$, and therefore $\hat{a} \cdot \hat{a}(\chi, y) = \top$.

Conversely, assume that (ii) does not hold. Thus, there are disjoint open subsets $U$ and $W$ of $X$ with $\overline{U} \cap \overline{W} \neq \emptyset$. Let $z$ be an ultrafilter containing $\overline{U} \cap \overline{W}$. Consider the ultrafilters $\chi$ and $y$ with filter bases

$$\mathcal{B}_\chi = \{A \mid A \text{ open and } A \in z \text{ or } A = U\},$$
$$\mathcal{B}_y = \{A \mid A \text{ open and } A \in z \text{ or } A = W\}.$$  

Then $\chi \leq z$ and $y \leq z$ but there is no $w$ with $w \leq \chi$ and $w \leq y$, that is, (iii) fails. □

2.4.5 Definition. A $(\mathbb{T}, \mathcal{V})$-space $(X, a)$ is extremally disconnected if

$$\hat{a} \circ \hat{a} \leq \hat{a} \cdot \hat{a};$$

that is,

$$\hat{a}(\chi, z) \otimes \hat{a}(y, z) \leq \bigvee_{w \in TX} (\hat{a}(w, x) \otimes \hat{a}(w, y))$$

for all $\chi, y, z \in TX$.  

2.4.6 Remark. A $\mathcal{V}$-space $(X, a)$ is normal if and only if $(X, a^\circ)$ is extremally disconnected.

Hence, using Proposition 2.4.3 we obtain:

2.4.7 Corollary. For a $(\mathbb{T}, \mathcal{V})$-space $(X, a)$, the following conditions are equivalent.

(i) $(X, a)$ is extremally disconnected;
(ii) $(TX, \hat{a})$ is an extremally disconnected $\mathcal{V}$-space;
(iii) $(TX, \hat{a}^\circ)$ is a normal $\mathcal{V}$-space;
(iv) $(TX, \hat{a} \cdot \hat{a}^\circ)$ is a $\mathcal{V}$-space.

2.4.8 Examples.

(1) Let $\mathbb{T} = \mathbb{I}$ be the identity monad (identically extended to $\mathcal{V}$-$\text{Rel}$). A $\mathcal{V}$-space $(X, a)$ is normal if and only if
\[
\forall x, y, z \in X \quad (a(x,y) \otimes a(x,z) \leq \bigvee_{s \in X} a(y,s) \otimes a(z,s)) \quad .
\] (2.4.ii)

If we write $x \xrightarrow{\beta} y$ whenever $a(x,y) = \beta$, diagrammatically this condition can be represented as follows:

Extremally disconnected $\mathcal{V}$-spaces are described in a similar way, with the arrows reversed.

In case $\mathcal{V} = 2$, an ordered set $(X, \leq)$, considered as a 2-space, is normal if and only if the order $\leq$ is confluent (see Exercise [II.5.A]). In particular, this shows a normal $(\mathbb{T}, \mathcal{V})$-space does not need to be regular (see also Exercise [2.H]).

A regular $\mathcal{V}$-space, that is a symmetric $\mathcal{V}$-space, is trivially normal and extremally disconnected.

(2) Let $\mathcal{B}$ be the ultrafilter monad. By Proposition 2.4.3 in $(\mathcal{B}, 2)$-$\text{Cat}$ a topological space $(X, a)$ is normal if and only if the order $\leq$ on $\beta X$ is confluent: for $\chi, y, z \in \beta X$, with $\chi \leq y, \chi \leq z$, there exists $w \in \beta X$ with $y \leq w$ and $z \leq w$:
Let \( \mathbb{T} = \mathbb{H} \) as in Example 2.3.2. Since \((\leq) = \hat{a}\) is transitive, normality of an \((\mathbb{H}, 2)\)-space is equivalent to the condition \(\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a} \). We recall that

\[
(\beta, y) \preceq (\alpha, x) \iff \exists \gamma \in H (\beta = \alpha \cdot \gamma \& y \xrightarrow{\gamma} x).
\]

According to this description, when \((\beta, y) \preceq (\alpha, x)\), we say that \((\beta, y)\) is a multiple of \((\alpha, x)\) and that \((\alpha, x)\) is a divisor of \((\beta, y)\).

For any \((\alpha, x), (\beta, y) \in H \times X\),

\[
(\alpha, x) (\hat{a} \cdot \hat{a}^\circ) (\beta, y) \iff \exists (\gamma, z) \in H \times X ((\alpha, x) \succeq (\gamma, z) \preceq (\beta, y)),
\]

that is, \((\alpha, x)\) and \((\beta, y)\) have a common multiple, and

\[
(\alpha, x) (\hat{a}^\circ \cdot \hat{a}) (\beta, y) \iff \exists (\delta, w) \in H \times X ((\alpha, x) \preceq (\delta, w) \succeq (\beta, y)),
\]

that is, \((\alpha, x)\) and \((\beta, y)\) have a common divisor. Hence \((X, a)\) is normal if and only if any pair of elements \((\alpha, x), (\beta, y)\) of \(H \times X\) with a common multiple has a common divisor. That is, for each \(x, y, z \in X, \alpha, \beta, \alpha_1, \beta_1 \in H\) such that \(\alpha \cdot \alpha_1 = \beta \cdot \beta_1\) and \(x \xleftarrow{\alpha_1} z \xrightarrow{\beta_1} y\), there exists \(w \in X, \delta, \alpha_2, \beta_2 \in H\) with \(\alpha = \delta \cdot \alpha_2, \beta = \delta \cdot \beta_2\) and \(x \xrightarrow{\alpha_2} w \xleftarrow{\beta_2} y\):

\[
\begin{array}{cc}
\alpha_1 & \beta_1 \\
\downarrow & \downarrow \\
x & y \\
\alpha_2 & \beta_2 \\
\downarrow & \downarrow \\
w & \\
\end{array}
\]

with \(\delta \cdot \alpha_2 \cdot \alpha_1 = \alpha \cdot \alpha_1 = \beta \cdot \beta_1 = \delta \cdot \beta_2 \cdot \beta_1\).

Reversing the arrows one obtains a description of extremally disconnected \((\mathbb{H}, 2)\)-spaces.

(4) If \( \mathbb{T} = \mathbb{L} \) is the list monad, then again we can use the condition \(\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a}\) to study normality of a multi-ordered set \((X, a)\). It is easy to check that \((X, a)\) is normal if and only if for all \((x_1, \ldots, x_n), (y_1, \ldots, y_m), (z_1, \ldots, z_l) \in LX\) the following condition holds: if there exist partitions \(1 \leq i_1 < \cdots < i_n = l\) and \(1 \leq j_1 < \cdots < j_m = m\) of \(l\) such that

\[
((z_1, \ldots, z_{i_1}), \ldots, (z_{i_n-1+1}, \ldots, z_l)) \vdash (x_1, \ldots, x_n) \&
((z_1, \ldots, z_{j_1}), \ldots, (z_{j_m-1+1}, \ldots, z_l)) \vdash (y_1, \ldots, y_m)
\]

then there exist \((w_1, \ldots, w_k) \in LX\) and \(k\)-partitions \(1 \leq n_1 < \cdots < n_k = n\) and \(1 \leq m_1 < \cdots < m_k = m\), of \(n\) and \(m\), such that

\[
((x_1, \ldots, x_{n_1}), \ldots, (x_{n_{k-1}+1}, \ldots, x_n)) \vdash (w_1, \ldots, w_k) \&
((y_1, \ldots, y_{m_1}), \ldots, (y_{m_{k-1}+1}, \ldots, y_m)) \vdash (w_1, \ldots, w_k)
\]

(where \(\vdash\) abbreviates \(\overline{LA}\)).
(5) In $\text{Met} \cong (I, P_+)-\text{Cat}$, a metric space $(X, a)$ is normal if and only if, for every $x, y, z \in X$,

$$a(z, x) + a(z, y) \geq \inf_{w \in X} a(x, w) + a(y, w).$$

Likewise in $\text{App} \cong (\beta, P_+)-\text{Cat}$, an approach space $(X, a)$ is normal if and only if for any $x, y, z \in \beta X$,

$$\hat{a}(z, x) + \hat{a}(z, y) \geq \inf_{w \in \beta X} \hat{a}(x, w) + \hat{a}(y, w),$$

where $\hat{a}(x, y) = \inf\{u \in [0, \infty] \mid \forall A \in x \ (A^{(u)} \in y)\}$.

Normal approach spaces will be investigated in [2.5] below. Here we give conditions on $\mathbb{T}$ and $\mathcal{V}$ in general for compact Hausdorff $(\mathbb{T}, \mathcal{V})$-spaces to be normal.

2.4.9 Proposition. If $\hat{T}$ is flat, then every $\mathbb{T}$-algebra is a normal $(\mathbb{T}, \mathcal{V})$-space.

Proof. First we remark that, when $a : TX \to X$ is a map, the $\mathcal{V}$-space $(TX, \hat{a} = Ta \cdot m_X^\circ)$ is completely determined by its underlying order (given by $B_p : \mathcal{V}\text{-Cat} \to 2\text{-Cat} = \text{Ord}$). Therefore, to show normality of the $\mathbb{T}$-algebra $(X, a)$ we have to check that $(\preceq) = \hat{a}$ is confluent. Let $x, y, z \in TX$ with $x \preceq y$ and $x \preceq z$; that is, there exist $\mathcal{Y}, \mathcal{Z} \in TTX$ such that $m_X(\mathcal{Y}) = m_X(\mathcal{Z}) = x$ and $Ta(\mathcal{Y}) = y$, $Ta(\mathcal{Z}) = z$. For $y = a(y)$ and $z = a(z)$, using the equality $a \cdot Ta = a \cdot m_X$ we conclude that

$$y = a \cdot Ta(\mathcal{Y}) = a \cdot m_X(\mathcal{Y}) = a(x) = a \cdot m_X(\mathcal{Z}) = a \cdot Ta(\mathcal{Z}) = z.$$

Now it is easy to conclude $y \preceq e_X(y)$, since, for $\mathcal{W} = e_{TX}(y)$, $m_X(\mathcal{W}) = y$ and $Ta(\mathcal{W}) = Ta(e_{TX}(y)) = e_X(a(y)) = e_X(y)$; an analogous argument shows that $z \preceq e_X(y)$. \hfill $\Box$

Using Proposition 1.2.1 we may now conclude.

2.4.10 Corollary. If $\mathcal{V}$ is integral and lean and $\hat{T}$ is flat, then every compact Hausdorff $(\mathbb{T}, \mathcal{V})$-space is normal.

2.5 Normal approach spaces. Normality of an approach space $(X, a)$ implies a strong separation property that can be expressed in terms of its approach distance. In preparation for that we first show the next result.

2.5.1 Lemma. For subsets $A, B$ of an approach space $X$ and any real $u > 0$, the following are equivalent:

(i) $\forall C \subseteq X \ (A \cap C^{(u)} \neq \emptyset)$ or $B \cap (X \setminus C)^{(u)} \neq \emptyset$;

(ii) $\exists x, y, z \in \beta X \forall C \in Z \ (A \cap C^{(u)} \in x)$ and $B \cap C^{(u)} \in y$.
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Proof. (i) \implies (ii): We must show the existence of an ultrafilter \( z \) such that both \( \{A\} \cup \{C^{(u)} | C \in z\} \) and \( \{B\} \cup \{C^{(u)} | C \in z\} \) generate proper filters on \( X \). But failing that, for every \( z \in \beta X \) we could choose \( D \in z \) with \( A \cap D^{(u)} = \emptyset \) or \( B \cap D^{(u)} = \emptyset \); moreover, by compactness of \( \beta X \), finitely many such sets \( D_1, \ldots, D_n \) could be found with \( \bigcup_i D_i = X \). Then, with

\[ C := \bigcup \{D_i | A \cap D_i^{(u)} = \emptyset \} \]

one would trivially have \( A \cap C^{(u)} = \emptyset \), and also \( B \cap (X \setminus C)^{(u)} = \emptyset \) since \( X \setminus C \subseteq \bigcup \{D_i | B \cap D_i^{(u)} = \emptyset \} \), in contradiction to (i).

(ii) \implies (i): For \( C \subseteq X \) one either has \( C \in z \) and then \( A \cap C^{(u)} \in \chi \), or \( X \setminus C \in z \) and then \( B \cap (X \setminus C)^{(u)} \in z \).

\[ \square \]

\textbf{2.5.2 Theorem.} For an approach space \( (X, a) \), each of the following statements implies the next:

(i) \( (X, a) \) is normal in \( (\beta, P_u)\)-Cat;

(ii) for all ultrafilters \( \chi, y, z \) on \( X \) and any real \( w > 0 \),

\[ \hat{a}(z, \chi) < w \land \hat{a}(z, y) < w \implies \exists w \in \beta X (\hat{a}(\chi, w) < 2w \land \hat{a}(y, w) < 2w) \]

\( \square w < w \hat{a}(z, \chi) \hat{a}(z, y) \hat{a}(\chi, w) \hat{a}(y, w) < 2w \]

(iii) for all \( A, B \subseteq X \) and any real \( v > 0 \),

\[ A^{(v)} \cap B^{(v)} = \emptyset \implies \exists u > 0 \exists C \subseteq X (A^{(u)} \cap C^{(u)} = \emptyset = B^{(u)} \cap (X \setminus C)^{(u)}) ; \]

(iv) for all ultrafilters \( \chi, y, z \) on \( X \),

\[ \hat{a}(z, \chi) = 0 = \hat{a}(z, y) \implies \exists w \in \beta X (\hat{a}(\chi, w) = 0 = \hat{a}(y, w)) . \]

Proof. (i) \implies (ii) is an immediate consequence of the defining property of normality:

\[ \hat{a}(z, \chi) + \hat{a}(z, y) \geq \inf_{w \in \beta X} (\hat{a}(\chi, w) + \hat{a}(y, w)) . \]

(ii) \implies (iii): Given \( v > 0 \), we claim that any \( u \leq \frac{v}{2} \) has the property described by (iii). Indeed, assuming failure of (iii), by Lemma \textbf{2.5.1} we would have \( A, B \subseteq X \) with \( A^{(v)} \cap B^{(v)} = \emptyset \) and ultrafilters \( \chi, y, z \) on \( X \) such that

\[ \forall C \in z (A^{(u)} \cap C^{(u)} \in \chi \land B^{(u)} \cap C^{(u)} \in y) . \]
Since $\hat{a}(z, \chi) = \inf \{ w : \forall C \in \mathcal{Z} \ (C^{(w)} \in \chi) \}$, one has $u \geq \hat{a}(z, \chi)$ and, likewise, $u \geq \hat{a}(z, y)$. From (ii) (with $w = \frac{3}{2} u$) one obtains $w \in \beta X$ with $\hat{a}(\chi, w) < 3u$ and $\hat{a}(y, w) < 3u$; therefore, since $A^{(u)} \in \chi$ and $B^{(u)} \in y$,

$$A^{(v)} \cap B^{(v)} \supseteq (A^{(u)})^{(3u)} \cap (B^{(u)})^{(3u)} \in w,$$

which contradicts $A^{(v)} \cap B^{(v)} = \emptyset$.

(iii) $\implies$ (iv): Consider $\chi, y, z \in \beta X$ with $\hat{a}(z, \chi) = 0 = \hat{a}(z, y)$. For any $A \in \chi$, $B \in y$ and $u > 0$ one has

$$\forall C \in \mathcal{Z} \ (A \cap C^{(u)} \neq \emptyset \text{ or } B \cap (X \setminus C)^{(u)} \neq \emptyset);$$

in particular

$$\forall C \in \mathcal{Z} \ (A^{(u)} \cap C^{(u)} \neq \emptyset \text{ or } B^{(u)} \cap (X \setminus C)^{(u)} \neq \emptyset).$$

By hypothesis (iii) then, $A^{(v)} \cap B^{(v)} \neq \emptyset$ for all $A \in \chi$, $B \in y$, $v > 0$. With an ultrafilter $w$ containing the filter base given by all these non-empty sets, one obtains

$$\forall v > 0 \forall A \in \chi \forall B \in y \ (A^{(v)} \in w \text{ and } B^{(v)} \in w),$$

that is, $\hat{a}(\chi, w) = 0 = \hat{a}(y, w)$. □

### 2.5.3 Remark.
None of the three implications of Theorem 2.5.2 is reversible, as the following three examples show. We point out that all the three examples are in particular metric spaces, showing that the four conditions are distinct already at the level of metric spaces.

1. Let $X = \{x, y, z, w\}$ with $a(z, x) = a(z, y) = a(z, w) = 1$, $a(x, w) = a(y, w) = 2$, $a(x', x') = 0$ for any $x' \in X$, and $a(x', y') = \infty$ elsewhere. Then it is straightforward to check that $(X, a)$ satisfies (ii) although $a(z, y) + a(z, x) < \inf_{w \in X} (a(x, w') + a(y, w')) = 4$.

2. Let $X = \{x, y, z, w\}$ with $a'(x', y') = a(x', y')$ except for $a'(x, w) = a'(y, w) = 3$. Then (ii) clearly fails. Since distinct points are at distance at least 1, for any $A \subseteq X$ and any $u < 1$, one has $A^{(u)} = A$, so (iii) holds.

3. Let $Y = \{x, y\} \cup \{x_n : n \in \mathbb{N}\}$, with $b(x_n, x) = \frac{1}{n} = b(x_n, y)$, $b(x', x') = 0$ for all $x' \in X$, and $b(x', y') = \infty$ elsewhere. Then $\{x\}^{(1)} \cap \{y\}^{(1)} = \{x\} \cap \{y\} = \emptyset$ although, for any $C \subseteq X$, either $C$ or $X \setminus C$ has an infinite number of $x_n$, and so, for any $u > 0$, either $x, y \in C^{(u)}$ or $x, y \in (X \setminus C)^{(u)}$; that is, (iii) does not hold. Moreover, $b(z, x) = 0$ only if $z = x$ and so (iv) holds.
Exercises

2.A **Preservation of low separation by algebraic functors.** Let $\alpha : (S, \hat{S}) \to (T, \hat{T})$ be a morphism of lax extensions and $A_\alpha : (T, V)$-$\text{Cat} \to (S, V)$-$\text{Cat}$ the induced algebraic functor. Prove that $(X, a)$ is T0, T1, or R0 if and only if $A_\alpha(X, a)$ has the respective property. Furthermore, if $(X, a)$ is R1 or Hausdorff, then $A_\alpha(X, a)$ has the respective property, with the converse statement holding when $\alpha_X$ is surjective. Exploit these statements for $\alpha = e$, that is, when $A_\alpha = A : (T, V)$-$\text{Cat} \to V$-$\text{Cat}$.

2.B **Low separation in $(H, 2)$-$\text{Cat}$.** Present $(H, 2)$-$\text{Cat}$ as $V$-$\text{Cat}$ with $V$ the powerset of $H$ as in Remark 1.4.3(1) and show that each of T0, T1, R0 and R1 changes its meaning under the change of presentation.

2.C **Compact T1-spaces for power-enriched monads.** When $T$ is power-enriched, prove that every compact T1-space in $(T, 2)$-$\text{Cat}$ has precisely one point.

2.D **Strengthening R0.** We say that a $(T, V)$-space is R0+ if $\hat{a} \cdot e_X \leq a$. Show that:

1. every R0+ space is R0;
2. every regular space is R0+.

2.E **Separated reflections and Kleisli monoids.** For a monad $T$ on Set with an associative lax extension $\hat{T}$ to $V$-$\text{Rel}$, show that the separated reflection of Theorem 2.1.2 defines also a left adjoint to the inclusion functor $(T, V)$-$\text{Cat}^{\hat{T}}_{\text{sep}} \to (T, V)$-$\text{Cat}^T$, where $T$ is being considered a monad on $(T, V)$-$\text{Cat}$ (see III.5.4). Conclude that if $T$ is power-enriched, then $T'X$ as defined in IV.4.2 is the separated reflection of $TX$.

*Hint.* See Exercise IV.4.C.

2.F **Regularity of $(X, a)$ does not imply regularity of $(TX, \hat{a})$.** Let $\mathcal{H}$ be as in Example 2.3.2(3).

1. Show that, when $\mathcal{H} = (\mathbb{N}, \times, 1)$ is the multiplicative monoid of natural numbers, the $(\mathcal{H}, 2)$-space $(\mathbb{N}, a)$, with $a((\alpha, x), y) = T$ only when $\alpha \cdot x = y$, is regular, although the ordered set $(HN, \hat{a})$ is not regular (that is, symmetric).

2. Show that, if $H$ is a group, then an $(\mathcal{H}, 2)$-space $(X, a)$ is regular if and only if $(HX, \hat{a})$ is regular.

2.G **Discrete $(T, V)$-spaces.** Show that, when $\hat{T}$ is flat, discrete $(T, V)$-spaces are both normal and extremally disconnected (see III.3.2).

2.H **The extended real half-line.** Show that, as a $\mathcal{P}_4$-space, the extended real half-line $([0, \infty], \odot)$ is normal and extremally disconnected, but not regular.
2.1 Normal Hausdorff spaces in \((\mathbb{L}, 2)-\text{Cat}\). Let \((X, \vdash)\) be a normal Hausdorff space in \((\mathbb{L}, 2)-\text{Cat}\). Then

\[ x \cdot y = z \iff (x, y) \vdash z \]

establishes a partially-defined binary operation \(\cdot\) on \(X\) such that, whenever \(x \cdot y, y \cdot z\) are defined, then \((x \cdot y) \cdot z, x \cdot (y \cdot z)\) are defined and equal, or \(x \cdot y = x\) and \(y \cdot z = z\).
3 Proper and open maps

The power of the notion of compact Hausdorff \((T, \mathcal{V})\)-space arises from its equational description as an Eilenberg–Moore algebra (see Proposition 1.2.1). In this subsection we consider the equationally defined classes of proper and of open maps in \((T, \mathcal{V})\)-Cat which in Section 4 \textit{inter alia}, lead us to equationally defined modifications of the notions of compact and Hausdorff \((T, \mathcal{V})\)-space, as defined in 1.1.

Throughout the section, \(\mathcal{V} = (\mathcal{V}, \otimes, k)\) is a quantale and \(T = (T, m, e)\) is a \textit{Set}-monad with a lax extension \(\hat{T}\) to \(\mathcal{V}\)-Rel.

3.1 Finitary stability properties. For \((T, \mathcal{V})\)-spaces \((X, a), (Y, b)\), \((T, \mathcal{V})\)-continuity of a map \(f : X \to Y\) is equivalently expressed by the inequalities

\[
f \cdot a \leq b \cdot Tf \quad \text{and} \quad a \cdot (Tf)^{\circ} \leq f^{\circ} \cdot b .
\]

Considering equalities in either case leads us to the two key notions of this subsection.

3.1.1 Definition. A \((T, \mathcal{V})\)-continuous map \(f : (X, a) \to (Y, b)\) is \textit{proper} if

\[
b \cdot Tf \leq f \cdot a ,
\]

and \(f : (X, a) \to (Y, b)\) is \textit{open} if

\[
f^{\circ} \cdot b \leq a \cdot (Tf)^{\circ} ,
\]

as in III.4.3.1.

Hence, \(f\) is proper if and only if

\[
\forall \chi \in TX, y \in Y \ (b(Tf(\chi), y) \leq \bigvee_{z \in f^{-1}y} a(\chi, z)) ,
\]

in which case the last inequality is actually an equality; and \(f\) is open if and only if

\[
\forall x \in X \forall y \in TY \ (b(y, f(x)) \leq \bigvee_{z \in (Tf)^{-1}y} a(z, x)) ,
\]

in which case the last inequality is again an equality.

In order to emphasize their dependency on \(T\) and \(\mathcal{V}\), whenever needed we speak more precisely of \((T, \mathcal{V})\)-\textit{proper} maps and \((T, \mathcal{V})\)-\textit{open} maps.
3.1.2 Remarks.

(1) For $\mathcal{V}$ commutative, one has the dualization functor

$(-)^{\text{op}} : \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}$,

$X = (X, a) \mapsto X^{\text{op}} = (X, a^{\circ})$,

which maps morphisms identically. For $\mathcal{T} = \mathbb{I}$ (identically extended to $\mathcal{V}\text{-Rel}$) and $f : (X, a) \to (Y, b)$ one has

$b \cdot f \leq f \cdot a \iff f^{\circ} \cdot b^{\circ} \leq a^{\circ} \cdot f^{\circ}$,

that is:

\[ f \text{ proper } \iff f^{\text{op}} \text{ open}. \]

Hence, open is dual to proper.

(2) When $\mathcal{V}$ is integral and superior (see 1.1.3), the $(\mathcal{T}, \mathcal{V})$-functor $(X, a) \to (1, \mathbb{T})$ is proper if and only if $(X, a)$ is compact.

(3) When $\mathcal{V}$ is integral and $T1 \cong 1$, then the $(\mathcal{T}, \mathcal{V})$-functor $(X, a) \to (1, \mathbb{T})$ is open. (See also Exercise 3.C)

3.1.3 Examples.

(1) In $\text{Ord} \cong 2\text{-Cat}$, a monotone map $f : (X, \leq) \to (Y, \leq)$ is proper if and only if

\[ f(x) \leq y \iff \exists z \in f^{-1}y \ (x \leq z), \quad x \leq z \quad \downarrow \quad f(x) \leq y \]

and it is open if and only if

\[ y \leq f(x) \iff \exists z \in f^{-1}y \ (z \leq x), \quad z \leq x \quad \downarrow \quad y \leq f(x). \]

In terms of the down- and up-closure operations (see II.1.7), one has

\[ f \text{ proper } \iff \forall x \in X \ (\uparrow_Y f(x) \subseteq f(\uparrow_X x)) \]

\[ \iff \forall A \subseteq X \ (\uparrow_Y f(A) \subseteq f(\uparrow_X A)) \]

\[ \iff \forall y \in Y \ (f^{-1}(\downarrow_Y y) \subseteq \downarrow_X (f^{-1}y)) \]

\[ \iff \forall B \subseteq Y \ (f^{-1}(\downarrow_Y B) \subseteq \downarrow_X f^{-1}(B)) \]

and all inclusions may equivalently be replaced by equalities. Openness of $f$ is characterized by the order-dual conditions.
3. PROPER AND OPEN MAPS

(2) If $\text{Ord}$ is presented as $(\mathbb{P}, 2)$-$\text{Cat}$ (with the powerset monad laxly extended by $\hat{\mathbb{P}}$, see Example [1.6.2(1)]), the meaning of proper map changes from the previous example. Indeed, in this presentation, an ordered set $(X, \leq)$ is considered as a $(\mathbb{P}, 2)$-space $(X, \ll)$ via

$$A \ll y \iff \forall x \in A (x \leq y)$$

for all $A \subseteq X$, $y \in X$, and with $\uparrow_X A := \{ x \in X | A \ll x \}$ one obtains for a monotone map $f$:

$$f \text{ is } (\mathbb{P}, 2)\text{-proper} \iff \forall A \subseteq X (\uparrow_Y f(A) \subseteq f(\uparrow_X A)).$$

Such maps must necessarily be surjective ($A = \emptyset$) and 2-proper ($A = \{x\}$), but not conversely. However, the meaning of openness stays the same as in the previous example:

$$f \text{ is } (\mathbb{P}, 2)\text{-open} \iff f \text{ is } 2\text{-open}.$$

(3) In $\text{Met} \cong \mathbb{P}_+\text{-Cat}$, a non-expansive map $f : (X, a) \to (Y, b)$ is proper if and only if

$$b(f(x), y) = \inf \{ a(x, z) | z \in X, f(z) = y \}$$

for all $x \in X$, $y \in Y$. Openness is characterized dually.

(4) For a monoid $H$ and $\mathbb{H}$ as in 1.4, an $(\mathbb{H}, 2)$-map $f : X \to Y$ is proper if

$$f(x) \xrightarrow{\alpha} y \implies \exists z \in f^{-1}y (x \xrightarrow{\alpha} z),$$

and open if

$$y \xrightarrow{\alpha} f(x) \implies \exists z \in f^{-1}y (z \xrightarrow{\alpha} x),$$

(for all $x \in X$, $y \in Y$, $\alpha \in H$).

(5) For the list monad $\mathbb{L}$ as in 1.4, an $(\mathbb{L}, 2)$-map $f : X \to Y$ is proper if

$$(f(x_1), \ldots, f(x_n)) \vdash y \implies \exists z \in f^{-1}y ((x_1, \ldots, x_n) \vdash z),$$

and open if

$$(y_1, \ldots, y_n) \vdash f(x) \implies \exists z_i \in f^{-1}(y_i) (i = 1, \ldots, n) ((z_1, \ldots, z_n) \vdash x),$$

(for all $x, x_i \in X$, $y, y_i \in Y$).

The more important examples $\text{Top} \cong (\mathbb{P}, 2)$-$\text{Cat}$ and $\text{App} \cong (\mathbb{P}, \mathbb{P}_+)$-$\text{Cat}$ will be discussed in 3.4 after we have established the following stability properties.

3.1.4 Proposition.

(1) The classes of proper maps and of open maps in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$ are both closed under composition and contain all isomorphisms.
(2) For \((T, V)\)-continuous maps \(f : (X, a) \to (Y, b), m : (Y, b) \to (Z, c)\), if \(m\) is injective and \(m \cdot f\) proper or open, \(f\) is also proper or open, respectively.

\(\circ\) (3) For \((T, V)\)-continuous maps \(e : (X, a) \to (Y, b), g : (Y, b) \to (Z, c)\), if \(e\) is surjective and \(g \cdot e\) proper or open, \(g\) is also proper or open, respectively.

(4) If \(V\) is cartesian closed, every pullback of a proper map is proper; if, in addition, \(T\) satisfies BC, every pullback of an open map is open.

Proof.

(1) Immediate.

(2) If \(m\) is injective, then \(m^\circ \cdot m = 1_Y\) and \((Tm)^\circ \cdot Tm = 1_{TY}\) (see Proposition [III]1.2.2 and Exercise [III]1.P). Hence, if \(m \cdot f\) is proper, then

\[
b \cdot Tf = b \cdot (Tm)^\circ \cdot Tm \cdot Tf \leq m^\circ \cdot c \cdot T(m \cdot f) = m^\circ \cdot m \cdot f \cdot a = f \cdot a
\]

follows, and if \(m \cdot f\) is open, then one obtains

\[
f^\circ \cdot b = f^\circ \cdot m^\circ \cdot m \cdot b \leq f^\circ \cdot m^\circ \cdot c \cdot Tm = c \cdot (Tf)^\circ \cdot (Tm)^\circ \cdot Tm = c \cdot (Tf)^\circ .
\]

\(\circ\) (3) If \(e\) is surjective, then \((Te) \cdot (Te)^\circ = 1_{TY}\), so that \(g \cdot e\) proper implies

\[
c \cdot Tg = c \cdot Tg \cdot Te \cdot (Te)^\circ \leq g \cdot e \cdot a \cdot (Te)^\circ \leq g \cdot e \cdot e^\circ \cdot b \leq g \cdot b .
\]

The case where \(g \cdot e\) is open is treated similarly.

(4) Pullback stability of openness was shown in Proposition [III]4.3.8. Keeping the notation used there one shows the same property for proper maps, but without requiring BC for \(T\), as follows:

\[
b \cdot Tq = (b \land b) \cdot Tq
\]

\[
\leq ((g^\circ \cdot c \cdot Tg) \land b) \cdot Tq
\]

\[
= (g^\circ \cdot c \cdot Tg \cdot Tq) \land b \cdot Tq
\]

\[
= (g^\circ \cdot c \cdot Tf \cdot Tp) \land b \cdot Tq
\]

\[
= (g^\circ \cdot f \cdot a \cdot Tp) \land b \cdot Tq \quad \text{\((f\) proper)}
\]

\[
= (q \cdot p^\circ \cdot a \cdot Tp) \land b \cdot Tq
\]

\[
= q \cdot ((p^\circ \cdot a \cdot Tp) \land (q^\circ \cdot b \cdot Tq)) \quad \text{(\(V\) cartesian closed by [III]4.3.7)}
\]

\[
= q \cdot d .
\]
3.2 First characterization theorems. In this subsection we show that openness of 
\((\mathbb{T}, \mathcal{V})\text{-Cat}\) is often described as openness in \(\mathcal{V}\text{-Cat}\) and then explore to which extent this 
reduction is possible also for proper maps. We then prove a first generalization of the 
characterization of proper maps in \(\text{Top}\) as the closed maps with compact fibres.

3.2.1 Remark. By Corollary III.1.4.4 every lax extension \(\hat{T}\) of \(T\) satisfies
\[
\hat{T}(r \cdot f) = \hat{T}r \cdot Tf \quad \text{and} \quad \hat{T}(g \circ r) = (Tg) \circ \hat{T}r
\]
for all \(f : X \to Y, r : Y \to Z, g : W \to Z\). We say that \(\hat{T}\) is left-whiskering if
\[
\hat{T}(h \cdot r) = Th \cdot \hat{T}r
\]
for all \(r : Y \to Z, h : Z \to W\). This condition implies in particular \(\hat{T}h = Th \cdot \hat{T}1_Z\), which is also sufficient for the general case when \(\hat{T}\) preserves composition and, a fortiori, when \(\hat{T}\) is associative (see Proposition III.1.9.4). Similarly one says that \(\hat{T}\) is right-whiskering if
\[
\hat{T}(s \cdot f^\circ) = \hat{T}s \cdot (Tf)^\circ
\]
for all \(f : X \to Y, s : X \to Z\), which implies \(\hat{T}(f^\circ) = \hat{T}1_X \cdot (Tf)^\circ\) without loss of information when \(\hat{T}\) preserves composition. Note that a flat associative lax extension is always left- and right-whiskering. Hence, the Barr extensions of \(\beta\) to \(\text{Rel}\) and to \(\mathcal{V}\text{-Rel}\) are left- and right-whiskering. The Kleisli extension \(\tilde{\beta}\) to \(\text{Rel}\) of the filter monad is right- but not left-whiskering: see Exercise 3.K in III.1.9.6 we gave an example of a non-associative flat lax extension which is left- and right-whiskering.

If \(m^\circ : \hat{T} \to \hat{T}\hat{T}\) is a natural transformation, in particular if \(\tilde{\beta}\) is associative, the problem of characterizing the proper and open maps may often be reduced to the \(\mathcal{V}\text{-Cat}\) case, via the composite functor
\[
\begin{align*}
(\mathbb{T}, \mathcal{V})\text{-Cat} & \xrightarrow{M} (\mathcal{V}\text{-Cat})^\mathbb{T} & \xrightarrow{G^\mathbb{T}} & \mathcal{V}\text{-Cat} \\
(X, a) & \xmapsto{\quad} (TX, \hat{T}a \cdot m_X^\circ, m_X) & \xmapsto{\quad} & (TX, \hat{T}a \cdot m_X^\circ)
\end{align*}
\]
of 2.3 and Theorem III.5.3.5 which, by abuse of notation, we denote by \(T\) again. Indeed, with \(\hat{T}\) left-whiskering or right-whiskering one obtains the following criteria.

3.2.2 Proposition. Assume \(m^\circ : \hat{T} \to \hat{T}\hat{T}\) to be a natural transformation and \(f : (X, a) \to (Y, b)\) to be \((\mathbb{T}, \mathcal{V})\text{-continuous}.

(1) If \(\hat{T}\) is left-whiskering, then
\[
f \text{ is } (\mathbb{T}, \mathcal{V})\text{-proper} \implies Tf \text{ is } \mathcal{V}\text{-proper}.
\]

(2) If \(\hat{T}\) right-whiskering, then
\[
f \text{ is } (\mathbb{T}, \mathcal{V})\text{-open} \iff Tf \text{ is } \mathcal{V}\text{-open}.
\]
Proof.

(1) From \( b \cdot Tf \leq f \cdot a \), one obtains
\[
\hat{T}b \cdot m_Y^\circ \cdot Tf \leq \hat{T}b \cdot \hat{T}f \cdot m_X^\circ \quad (\text{\( m^\circ \) natural})
\]
\[
\leq \hat{T}(b \cdot \hat{T}f) \cdot m_X^\circ \quad (\hat{T} \text{ lax functor})
\]
\[
= \hat{T}(b \cdot Tf) \cdot m_X^\circ \quad (b \text{ right unitary})
\]
\[
= \hat{T}(f \cdot a) \cdot m_X^\circ \quad (f \text{ is } (\mathbb{T}, \mathcal{V})\text{-proper})
\]
\[
= Tf \cdot \hat{T}a \cdot m_X^\circ \quad (\hat{T} \text{ left-whiskering.})
\]

(2) From \( f \circ b \leq a \cdot (Tf)^\circ \), one derives
\[
(Tf)^\circ \cdot \hat{T}b \cdot m_Y^\circ = \hat{T}(f \circ b) \cdot m_Y^\circ
\]
\[
\leq \hat{T}(a \cdot (Tf)^\circ) \cdot m_Y^\circ \quad (f \text{ is } (\mathbb{T}, \mathcal{V})\text{-open})
\]
\[
= \hat{T}a \cdot (TTf)^\circ \cdot m_Y^\circ \quad (\hat{T} \text{ right-whiskering})
\]
\[
= \hat{T}a \cdot m_X^\circ \cdot (Tf)^\circ.
\]

Conversely, from \((Tf)^\circ \cdot \hat{T}b \cdot m_Y^\circ \leq \hat{T}a \cdot m_X^\circ \cdot (Tf)^\circ\) we can derive
\[
f \circ b \leq f \circ b \cdot e_Y^\circ \cdot e_Y
\]
\[
\leq e_X^\circ \cdot \hat{T}(f \circ b) \cdot e_Y
\]
\[
\leq e_X^\circ \cdot (Tf)^\circ \cdot \hat{T}b \cdot m_Y^\circ
\]
\[
\leq e_X^\circ \cdot \hat{T}a \cdot m_X^\circ \cdot (Tf)^\circ
\]
\[
= (e_X^\circ \circ a) \cdot (Tf)^\circ = a \cdot (Tf)^\circ,
\]

since \( a \) is left unitary. \(\square\)

Lax extensions for which the implication of Proposition 3.2.2(1) becomes a logical equivalence are, from a topological perspective, rare, but can be fully characterized, as follows.

3.2.3 Proposition. Assume \( m^\circ : \hat{T} \to \hat{T} \hat{T} \) to be a natural transformation and \( \hat{T} \) left-whiskering. Consider the following assertions:

(i) every \((\mathbb{T}, \mathcal{V})\)-continuous map \( f : (X,a) \to (Y,b) \) such that \( Tf \) is \( \mathcal{V} \)-proper is \((\mathbb{T}, \mathcal{V})\)-proper;

(ii) every function \( f : X \to Y \) gives a \((\mathbb{T}, \mathcal{V})\)-proper map \( f : (X, 1_X^\circ) \to (Y, 1_Y^\circ) \);

(iii) \( e : 1 \to T \) satisfies the Beck–Chevalley Condition.

Then (i) \(\iff\) (ii) \(\iff\) (iii), and all are equivalent when \( \hat{T} \) is flat.
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Proof. (i) $\implies$ (ii): We first determine how $T : (\mathbb{T}, \mathcal{V})\text{-}\text{Cat} \to \mathcal{V}\text{-}\text{Cat}$ maps discrete $(\mathbb{T}, \mathcal{V})$-spaces and prove $T(X, 1^X_T) = (TX, \hat{T}1_X)$. Indeed, since $m^o$ is a natural transformation,

$$\hat{T}^o_X = \hat{T}(e_X^o \cdot \hat{T}1_X) \cdot m_X^o = (Te_X)^o \cdot \hat{T}1_X \cdot m_X^o = (Te_X)^o \cdot m_X^o \cdot \hat{T}1_X = \hat{T}1_X .$$

Since $\hat{T}$ is left-whiskering, $Tf \cdot \hat{T}1_X = \hat{T}f = \hat{T}1_Y \cdot Tf$, so that

$$Tf : (TX, \hat{T}1_X) \to (TY, \hat{T}1_Y)$$

is $\mathcal{V}$-proper, for every function $f : X \to Y$. By hypothesis (i), the map $f : (X, 1^X_T) \to (Y, 1^Y_T)$ is $(\mathbb{T}, \mathcal{V})$-proper.

(ii) $\implies$ (i): For $f : (X, a) \to (Y, b)$, let $Tf : (TX, \hat{a}) \to (TY, \hat{b})$ be $\mathcal{V}$-proper, and $f : (X, 1^X_T) \to (Y, 1^Y_T)$ be $(\mathbb{T}, \mathcal{V})$-proper. Since $\hat{T}$ is left-whiskering we obtain

$$b \cdot Tf = (e_Y^o \circ b) \cdot Tf$$

$$= e_Y^o \cdot \hat{T}b \cdot m_Y^o \cdot Tf$$

$$= e_Y^o \cdot Tf \cdot \hat{T}a \cdot m_X^o$$

$$= e_Y^o \cdot Tf \cdot \hat{T}1_X \cdot \hat{T}a \cdot m_X^o$$

$$= e_Y^o \cdot \hat{T}1_X \cdot Tf \cdot \hat{T}a \cdot m_X^o$$

$$= 1^Y_T \cdot Tf \cdot \hat{T}a \cdot m_X^o$$

$$= f \cdot 1^X_T \cdot \hat{T}a \cdot m_X^o$$

$$= f \cdot (e_X^o \circ a)$$

$$= f \cdot a .$$

(iii) $\iff$ (ii): $f \cdot e_X^o = e_Y^o \cdot Tf$ implies $f \cdot e_X^o \cdot \hat{T}1_X = e_Y^o \cdot Tf \cdot \hat{T}1_X = e_Y^o \cdot \hat{T}1_Y \cdot Tf$ since $\hat{T}$ left-whiskering, and the reverse implication holds if $\hat{T}$ is flat.

\[ \square \]

3.2.4 Remarks.

1. For a monoid $H$, consider the flat extension $\mathcal{H}$ of the associated monad $\mathbb{H}$ (see [1.4]). An $(\mathbb{H}, 2)$-functor $f : (X, a) \to (Y, b)$ is proper if it satisfies

$$f(x) \xrightarrow{\alpha} y \implies \exists z \in f^{-1}y \ (x \xrightarrow{\alpha} z)$$

for all $x \in X$, $y \in Y$, $\alpha \in H$. The lax extension satisfies the hypothesis of Proposition 3.2.3 and the unit morphism of the monad satisfies BC. Proper $(\mathbb{H}, 2)$-functors $f$ are therefore equivalently described as those $(\mathbb{H}, 2)$-functors for which $1_H \times f$ is proper.

2. For the ultrafilter monad $\mathbb{B}$, the unit fails to satisfy BC; see Proposition [III1.12.4]. indeed, the naturality square for the map $X \to 1$ with $X$ infinite fails to be a
CHAPTER V. LAX ALGEBRAS AS SPACES

BC-square. Consequently, by Proposition \ref{prop:3.2.3}, there are maps \( f : X \to Y \) in \( \text{Top} \cong (\beta, 2)-\text{Cat} \) for which \( \beta f \) is proper, but \( f \) is not. We will see in Proposition \ref{prop:3.4.5} that \ref{prop:3.2.3} gives the \((\mathbb{T}, \mathcal{V})\)-categorical reason for the existence of closed maps that are not stable under pullback.

We can now give a complete characterization of \((\mathbb{T}, \mathcal{V})\)-proper maps in terms of the condition that \( \mathbb{T}f \) be \( \mathcal{V} \)-proper which, as we will show in Proposition \ref{prop:3.4.5}, generalizes the characterization of proper maps in \( \text{Top} = (\beta, 2)-\text{Cat} \) as the closed maps with compact fibres, and similarly in \( \text{App} \cong (\beta, P_+)-\text{Cat} \). But in order to be able to talk about fibres of \( f \), we should first clarify that very term. For each \( y \in Y \), the assignment \(* \mapsto x \mapsto y\) defines a \((\mathbb{T}, \mathcal{V})\)-functor \( y : (1, 1) \to (Y, b) \), where \( 1^\sharp = e_1 \circ 1 \cdot \hat{T}1_1 \) is the discrete structure on \( 1 = \{\ast\} \) (see \ref{sec:3.2}); explicitly, for \( w \in T1 \),

\[
1^\sharp(w, \ast) = \hat{T}1_1(w, e_1(\ast)) .
\]

By fibre of \( f \) on \( y \) we mean the pullback \((f^{-1}y, \hat{a}) \to (1, 1^\sharp)\) of \( f \) along the \((\mathbb{T}, \mathcal{V})\)-functor \( y : (1, 1^\sharp) \to (Y, b) \). We note that \((f^{-1}y, \hat{a}) \to (X, a)\) is a monomorphism, but in general not a regular monomorphism, that is, \( \hat{a} \) does not need to be the restriction of \( a : TX \times X \to V \) to \( T(f^{-1}y) \times f^{-1}y \):

\[
\hat{a}(\chi, x) = a(\chi, x) \wedge 1^\sharp(T!(\chi), \ast) \quad \text{(with } ! : f^{-1}y \to 1) \]

\[
= a(\chi, x) \wedge \hat{T}1_X(T!(\chi), e_1(\ast)) ,
\]

for every \( \chi \in T(f^{-1}y) \) and \( x \in f^{-1}y \). However, when \( T1 \cong 1 \) and \( \mathcal{V} \) is integral, from \( e_1^{-1} = e_1^\sharp \leq e_1^\sharp \cdot \hat{T}1 \) one obtains \( 1^\sharp = \mathbb{T} \), and every \((f^{-1}y, \hat{a})\) becomes a subspace of \((X, a)\).

Cartesian closedness of \( \mathcal{V} \) suffices to insure that proper \((\mathbb{T}, \mathcal{V})\)-functors have proper fibres, see Proposition \ref{prop:3.1.4}

We can now prove a first characterization theorem.

\textbf{3.2.5 Theorem.} Let \( \mathcal{V} \) be cartesian closed and \( T \) be taut, with \( \hat{T} \) left-whiskering and \( m^\circ \) a natural transformation. Then a \((\mathbb{T}, \mathcal{V})\)-functor \( f : (X, a) \to (Y, b) \) is proper if and only if all of its fibres are proper, and the \( \mathcal{V} \)-functor \( Tf : (TX, \hat{a}) \to (TY, \hat{b}) \) is proper.

\textbf{Proof.} If \( f \) is proper, then the fibres of \( f \) are proper by Proposition \ref{prop:3.1.4} and \( Tf \) is proper by Proposition \ref{prop:3.2.2}.

Conversely, assume that all fibres of \( f \) are proper in \((\mathbb{T}, \mathcal{V})\)-\text{Cat} and \( Tf \) is proper in \( \mathcal{V}\text{-Cat} \). Since

\[
b = b \cdot e_{TY}^\circ \cdot m_{TY}^\circ \leq e_Y^\circ \cdot \hat{T}b \cdot m_{TY}^\circ = e_Y^\circ \cdot \hat{b} ,
\]
for all \( x \in TX, y \in Y \) one obtains:

\[
\begin{align*}
\beta \cdot Tf(x,y) &= b(Tf(x),y) \\
&\leq b(Tf(x),e_Y(y)) \\
&= \forall z \in (Tf)^{-1}(e_Y(y)) \hat{a}(x,z) \\
&= \forall z \in (Tf)^{-1}(e_Y(y))(\hat{T}a \cdot m^0_X)(x,z) \\
&= \forall z \in (Tf)^{-1}(e_Y(y)) \forall x \in m^1_X \hat{T}a(x,z) \otimes k.
\end{align*}
\]

Since tautness of \( T \) guarantees that the following diagram is a pullback

\[
\begin{array}{ccc}
T(f^{-1}y) & \xrightarrow{T^1} & T1 \\
\downarrow & & \downarrow T_y \\
TX & \xrightarrow{Tf} & TY,
\end{array}
\]

every \( z \in (Tf)^{-1}(e_Y(y)) = (Tf)^{-1}(Ty(e_1(\ast))) \) satisfies \( z \in T(f^{-1}y) \) and \( T!(z) = e_1(\ast) \). Using propriety of \((f^{-1}y, \hat{a}) \to (1, 1^2)\) one gets:

\[
\begin{align*}
\forall z \in (Tf)^{-1}(e_Y(y)) \forall x \in m^1_X \hat{T}a(x,z) \otimes k &\leq \forall z \in (Tf)^{-1}(e_Y(y)) \forall x \in m^1_X \hat{T}a(x,z) \otimes a(z,x) \\
&\leq \forall z \in (Tf)^{-1}(e_Y(y)) \forall x \in (f^{-1}y a)(m_X(x),x) \\
&\leq \forall z \in (Tf)^{-1}(e_Y(y)) (f \cdot a)(x,y).
\end{align*}
\]

Hence, \( f \) is proper. \( \square \)

Next we show that propriety of fibres trivializes whenever the unit \( e \) of the monad \( T \) satisfies BC—a rather restrictive condition, as we have seen in [III 1.12] and Exercise [III 1.12].

3.2.6 Proposition. If \( V \) is integral and \( e^\circ : \hat{T} \to 1 \) is a natural transformation, then any \((\mathcal{T}, V)\)-functor has proper fibres.

Proof. For a \((\mathcal{T}, V)\)-functor \( f : (X, a) \to (Y, b) \) and \( y \in Y \), we must show that the diagram

\[
\begin{array}{ccc}
T(f^{-1}y) & \xrightarrow{T^1} & T1 \\
\hat{a} \downarrow & & \downarrow 1^1 \\
(f^{-1}y) & \xrightarrow{1} & 1
\end{array}
\]

commutes; for that, it suffices to consider \( \chi \in T(f^{-1}y) \) with \( 1^\sharp(T!(\chi),*) = \hat{T}1(T!(\chi), e_1(*)) > \bot \) and show \( \tilde{a}(\chi,*) = \top \). From the commutativity of the diagram

\[
\begin{array}{ccc}
T(f^{-1}y) & \xrightarrow{T!} & T1 \\
\downarrow e^o & & \downarrow e^o_1 \\
f^{-1}y & \xrightarrow{!} & 1
\end{array}
\]

we first obtain

\[
\bot < e^o_1 \cdot \hat{T}1 \cdot T!(\chi,*) = e^o_1 \cdot T!(\chi,*) = ! \cdot e^o(\chi,*) = \bigvee_{x\in f^{-1}y} e^o(\chi, x) = k ,
\]

and then

\[
! \cdot \tilde{a}(\chi,*) \geq ! \cdot e^o(\chi, x) = k = \top .
\]

3.2.7 Corollary. Under the hypotheses of Theorem 3.2.5 and Proposition 3.2.6, a \((\mathbb{T}, \mathcal{V})\)-functor \( f : (X,a) \to (Y,b) \) is proper if and only if the \( \mathcal{V}\)-functor \( Tf \) is proper.

3.3 Notions of closure. Our next aim is to characterize \((\mathbb{T}, \mathcal{V})\)-proper and \((\mathbb{T}, \mathcal{V})\)-open maps in terms of suitable notions of closure. Since the functor \( T \) of the monad \( \mathbb{T} \) preserves injections (see Exercise III.1.P), for a subset \( A \subseteq X \) we may assume \( TA \subseteq TX \); with \( A \subseteq TX, v \in \mathcal{V} \) we define:

\[
\mathcal{A}[v] := \{ x \in X | \forall \chi \in A a(\chi, x) \geq v \} ,
\]

\[
A^{(v)} := TA[v] ,
\]

\[
\overline{A} := \bigcup_{v > \bot} A^{(v)} = \{ x \in X | \exists \chi \in TA (a(\chi, x) > \bot) \} ,
\]

and call \( A^{(v)} \) and \( \overline{A} \) the \( v \)-closure and grand closure of \( A \), respectively.

3.3.1 Proposition. The grand closure defines a hereditary \( \mathcal{M} \)-closure operator on \((\mathbb{T}, \mathcal{V})\)-Cat, with \( \mathcal{M} \) the class of embeddings, and so does the \( v \)-closure, for any fixed \( v \leq k \) in \( \mathcal{V} \). In general, neither operator is idempotent.

Proof. See Exercise 3.A. \( \square \)

We can now analyse to which extent these closures help us characterize propriety and openness of maps.

3.3.2 Proposition. The following statements on a \((\mathbb{T}, \mathcal{V})\)-continuous map \( f : (X,a) \to (Y,b) \) satisfy the implications

\[
(i) \implies (iii) \implies (v) \\
\downarrow \quad \downarrow \\
(ii) \implies (iv);
\]
furthermore, the vertical implications are equivalences if \( \mathcal{V} \) is completely distributive, and one has (ii) \( \iff \) (iv) if \( \mathbb{1} = \mathbb{I} \) is the identity monad, and (iii) \( \iff \) (v) if \( \mathcal{V} = 2 \).

(i) \( b \cdot Tf \leq f \cdot a \), that is, \( f \) is proper;
(ii) \( \forall A \subseteq TX \forall u \ll v \in \mathcal{V} \ (Tf(A)^v \subseteq f(A)^u) \);
(iii) \( \forall A \subseteq X \ (b \cdot Tf \cdot Ti_{A^!} \leq f \cdot a \cdot Ti_{A^!}) \);
(iv) \( \forall A \subseteq X \forall u \ll v \in \mathcal{V} \ (f(A)^v \subseteq f(A)^u) \);
(v) \( \forall A \subseteq X \ (f(A)^f \subseteq f(A)) \);

where \( i_A : A \hookrightarrow X \) is the inclusion and \( !_{TA} : TA \to 1 \).

Proof. The implications (i) \( \implies \) (iii) and (ii) \( \implies \) (iv) are trivial while the remaining generally valid implications become obvious once transcribed in elementwise terms, as follows:

(i) \( \forall \chi \in TX, y \in Y \ (b(Tf(\chi), y) \leq \bigvee_{x \in f^{-1}y} a(\chi, x)) \);
(ii) \( \forall A \subseteq TX, u \ll v \in \mathcal{V}, y \in Y \ (v \leq \bigvee_{x \in A} b(Tf(\chi), y) \implies \exists x \in f^{-1}y \ (u \leq \bigvee_{x \in A} a(z, x))) \);
(iii) \( \forall A \subseteq X, y \in Y \ (\bigvee_{x \in TA} b(Tf(\chi), y) \leq \bigvee_{x \in f^{-1}y} \bigvee_{z \in TA} a(z, x)) \);
(iv) \( \forall A \subseteq X, u \ll v \in \mathcal{V}, y \in Y \ (v \leq \bigvee_{x \in TA} b(Tf(\chi), y) \implies \exists x \in f^{-1}y \ (u \leq \bigvee_{z \in TA} a(z, x))) \);
(v) \( \forall A \subseteq X, y \in Y \ (\exists \chi \in TA \ (b(Tf(\chi), y) > \bot) \implies \exists x \in f^{-1}y, z \in TA \ (a(z, x) > \bot)) \).

Here, for the reformulation of (iii) and (iv), note that \( T(f(A)) = Tf(TA) \) since \( T \) preserves the surjection \( A \to f(A) \).

To see (ii) \( \implies \) (i) if \( \mathcal{V} \) is completely distributive, put \( A := \{ \chi \} \) and \( v := b(Tf(\chi), y) \), and for (iv) \( \implies \) (iii) put \( v := \bigvee_{x \in TA} b(Tf(\chi), y) \). One trivially has (iv) \( \implies \) (ii) when \( T \) is the identity functor, and the elementwise formulations also show (v) \( \implies \) (iii) for \( \mathcal{V} = 2 \) where \( \forall_i v_i = \top \) equivalently means \( v_i > \bot \) for some \( i \).

\[ \square \]

3.3.3 Remarks.

(1) With the reverse inequalities in Proposition 3.3.2(i) and (iii) holding by \( (\mathbb{1}, \mathcal{V}) \)-continuity of \( f \), these inequalities may also be replaced by equalities. The same is true for the inclusion in (v), and when \( \mathcal{V} \) is completely distributive, statements (ii) and (iv) may respectively be replaced by:

(iii') \( \forall A \subseteq TX, v \in \mathcal{V} \ (Tf(A)^v = \bigcap_{u \ll v} f(A)^u) \);
(iv') \( \forall A \subseteq X, v \in \mathcal{V} \) \( (f(A)^{(v)} = \bigcap_{u \ll v} f(A^{(u)}) \)).

Briefly, all five conditions considered in 3.3.2 are *equationally defined*.

(2) While conditions (i)–(iv) are equivalent in \( \mathcal{V}\text{-Cat} \) for \( \mathcal{V} \) completely distributive (and \( \mathcal{T} = \mathbb{1} \)), condition (v) is in general considerably weaker: in \( \text{Met} = \mathcal{P}_+\text{-Cat} \), every surjective map \( (X, a) \to (Y, b) \) with \( a, b \) finite satisfies condition (v)!

By contrast, *in \( \text{Top} = (\beta, 2)\text{-Cat} \) the grand closure describes the ordinary Kuratowski-closure, so that each of (iii) \( \iff \) (iv) \( \iff \) (v) describes closed continuous maps (continuous maps which preserve closed subsets), while each of (i) \( \iff \) (ii) describes stably-closed maps, as we will show in Proposition 3.4.5 and Theorem 3.4.6.

(3) For an embedding \( f : (X, a) \to (Y, b) \) in \( (\mathcal{T}, \mathcal{V})\text{-Cat} \) one has (i) \( \iff \) (iii) \( \iff \) (v) in 3.3.2; these equivalent conditions are satisfied precisely when the set \( X \) is closed in \( Y \) with respect to the grand closure, that is, when \( \overrightarrow{X} = X \): see Exercise 3.A.

While Proposition 3.3.2 compares propriety with the behavior of closures under taking images, we now similarly compare openness with the behavior under taking inverse images.

In order to do so, we need that \( T(f^{-1}B) = (Tf)^{-1}(TB) \) for all \( B \subseteq Y \) and \( f : X \to Y \), that is, that \( T \) is taut (see III.4.3.5).

**3.3.4 Proposition.** Let the functor \( T \) of the monad \( \mathcal{T} \) be taut. The following statements on a \( (\mathcal{T}, \mathcal{V})\text{-continuous map} f : (X, a) \to (Y, b) \) satisfy the implications

\[
\begin{align*}
(i) \implies (iii) \implies (v) \\
(ii) \implies (iv)
\end{align*}
\]

Furthermore, the vertical implications are equivalences if \( \mathcal{V} \) is completely distributive, and one has (ii) \( \iff \) (iv) if \( \mathcal{T} = \mathbb{1} \) is the identity monad, and (iii) \( \iff \) (v) if \( \mathcal{V} = 2 \).

(i) \( f \circ b \leq a \cdot (Tf)^o \), that is, \( f \) is open;

(ii) \( \forall B \subseteq TY, v \in \mathcal{V} \) \( (\bigcap_{u \ll v} f^{-1}(B^{(u)}) \subseteq (Tf)^{-1}(B^{(v)}) \));

(iii) \( \forall B \subseteq Y \) \( (f^o \cdot b \cdot Ti_B \cdot l_B^T \leq a \cdot (Tf)^o \cdot Ti_B \cdot l_B^T) \);

(iv) \( \forall B \subseteq Y, v \in \mathcal{V} \) \( (\bigcap_{u \ll v} f^{-1}(B^{(u)}) \subseteq f^{-1}(B^{(v)}) \));

(v) \( \forall B \subseteq Y \) \( (f^{-1}(B) \subseteq f^{-1}(B)) \).

**Proof.** One proceeds schematically as in the proof for Proposition 3.3.2 by transcribing the five statements in elementwise terms:

(i) \( \forall x \in X, y \in TY \) \( (b(y, f(x)) \leq \bigvee_{z \in (Tf)^{-1}y} a(z, x)) \);
(ii) \( \forall B \subseteq TY, v \in V, x \in X ((\forall u \ll v (u \leq \bigvee_{y \in B} b(y, f(x)))) \implies v \leq \bigvee_{z \in (TF)^{-1}(B)} a(z, x)) \);

(iii) \( \forall B \subseteq Y, x \in X ( \bigvee_{y \in TB} b(y, f(x)) \leq \bigvee_{z \in (TF)^{-1}(TB)} a(z, x)) \);

(iv) \( \forall B \subseteq Y, v \in V, x \in X ((\forall u \ll v (u \leq \bigvee_{y \in TB} b(y, f(x)))) \implies v \leq \bigvee_{z \in (TF)^{-1}(TB)} a(z, x)) \);

(v) \( \forall B \subseteq Y, x \in X (\exists y \in TB (b(y, f(x)) > \bot) \implies \exists z \in (TF)^{-1}(TB) (a(z, x) > \bot)) \).

3.3.5 Remarks.

(1) Because of the \((T, V)\)-continuity of \(f\), the inequalities in (i), (iii) and the inclusion in Proposition 3.3.4(v) may equivalently be replaced by equalities; likewise in (ii) and (iv) if \(V\) is completely distributive.

(2) As in Proposition 3.3.2, conditions (i)–(iv) of 3.3.4 are equivalent in \(V\)-\textit{Cat} for \(V\) completely distributive (and \(T = I\)), but condition (v) is generally weaker: again, any surjective map \((X, a) \to (Y, b)\) in \textit{Met} = \(P_\rightarrow\text{-Cat}\) with \(a, b\) finite satisfies condition (v).

However, in \(\text{Top} = (\beta, 2)\)-\textit{Cat} all five conditions are equivalent and describe open maps \textit{in the usual sense}, that is, continuous maps which preserve open subsets. Indeed, such a map \(f : X \to Y\) satisfies condition (i) which reads as:

\[
\forall x \in X, y \in \beta Y \ (y \to f(x) \implies \exists z \in \beta X \ (f[z] = y \& z \to x)), \quad y \to f(x).
\]

(One simply takes for \(z\) an ultrafilter on \(X\) containing the filter base \(\{f^{-1}B \mid B \in y\}\).)

Since (i) implies (v), it suffices to show that (v) makes \(f\) open in the usual sense. But for \(A \subseteq X\) open one obtains

\[
f^{-1}(Y \setminus f(A)) = f^{-1}(Y \setminus f(A)) \subseteq X \setminus A = X \setminus A
\]

and then \(Y \setminus f(A) \subseteq Y \setminus f(A)\), so that \(f(A)\) is open.

(3) As in 3.3.2 one has (i) \(\iff\) (iii) \(\iff\) (v) in 3.3.4 for an embedding \(f : (X, a) \hookrightarrow (Y, b)\), where now (v) reads as:

\[
\forall B \subseteq Y \ (B \cap X = B \cap X).
\]

Letting \(\text{Top} = (\beta, 2)\)-\textit{Cat} again guide our terminology in the general context, we consider properties 3.3.2(iii) and 3.3.4(iii) and define:

3.3.6 Definition. A \((T, V)\)-continuous map \(f : (X, a) \to (Y, b)\) is \textit{closed} if

\[
b \cdot Tf \cdot Ti_A \cdot 1_T^A \leq f \cdot a \cdot Ti_A \cdot 1_T^A
\]
for all \( A \subseteq X \), and \( f : (X, a) \to (Y, b) \) is *inversely closed* if

\[
f^\circ \cdot b \cdot T i_B \cdot i_T B \leq a \cdot (Tf)^\circ \cdot Ti_B \cdot i_T B
\]

for all \( B \subseteq Y \).

Emphasizing the dependency on the parameters, we often add the prefixes \((\mathbb{T}, \mathbb{V})\) or \(\mathbb{V}\) (when \(\mathbb{T} = \mathbb{I}\)). One trivially has:

\[
f \text{ proper} \implies f \text{ closed}, \quad f \text{ open} \implies f \text{ inversely closed},
\]

with the reversed implications holding for \(\mathbb{T} = \mathbb{I}\) (see Propositions 3.3.2 and 3.3.4). With Propositions 3.2.2 and 3.2.3 we obtain:

3.3.7 Corollary. Assume \( m^\circ : \hat{T}\hat{T} \to \hat{T} \) to be a natural transformation, and let \( f : (X, a) \to (Y, b) \) be \((\mathbb{T}, \mathbb{V})\)-continuous.

1. If \( \hat{T} \) is left-whiskering, then

\[
f \text{ is proper} \implies Tf : (TX, \hat{a}) \to (TY, \hat{b}) \text{ is closed},
\]

with the reverse implication holding when \( e : 1 \to T \) satisfies BC.

2. If \( \hat{T} \) is right-whiskering, then

\[
f \text{ is open} \iff Tf : (TX, \hat{a}) \to (TY, \hat{b}) \text{ is inversely closed}.
\]

It is essential to consider \( Tf \) on the right-hand sides, not just \( f \), as the following example shows.

3.3.8 Example. For a monoid \( H \) and the flat extension on the associated monad \( \mathbb{H} = (H \times (-), e, m) \), \( m^\circ \) is natural and \( e \) satisfies BC. However, it is easy to check that the identity map \( f : (\{0, 1\}, a) \to (\{0, 1\}, b) \), where \( a((\alpha, 0), i) = \mathbb{T} \) and \( a((\alpha, 1), 1) = \mathbb{T} \) only if \( \alpha = 1 \), and \( b((\alpha, i), j) = \mathbb{T} \) for every \( i, j \in \{0, 1\}, \alpha \in H \), is closed and inversely closed but neither proper nor open.

Nevertheless, in Proposition 3.4.5 below we will show that, in \( \text{Top} = (\mathbb{B}, 2)\)-Cat and \( \text{App} = (\mathbb{B}, \mathbb{P}_+)\)-Cat, the condition that \( Tf \) be (inversely) closed may be replaced by the condition that \( f \) be (inversely) closed.

3.4 The Kuratowski–Mrówka Theorem. We are now ready to characterize proper maps in terms of closure properties. The Kuratowski–Mrówka Theorem is the object version of that characterization and facilitates the proof in the general case. For its proof we rely in turn on Proposition 4.9.1 which asserts that, under the hypotheses that \( T \) preserve
disjointness, \( \hat{T} \) be flat and \( e^o \) finitely strict, for any set \( X \) and \( x \in TX \), one can define a \((\mathbb{T}, \mathcal{V})\)-space structure \( c = c_x \) on \( Z = X + 1 \) by

\[
c(z, z) = \begin{cases} 
    k & \text{if } z = e_Z(z), \\
    \top & \text{if } z = x \text{ and } z \in 1, \\
    \bot & \text{otherwise},
\end{cases}
\]

for all \( z \in Z, z \in T \hat{Z} \). Whenever all \((X + 1, c)\) are \((\mathbb{T}, \mathcal{V})\)-spaces, in particular under the hypotheses above, we say that \((\mathbb{T}, \mathcal{V})\)-\textit{Cat} has enough KM-test spaces.

3.4.1 Theorem. Let \( \mathcal{V} \) be cartesian closed and let \((\mathbb{T}, \mathcal{V})\)-\textit{Cat} have enough KM-test spaces. Then the following assertions for a \((\mathbb{T}, \mathcal{V})\)-space \((X, a)\) are equivalent:

(i) \((X, a) \to (1, \top)\) is proper;

(ii) the product projection \( X \times Y \to Y \) is closed for any \((\mathbb{T}, \mathcal{V})\)-space \((Y, b)\).

When, in addition, \( \mathcal{V} \) is integral and superior, we may extend the list by:

(iii) \((X, a)\) is compact.

Proof. (i) \(\implies\) (ii) follows from Proposition 3.1.4(3) since \( X \times Y \to Y \) is a pullback of \( X \to 1 \). (i) \(\iff\) (iii) re-states Remark 3.1.2(2). For (ii) \(\implies\) (i) we must show

\[
\bigvee_{x \in X} a(x, x) = \top
\]

for all \( \chi \in TX \). Hence we exploit the hypothesis for \((\mathcal{Y}, b) = (Z, c_\chi)\) and obtain, with \( \Delta = \{(x, x) \mid x \in X\} \subseteq X \times Z \) and \( q: (X \times Z, d) \to (Z, c) \) denoting the second product projection,

\[
c \cdot Tq \cdot Ti_{\Delta} \cdot !_{\Delta} \leq q \cdot d \cdot Ti_{\Delta} \cdot !_{\Delta}.
\]

With the natural bijection \( \delta: X \to \Delta \) every \( z \in T \Delta \) is of the form \( z = T \delta(y) \) with \( y \in TX \), and \( Tp(z) = y = Tq(z) \in TX \subseteq T \hat{Z} \) with \( p: X \times Z \to X \) the first projection. Hence, an exploitation of the elementwise form of (3.4.i) gives (with \(* \in 1 \subseteq Z\) )

\[
\top = c(\chi, \cdot) \leq \bigvee_{x \in X} \bigvee_{y \in TX} d(T \delta(y), (x, \cdot)) = \bigvee_{x \in X} \bigvee_{y \in TX} a(y, x) \land c(y, \cdot)
\]

for every \( y \neq \chi \) one has \( c(y, \cdot) = \bot \).

With Theorem 3.2.5 we conclude from Theorem 3.4.1 the desired characterization of proper maps, calling a map \textit{stably closed} if all of its pullbacks are closed.

3.4.2 Corollary. Let \( \mathcal{V} \) be cartesian closed, \( \hat{T} \) be taut with \( T\emptyset = \emptyset \), \( \hat{T} \) flat and left-whiskering, \( e^o \) finitely strict and \( m^o \) a natural transformation. Then the following assertions on a \((\mathbb{T}, \mathcal{V})\)-continuous map \( f: (X, a) \to (Y, b) \) are equivalent:

(i) \( f \) is proper;
(ii) \( f \) is stably closed, and \( T f : (TX, \hat{a}) \to (TY, \hat{b}) \) is closed;

(iii) \( f^{-1}y \to 1 \) is proper for all \( y \in Y \), and \( T f \) is closed.

When, in addition, \( \mathcal{V} \) is integral and superior, we may extend the list by:

(iv) \( f^{-1}y \) is compact for all \( y \in Y \), and \( T f \) is closed.

**Proof.** (i) \( \Rightarrow \) (ii): From Propositions 3.1.4(4) and 3.2.2(1).

(ii) \( \Rightarrow \) (iii): As a pullback of \( f \), each fibre of \( f \) is stably closed and therefore proper, by Theorem 3.4.1.

(iii) \( \Rightarrow \) (i): From Theorem 3.2.5.

(iii) \( \iff \) (iv): From Remark 3.1.2(2).

3.4.3 **Remark.** The condition that \( T f \) be closed can neither be removed from 3.4.2(ii) nor be substituted in (iii) by the requirement that \( f \) be closed. Indeed, going back to Example 3.3.8 one easily checks that \( f \) is stably closed but not proper, and Exercise 3.D gives an example of a closed \((\mathcal{T}, \mathcal{V})\)-continuous map with compact fibres which is not proper.

When \( \mathcal{T} = \mathcal{B} \) is the ultrafilter monad, with its Barr extension to \( \mathcal{V} \)-Rel (see Corollary IV.2.4.5), for a \((\mathcal{B}, \mathcal{V})\)-space \((X, \alpha)\) and \( x, y \in \beta X \) one has, by definition,

\[
\hat{a}(\chi, y) = \bigvee_{x \in m_X^{-1}_X} \beta a(x, y) = \bigvee_{x \in m_X^{-1}_X} \bigwedge_{A \in \chi, B \in y} \bigvee_{z \in A, y} a(z, y).
\]

3.4.4 **Lemma.** If \( \mathcal{V} \) is a chain (and, thus completely distributive), then

\[
\hat{a}(\chi, y) = \bigvee \{ u \in V \mid \forall A \in \chi \ (A^{(u)} \in y) \}.
\]

**Proof.** For “\( \leq \)”, consider any \( X \in \beta \beta X \) with \( m_X(X) = \chi \). It suffices to show that every \( u \ll \bigwedge_{A \in \chi, B \in y} \bigvee_{z \in A, y} a(z, y) \) has the property that \( A^{(u)} \in y \) for all \( A \in \chi \). But if for \( A \in \chi \) we assume \( A^{(u)} \not\in y \), so that \( B := X \setminus A^{(u)} \in y \), considering

\[
A := A^\emptyset = \{ z \in \beta X \mid A \in z \} \in X \quad (\text{since } A \in \chi)
\]

we would conclude

\[
u \ll \bigvee_{z \in A, y} a(z, y)
\]

and therefore \( A^{(u)} \cap B \neq \emptyset \), a contradiction.

For “\( \geq \)”, consider \( v \ll \bigvee \{ u \in V \mid \forall A \in \chi \ (A^{(u)} \in y) \} \) in \( \mathcal{V} \). For all \( A \in \chi, B \in y \), the ultrafilter \( y \) contains \( A^{(v)} \cap B \neq \emptyset \), so that \( v \leq \bigvee_{A \in \mathcal{A}} a(z, y) \) for some \( y \in B \), and

\[
v \leq \bigwedge_{B \in y} \bigvee_{z \in A, y} a(z, y)
\]
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follows for every $A \in \chi$. Now,

$$\mathcal{F} = \{ A \subseteq \beta X \mid A^\beta \subseteq A \text{ for some } A \in \chi \}$$

is a filter on $\beta X$, and

$$\mathcal{J} := \{ B \subseteq \beta X \mid v > \bigwedge_{B \in y} \bigvee_{z \in B, y \in B} a(z, y) \}$$

is an ideal on $\beta X$ that is disjoint from $\mathcal{F}$. In order to establish closure of $\mathcal{J}$ under binary union, we use the fact that the order of $\mathcal{V}$ is total, as follows: if $B$ and $C$ belong to $\mathcal{J}$, then $v > \bigvee_{z \in B, y \in B} a(z, y)$ and $v > \bigvee_{z \in C, y \in C} a(z, y)$ for some $B, C \in y$, hence

$$v > \left( \bigvee_{z \in B, y \in B} a(z, y) \right) \lor \left( \bigvee_{z \in C, y \in C} a(z, y) \right) \geq \bigvee_{z \in B \cup C, y \in B \cap C} a(z, y),$$

and then

$$v > \bigwedge_{D \in y} \bigvee_{z \in D, y \in D} a(z, y)$$

since $B \cap C$ belongs to $y$. The filter $\mathcal{F}$ must be contained in an ultrafilter $X$ which does not meet the ideal $\mathcal{J}$; by definition of $\mathcal{F}$ one has $\chi = m_X(X)$, and by definition of $\mathcal{J}$

$$v \leq \bigwedge_{A \in \chi, B \in y} \bigvee_{z \in A, y \in B} a(z, y) \leq \hat{a}(\chi, y)$$

follows. \hfill \Box

Since the structure $a$ can be recovered from $\hat{a}$ as

$$a(\chi, x) = \hat{a}(\chi, e_X(x)),$$

the equality (3.4.ii) shows that $v$-closures on subsets of $X$ encode completely the structure $a$.

3.4.5 Proposition. Let $\mathcal{V}$ be a chain and let $f : (X, a) \to (Y, b)$ be a $($$\beta, \mathcal{V})$-continuous map. Then the following hold.

1. $f$ is closed $\iff$ $\beta f$ is closed $\iff$ $\beta f$ is proper.

2. $f$ is inversely closed $\iff$ $\beta f$ is inversely closed $\iff$ $f$ is open.

Proof. To see (1), we observe that with [3.3.7](1) it suffices to show that $f$ is closed if and only if $\beta f$ is proper. First assuming $f$ to be closed, we must verify

$$\hat{b}(f[\chi], y) \leq \bigvee \{ \hat{a}(\chi, z) \mid z \in \beta X, f[z] = y \}$$

for all $\chi \in \beta X$, $y \in \beta Y$, where

$$\hat{b}(f[\chi], y) = \bigvee \{ v \in \mathcal{V} \mid \forall B \in f[\chi] \ (B^{(v)} \in y) \}.$$
Since $f$ is closed, for any $v \in \mathcal{V}$ contributing to the join on the right, one has $f(A^{(u)}) \in y$ for all $A \in \chi$ and every $u \ll v$. Consequently, with an ultrafilter $\mathcal{z}_u$ on $X$ containing the filter base 

$$\{A^{(u)} \cap f^{-1}(B) \mid A \in \chi, B \in y \} ,$$

one has $f[\mathcal{z}_u] = y$ and $A^{(u)} \in \mathcal{z}_u$ for all $A \in \chi$, that is, $u \leq \hat{a}(\chi, \mathcal{z}_u)$. Therefore $v \leq \bigvee_{u \ll v} u \leq (\beta f \cdot \hat{a})(\chi, y)$, as desired.

Conversely, assuming $\beta f$ to be proper we consider $y \in f(A^{(u)})$ for $A \subseteq X, v \in \mathcal{V}$, so that $v \leq \bigvee_{y \in f(A^{(u)})} b(y, y) = \bigvee_{\chi \in A} b(f[\chi], y)$.

Since $b(f[\chi], y) = \tilde{b}(f[\chi], e_Y(y)) = \bigvee \{\hat{a}(\chi, \mathcal{z}) \mid \mathcal{z} \in \beta_X, f[\mathcal{z}] = e_Y(y)\}$ with $\hat{a}(\chi, \mathcal{z})$ as in (3.4.ii), for all $u \ll v$ one obtains $\chi_\mathcal{u} \in \beta_X$ with $f[\chi_\mathcal{u}] = e_Y(y)$ and $A^{(u)} \in \chi_\mathcal{u}$, hence $y \in f(A^{(u)})$.

The proof of (2) is similar.

Finally we get a characterization theorem for $(\beta, \mathcal{V})$-spaces that has as particular instances the well-known characterizations of proper maps in $\text{Top} \cong (\beta, 2)$-$\text{Cat}$ and $\text{App} \cong (\beta, \mathcal{P}^+)$-$\text{Cat}$.

3.4.6 Theorem. Let $\mathcal{V}$ be totally ordered, integral and superior. The following assertions are equivalent, for $f : (X, a) \to (Y, b)$ $(\beta, \mathcal{V})$-continuous:

(i) $f$ is proper;

(ii) $f$ is stably closed;

(iii) $f$ is closed and, for every $y \in Y$, the $(\beta, \mathcal{V})$-functor $f^{-1}(y) \to 1$ is proper;

(iv) $f$ is closed with compact fibres.

3.4.7 Examples.

(1) Proper maps in $\text{Top}$, introduced as the stably-closed maps by Bourbaki [1989] and characterized as the closed maps with compact fibres, have a description dual to open maps (see Remark 3.3.5(2)) in $(\beta, 2)$-$\text{Cat}$:

$$\forall \chi \in \beta_X, y \in Y (f[\chi] \to y \implies \exists \in X (f(z) = y \& \chi \to z)) , \quad \chi \leftarrow z \quad f[\chi] \to y .$$

(2) When $\text{Top}$ is presented as $(\mathcal{F}_p, 2)$-$\text{Cat}$ as in 1.1.4(4), then $(\mathcal{F}_p, 2)$-proper maps are called superproper and characterized by the same property as in (1), except that $\chi$ is now allowed to be any proper filter. As compact spaces need not be supercompact, superproper is considerably stronger than proper.
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(3) Closed maps $f : (X, a) \to (Y, b)$ in ($\beta, P_+$)-Cat $\cong$ App as introduced in 3.3.6 are characterized in terms of approach distances by

$$\forall A \subseteq X, y \in Y \left( \inf_{x \in f^{-1}y} \delta^X(x, A) \leq \delta^Y(y, f(A)) \right)$$

(see Proposition 3.3.2), and this is the description used in approach space theory. Hence, proper maps are also here characterized as the “classically” stably-closed maps, or the closed maps with 0-compact fibres.

(4) By Proposition 3.4.5, open maps $f : (X, a) \to (Y, b)$ in ($\beta, P_+$)-Cat $\cong$ App are equivalently described as the inversely closed maps which, in terms of approach distances, leads to the classical description of open maps in App:

$$\forall B \subseteq Y, x \in X \left( \delta^X(x, f^{-1}(B)) \leq \delta^Y(f(x), B) \right).$$

3.5 Products of proper maps. Our goal is to prove that the product $\prod_{i \in I} g_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ of proper maps $g_i : (X_i, a_i) \to (Y_i, b_i)$ is proper, subject to a condition on $V$ that entails its cartesian closedness and therefore makes proper maps stable under pullback in ($\mathbb{T}, V$)-Cat (see Proposition 3.1.4). (Throughout this subsection, $I$ is assumed to be small, a set.)

Hence, we first point out that for any class $\mathcal{P}$ of morphisms in a small-complete category $X$ that contains all isomorphisms and is stable under pullback, one has the following result.

3.5.1 Proposition. $\mathcal{P}$ is closed under small products if and only if $\mathcal{P}$ is stable under small multiple pullback in $X$, that is: for any small multiple pullback diagram

$$\begin{array}{ccc}
P & \xrightarrow{f} & Y \\
\downarrow{p_i} & \quad & \quad \\
X_i & \xrightarrow{f_i} & Y \\
\end{array} \quad (i \in I)$$

in $X$, if $f_i \in \mathcal{P}$ for all $i \in I$, then $f \in \mathcal{P}$.

Proof. The multiple pullback $f$ in (3.5.i) is a pullback of $\prod_{i} f_i$ along $\delta_Y : Y \to Y_I$, as in

$$\begin{array}{ccc}
P & \xrightarrow{f} & Y \\
\downarrow{p_i} & \quad & \quad \\
\prod_{i} X_i & \xrightarrow{\prod_{i} f_i} & Y_I \\
\downarrow{\delta_Y} & \quad & \quad \\
X_i & \xrightarrow{f_i} & Y. \\
\end{array}$$
Conversely, the product $\prod_i g_i$ of morphisms $g_i : X_i \to Y_i$ is a multiple pullback of the family of pullbacks $h_i : P_i \to \prod_i Y_i = Y$ of $g_i$ along the projection $Y \to Y_i$, as in

\[
\begin{array}{c}
\prod_i X_i \xrightarrow{\prod_i g_i} \prod_i Y_i \\
\downarrow \pi_i \downarrow \pi_i \\
X_i \xrightarrow{g_i} Y_i.
\end{array}
\]

Hence, to verify closure of $\mathcal{P}$ under products, it suffices to prove stability under multiple pullbacks. Let us call $\mathcal{V}$-$\text{Rel}$ widely modular if for every non-empty family $r_i : Z \to X_i$ ($i \in I$) of $\mathcal{V}$-relations and every multiple pullback diagram (3.5.i) in $\text{Set}$, one has

\[
f \cdot \bigwedge_{i \in I} p_i^\circ \cdot r_i = \bigwedge_{i \in I} f_i \cdot r_i \tag{3.5.ii}
\]

in $\mathcal{V}$-$\text{Rel}$.

3.5.2 Proposition. If $\mathcal{V}$-$\text{Rel}$ is widely modular, the class of proper maps is stable under multiple pullback in $(\mathbb{T}, \mathcal{V})$-Cat.

Proof. The multiple pullback of $f_i : (X_i, a_i) \to (Y, b)$ in $(\mathbb{T}, \mathcal{V})$-Cat is formed by providing the limit $P$ of the multiple-pullback diagram (3.5.i) in $\text{Set}$ with the structure $d = \bigwedge_{i \in I} p_i^\circ \cdot a_i \cdot Tp_i$. For $I = \emptyset$, the map $f = 1_Y$ is trivially proper. Otherwise, we may invoke the wide modularity of $\mathcal{V}$ and obtain

\[
f \cdot d = f \cdot \bigwedge_{i \in I} a_i \cdot Tp_i = \bigwedge_{i \in I} f_i \cdot a_i \cdot Tp_i = \bigwedge_{i \in I} b \cdot T f_i \cdot Tp_i = b \cdot T f
\]

when all $f_i$ are proper. Hence, $f$ is proper. \qed

We must now analyze the status of the hypothesis on $\mathcal{V}$-$\text{Rel}$ in Proposition 3.5.2. We first note that this hypothesis constitutes no restriction if all $f_i$ are injective.

3.5.3 Remark. If all maps $f_i : X_i \to Y$ are injective, then (3.5.ii) holds for all $\mathcal{V}$-relations $r_i : Z \to X_i$ ($i \in I \neq \emptyset$). Indeed, in that case we may assume that all maps of (3.5.i) are inclusion maps, with $P = \bigcap_{i \in I} X_i \subseteq Y$, and for all $z \in Z$, $y \in Y$ one has

\[
(f \cdot \bigwedge_{i} p_i^\circ \cdot r_i)(z, y) = \begin{cases} 
\bigwedge_i r_i(z, y) & \text{if } y \in P = \bigcap_i X_i, \\
\bot & \text{otherwise} 
\end{cases} 
= (\bigwedge_i f_i \cdot r_i)(z, y).
\]

Here is what unrestricted wide modularity of $\mathcal{V}$-$\text{Rel}$ means for $\mathcal{V}$.

© 3.5.4 Proposition. $\mathcal{V}$-$\text{Rel}$ is widely modular if and only if $\mathcal{V}$ is completely distributive.
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Proof. Given $f_i : X_i \to Y$ and $r_i : Z \to X_i$ ($i \in I \neq \emptyset$) as in (3.5.ii), for all $z \in Z$, $y \in Y$ one has

$$(f \cdot \bigwedge_{i \in I} p_i \cdot r_i)(z, y) = \bigvee_{w \in f^{-1}y} \bigwedge_{i \in I} r_i(z, p_i(w)) = \bigvee_{(w_i) \in \prod_{i \in I} f_i^{-1}y} \bigwedge_{i \in I} r_i(z, w_i)$$

and

$$(\bigwedge_{i \in I} f_i \cdot r_i)(z, y) = \bigvee_{w_i \in f_i^{-1}y} r_i(z, w_i).$$

Complete distributivity of $\mathcal{V}$ makes the right-hand sides equal (see II.1.11). Conversely, let us assume $\mathcal{V}$-Rel to be widely modular and consider subsets $A_i \subseteq V$, $i \in I$.

For $I = \emptyset$ one trivially has

$$\bigvee_{(a_i) \in \prod_i A_i} \bigwedge_{i \in I} a_i = \bigwedge_{i \in I} \bigvee A_i,$$

and for $I \neq \emptyset$ one obtains this identity by considering the unique maps $f_i : A_i \to 1$ and the $\mathcal{V}$-relations $r_i : 1 \to A_i$ with $r_i(\ast, a_i) = a_i$ for $a_i \in A_i$, $\ast \in 1$.

3.5.5 Remarks.

1. If $\mathcal{V}$ satisfies (3.5.iii) (for all $A_i \subseteq V$, $i \in I$), so does its complete sublattice $2 = \{\bot, \top\}$. But validity of (3.5.iii) for $2$ is, in Zermelo–Fraenkel set theory, equivalent to the Axiom of Choice [Herrlich, 2006]. Hence, in ZF one can state Proposition 3.5.4 more precisely as:

$$\mathcal{V}$-Rel is widely modular $\iff$ AC holds & $\mathcal{V}$ is completely distributive.$$

2. An injective proper $(\mathbb{T}, \mathcal{V})$-continuous map $f : (X, a) \to (Y, b)$ is necessarily $O$-initial (for $O : (\mathbb{T}, \mathcal{V})$-Cat $\to$ Set, see Exercises 3.A and 3.E), that is, an embedding. Proper embeddings are more commonly characterized as the closed embeddings (see Remark 3.3.5(3)), and a multiple pullback of embeddings is better known as an intersection (see Theorem II.5.3.2).

3.5.6 Theorem.

1. Closed embeddings in $(\mathbb{T}, \mathcal{V})$-Cat are stable under intersections.

2. If $\mathcal{V}$ is cartesian closed, then every product of closed embeddings is a closed embedding in $(\mathbb{T}, \mathcal{V})$-Cat.

3. If $\mathcal{V}$ is completely distributive, then every product of proper $(\mathbb{T}, \mathcal{V})$-continuous maps is proper.

Proof. Combine 3.3.3(3) with 3.5.1–3.5.5, observing also that complete distributivity implies that $\mathcal{V}$, as a lattice, is a frame that is cartesian closed.

3.5.7 Corollary. Products of proper maps in Top and in App are proper.

Proof. Apply Theorem 3.5.6(2), noting that every chain is completely distributive; in particular, $2$ and $P_+$ are so.
3.6 Coproducts of open maps. Throughout this subsection we assume that

- \( m^\circ : \hat{T} \to \hat{T} \hat{T} \) is a natural transformation;
- \( \hat{T} \) is right-whiskering.

By Remarks III.4.3.4, these hypotheses guarantee that for a coproduct

\[ t_i : (X_i, a_i) \to \coprod_{i \in I} (X_i, a_i) = (X, a) \quad (i \in I) \]

in \((\mathcal{T}, \mathcal{V})\text{-Cat}\) one has

\[ a = \bigvee_{i \in I} t_i \cdot a_i \cdot (Tt_i)^\circ , \]

with each \( t_i \) open. These facts give us stability of openness under coproducts, as follows.

3.6.1 Proposition.

(1) If all \( f_i : (X_i, a_i) \to (Y, b) \) in \((\mathcal{T}, \mathcal{V})\text{-Cat}\) are open, then the induced \( f : \coprod_{i \in I} (X_i, a_i) \to (Y, b) \) is open.

(2) If all \( f_i : (X_i, a_i) \to (Y_i, b_i) \) in \((\mathcal{T}, \mathcal{V})\text{-Cat}\) are open, then

\( \coprod_{i \in I} f_i : \coprod_{i \in I} (X_i, a_i) \to \coprod_{i \in I} (Y_i, b_i) \)

is open.

(3) Open embeddings are closed under set-theoretic union.

Proof.

(1) Since \( V\text{-Rel} \) is a quantaloid one has

\[
  f^\circ \cdot b = (\bigvee_{i \in I} t_i \cdot t_i^\circ) \cdot f^\circ \cdot b = \bigvee_{i \in I} t_i \cdot t_i^\circ \cdot f^\circ \cdot b = \bigvee_{i \in I} t_i \cdot f_i^\circ \cdot b = \bigvee_{i \in I} t_i \cdot a_i \cdot (Tf_i)^\circ \quad (f_i \text{ open})
\]

(2) One applies (1) to the open maps

\[
  (X_i, a_i) \xrightarrow{f_i} (Y_i, b_i) \xrightarrow{\coprod_{i \in I}} \coprod_{i \in I} (Y_i, b_i) .
\]

(3) If in (1) all \( f_i \) are inclusion maps, then the image of the open map \( f \) is precisely the union \( \bigcup_{i \in I} X_i \) which (when provided with the \( O \)-initial structure of \( (Y, b) \)) is open in \( Y \). \( \square \)
The inclusion map $A \hookrightarrow X$ of a subset of a $(\mathbb{T}, \mathcal{V})$-space $(X, a)$ provided with the $O$-initial structure is open if and only if
\[ \forall x \in X, \chi \in TX \quad (a(\chi, x) > \perp \& x \in A \implies \chi \in TA), \quad (3.6.i) \]
where we assume $T(A \hookrightarrow X) = (TA \hookrightarrow TX)$ (see Exercise [III.1.P]). The open subsets of $X$ defined in this way form a topology on $X$ if $T$ is taut. Denoting by $\Omega(X, a)$ the resulting topological space, we obtain:

3.6.2 Corollary. If $T$ is taut, then there is a functor
\[ \Omega : (\mathbb{T}, \mathcal{V})\text{-Cat} \rightarrow \text{Top} \]
that preserves the underlying sets and sends $(\mathbb{T}, \mathcal{V})$-open maps to open maps. $\Omega$ also preserves and reflects coproducts.

Proof. Since $T$ is taut, open embeddings in $(\mathbb{T}, \mathcal{V})\text{-Cat}$ are stable under pullbacks by Proposition [III.4.3.8] so they are stable under finite intersection. Since open embeddings are also closed under unions (see Proposition 3.6.1), the set $\Omega(X, a)$ is in fact a topological space with its open sets $A$ defined by (3.6.i). Stability of open embeddings under pullback also shows that a $(\mathbb{T}, \mathcal{V})$-continuous map $f : (X, a) \rightarrow (Y, b)$ gives a continuous map $\Omega f = f : \Omega(X, a) \rightarrow \Omega(Y, b)$. As for the preservation of open subsets by $\Omega f$ when $f$ is open in $(\mathbb{T}, \mathcal{V})\text{-Cat}$, we note that for $A \subseteq X$ open the composite morphism
\[ (A \longrightarrow f(A) \hookrightarrow (Y, b)) = (A \hookrightarrow (X, a) \xrightarrow{f} (Y, b)) \]
is open, so that $f(A)$ is open in $(Y, b)$, by Proposition 3.1.4(2). Since $\Omega$ preserves open maps, it preserves in particular open embeddings (see Exercise 3.E(4)) and therefore the open injections of a coproduct in $(\mathbb{T}, \mathcal{V})\text{-Cat}$. Preservation of coproducts follows, and reflection thereof likewise. \qed

3.6.3 Examples.

(1) For $(\mathbb{T}, \mathcal{V}) = (\mathbb{I}, 2)$, the functor $\Omega$ describes the full coreflective embedding $\text{Ord} \hookrightarrow \text{Top}$ which provides an ordered set with its Alexandroff topology, that is, its open sets are the down-closed sets (see Examples [III.3.5.2]). For $(\mathbb{T}, \mathcal{V}) = (\beta, 2)$, since (3.6.1) describes open sets in terms of ultrafilter convergence, $\Omega$ is the isomorphism $(\mathbb{T}, \mathcal{V})\text{-Cat} \cong \text{Top}$ of Theorem [III.2.2.5] expressed in terms of open sets.

(2) For $(\mathbb{T}, \mathcal{V}) = (\mathbb{I}, \mathbb{P}_+)$, $\Omega$ provides a metric space $(X, a)$ with a rather crude topology: $A \subseteq X$ is open if and only if
\[ \forall x, y \in X \quad (a(x, y) < \infty \& y \in A \implies x \in A) . \]
In particular, this choice of $(\mathbb{T}, \mathcal{V})$ shows that openness of $\Omega f$ does not imply openness of $f$ in general. In fact, if the metric $a$ is finite, then $\Omega X$ is indiscrete. Hence,
openness of $\Omega f$ for $f : (X,a) \rightarrow (Y,b)$ in $\text{Met}$ with $a,b$ finite and $X \neq \emptyset$ just means that $f$ is surjective which, in general, does not guarantee openness in $\text{Met}$ (see Proposition 2.3.3(3)).

(3) A set $C \subseteq X$ is open in $(X,a) \in \text{App} \cong (\mathcal{F}, P_+)-\text{Cat}$ precisely when

$$\forall B \subseteq X, x \in C \ (\delta(x, C \cap B) \leq \delta(x, B)),$$

where $\delta$ is the associated approach distance of $(X,a)$ (see 3.4.7(4)). Choosing $B = \{y\}$ for $y \in X$, we see that openness of $C$ in $X$ in $\text{App}$ implies openness of $C$ in $X$ in $\text{Met}$, where $X$ carries the metric $a \cdot e_X$ (see III.3.5.2(2)). Consequently, for an approach space $X$ sent by the algebraic functor $A_e : \text{App} \rightarrow \text{Met}$ (and coreflector of $\text{Met} \rightarrow \text{App}$) to a space with a finite metric, we obtain from (2) that the topological space $\Omega X$ is indiscrete.

While Examples (2) and (3) indicate the limitations of the functor $\Omega$, we will use $\Omega$ as an essential tool in 5.3 below, especially when $\mathcal{V} = 2$.

3.6.4 Remark. Taking as its closed sets those subsets $A \subseteq X$ for which $A \rightarrow (X,a)$ is proper, one obtains a functor

$$\Gamma : (\mathcal{T}, \mathcal{V})-\text{Cat} \rightarrow \text{Top}$$

that sends $(\mathcal{T}, \mathcal{V})$-proper maps to closed maps in $\text{Top}$, provided that $T : \text{Set} \rightarrow \text{Set}$ preserves finite coproducts: see Exercise 3.F.

3.7 Preservation of space properties. For a $(\mathcal{T}, \mathcal{V})$-continuous map $f : (X,a) \rightarrow (Y,b)$, we briefly discuss cases when the codomain inherits a special property from the domain, and vice versa. We remind the reader that injectivity of $f$ is always inherited by $Tf$, and the same is true for surjectivity if the Axiom of Choice is assumed.

3.7.1 Proposition.

(1) If $f$ is injective and proper and $(Y,b)$ is compact, then $(X,a)$ is compact.

(2) If $Tf$ is surjective and $(X,a)$ is compact, then $(Y,b)$ is compact.

Proof. (1) follows from Proposition 1.1.5. For (2), note simply

$$1_{TY} = Tf \cdot (Tf)^\circ \leq Tf \cdot a^\circ \cdot a \cdot (Tf)^\circ \leq b^\circ \cdot f \cdot f^\circ \cdot b \leq b^\circ \cdot b.$$ 

3.7.2 Proposition.

(1) If $f$ is injective and $(Y,b)$ Hausdorff, then $(X,a)$ is Hausdorff.

(2) If $Tf$ is surjective, $f$ proper and $(X,a)$ Hausdorff, then $(Y,b)$ is Hausdorff.
Proof. (1) follows from Proposition 3.1.5. For (2), we have

\[ 1_Y \geq f \cdot f^o \geq f \cdot a \cdot a^o \cdot f^o = b \cdot Tf \cdot (Tf)^o \cdot b^o = b \cdot b^o. \]

For preservation of normality and extremal disconnectedness, we first consider the case \( T = I \).

3.7.3 Proposition. Let \((X, a), (Y, b)\) be \( \mathcal{V} \)-spaces and \( f : (X, a) \to (Y, b) \) a proper \( \mathcal{V} \)-continuous map.

(1) If \( f \) is injective and \((Y, b)\) normal, then \((X, a)\) is normal.

(2) If \( f \) is surjective and \((X, a)\) normal, then \((Y, b)\) is normal.

Proof. (1) follows from

\[ a \cdot a^o \leq f^o \cdot b \cdot f \cdot f^o \cdot b^o \cdot f \leq f^o \cdot b^o \cdot b \cdot f = a^o \cdot f^o \cdot f \cdot a = a^o \cdot a. \]

(2) is proved analogously, using the equality \( f \cdot f^o = 1_Y \).

3.7.4 Corollary. Let \((X, a), (Y, b)\) be \( \mathcal{V} \)-spaces and \( f : (X, a) \to (Y, b) \) an open \( \mathcal{V} \)-continuous map.

(1) If \( f \) is injective and \((Y, b)\) extremally disconnected, then \((X, a)\) is extremally disconnected.

(2) If \( f \) is surjective and \((X, a)\) extremally disconnected, then \((Y, b)\) is extremally disconnected.

Proof. Apply Proposition 3.7.3 to the \( \mathcal{V} \)-functor \( f : (X, a^o) \to (Y, b^o) \).

3.7.5 Theorem. Let \( m^o : \hat{T} \to \hat{T^T} \) be a natural transformation, \( \hat{T} \) right-whiskering, and \( f : (X, a) \to (Y, b) \) a \((\mathcal{T}, \mathcal{V})\)-continuous map.

(1) (a) If \( f \) is proper, \( Tf \) is injective and \((Y, b)\) is normal, then \((X, a)\) is normal.

(b) If \( f \) is proper, \( Tf \) is surjective and \((X, a)\) is normal, then \((Y, b)\) is normal.

(2) (a) If \( f \) is open, \( Tf \) is injective and \((Y, b)\) is extremally disconnected, then \((X, a)\) is extremally disconnected.

(b) If \( f \) is open, \( Tf \) is surjective and \((X, a)\) is extremally disconnected, then \((Y, b)\) is extremally disconnected.
Proof. We observe that the claims of (1) follow from Proposition 3.7.3 applied to the \( \mathcal{V} \)-continuous map \( Tf : (TX, \hat{a}) \to (TY, \hat{b}) \), using commutativity of the diagram below:

\[
\begin{array}{c}
TX \xrightarrow{Tf} TY \\
\downarrow m_X \quad \downarrow m_Y \\
TTX \xrightarrow{TTf} TTY \\
\downarrow \hat{a} \quad \downarrow \hat{b} \\
TX \xrightarrow{Tf} TY
\end{array}
\]

(2) is proved analogously. \( \square \)

Exercises

3.A Closure operators.

(1) Show that, for \( v \leq k \) in \( \mathcal{V} \), the \( v \)-closure and the grand closure provide hereditary but generally non-idempotent \( M \)-closure operators on \((\mathbb{T}, \mathcal{V})\)-Cat, with \( M \) the class of embeddings. (For non-idempotency of the grand closure, see Exercise 3.B.)

(2) Show that proper and open embeddings can be characterized using the grand closure as stated in Remarks 3.3.3(3) and 3.3.5(3).


(1) For a monad \( \mathbb{T} \) on \( \text{Set} \) with an associative lax extension to \( \mathcal{V} \)-Rel, a \((\mathbb{T}, \mathcal{V})\)-space \((X, a)\), and \( A \subseteq X \) with

\[
T\overline{A} \subseteq \overline{T}A = \{ y \in TX \mid \exists x \in TA \ (\hat{a}(x, y) > \perp) \},
\]

show \( \overline{\overline{A}} = \overline{A} \). Conclude that the grand closure is idempotent whenever \( T\overline{A} \subseteq \overline{T}A \) for all \( A \subseteq X \).

(2) In \( \text{App} \cong (\beta, P_+)-\text{Cat} \), consider \( ([0, \infty], a) \) with structure

\[
a(u, v) = v \odot \xi(u),
\]

where \( \xi(u) = \sup_{C \in u} \inf_{u \in C} u \) for all \( u \in \beta[0, \infty], v \in [0, \infty] \) (see Exercise III.5.J). Show for \( A = \{0\} \):

(a) \( \overline{A} = [0, \infty) \),
(b) \( \beta\overline{A} \not\subseteq \beta\overline{A} \),
(c) \( \overline{\overline{A}} = [0, \infty] \).
3. PROPER AND OPEN MAPS

Hint. Consider an ultrafilter \( y \) on \([0, \infty)\) containing \([0, \infty)\) and all \([v, \infty), v < \infty\). Then \( y \in \beta A \), but \( y \not\in \beta A \).

3.C Openness of product projections. Suppose that the functor \( T \) satisfies BC.

(1) Prove that, if \( V \) is integral and \( T1 \cong 1 \), then product projections are open in \((\mathcal{T}, V)\)-Cat.

(2) Show that both conditions are essential in the previous statement.

3.D Failure of the classical Kuratowski–Mrówka Theorem. Let \( H = \mathbb{N} \setminus \{0\} \) be the multiplicative monoid of positive integers, and \( \mathcal{H} \) the flat lax extension to \( \mathcal{R}el \) of the associated monad \( \mathcal{H} \) (see 1.4). Consider the \((\mathcal{H}, 2)\)-spaces \((X, a), (2, \top)\), with \( X = \mathbb{N} \cup \{\infty\}, 2 = \{0, 1\}\), and the only \( a \)-relations that hold are:

\[
(\alpha, \infty) \ a \infty, \quad (\alpha, 0) \ a \infty, \quad (\alpha, 0) \ a 0, \quad (\alpha, 0) \ a \alpha, \quad (\alpha, n) \ a (\alpha \cdot n)
\]

for \( \alpha, n \in H \subseteq X \). Show that the \((\mathcal{H}, 2)\)-continuous map \( f : (X, a) \to (2, \top) \), defined by \( f(0) = f(\infty) = 0 \) and \( f(n) = 1 \) for \( n \geq 1 \), is closed, has compact fibres, but is not proper.

3.E Injective and surjective proper and open maps. For \( f : (X, a) \to (Y, b) \) in \((\mathcal{T}, V)\)-Cat and \( O \) the underlying-set functor:

(1) \( f \) proper, \( f \) injective \( \implies \) \( f \) is \( O \)-initial,

(2) \( f \) proper, \( Tf \) surjective \( \implies \) \( f \) is \( O \)-final,

(3) \( f \) open, \( f \) surjective \( \implies \) \( f \) is \( O \)-final,

(4) \( f \) open, \( Tf \) injective \( \implies \) \( f \) is \( O \)-initial.

Furthermore, the \( O \)-final structure in (2) and (3) is described by \( b = f \cdot a \cdot (Tf)^\circ \).

3.F The functor \( \Gamma \). For a \((\mathcal{T}, V)\)-space \((X, a)\), declare \( A \subseteq X \) to be closed if \( A \hookrightarrow (X, a) \) is proper (where \( A \) carries the \( O \)-initial \((\mathcal{T}, V)\)-structure). If \( T : \mathbf{Set} \to \mathbf{Set} \) preserves finite products, this defines the object part of a functor

\[
\Gamma : (\mathcal{T}, V)\text{-Cat} \to \mathbf{Top}
\]

that sends \((\mathcal{T}, V)\)-proper maps to closed continuous maps. Describe this functor for \( T = \mathbb{I} \) or \( T = \beta \) and \( V = 2 \) or \( V = \mathcal{P}_+ \).

3.G Closed subspaces of normal spaces. For \( \mathcal{T} \) with an associative and left-whiskering lax extension to \( \mathcal{V}\mathcal{R}el \), prove that closed subspaces of normal \((\mathcal{T}, V)\)-spaces are normal.

3.H Quasi-proper maps. A \((\mathcal{T}, V)\)-continuous map \( f : (X, a) \to (Y, b) \) is quasi-proper if \( b \cdot Tf \leq b \cdot e_V \cdot f \cdot a \).

(1) Every quasi-proper map \( f : (X, a) \to (Y, b) \) satisfies \( b \cdot Tf = b \cdot e_V \cdot f \cdot a \).
(2) Every proper map is quasi-proper.

(3) If $\hat{T}$ is associative, then every left adjoint map in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$ (see Exercise III.3.F) is quasi-proper.

(4) In $\text{Ord} \cong (\mathbb{I}, 2)$-$\text{Cat}$, the embedding $\{0, 1\} \hookrightarrow \{0, 1, 2\}$ is left adjoint but not proper, and $\{1, 2\} \hookrightarrow \{0, 1, 2\}$ is proper but not left adjoint.

(5) A monotone map $f : X \to Y$ of ordered sets is quasi-proper as a morphism in $(\mathbb{P}, 2)$-$\text{Cat}$ (see Example III.1.6.4(1)) if and only if $f$ is left adjoint as a morphism in $(\mathbb{I}, 2)$-$\text{Cat}$.

3.I Closure under composition. Show that the classes of closed maps, of inversely closed maps and of quasi-proper maps are all closed under composition.

3.J Openness and near openness with respect to the filter monad. Show that in $(\mathbb{F}, 2)$-$\text{Cat}$ and $(\mathbb{F}_p, 2)$-$\text{Cat}$ the notions of open map coincide with the usual topological notion. Nearly open maps in $(\mathbb{F}_p, 2)$-$\text{Cat}$ (see Definition III.4.3.1) are also open while in $(\mathbb{F}, 2)$-$\text{Cat}$ every map is nearly open.


(1) Prove that the lax extension $\hat{\mathbb{P}}$ of the power-set monad to $\text{Rel}$ (see Examples III.1.4.2) is left-, but not right-whiskering, and that the lax extension $\check{\mathbb{P}}$ behaves conversely.

(2) The Kleisli extension $\check{\mathbb{F}}$ of the filter monad is right- but not left-whiskering.
4 Topologies on a category

In this subsection we give an axiomatic approach to considering objects in a category as spaces where the category comes equipped with a class of “proper maps”. This class, called the topology of the category, determines notions of compactness and separation and allows us to exhibit their interrelations at both the object and morphism levels.

4.1 Topology, fibrewise topology, derived topology. The finitary stability properties of the classes of proper and of open maps in \((\mathcal{T}, \mathcal{V})\)-Cat studied in \([3.1]\) lead us to considering classes of morphisms \(\mathcal{P}\) in an arbitrary finitely complete category \(X\) and to call \(\mathcal{P}\) a topology on \(X\) if

1. \(\mathcal{P}\) contains all isomorphisms,
2. \(\mathcal{P}\) is closed under composition,
3. \(\mathcal{P}\) is stable under pullback.

For another morphism class \(\mathcal{E}\) in \(X\) we call \(\mathcal{P}\) an \(\mathcal{E}\)-topology if in addition

4. \(p \cdot e \in \mathcal{P}\) with \(e \in \mathcal{E}\) implies \(p \in \mathcal{P}\).

Throughout this subsection we require that the class \(\mathcal{E}\) itself is an \(\mathcal{E}\)-topology. The presence of such a class \(\mathcal{E}\) constitutes no restriction of generality since every topology is an \((\text{Iso} X)\)-topology. Note also that any pullback-stable class \(\mathcal{E}\) that belongs to a factorization system \((\mathcal{E}, \mathcal{M})\) is an \(\mathcal{E}\)-topology (see Proposition \([11.5.1.1]\)).

4.1.1 Examples. Let \(X = (\mathcal{T}, \mathcal{V})\)-Cat and \(\mathcal{E}\) be the class of epimorphisms (that is, surjective \((\mathcal{T}, \mathcal{V})\)-continuous maps). Then \(\mathcal{E}\) is an \(\mathcal{E}\)-topology, with respect to which we consider the following classes \(\mathcal{P}\).

1. The class \(\mathcal{P} = \text{Prop}(\mathcal{T}, \mathcal{V})\) of proper maps is an \(\mathcal{E}\)-topology if \(\mathcal{V}\) is cartesian closed: see Proposition \([3.1.4]\).
2. The class \(\text{Open}(\mathcal{T}, \mathcal{V})\) of open maps is an \(\mathcal{E}\)-topology if \(\mathcal{V}\) is cartesian closed and \(\mathcal{T}\) satisfies BC: see \([3.1.4]\).
3. For the forgetful functor \(O : (\mathcal{T}, \mathcal{V})\)-Cat \(\to\) Set, the class \(\text{Ini} O\) of \(O\)-initial morphisms is an \(\mathcal{E}\)-topology, and so are the classes \(\text{Mono}((\mathcal{T}, \mathcal{V})\)-Cat) and \(\text{RegMono}((\mathcal{T}, \mathcal{V})\)-Cat) = \(\text{Ini} O \cap \text{Mono}((\mathcal{T}, \mathcal{V})\)-Cat).

The class of closed continuous maps in \(\text{Top}\) satisfies properties (1), (2), (4) of an \(\mathcal{E}\)-topology \((\mathcal{E} = \text{Epi Top})\), but not (3) (for example, \(\mathbb{R} \to 1\) is closed, but its pullback \(\mathbb{R} \times \mathbb{R} \to \mathbb{R}\)
along $\mathbb{R} \to 1$ is not). However, it contains a greatest pullback-stable subclass, the class of all proper maps.

In general, we call a class $\mathcal{P}$ in $X$ an $\mathcal{E}$-pretopology if it satisfies properties (1), (2), (4). The class $\mathcal{P}$ is hereditary if every pullback of a morphism in $\mathcal{P}$ along an embedding lies in $\mathcal{P}$.

4.1.2 Proposition. An $\mathcal{E}$-pretopology $\mathcal{P}$ on $X$ contains a largest $\mathcal{E}$-topology $\mathcal{P}^* \subseteq \mathcal{P}$. A morphism $f$ lies in $\mathcal{P}^*$ precisely when every pullback of $f$ lies in $\mathcal{P}$. If $\mathcal{P}$ is stable under pullback along split monomorphisms, in particular if $\mathcal{P}$ is hereditary, then $f$ lies in $\mathcal{P}^*$ if and only if $f \times 1_Z$ lies in $\mathcal{P}$ for every object $Z$ in $X$.

Proof. For the “if” part of the last claim, we consider a pullback diagram

$$
\begin{array}{ccc}
P & \xrightarrow{f'} & Z \\
g' \downarrow & & \downarrow g \\
X & \xrightarrow{f} & Y
\end{array}
= 
\begin{array}{ccc}
P & \xrightarrow{f'} & Z \\
\langle g', f' \rangle \downarrow & & \downarrow \langle g, 1_Z \rangle \\
X \times Z & \xrightarrow{f \times 1_Z} & Y \times Z \\
\end{array}

\text{(4.1.i)}
$$

and its decomposition into two pullback diagrams. The hypotheses yield

$$
f \in \mathcal{P} \implies f \times 1_Z \in \mathcal{P} \implies f' \in \mathcal{P}
$$

since $\langle g, 1_Z \rangle$ is split mono. \qed

4.1.3 Corollary. The class $\text{Clo}(\mathbb{T}, \mathcal{V})$ of closed maps in $(\mathbb{T}, \mathcal{V})$-Cat is a hereditary $\mathcal{E}$-pretopology, with $\mathcal{E}$ the class of epimorphisms. One has

$$
f \in \text{Clo}(\mathbb{T}, \mathcal{V})^* \iff \forall (Z, c) \in (\mathbb{T}, \mathcal{V})$Cat (f \times 1_Z \text{ is closed}),
$$

and if $\mathbb{T} = \mathbb{B}$ and $\mathcal{V}$ cartesian closed, then $\text{Clo}(\mathbb{B}, \mathcal{V})^* = \text{Prop}(\mathbb{B}, \mathcal{V})$.

Proof. For closure of $\text{Clo}(\mathbb{T}, \mathcal{V})$ under composition, see Exercise 3.1. If the composite map

$$
(Z, c) \xrightarrow{e} (X, a) \xrightarrow{f} (Y, b)
$$

is closed with $e$ surjective, then for all $A \subseteq X$ and $C = e^{-1}(A)$, one has

$$
b \cdot Tf \cdot Ti_A \cdot 1_T^A = b \cdot Tf \cdot Te \cdot Ti_C \cdot 1_T^C \quad \text{(Te is surjective)}
$$

$$
= f \cdot e \cdot c \cdot Ti_C \cdot 1_T^C
$$

$$
\leq f \cdot b \cdot Te \cdot Ti_C \cdot 1_T^C
$$

$$
= f \cdot b \cdot Ti_A \cdot 1_T^A
$$
and therefore $f$ is closed. For a pullback diagram
\[
(W = f^{-1}(Z), d) \overset{f'}{\rightarrow} (Z, c) \overset{f}{\rightarrow} (X, a) \rightarrow (Y, b)
\]
with $f$ closed and $c = i_Z^* \cdot b \cdot Ti_Z$, one has $d = i_W^* \cdot a \cdot Ti_W$, and for all $A \subseteq W$ one obtains
\[
c \cdot Tf' \cdot Ti_A \cdot !_{TA}^Z = i_Z^* \cdot b \cdot Tf \cdot Ti_W \cdot Ti_A \cdot !_{TA}^Z
\]
\[
= i_Z^* \cdot f \cdot a \cdot Ti_W \cdot Ti_A \cdot !_{TA}^Z
\]
\[
= f' \cdot i_W^* \cdot a \cdot Ti_W \cdot Ti_A \cdot !_{TA}^Z \quad (4.1.iii) \text{ is a BC-square}
\]
\[
= f' \cdot d \cdot Ti_A \cdot !_{TA}^Z,
\]
so that $f'$ is closed. Consequently, $\text{Clo}(\mathbb{T}, \mathbb{V})$ is a hereditary $E$-pretopology. The remaining assertions follow from Proposition 4.1.2 and Theorem 3.4.6.

There are two ways of creating “new topologies from old” that are of particular importance to us now. First, given an object $Z$, we can “slice at $Z$” any topology $P$ of $X$, by considering the class $P_Z := \{ f : X \to \mathbb{V} \mid \delta_f = \langle 1_X, 1_X \rangle : X \to X \times \mathbb{V} X \text{ lies in } P \}$. The following result is easy to prove.

\begin{proposition}
If $P$ is an $E$-topology of $X$, then $P_Z$ is an $E_Z$-topology of $X/Z$, for every object $Z$ in $X$. The statement still holds if “$E$-topology” is replaced everywhere by “$E$-pretopology”.
\end{proposition}

We call $P_Z$ the fibrewise topology of $P$ at $Z$ (or the fibrewise pretopology accordingly).

Less trivially, for any class $P$, one may consider the derived class
\[
P' = \{ f : X \to Y \mid \delta_f = \langle 1_X, 1_X \rangle : X \to X \times_Y X \text{ lies in } P \}.
\]

\begin{proposition}
For a topology $P$ of $X$, the derived class $P'$ is also a topology of $X$ that moreover contains all monomorphisms and satisfies the cancellation condition
\[
g \cdot f \in P' \implies f \in P'.
\]
If $P$ is an $E$-topology, then $P'$ is an $(E \cap P)$-topology.
\end{proposition}

We call $P'$ the derived topology of $P$. 
Proof. A morphism $f$ is a monomorphism if and only if $\delta_f$ is an isomorphism. Hence, $\mathcal{P}'$ contains all monomorphisms. For $f : X \to Y$ and $g : Y \to Z$ in $X$, let $h = g \cdot f$. There is a unique morphism $t : X \times_Y X \to X \times Z$ with $h_1 \cdot t = f_1$, $h_2 \cdot t = f_2$, where $(f_1, f_2)$ and $(h_1, h_2)$ form the kernel pairs of $f$ and $h$, respectively. The right square of

$\begin{array}{ccc}
X & \xrightarrow{\delta_f} & X \times_Y X \\
\downarrow{t} & & \downarrow{\delta_g} \\
X & \xrightarrow{\delta_h} & X \times_Z X
\end{array}$

is a pullback diagram since $t$ is the equalizer of $f \cdot h_1$ and $f \cdot h_2$. Consequently, one has the implications

$$f, g \in \mathcal{P}' \implies \delta_f, \delta_g \in \mathcal{P} \implies \delta_f, t \in \mathcal{P} \implies \delta_h = t \cdot \delta_f \in \mathcal{P} \implies h \in \mathcal{P}' .$$

Since $t$ is monic, the left square of (4.1.iii) is also a pullback diagram, and one concludes

$$g \cdot f \in \mathcal{P}' \implies \delta_h \in \mathcal{P} \implies \delta_f \in \mathcal{P} \implies f \in \mathcal{P}' .$$

The pullback diagram of $(f, g)$ can be factorized as the following outer pullback diagram:

$\begin{array}{ccc}
P & \xrightarrow{\delta_f'} & P \times_Z P \\
g' \downarrow & & \downarrow{g'} \\
X & \xrightarrow{\delta_f} & X \times_Y X
\end{array}$

$\begin{array}{ccc}
P & \xrightarrow{\delta_f'} & P \times_Z P \\
g' \downarrow & & \downarrow{g'} \\
X & \xrightarrow{\delta_f} & X \times_Y X
\end{array}$

is a pullback diagram, the left square is also one, so

$$f \in \mathcal{P} \implies f' \in \mathcal{P} .$$

Finally, when $\mathcal{P}$ is an $\mathcal{E}$-topology, set $h = g \cdot f \in \mathcal{P}'$ with $f \in \mathcal{E} \cap \mathcal{P}$. From (4.1.iii) one has $\delta_g \cdot f = (f \times f) \cdot \delta_h$, with $\delta_h \in \mathcal{P}$ and $f \times f = (f \times 1) \cdot (1 \times f) \in \mathcal{P}$ (as the composite of two pullbacks of $f$). Since $f \in \mathcal{E}$, one has $\delta_g \in \mathcal{P}$, that is, $g \in \mathcal{P}'$. \hfill \Box

4.1.6 Remark. One has:

$$(\text{Iso} X)' = \text{Mono} X \quad \text{and} \quad (\text{SplitMono} X)' = \text{mor} X .$$

In particular, for any class $\mathcal{P}$ containing $\text{Iso} X$,

$$(\mathcal{P}')' = \text{mor} X ,$$

and for the forgetful functor $O : (\mathcal{T}, \mathcal{V})\text{-Cat} \to \text{Set}$,

$$(\text{Ini} O)' = \text{mor}(\mathcal{T}, \mathcal{V})\text{-Cat} .$$
4. TOPOLOGIES ON A CATEGORY

4.2 \( P \)-compactness, \( P \)-Hausdorffness. Let \( P \) be a topology of the finitely complete category \( X \). An object \( X \) is \( P \)-compact if

\[
(X \rightarrow 1) \in P
\]

(with 1 a terminal object in \( X \)), and \( X \) is \( P \)-Hausdorff if

\[
(X \rightarrow 1) \in P',
\]

that is, if \( (\delta_X : X \rightarrow X \times X) \in P \).

A morphism \( f : X \rightarrow Y \) is \( P \)-proper if \( f \) is \( P_Y \)-compact in \( X/Y \) (see Proposition [4.1.2]), \( f \) is \( P \)-Hausdorff if \( f \) is \( P_Y \)-Hausdorff in \( X/Y \), and \( f \) is \( P \)-perfect if \( f \) is both \( P \)-proper and \( P \)-Hausdorff.

Since \( 1_Y \) is a terminal object in \( X/Y \) and \( f \) is the unique morphism \( f \rightarrow 1_Y \) in \( X/Y \), one sees immediately that

\[
f \text{ is } P \text{-proper} \iff f \in P,
\]

\[
f \text{ is } P \text{-Hausdorff} \iff f \in P' \iff (\delta_f : X \rightarrow X \times_Y X) \in P.
\]

We examine these notions first in terms of the principal topologies for \( (\mathbb{T}, V) \)-Cat as considered in Examples [4.1.1] and compare them with those introduced in [1.1].

4.2.1 Proposition. Let \( V \) be a cartesian closed quantale, and \( (X, a) \) a \( (\mathbb{T}, V) \)-space.

(1) \( (X, a) \) is \( \text{Prop}(\mathbb{T}, V) \)-compact if and only if

\[
\forall x \in TX \ (\top \leq \bigvee_{z \in X} a(x, z)).
\]

(2) Suppose that \( V \) is integral; if \( (X, a) \) is compact then \( (X, a) \) is \( \text{Prop}(\mathbb{T}, V) \)-compact, with the converse statement holding when \( V \) is superior.

(3) \( (X, a) \) is \( \text{Prop}(\mathbb{T}, V) \)-Hausdorff if and only if

\[
\forall x, y \in X \forall z \in TX \ (\perp < a(z, x) \land a(z, y) \implies x = y).
\]

(4) Suppose that \( V \) is integral; if \( (X, a) \) is \( \text{Prop}(\mathbb{T}, V) \)-Hausdorff then \( (X, a) \) is Hausdorff, with the converse statement holding if the map \( o : V \rightarrow 2 \) with \( o(v) = \perp \iff v = \perp \) is a lax homomorphism of quantales, that is, if \( V \) satisfies \( u \otimes v = \perp \implies u = \perp \) or \( v = \perp \).

(5) A \( (\mathbb{T}, V) \)-continuous map \( f : (X, a) \rightarrow (Y, b) \) is \( \text{Prop}(\mathbb{T}, V) \)-Hausdorff if and only if

\[
\forall x, y \in X \forall z \in TX \ (f(x) = f(y) \& (\perp < a(z, x) \land a(z, y) \implies x = y)).
\]
4.2.3 Proposition. Suppose that

\[ (\delta_X : (X, a) \to (1, 1)) \in \text{Prop}(\mathbb{T}, \mathcal{V}) \text{ means } \top \cdot T \delta_X \leq !_X \cdot a; \]
this, when stated elementwise, reads as claimed.

For (3), consider the structure \( b = (p^e \cdot a \cdot Tp) \land (q^e \cdot a \cdot Tq) \) of \( X \times X \) (with projections \( p, q \)); then \((\delta_X : (X, a) \to (X \times X, b)) \in \text{Prop}(\mathbb{T}, \mathcal{V}) \) means \( b \cdot T\delta_X = \delta_X \cdot a \). Since for all \( x, y \in X \), \( z \in TX \),

\[
(b \cdot T\delta_X(z, (x, y))) = a(Tp \cdot T\delta_X(z), x) \land a(Tq \cdot T\delta_X(z), y) = a(z, x) \land a(z, y),
\]

\[
(\delta_X \cdot a)(z, (x, y)) = \bigvee_{z \in \delta_X^{-1}(x, y)} a(z, z) = \begin{cases} a(z, x) & \text{if } x = y, \\ \bot & \text{otherwise,} \end{cases}
\]

the criterion follows.

Finally, (2) and (4) follow from Proposition 1.1.2 and (5) is proved as (3). \( \square \)

4.2.2 Examples.

1. Since 2 and \( P_+ \) satisfy all additional hypotheses used in Proposition 4.2.1 in all examples presented in 1.1.4, the notions of Prop(\( \mathbb{T}, \mathcal{V} \))-compactness and Prop(\( \mathbb{T}, \mathcal{V} \))-Hausdorffness for objects are equivalent to the notions of compactness and Hausdorffness of 1.1.

2. For \( \mathcal{V} \) integral and \( \mathbb{T} = \emptyset \) (identically extended to \( \mathcal{V} \)-Rel), a \( \mathcal{V} \)-functor \( f : (X, a) \to (Y, b) \) is Prop(\( \mathbb{T}, \mathcal{V} \))-Hausdorff if every fibre of \( f \) (as a subspace of \( (X, a) \)) is discrete.

3. In \( \text{Top} \cong (\beta, 2)\text{-Cat} \), \( f : X \to Y \) is Prop(\( \beta, 2 \))-Hausdorff if and only if any two distinct points in the same fibre of \( f \) may be separated by disjoint open neighborhoods in \( X \). Such maps are usually called separated in the literature on fibred topology (see [James, 1989]). In \( \text{App} \cong (\beta, P_+)\text{-Cat} \), a map \( f \) is Prop(\( \beta, P_+ \))-Hausdorff if and only if, for all \( x, y \in X \), \( z \in \beta X \) with \( f(x) = f(y) \), \( a(z, x) < \infty \) and \( a(z, y) < \infty \), one has \( x = y \).

4.2.3 Proposition. Suppose that \( \mathcal{V} \) is cartesian closed and that \( T \) satisfies BC.

1. A \( (\mathbb{T}, \mathcal{V}) \)-space \( (X, a) \) is Open(\( \mathbb{T}, \mathcal{V} \))-compact if and only if

\[
\forall a \in T1, x \in X \ (\top \leq \bigvee_{x \in (T1)_X} \!_X a(x, x)) ,
\]

where \( \!_X : X \to 1 \).

2. If \( \mathcal{V} \) is integral and \( T1 \cong 1 \), then every \( (\mathbb{T}, \mathcal{V}) \)-space is Open(\( \mathbb{T}, \mathcal{V} \))-compact.

3. An Open(\( \mathbb{T}, \mathcal{V} \))-continuous map \( f : (X, a) \to (Y, b) \) is Open(\( \mathbb{T}, \mathcal{V} \))-Hausdorff if and only if

\[
\forall x \in X \forall w \in T(X \times Y X) (\bot < a(Tp(w), x) \land a(Tq(w), x) \implies w \in T\Delta_X) ,
\]

where \( p, q : X \times Y X \to X \) are the projections, and \( w \in T\Delta_X \) means \( w = T\delta_f(\chi) \) for some (uniquely determined) \( \chi \in TX \).
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Proof.

(1) The given criterion expresses $!_X \cdot \top \leq a \cdot (T !_X) \circ$ in pointwise form.

(2) If $T 1 \cong 1$, then $(T !_X)(e_X(x))$ is the only element in $T 1$.

(3) Formulating $\delta \circ f \cdot b \leq a \cdot (T \delta f)$ with $b = (p \circ a \cdot T p) \wedge (q \circ a \cdot T q)$ ($p, q : X \times_X X \to X$ projections) in pointwise form, one obtains

\[
a(T p(w), x) \wedge a(T q(w), x) = \begin{cases} 
  a(x, x) & \text{if } \exists x \in TX \ (T \delta f(x) = w), \\
  \bot & \text{otherwise},
\end{cases}
\]

for all $w \in T(X \times_X X), x \in X$. Since in the first case (that is, when $w \in T \Delta_X$) the equality holds trivially, one obtains the criterion as stated.

4.2.4 Examples.

(1) For $\mathcal{V}$ integral and $\top = 1$, a $\mathcal{V}$-functor is $\text{Open}(\mathbb{I}, \mathcal{V})$-Hausdorff if and only if its fibres are discrete. (Hence, $\text{Open}(\mathbb{I}, \mathcal{V})$-Hausdorffness is equivalent to $\text{Prop}(\mathbb{I}, \mathcal{V})$-Hausdorffness, see Example 4.2.2(2).)

(2) In $\text{Top} \cong (\beta, 2)$-$\text{Cat}$, a continuous map $f : X \to Y$ is $\text{Open}(\beta, 2)$-Hausdorff if and only if the diagonal $\Delta_X \subseteq X \times_X X$ is open, and this means equivalently that for every point in $X$ there is a neighborhood $U$ of $x$ such that $f|_U : U \to X$ is injective; such maps are called locally injective. When applied to $X \to 1$, this yields that $X$ is $\text{Open}(\beta, 2)$-Hausdorff if and only if $X$ is discrete.

(3) An $\text{Open}(\beta, \mathbb{P}^+)$-Hausdorff morphism $f : (X, a) \to (Y, b)$ has the property that for every ultrafilter $w$ on $X \times_X X$ that converges to $(x, x)$ in the induced pretopology (see [III.3.6] and Example [III.1.3]), so that $p[w] \to x$ and $q[w] \to x$, one has $\Delta_X \in w$. Consequently, the neighborhood filter of any diagonal point $(x, x)$ in the pseudotopological space $X \times_X X$ contains $\Delta_X$. Hence, $\text{Open}(\beta, \mathbb{P}^+)$-Hausdorffness is characterized in $\text{App}$ by local injectivity, with “local” referring to the induced pretopology. In particular, $\text{Open}(\beta, \mathbb{P}^+)$-Hausdorff approach spaces are discrete.

4.2.5 Remark. Only indiscrete $(\top, \mathcal{V})$-spaces are $\text{Ini} \text{O}$-compact, but every $(\top, \mathcal{V})$-space is $\text{Ini} \text{O}$-Hausdorff.

4.3 A categorical characterization theorem. The following easy-to-prove theorem collects important characteristic properties of $\mathcal{P}$-compact objects in any finitely complete category $X$ with an $\mathcal{E}$-topology $\mathcal{P}$:

4.3.1 Theorem. The following assertions for an object $X$ are equivalent:

(i) $X$ is $\mathcal{P}$-compact;
(ii) every morphism \( f : X \to Y \) with \( Y \) \( \mathcal{P} \)-Hausdorff is \( \mathcal{P} \)-proper;

(iii) there is a \( \mathcal{P} \)-proper morphism \( f : X \to Y \) such that \( Y \) is \( \mathcal{P} \)-compact;

(iv) the projection \( X \times Y \to Y \) is \( \mathcal{P} \)-proper for all objects \( Y \);

(v) \( X \times Y \) is \( \mathcal{P} \)-compact for every \( \mathcal{P} \)-compact object \( Y \);

(vi) for every morphism \( e : X \to Y \) in \( \mathcal{E} \), the object \( Y \) is \( \mathcal{P} \)-compact.

Proof. (i) \( \Rightarrow \) (ii): In the graph factorization

\[
\begin{array}{ccc}
X 	imes Y & \xrightarrow{p} & Y \\
\langle 1_X, f \rangle & \downarrow & \\
X & \xrightarrow{f} & Y
\end{array}
\]

\( \langle 1_X, f \rangle \) is in \( \mathcal{P} \) as a pullback of \( \delta_Y \), and \( p \) is in \( \mathcal{P} \) as a pullback of \( X \to 1 \).

(ii) \( \Rightarrow \) (iii): Consider \( Y = 1 \).

(iii) \( \Rightarrow \) (i): \( (X \to 1) = (X \xrightarrow{f} Y \to 1) \).

(i) \( \Rightarrow \) (iv): \( p \) is a pullback of \( X \to 1 \).

(iv) \( \Rightarrow \) (v): \( (X \times Y \to 1) = (X \times Y \to Y \to 1) \).

(v) \( \Rightarrow \) (i): Consider \( Y = 1 \).

(i) \( \Rightarrow \) (vi): Apply the \( \mathcal{E} \)-topology property to \( !_X = !_Y \cdot e \).

(vi) \( \Rightarrow \) (i): Consider \( e = 1_X \).

4.3.2 Corollary. Let the composite morphism \( g \cdot f \) be \( \mathcal{P} \)-proper, with \( g \) \( \mathcal{P} \)-Hausdorff. Then \( f \) is \( \mathcal{P} \)-proper.

Proof. Using Proposition 4.1.4, apply Theorem 4.3.1 (i) \( \Rightarrow \) (ii) to the morphism \( f : g \cdot f \to g \) in \( X/\text{cod}(g) \).

Next we apply Theorem 4.3.1 with \( \mathcal{P}' \) in lieu of \( \mathcal{P} \). With Proposition 4.1.5 we obtain the following characterization of \( \mathcal{P} \)-separated objects.

4.3.3 Corollary. The following assertions for an object \( X \) are equivalent:

(i) \( X \) is \( \mathcal{P} \)-Hausdorff;

(ii) every morphism \( f : X \to Y \) is \( \mathcal{P} \)-Hausdorff;

(iii) there is a \( \mathcal{P} \)-Hausdorff morphism \( f : X \to Y \) such that \( Y \) is \( \mathcal{P} \)-Hausdorff;

(iv) the projection \( X \times Y \to Y \) is \( \mathcal{P} \)-Hausdorff for all objects \( Y \);

\[ \square \]
(v) \( X \times Y \) is \( \mathcal{P} \)-Hausdorff for every \( \mathcal{P} \)-Hausdorff object \( Y \);
(vi) for every \( \mathcal{P} \)-proper morphism \( e : X \to Y \) in \( \mathcal{E} \), the object \( Y \) is \( \mathcal{P} \)-Hausdorff;
(vii) for every equalizer diagram \( E \xrightarrow{u} Z \xrightarrow{\rightarrow} X \), the morphism \( u \) is \( \mathcal{P} \)-proper.

Proof. We just observe that \( \mathcal{P}' \)-compact translates to \( \mathcal{P} \)-Hausdorff for objects, and \( \mathcal{P}' \)-proper to \( \mathcal{P} \)-Hausdorff for morphisms. Furthermore, since \( \mathcal{P}' \) contains all monomorphisms, all objects and morphisms are \( \mathcal{P}' \)-Hausdorff. Hence, the equivalence of (i)–(vi) follows from Theorem 4.3.1. For the equivalence (i) \( \iff \) (vii), we just note that equalizers as in (vii) are precisely the pullbacks of \( \delta_X : X \to X \times X \).

4.3.4 Corollary. For a \( \mathcal{P} \)-perfect morphism \( f : X \to Y \), one has

\[ Y \text{ is } \mathcal{P} \text{-compact and } \mathcal{P} \text{-Hausdorff} \implies X \text{ is } \mathcal{P} \text{-compact } \mathcal{P} \text{-Hausdorff}, \]

with the converse implication holding when \( f \) lies in \( \mathcal{E} \).

Proof. Use (iii) \( \implies \) (i) \( \implies \) (vi) of both Theorem 4.3.1 and Corollary 4.3.3.

4.3.5 Corollary. The full subcategory of \( \mathcal{P} \)-compact objects is closed under finite products in \( X \), and the full subcategories of \( \mathcal{P} \)-Hausdorff objects and of \( \mathcal{P} \)-compact \( \mathcal{P} \)-Hausdorff objects are both closed under finite limits in \( X \).

Proof. Closure under finite products for all three subcategories in question follows from (i) \( \iff \) (v) of 4.3.1 and 4.3.3. Closure under equalizers for the two subcategories mentioned follows from (i) \( \iff \) (iii) \( \iff \) (vii) of 4.3.3 and (i) \( \iff \) (iii) of 4.3.1.

4.3.6 Remarks.

(1) \( \mathcal{P} \)-Hausdorffness generally fails to be closed under infinite products, also in conjunction with \( \mathcal{P} \)-compactness, as the example \( X = \mathbb{Top} \) and \( \mathcal{P} = \text{Open}(\mathbb{B}, 2) \) shows: a countable power of a two-point discrete space is no longer discrete.

(2) Theorem 4.3.1 and its corollaries may, of course, be readily applied to the topologies \( \mathcal{P} = \text{Prop}(\mathbb{T}, \mathbb{V}) \) and \( \mathcal{P} = \text{Open}(\mathbb{T}, \mathbb{V}) \), provided that \( \mathbb{V} \) is cartesian closed and (for \( \mathcal{P} = \text{Open}(\mathbb{T}, \mathbb{V}) \)) that \( T \) satisfies BC. We do not formulate explicitly the emerging finitary stability properties in these two important cases. However, in what follows we highlight the most important infinite stability properties available for these two general choices of \( \mathcal{P} \).

4.3.7 Proposition. For \( \mathbb{V} \) cartesian closed, the full subcategory of \( \text{Prop}(\mathbb{T}, \mathbb{V}) \)-Hausdorff spaces is closed under mono-sources in \( (\mathbb{T}, \mathbb{V})\text{-Cat} \), hence is strongly epireflective in \( (\mathbb{T}, \mathbb{V})\text{-Cat} \).
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Proof. Let \((f_i : (X, a))_{i \in I}\) be a point-separating source in \((\text{T}, \text{V})\)-Cat, with all \((Y_i, b_i)\) Prop(\(\text{T}, \text{V}\))-Hausdorff. Hence, for \(x \neq y\) in \(X\) there is \(j \in I\) with \(f_j(x) \neq f_j(y)\). Consequently, for all \(z \in TX\), one has
\[a(z, x) \land a(z, y) \leq b_j(Tf_j(z), f_j(x)) \land b_j(Tf_j(z), f_j(y)) = \bot\,.
\]
Hence, by Proposition 4.2.1(3), the space \((X, a)\) is Prop(\(\text{T}, \text{V}\))-Hausdorff.

With Theorem 3.5.6 we obtain, of course, a Tychonoff Theorem for Prop(\(\text{T}, \text{V}\))-compactness, and then with Proposition 4.2.1 also for compactness.

4.3.8 Corollary. Let \(\text{V}\) be completely distributive. Then every product of Prop(\(\text{T}, \text{V}\))-compact \((\text{T}, \text{V})\)-spaces is Prop(\(\text{T}, \text{V}\))-compact. If \(\text{V}\) is also integral and superior, then \((\text{T}, \text{V})\)-Cat\(_{\text{Comp}}\) is closed under products in \((\text{T}, \text{V})\)-Cat, and likewise for \((\text{T}, \text{V})\)-Cat\(_{\text{Haus}}\) and \((\text{T}, \text{V})\)-Cat\(_{\text{CompHaus}}\).

4.4 \(\mathcal{P}\)-dense maps, \(\mathcal{P}\)-open maps. Given an \(\mathcal{E}\)-topology \(\mathcal{P}\) on a finitely complete category \(\text{X}\), we wish to develop an internal notion of \(\mathcal{P}\)-open map in \(\text{X}\). A crucial step towards this goal is the introduction of \(\mathcal{P}\)-dense maps which, in the case of the principal example \(\mathcal{P} = \text{Prop}(\text{T}, \text{V})\), will be described via suitable closure operators.

4.4.1 Definition. A morphism \(f\) in \(\text{X}\) is \(\mathcal{P}\)-dense if in any factorization \(f = p \cdot h\) with \(p \in \mathcal{P}\) one has \(p \in \mathcal{E}\). The class of all \(\mathcal{P}\)-dense maps in \(\text{X}\) is denoted by \(\mathcal{P}^d\).

4.4.2 Proposition. Consider a composite morphism \(g \cdot f\) in \(\text{X}\).

1. If \(g \cdot f \in \mathcal{P}^d\), then \(g \in \mathcal{P}^d\).
2. If \(f \in \mathcal{P}^d\) and \(g \in \mathcal{E}\), then \(g \cdot f \in \mathcal{P}^d\).
3. If \(\text{X}\) has pushouts of morphisms in \(\mathcal{E}\) and these belong to \(\mathcal{E}\) again, then \(f \in \mathcal{E}\) and \(g \in \mathcal{P}^d\) implies \(g \cdot f \in \mathcal{P}^d\).
4. \(\mathcal{E} \subseteq \mathcal{P}^d\) if and only if \(\mathcal{P} \cap \text{SplitEpiX} \subseteq \mathcal{E}\).
5. If \((\mathcal{E}, \mathcal{M})\) is a factorization system, then \(f \in \mathcal{P}^d\) if and only if in any factorization \(f = p \cdot h\) with \(p \in \mathcal{P} \cap \mathcal{M}\), \(p\) is an isomorphism.

Proof. (1) is trivial. For (2) and (3), consider the diagrams
\[(\text{4.4.i})\]
with \( p \in P \), the square being a pullback in the left diagram and a pushout in the right one, and with \( s, t \) induced morphisms. From \( f \in P^d \), \( p' \in P \), one has \( p' \in E \), and from \( g \in E \) follows \( g' \in E \) and, hence, \( p \cdot g' \in E \) so finally \( p \in E \) (since \( E \) is an \( E \)-topology). Similarly, from \( f \in E \) one obtains \( f' \in E \) and then \( t \in P \) (since \( P \) is an \( E \)-topology), which implies \( t \in E \) when \( g' \in E \) and, hence, \( p \in E \).

To see (4), remark that if \( E \subseteq P^d \), then every identity morphism is \( P \)-dense. Hence, if \( p \cdot s = 1 \) with \( p \in P \), we can conclude \( p \in E \). Conversely, \( P \cap \text{SplitEpi} X \subseteq E \) implies that identity morphisms lie in \( P^d \) which, with (2), implies \( E \subseteq P^d \).

(5) follows easily from \( E \setminus M = \text{Iso} X \) and the fact that, in an \( (E, M) \)-factorization \( p = m \cdot e \) of \( p \in P \), \( m \) is also in \( P \).

4.4.3 Lemma. An embedding \( A \hookrightarrow (X, a) \) is \( \text{Prop}(\mathbb{T}, \mathbb{V}) \)-dense if and only if \( \overline{A} = X \).

Proof. The embedding \( A \hookrightarrow X \) factors through \( \overline{A} \hookrightarrow X \). If \( A \hookrightarrow X \) is \( \text{Prop}(\mathbb{T}, \mathbb{V}) \)-dense, then \( \overline{A} = X \) follows. Conversely, assuming \( \overline{A} = X \) and considering any \( (\mathbb{T}, \mathbb{V}) \)-proper \( B \hookrightarrow X \) with \( A \subseteq B \), we have \( B = \overline{B} \) and, hence, \( A = \overline{A} \subseteq B = B \). \( \square \)

4.4.4 Corollary. A \( (\mathbb{T}, \mathbb{V}) \)-continuous map \( f : (X, a) \rightarrow (Y, b) \) is \( \text{Prop}(\mathbb{T}, \mathbb{V}) \)-dense if and only if \( \overline{f(X)} = Y \).

Guided by Proposition 3.3.4, we can now introduce \( P \)-open maps, as follows.

4.4.5 Definition. A morphism \( f : X \rightarrow Y \) in \( X \) is \( P \)-open if, for every pullback \( f' \) of \( f \), pulling back along \( f' \) preserves \( P \)-density, that is: if for all stacked pullback diagrams

\[
\begin{array}{ccc}
d' & \rightarrow & d \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
f' & \rightarrow & f \\
\end{array}
\]

\( d \in P^d \) implies \( d' \in P^d \). We let \( P^o \) denote the class of \( P \)-open maps in \( X \).
4.4.6 Proposition. For an \( \mathcal{E} \)-topology \( \mathcal{P} \) on \( X \), \( \mathcal{P}^o \) is also an \( \mathcal{E} \)-topology on \( X \).

Proof. \( \mathcal{P}^o \) is clearly a topology. Since any pullback of \( f \cdot e \) with \( e \in \mathcal{E} \) is of the form \( f' \cdot e' \) with \( e' \in \mathcal{E} \), it suffices to show that \( \mathcal{P} \)-density pulls back along \( f \) if it does so along \( f \cdot e \).

But for the pullback diagrams

\[
\begin{array}{ccc}
d' & \xrightarrow{e'} & f' \\
\downarrow{\mathrlap{d''}} & & \downarrow{\mathrlap{d'}} \\
ed & \xrightarrow{f} & d
\end{array}
\]

\( d \in \mathcal{P}^d \) implies \( d'' \in \mathcal{P}^d \), which gives \( e \cdot d'' = d' \cdot e' \in \mathcal{P}^d \) by Proposition 4.4.2(2) and therefore \( d' \in \mathcal{P}^d \).

Our goal must now be to describe the Prop(\( T \), \( \mathcal{V} \))-open maps, for a cartesian closed and integral quantale \( \mathcal{V} \) and a lax extension \( \hat{T} \) of the Set-monad \( T = (T, m, e) \). For convenience we also assume \( T \) to satisfy the Beck–Chevalley condition, so that not only Prop(\( T \), \( \mathcal{V} \)) but also Open(\( T \), \( \mathcal{V} \)) is an \( \mathcal{E} \)-topology on (\( (T, \mathcal{V}) \)-Cat (see Examples 4.1.1). Since the class \( \mathcal{P}^o \) is fully determined by the class \( \mathcal{P}^d \) which, in turn, is fully determined by the grand closure when \( \mathcal{P} = \text{Prop}(T, \mathcal{V}) \) (see Remarks 3.3.3), Prop(\( T \), \( \mathcal{V} \))-open maps retrieve only very little information of the quantale \( \mathcal{V} \), unless \( \mathcal{V} = 2 \). To make this statement more precise, let us introduce the following auxiliary notion.

4.4.7 Definition. A (\( T \), \( \mathcal{V} \))-continuous map \( f : (X, a) \to (Y, b) \) is called 2-open if, for all \( x \in X \), \( y \in TY \),

\[
b(y, f(x)) > \bot \implies \exists x \in TX \ (Tf(x) = y \& a(x, x) > \bot) .
\]

The following remarks explain this terminology.

4.4.8 Remarks.

(1) Since \( \mathcal{V} \) is integral, the quantale homomorphism \( \iota : 2 \to \mathcal{V} \) has (as a monotone map) a left adjoint \( o \) (with \( o(v) = \top \iff v > \bot \)) which is a quantale homomorphism if and only if \( \mathcal{V} \) satisfies

\[
u \otimes v = \bot \implies u = \bot \text{ or } v = \bot \quad (4.4.\text{ii})
\]

(see Example III.3.5.2(1)).

(2) The lax extension \( \hat{T} : \mathcal{V} \text{-Rel} \to \mathcal{V} \text{-Rel} \) may be restricted to \( \text{Rel} = 2 \text{-Rel} \); one simply considers the composite lax functor

\[
\begin{array}{ccccccc}
\hat{T} & : & (\text{Rel} \xrightarrow{r} \mathcal{V} \text{-Rel} \xrightarrow{\hat{T}} \mathcal{V} \text{-Rel} \xrightarrow{o} \text{Rel}) \\
& & (r \xrightarrow{ir} \hat{T}(ir) \xrightarrow{o\hat{T}(ir)} ,)
\end{array}
\]
that is,

\[ x(\tilde{T}r) y \iff \tilde{T}(ur)(x, y) > \perp. \]

Although \( \tilde{T} \) is a lax extension of \( T \) (in the sense of III.1.4.1) which makes \( e : 1_{\text{Rel}} \to \tilde{T} \) oplax, \( \tilde{T} \) generally fails to make the monad multiplication \( m \) oplax. Still, with respect to \( \tilde{T} \), we can form the category \((T, 2)\)-Gph of \((T, 2)\)-graphs as in III.4.1 and obtain the functor

\[(\mathbb{T}, \mathcal{V})\text{-Cat} \to (\mathbb{T}, 2)\text{-Gph}, \quad (X, a) \mapsto (X, oa).\]

For example, for the Barr extension of the ultrafilter monad to \( P^+\text{-Rel} \), one obtains the “induced-pseudotopology” functor

\[ \text{App} \to \text{PsTop}, \]

which we encountered in III.3.6.

(3) Defining a morphism of \((\mathbb{T}, 2)\)-graphs to be open as in the case of \((\mathbb{T}, \mathcal{V})\)-categories, one can now state that, for a \((\mathbb{T}, \mathcal{V})\)-functor \( f : (X, a) \to (Y, b) \),

\[ f \text{ is 2-open } \iff f \text{ is open in } (\mathbb{T}, 2)\text{-Gph}. \]

(4) If \( \mathcal{V} = 2 \), then \( \tilde{T} = \hat{T} \), and the notion of 2-openness returns precisely the notion of openness in \((\mathbb{T}, 2)\)-Cat.

**4.4.9 Proposition.** Let \( \mathcal{V} \) be cartesian closed and integral, satisfying (4.4.ii), and let \( T \) satisfy BC. Then, for the following statements on a \((\mathbb{T}, \mathcal{V})\)-continuous map \( f \) of \((\mathbb{T}, \mathcal{V})\)-categories, one has (i) \( \implies \) (ii) \( \implies \) (iii), with (i) \( \iff \) (ii) holding if \( \mathcal{V} = 2 \), and (ii) \( \iff \) (iii) if \( \mathbb{T} = 1 \):

(i) \( f \in \text{Open}(\mathbb{T}, \mathcal{V}); \)

(ii) \( f \) is 2-open;

(iii) \( f \) is \( \text{Prop}(\mathbb{T}, \mathcal{V})\)-open.

**Proof.** The claims about (i), (ii) are obvious (see Remarks 4.4.8). Now we consider a 2-open \((\mathbb{T}, \mathcal{V})\)-functor \( f : (X, a) \to (Y, b) \) and first show (using the grand closure of 3.3)

\[ f^{-1}(N) \subseteq \overline{f^{-1}(N)} \]

for all \( N \subseteq Y \). Indeed, \( x \in f^{-1}(N) \) implies \( b(y, f(x)) > \perp \) for some \( y \in TN \subseteq TY \), and 2-openness gives some \( \chi \in TX \) with \( Tf(\chi) = y \) and \( a(\chi, x) > \perp \). Moreover, since \( T \) is taut, \( \chi \in (Tf)^{-1}(TN) = Tf^{-1}(N) \), which implies \( x \in \overline{f^{-1}(N)} \).

By ordinal recursion one concludes

\[ f^{-1}(N^\infty) \subseteq \overline{f^{-1}(N)}^\infty \]
for all \( N \subseteq Y \), which implies that pulling back along \( f \) preserves Prop(\( \mathbb{T}, \mathcal{V} \))-density (see Corollary 4.4.4). This property holds in fact for every pullback \( f' \) of the 2-open map \( f \), since 2-openness is stable under pullback, thanks to \( T \) satisfying BC, as one easily shows as in Proposition 3.1.4(4). Consequently, \( f \) is Prop(\( \mathbb{T}, \mathcal{V} \))-open.

Let now \( \mathbb{T} = \mathbb{I} \), identically extended to \( \mathcal{V} \)-Rel. Condition 4.4.ii guarantees that the grand closure is idempotent in this case. Assuming now \( f : (X, a) \to (Y, b) \) in \( \mathcal{V} \)-Cat to be Prop(\( \mathbb{I}, \mathcal{V} \))-open, we consider \( x \in X \), \( y \in Y \) with \( b(y, f(x)) > \perp \) and let \( f' \) be the restriction \( f^{-1}Z \to Z \) of \( f \), with the subspace \( Z = \{ y, f(x) \} \) of \( Y \). Trivially, as a subspace of \( Z \), \( \{ y \} \) is dense in \( Z \), so that \( f^{-1}y \) is dense in \( f^{-1}Z \). Hence, \( x \in f^{-1}Z = \overline{f^{-1}y} \) gives \( z \in f^{-1}y \) with \( a(z, x) > \perp \), which shows that \( f \) is 2-open.

4.4.10 Examples.

1. A 2-open map may fail to be open, even when \( \mathbb{T} = \mathbb{I} \). Indeed, for \( \mathbb{R} \) with its Euclidean metric \( d \), all surjective non-expansive maps \( f : \mathbb{R} \to \mathbb{R} \) are 2-open in \( \text{Met} = \mathbb{P} \_	ext{-Cat} \), but among them only those with

\[
d(y, f(x)) = \inf_{z \in f^{-1}y} d(z, x)
\]

are also open. Already, \( f(x) = \frac{1}{2}x \) fails to be open in \( \mathbb{P} \_	ext{-Cat} \).

2. For \( \mathbb{T} = \mathbb{B} \) and \( \mathcal{V} = 2 \), so that \( (\mathbb{T}, \mathcal{V} \text{-Cat}) \cong \text{Top} \), all three conditions of Proposition 4.4.9 are equivalent. The only critical implication left to be shown is \( (iii) \implies (ii) \). We already know that \( (i) \) (or, equivalently, \( (ii) \)) describes open maps in \( \text{Top} \), in the usual sense that images of open sets are open: see Remarks 3.3.5. Hence, assuming \( (iii) \), consider an open set \( A \subseteq X \) and let \( N = Y \setminus f(A) \). Since the density of \( N \) in \( Y \) pulls back along the restriction \( f' : f^{-1}(N) \to N \) of \( f \), one has that \( f^{-1}(N) \) is dense in \( f^{-1}(N) \), that is: \( f^{-1}(N) = \overline{f^{-1}(N)} \subseteq X \setminus A = X \setminus f(A) \) and therefore \( Y \setminus f(A) \subseteq Y \setminus f(A) \), so that \( f(A) \) is open in \( Y \).

3. Consider the monad \( \mathbb{H} \) with a monoid \( H \) as in 1.4. A 2-open morphism \( f : X \to Y \) in \( (H, 2 \text{-Cat}) \) is (in the notation of 1.4) easily characterized by

\[
y \xrightarrow{\alpha} f(x) \implies \exists z \in f^{-1}y \ (z \xrightarrow{\alpha} x),
\]

whereas a Prop(\( \mathbb{H}, 2 \))-open map is described by

\[
y \xrightarrow{\alpha} f(x) \implies \exists z \in f^{-1}y, \beta \in H \ (z \xrightarrow{\beta} x),
\]

as one shows similarly to the proof of 4.4.9 \((iii) \implies (ii) \). However the two notions are equivalent when \( f \) is an embedding.

4. For the list monad \( \mathbb{L} \) (see 1.4), a Prop(\( \mathbb{L}, 2 \))-open map may fail to be open as well: see Exercise 1.1. However, Prop(\( \mathbb{L}, 2 \))-open embeddings are open in \( (\mathbb{L}, 2 \text{-Cat}) \).
Returning to the general setting of an $\mathcal{E}$-topology $\mathcal{P}$ in a category $X$, with the “new” $\mathcal{E}$-topology $\mathcal{P}^o$ at hand, one may now consider the derived topology $(\mathcal{P}^o)'$ (see Proposition 4.1.5) and explore the notions of $\mathcal{P}^o$-compactness and $\mathcal{P}^o$-Hausdorffness. While we must leave it to the reader to pursue this program in general, let us take a glimpse at $X = \text{Top} \cong (\mathbb{B}, 2)$-Cat with $\mathcal{P} = \{\text{proper maps}\} = \text{Prop}(\mathbb{B}, 2)$ again, so that $\mathcal{P}^o = \{\text{open maps}\} = \text{Open}(\mathbb{B}, 2)$. In this case, every object is $\mathcal{P}^o$-compact, while $\mathcal{P}^o$-Hausdorffness means discreteness (see Example 4.2.4(2)). Furthermore, for a continuous map $f : X \rightarrow Y$ one has:

\[ f \text{ is } \mathcal{P}^o\text{-Hausdorff } \iff \text{f is locally injective (see Example 4.2.4(2))}, \]
\[ f \text{ is } \mathcal{P}^o\text{-perfect } \iff \text{f is } \mathcal{P}^o\text{-proper and } \mathcal{P}^o\text{-Hausdorff} \iff \text{f is open and locally injective} \iff \text{f is a local homeomorphism}, \]

where $f$ is a local homeomorphism if every point in $X$ has an open neighborhood $U$ such that $f(U)$ is open and the restriction $U \rightarrow f(U)$ of $f$ is a homeomorphism.

### 4.5 $\mathcal{P}$-Tychonoff and locally $\mathcal{P}$-compact Hausdorff objects.

We continue to work in a finitely complete category $X$ equipped with an $\mathcal{E}$-topology; furthermore, we now assume that $\mathcal{E}$ belongs to a proper $(\mathcal{E}, \mathcal{M})$-factorization system of $X$, with $\mathcal{E}$ stable under pullback. (In this case, $\mathcal{E}$ is automatically an $\mathcal{E}$-topology.)

We consider the classes

\[ (\mathcal{P} \cap \mathcal{P}') \cdot \mathcal{M} = \{ p \cdot m \mid m \in \mathcal{M}, \ p \text{ is } \mathcal{P}\text{-perfect} \}, \]
\[ (\mathcal{P} \cap \mathcal{P}') \cdot (\mathcal{M} \cap \mathcal{P}^o) = \{ p \cdot m \mid m \in \mathcal{M} \text{ is } \mathcal{P}\text{-open}, \ p \text{ is } \mathcal{P}\text{-perfect} \} \]

and note that they contain all isomorphisms and are stable under pullback, but are not necessarily closed under composition.

For an object $X$ and the unique morphism $!_X : X \rightarrow 1$, one has

\[ !_X \in (\mathcal{P} \cap \mathcal{P}') \cdot \mathcal{M} \iff \exists m : X \rightarrow K \ (m \in \mathcal{M}, K \text{ is } \mathcal{P}\text{-compact } \& \mathcal{P}\text{-Hausdorff}) , \]
\[ !_X \in (\mathcal{P} \cap \mathcal{P}') \cdot (\mathcal{M} \cap \mathcal{P}^o) \iff \exists m : X \rightarrow K \ (m \in \mathcal{M}, K \text{ is } \mathcal{P}\text{-compact } \& \mathcal{P}\text{-Hausdorff}) . \]

Guided by the role model $\text{Top}$ with $\mathcal{P} = \{\text{proper maps}\}$ and $\mathcal{E} = \{\text{surjective maps}\}$, where for a space $X$,

\[ X \text{ is Tychonoff } \iff X \text{ subspace of a compact Hausdorff space,} \]
\[ X \text{ is locally compact Hausdorff } \iff X \text{ open subspace of a compact Hausdorff space,} \]

in our abstract category $X$ we say that a morphism

\[ f \text{ is } \mathcal{P}\text{-Tychonoff } \iff f \in (\mathcal{P} \cap \mathcal{P}') \cdot \mathcal{M} , \]
\[ f \text{ is locally } \mathcal{P}\text{-perfect } \iff f \in (\mathcal{P} \cap \mathcal{P}') \cdot (\mathcal{M} \cap \mathcal{P}^o) , \]
and an object

\[ X \text{ is } \mathcal{P}-\text{Tychonoff} \iff !_X \text{ is } \mathcal{P}-\text{Tychonoff}, \]

\[ X \text{ is locally } \mathcal{P}\text{-compact Hausdorff} \iff !_X \text{ is locally } \mathcal{P}\text{-perfect}. \]

Although the two classes of morphisms under consideration generally fail to be topologies, one may establish characteristic properties similar to those of Theorem 4.3.1, as follows.

**4.5.1 Proposition.** The following assertions for an object \( X \) are equivalent:

(i) \( X \) is \( \mathcal{P}\)-Tychonoff;

(ii) every morphism \( f : X \to Y \) is \( \mathcal{P}\)-Tychonoff;

(iii) there is a \( \mathcal{P}\)-Tychonoff morphism \( f : X \to Y \) with \( Y \) \( \mathcal{P}\)-compact \( \mathcal{P}\)-Hausdorff;

(iv) the projection \( X \times Y \to Y \) is \( \mathcal{P}\)-Tychonoff for all objects \( Y \);

(v) \( X \times Y \) is \( \mathcal{P}\)-Tychonoff for every \( \mathcal{P}\)-Tychonoff object \( Y \).

**Proof.** (i) \( \implies \) (ii): With \( m : X \to K \) in \( \mathcal{M} \) and \( K \) \( \mathcal{P}\)-compact \( \mathcal{P}\)-Hausdorff, we consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{m} & K \\
\downarrow{f} & & \downarrow{p_2} \\
Y \times K & \xrightarrow{(f,m)} & K
\end{array}
\]

Since \( p_2 \cdot (f,m) = m \in \mathcal{M} \) also \( (f,m) \in \mathcal{M} \) (see Proposition II.5.1.1(3)), and \( p_1 \in \mathcal{P} \cap \mathcal{P}' \) by Theorem 4.3.1 and Corollary 4.3.3.

(ii) \( \implies \) (iii), (iv) \( \implies \) (i), (v) \( \implies \) (i): Choose \( Y = 1 \).

(iii) \( \implies \) (i): By hypothesis, \( f = p \cdot m \) with \( m \in \mathcal{M} \) and \( p : Z \to Y \) in \( \mathcal{P} \cap \mathcal{P}' \). By Theorem 4.3.1 and Corollary 4.3.3 \( p \) transfers the needed properties from \( Y \) to \( Z \).

(i) \( \implies \) (iv): \( X \times Y \to Y \) is a pullback of \( X \to 1 \).

(i) \( \implies \) (v): If \( m : X \to K, n : Y \to L \) are in \( \mathcal{M} \), and \( K, L \) are \( \mathcal{P}\)-compact \( \mathcal{P}\)-Hausdorff, then \( m \times n = (m \times 1_L) \cdot (1_X \times n) \) is in \( \mathcal{M} \), and \( K \times L \) is \( \mathcal{P}\)-compact \( \mathcal{P}\)-Hausdorff.

**4.5.2 Corollary.** If the composite morphism \( g \cdot f \) is \( \mathcal{P}\)-Tychonoff, \( f \) is also \( \mathcal{P}\)-Tychonoff.

**Proof.** One argues as in the proof of Corollary 4.3.2.

In order to establish analogous properties for locally \( \mathcal{P}\)-compact Hausdorff objects we need an additional hypothesis, as follows. We say that \( X \) has the \( \mathcal{P}\)-open-closed interchange property if every composite morphism \((X \xrightarrow{m} Y \xrightarrow{n} Z)\) with \( m \in \mathcal{M} \cap \mathcal{P} \) and \( n \in \mathcal{M} \cap \mathcal{P}' \) may be rewritten as \((X \xrightarrow{n'} W \xrightarrow{m'} Z)\) with \( n' \in \mathcal{M} \cap \mathcal{P} \) and \( m' \in \mathcal{M} \cap \mathcal{P}' \). In the role
model \textbf{Top}, if \( X \subseteq Y \) is a closed subspace and \( Y \subseteq Z \) an open subspace, one may choose \( W = \overline{X} \) as the closure of \( X \) in \( Z \).

4.5.3 \textbf{Proposition.} If \( X \) has the \( \mathcal{P} \)-open-closed interchange property, the following conditions are equivalent for an object \( X \):

(i) \( X \) is locally \( \mathcal{P} \)-compact Hausdorff;

(ii) every morphism \( f : X \to Y \) with \( Y \) \( \mathcal{P} \)-Hausdorff is locally \( \mathcal{P} \)-perfect;

(iii) there is a locally \( \mathcal{P} \)-perfect morphism \( f : X \to Y \) with \( Y \) \( \mathcal{P} \)-compact \( \mathcal{P} \)-Hausdorff;

(iv) the projection \( X \times Y \to Y \) is locally \( \mathcal{P} \)-perfect for all objects \( Y \);

(v) \( X \times Y \) is locally \( \mathcal{P} \)-compact Hausdorff for every locally \( \mathcal{P} \)-compact Hausdorff object \( Y \).

\textbf{Proof.} \( (i) \Rightarrow (ii) \): Revisiting the proof of 4.5.1 \( (i) \Rightarrow (ii) \) one decomposes \( \langle f, \mu \rangle \in \mathcal{M} \) as \( X \langle f, 1_X \rangle \to Y \times X \xrightarrow{1_Y \times \mu} Y \times K \).

Then \( \langle f, 1_X \rangle \in \mathcal{M} \) (since \( \mathcal{E} \subseteq \text{Epi} X \)) and \( \langle f, 1_X \rangle \in \mathcal{P} \) as a pullback of \( \delta_Y : Y \to Y \times Y \) (since \( Y \) is \( \mathcal{P} \)-Hausdorff); furthermore, \( 1_Y \times m \in \mathcal{M} \cap \mathcal{P}^{o} \) as a pullback of \( m \in \mathcal{M} \cap \mathcal{P}^{o} \).

With the \( \mathcal{P} \)-open-closed interchange property and \( \mathcal{M} \subseteq \text{Mono} X \), \( \langle f, m \rangle \) is locally \( \mathcal{P} \)-perfect, and so is \( f = p_1 \cdot \langle f, m \rangle \) because \( p_1 \in \mathcal{P} \cap \mathcal{P}' \).

All other steps can be taken as in the proof of Proposition 4.5.3. \hfill \Box

4.5.4 \textbf{Corollary.} Let \( X \) have the \( \mathcal{P} \)-open-closed interchange property. If the composite morphism \( g \cdot f \) is locally \( \mathcal{P} \)-perfect and \( g \) is \( \mathcal{P} \)-perfect, then \( f \) is locally \( \mathcal{P} \)-perfect.

\textbf{Proof.} Exploit Proposition 4.5.3 \( (i) \Rightarrow (ii) \) in \( X/\text{cod}(g) \). \hfill \Box

Let us now consider \( X = (\mathbb{T}, \mathcal{V})\text{-Cat} \) and \( \mathcal{P} = \text{Prop}(\mathbb{T}, \mathcal{V}) \) with \( \mathcal{V} \) cartesian closed and integral. Since every \( \mathcal{V} \)-space is \( \mathcal{P} \)-compact and \( \mathcal{P} \)-Hausdorffness means discreteness, we note that for \( \mathbb{T} = \emptyset \), being \( \mathcal{P} \)-Tychonoff or locally \( \mathcal{P} \)-compact Hausdorff also amounts to being discrete. For a general monad \( \mathbb{T} \) we remark that, trivially, subspaces of \( \mathcal{P} \)-Tychonoff spaces and \( \mathcal{P} \)-open subspaces of locally \( \mathcal{P} \)-compact Hausdorff spaces maintain the respective properties.

4.5.5 \textbf{Proposition.} For \( \mathcal{V} \) lean and superior and \( \mathbb{T} \) flat, every \( \text{Prop}(\mathbb{T}, \mathcal{V}) \)-Tychonoff space is regular and Hausdorff in \( (\mathbb{T}, \mathcal{V})\text{-Cat} \).

\textbf{Proof.} A \( \mathcal{P} \)-Tychonoff space \( X \) is embeddable into a \( \mathcal{P} \)-compact \( \mathcal{P} \)-Hausdorff space \( K \) which, by Proposition 4.2.1, is a compact Hausdorff object in \( (\mathbb{T}, \mathcal{V})\text{-Cat} \) and therefore regular by Proposition 2.3.4. Regularity and Hausdorffness are both inherited by subspaces. \hfill \Box
Let us now restrict our attention to $P$-Tychonoff maps among $P$-Hausdorff spaces; these are simply restrictions of proper maps in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}_{\text{Haus}}$, and one may exploit Proposition 4.5.1 and Corollary 4.5.2 in this case. To be able to apply Proposition 4.5.3 to $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$, we must secure the $\mathcal{P}$-open-closed interchange property:

**4.5.6 Proposition.** *If the grand closure is idempotent, then $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$ has the Prop$(\mathbb{T}, \mathcal{V})$-open-closed interchange property.*

**Proof.** For a closed subspace $X$ of a $\mathcal{P}$-open subspace $Y$ of $Z$, consider the closure $W = \overline{X}^Z$ of $X$ in $Z$. Then $W$ is closed in $Z$ by the idempotency hypothesis, and since hereditariness of the grand closure makes the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\downarrow & & \downarrow \\
W & \xleftarrow{\alpha} & Z
\end{array}
\]

a pullback, $\mathcal{P}$-openness of $Y$ in $Z$ gives the same property for $X$ in $W$. \qed

**4.5.7 Remark.** The idempotency hypothesis is certainly restrictive, as the case $\mathbb{T} = \beta$, $\mathcal{V} = \mathcal{P}_+$ shows (see Exercise 3.B). Trying to strengthen Proposition 4.5.6 one may be tempted to consider the idempotent hull of the grand closure. However, already in the general case, the idempotent hull of the grand closure in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$ can be hereditary only if the grand closure is idempotent. Indeed, assume that for $A \subseteq X$ we have $x \in \overline{A}^X$, but $x \notin \overline{A}^X$. For the subspace $Y := A \cup \{x\}$ one trivially has $\overline{A}^Y \subseteq \overline{A}^X \cap Y$ and therefore $x \notin \overline{A}^Y$. Consequently, $\overline{A}^Y = A$ is closed in $Y$. Denoting the idempotent hull by $\tilde{A} = \overline{A}^\infty$ and assuming its hereditariness one obtains $A = \tilde{A}^Y = \tilde{A}^X \cap Y \supseteq \overline{A}^X \cap Y \ni x$, a contradiction.

**4.5.8 Examples.**

1. In $\text{Top} = (\beta, 2)$-$\text{Cat}$ with $\mathcal{P} = \{\text{proper maps}\}$, $\mathcal{P}$-Tychonoff spaces are characterized as the completely regular Hausdorff spaces (or Tychonoff spaces), that is, as $T_1$-spaces $X$ with the property that for every closed set $A \subseteq X$ and every $x \in X \setminus A$ there is a continuous map $f : X \to [0, 1]$ with $f(x) = 0$ and $f(y) = 1$ for all $y \in A$. Such spaces are easily seen to be embeddable into powers of the unit interval and, hence, into a compact Hausdorff space. Locally $\mathcal{P}$-compact Hausdorff spaces are locally compact Hausdorff spaces, that is, Hausdorff spaces that are locally compact (see III.5.7); in the presence of Hausdorff separation, these are the spaces in which every point has a compact neighborhood. Such spaces are Tychonoff spaces and, in fact, openly embeddable into a compact Hausdorff space (see Engelking [1989]).

2. For a monoid $H$ and $\mathbb{H}$ as in 1.4 and $\mathcal{P} = \{\text{proper maps}\}$ as described in Example 3.1.3(4), an $(\mathbb{H}, 2)$-space $(X, \rightarrow)$ is $\mathcal{P}$-Tychonoff if and only if it is embeddable into an $H$-action $Y$, that is,

\[x \xrightarrow{\alpha} y \text{ in } X \iff \alpha \cdot x = y \text{ in } Y\]
for all $x, y \in X$, $\alpha \in H$ (with $\alpha \cdot x$ denoting the action of $H$ on $Y$). Since $P$-open maps are open (see Example 4.4.10(3)), locally $P$-compact Hausdorff spaces in $(\mathcal{H}, 2)\text{-Cat}$ have the additional property that the embedding $X \hookrightarrow Y$ can be chosen to be open, that is, if $\alpha \cdot x \in X$ for $x \in Y$, then $x \in X$. Finally, we note that $(\mathcal{H}, 2)\text{-Cat}$ has the $P$-open-closed interchange property since the grand closure is idempotent.

(3) In $(\mathcal{L}, 2)\text{-Cat}$ with $P = \text{Prop}(\mathcal{L}, 2)$, $P$-Tychonoff spaces are those $(X, \vdash)$ that are embeddable into monoids $Y$, that is, $(x_1, \ldots, x_n) \vdash y$ in $X \iff x_1 \cdot x_2 \cdot \ldots \cdot x_n = y$ in $Y$.

If the embedding $X \hookrightarrow Y$ is open, one has the additional property that $\forall x, y \in Y (x \cdot y \in X \implies x, y \in X)$, and this property makes $X \hookrightarrow Y$ $P$-open (see 4.4.10(4)) and therefore characterizes locally $P$-compact Hausdorff spaces in $(\mathcal{L}, 2)\text{-Cat}$. In addition, since the grand closure is idempotent in $(\mathcal{L}, 2)\text{-Cat}$, Proposition 4.5.3 is applicable in this category as well.

Exercises

4.A $P$-discrete objects. For an $\mathcal{E}$-topology $P$ on a finitely complete category $X$, call a morphism $f : X \to Y$ locally $P$-injective if $\delta_f : X \to X \times_Y X$ is $P$-open, and call an object $X$ $P$-discrete if $!_X$ is locally $P$-injective. Show the equivalence of the following statements on $X$:

(i) $X$ is $P$-discrete;
(ii) every morphism $f : X \to Y$ is locally $P$-injective;
(iii) there is a locally $P$-injective morphism $f : X \to Y$ with $Y$ $P$-discrete;
(iv) the projection $X \times Y \to Y$ is locally $P$-injective for all objects $Y$;
(v) $X \times Y$ is $P$-discrete for every $P$-discrete object $Y$;
(vi) $Y$ is $P$-discrete for every $P$-open morphism $f : X \to Y$ in $\mathcal{E}$.


Let $X$ be extensive, and let the class $\mathcal{E}$ be closed under coproducts (so that $\amalg_{i \in I} p_i : \amalg_{i \in I} X_i \to \amalg_{i \in I} Y_i$ lies in $\mathcal{E}$ whenever all $p_i \in \mathcal{E}$). Show that $P^o$ is closed under coproducts, for any $\mathcal{E}$-topology $P$. 

(2) For an $\mathcal{E}$-topology $\mathcal{P}$, assume $\mathcal{P}^d$ to be closed under coproducts and let $X$ be an object such that, for all objects $U$, the morphism

$$e_U : \coprod_{x:1} U \to X \times U,$$

whose $x$-th restriction to $U$ is $\langle x, 1_U \rangle : U \to X \times U$, lies in $\mathcal{E}$. Show that $X$ is $\mathcal{P}^o$-compact. Conclude that every topological space is $\{\text{open map}\}$-compact.

4.C The left adjoint left-inverse topology. Recall that $f : (X, a) \to (Y, b)$ is left adjoint to $g : (Y, b) \to (X, a)$ in $(\mathcal{T}, \mathcal{V})$-$\text{Cat}$ if $g^\circ \cdot a = b \cdot Tf$; $f$ is left adjoint left-inverse to $g$ if, in addition, $f \cdot g = 1_Y$. Let $\mathcal{L}$ be the class of all left adjoint left-inverse maps. Show:

(1) $\mathcal{L}$ is a topology on $(\mathcal{T}, \mathcal{V})$-$\text{Cat}$. For $\mathcal{T} = \mathbb{I}$, $\mathcal{V} = 2$, a monotone map $f : X \to Y$ with $Y$ separated is left adjoint left-inverse if and only if $f$ is proper and left adjoint.

(2) A $(\mathcal{T}, \mathcal{V})$-space $(X, a)$ is $\mathcal{L}$-compact if and only if there is a point $x_0 \in X$ with $a(x, x_0) = \top$ for all $x \in TX$. In particular, for $\mathcal{T} = \beta$ and $\mathcal{V} = 2$, a topological space is $\mathcal{L}$-compact if and only if it contains a point whose only neighborhood is the space itself.

(3) If $\mathcal{V}$ is cartesian closed and $T$ satisfies BC, then $\mathcal{L}$ is a $\mathcal{Q}$-topology on $(\mathcal{T}, \mathcal{V})$-$\text{Cat}$, where $\mathcal{Q}$ is the class of open surjections in $(\mathcal{T}, \mathcal{V})$-$\text{Cat}$.

4.D The exponentiable topology. Show that in a finitely complete category $\mathcal{X}$, the class $\text{Exp} \mathcal{X}$ of exponentiable morphisms of $\mathcal{X}$ forms a topology. The $\text{Exp} \mathcal{X}$-compact objects are the exponentiable objects. For $\mathcal{X} = \text{Top}$, prove that a space $X$ is $\text{Exp} \mathcal{X}$-Hausdorff if and only if every point in $X$ has a Hausdorff neighborhood.

4.E $\mathcal{P}$-open versus open. Show that in $(\mathcal{H}, 2)$-$\text{Cat}$ and $(\mathcal{L}, 2)$-$\text{Cat}$ there are $\{\text{proper maps}\}$-open maps which fail to be open.

4.F Nearly open maps. Prove that the class of nearly open maps (see Definition 4.3.1) forms an $\mathcal{E}$-pretopology in $(\mathcal{T}, \mathcal{V})$-$\text{Cat}$ with $\mathcal{E}$ the class of all epimorphisms.
5. Connectedness

An object in a category is connected if it has no non-trivial decomposition into a coproduct. This property becomes particularly powerful when the ambient category is extensive. After a brief review of extensive categories we explore the notion of connectedness in \((\mathbb{T}, \mathcal{V})\text{-Cat}\) and exhibit the pivotal role of the category \(\text{Top}\) in this context. Stability under products is discussed at the end.

5.1 Extensive categories. In Corollary \[\text{III.4.3.10}\] we gave an ad-hoc definition of extensive category and provided sufficient conditions for \((\mathbb{T}, \mathcal{V})\text{-Cat}\) to be extensive. Here we investigate this notion more systematically.

For every small family \((Y_i)_{i \in I}\) of objects in a category \(X\) with small-indexed coproducts and pullbacks one has the adjunction

\[
\prod_{i \in I} X / Y_i \xrightarrow{\perp} X / \coprod_{i \in I} Y_i \tag{5.1.i}
\]

with the left adjoint given by coproduct (mapping \((f_i : X_i \rightarrow Y_i)_{i \in I}\) to \(\coprod_{i \in I} f_i : \coprod_{i \in I} X_i \rightarrow \coprod_{i \in I} Y_i\)) and the right adjoint by pullback along the coproduct injection \(t_j\) of \(\coprod_{i \in I} Y_i\):

\[
\begin{array}{ccc}
X_j & \xrightarrow{s_j} & X \\
\downarrow{f_j} & & \downarrow{f} \\
Y_j & \xrightarrow{t_j} & \coprod_{i \in I} Y_i \\
\end{array} \tag{5.1.ii}
\]

5.1.1 Definition. A category \(X\) with small-indexed coproducts and pullbacks is extensive if the adjunction \[5.1.i\] is an equivalence of categories; equivalently, if both the counits and the units are isomorphisms, that is: if

1. **small coproducts are universal** in \(X\), so that \(X\) is a coproduct of \((X_i)_{i \in I}\) with injections \(s_j\) if all diagrams \[5.1.ii\] are pullbacks, and

2. **small coproducts are totally disjoint** in \(X\), so that all commutative diagrams \[5.1.ii\] are pullbacks if \(X \cong \coprod_{i \in I} X_i\) (and therefore \(f = \coprod_{i \in I} f_i\)).

\(X\) is finitely extensive if instead of small coproducts we consider only finite coproducts everywhere.

Note that universality of finite coproducts entails in particular that the initial object 0 must be strict, that is, any morphism \(f : X \rightarrow 0\) must be an isomorphism (just consider \(I = \emptyset\)). In fact, strictness of 0 follows already from the universality of binary coproducts, as the following result shows.

5.1.2 Proposition. **Finite coproducts are universal in \(X\) if binary coproducts are.**
Proof. It suffices to show that the initial object 0 in $X$ is strict. For any morphism $f : X \to 0$,

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow & & \downarrow f \\
0 & \xleftarrow{1} & 0
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow & & \downarrow f \\
0 & \xleftarrow{1} & 0
\end{array}
$$

are pullback diagrams, with the bottom arrows representing the injections of a binary coproduct. The same is therefore true for the top arrows. To say that $X \cong X \times X$ with isomorphic injections means equivalently that $X$ is pre-initial, that is: $|X(X, Y)| \leq 1$ for all objects $Y$. In particular, the split epimorphism $f$ must be inverse to $0 \to X$.

5.1.3 Proposition. If binary coproducts are universal in $X$, then the two injections of a binary coproduct are monomorphic and their pullback is pre-initial.

Proof. Let

$$
\begin{array}{ccc}
P & \xrightarrow{p} & X \\
\downarrow & & \downarrow s \\
X & \xrightarrow{s} & X + Y
\end{array}
\quad
\begin{array}{ccc}
P & \xrightarrow{q} & Q \\
\downarrow & & \downarrow t \\
X & \xleftarrow{s} & X + Y
\end{array}
$$

be pullback diagrams and $s, t$ coproduct injections. By hypothesis, $p, q$ are coproduct injections as well, and since the left pullback property makes $p$ a split epimorphism, the coproduct property makes it actually an isomorphism. Consequently, having an isomorphic projection in its kernel pair, $s$ is a monomorphism.

This shows that coproduct injections are monic; in particular, $q$ is monic. With the trivial pullback diagrams

$$
\begin{array}{ccc}
Q & \xrightarrow{1Q} & Q \\
\downarrow q & & \downarrow q \\
X & \xrightarrow{1x} & X
\end{array}
\quad
\begin{array}{ccc}
Q & \xleftarrow{1Q} & \xrightarrow{1Q} Q \\
\downarrow q & & \downarrow q \\
X & \xrightarrow{q} & Q
\end{array}
$$

we see that, since the bottom arrows are the injections of $X + Q \cong X$, universality gives $Q + Q \cong Q$ with isomorphic injections, that is, $Q$ is pre-initial.

5.1.4 Lemma. Let $X$ be a category with finite coproducts and pullbacks. If binary coproducts in $X$ are universal, and pre-initial objects are initial, then the functor

$$
(-) + C : X \to X
$$

reflects isomorphisms, for all objects $C$ in $X$.

Proof. For $f : A \to B$ in $X$, assume that $f + 1_C : A + C \to B + C$ is an isomorphism. Since coproduct injections are monic, the functor $(-) + C$ is faithful and therefore reflects mono-
5. CONNECTEDNESS

and epimorphisms, so that \( f \) is both monic and epic. We form the pullbacks \( D, E \) as in

\[
\begin{array}{ccc}
D & \xrightarrow{u} & B \\
\downarrow_{a} & ^{w} & \downarrow_{c} \\
A & \xrightarrow{s} & A + C
\end{array}
\]

\[
\begin{array}{ccc}
& & & \downarrow_{(f+1c)^{-1}} \\
& & & \downarrow_{t} \\
B + C & \xleftarrow{f} & E
\end{array}
\]

where \( s, t, w \) are coproduct injections and \( a, u, c, v \) pullback projections. One obtains \( g : A \rightarrow D \) with \( a \cdot g = 1_{A} \) and \( u \cdot g = f \), making \( a \) an isomorphism as a pullback of the monomorphism \( (f+1c)^{-1} \cdot w \). Consequently, with \( u, v \) being coproduct injections by universality, the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow_{1_{A}} & & \downarrow_{1_{A}} \\
A + (E + E) & \xleftarrow{n} & E + E
\end{array}
\]

\[
\begin{array}{ccc}
& & \downarrow_{1_{A}+i} \\
& & \downarrow_{1_{A}+j} \\
& & \downarrow_{i} \\
A & \xrightarrow{m} & A + E \\
\downarrow_{f} & & \downarrow_{i} \\
E & \xleftarrow{v} & E
\end{array}
\]

where \( 1_{A} + i = 1_{A} + j \) since \( f \) is epic. But then \( i = j \) because \( n \) is monic, which makes \( E \) pre-initial and thus \( E \cong 0 \), by hypothesis. Consequently, the coproduct injection \( f \) becomes an isomorphism.

\[\square\]

5.1.5 Theorem. The following conditions on a category \( X \) with finite coproducts and pullbacks are equivalent:

(i) \( X \) is finitely extensive, that is, finite coproducts in \( X \) are universal and totally disjoint;

(ii) binary coproducts in \( X \) are universal and pre-initial objects are initial;

(iii) binary coproducts in \( X \) are universal and disjoint (so that the pullback of the two coproduct injections is the initial object).

Proof. (i) \( \implies \) (ii): For \( Q \) pre-initial, both rows of

\[
\begin{array}{ccc}
0 & \xrightarrow{1} & Q \\
\downarrow & & \downarrow \scriptstyle{1} \\
Q & \xleftarrow{1} & Q
\end{array}
\]

\[
(5.1.iii)
\]

represent binary coproducts. Total disjointness implies in particular that the left-hand side is a pullback diagram, so that we must have \( Q \cong 0 \).
(ii) \implies (iii) follows from Proposition 5.1.3.

(iii) \implies (ii): If \( Q \) is pre-initial, the left-hand side of (5.1.iii) is a pullback diagram under the disjointness hypothesis, and \( Q \cong 0 \) follows.

(ii) \implies (i): By Proposition 5.1.2 it suffices to show total disjointness of binary coproducts. Given the outer commutative diagram of

\[
\begin{array}{ccc}
A & \rightarrow & A + B \\
\downarrow^h & & \downarrow^p \\
X & \rightarrow & X + Y \\
\end{array}
\begin{array}{ccc}
\rightarrow & & \leftarrow \\
^l & & ^q \\
\rightarrow & & \rightarrow \\
B & \leftarrow & Y \\
\end{array}
\]

(with the horizontal arrows coproduct injections), one forms the pullbacks \( P \) and \( Q \) and obtains the comparison morphisms \( h, l \) making (5.1.iv) commutative. By hypothesis, \( P + Q \cong A + B \) (with coproduct injections \( p, q \)), so that \( h + l : A + B \to P + Q \) is an isomorphism. Since

\[ h + l = (h + 1_Q) \cdot (1_A + l) = (1_P + l) \cdot (h + 1_B), \]

\( h + 1_Q \) is split epic and \( h + 1_B \) is split monic. Consequently,

\[ (h + 1_Q) + 1_B \cong h + 1_{B+Q} \cong (h + 1_B) + 1_Q \]

is both split epic and split monic, that is an isomorphism. With Lemma 5.1.4 we deduce that \( h \) (as well as \( l \), by symmetry) is an isomorphism, as desired. \( \square \)

5.1.6 Corollary. The following conditions on a category \( X \) with small-indexed coproducts and pullbacks are equivalent:

(i) \( X \) is extensive;

(ii) non-empty coproducts in \( X \) are universal and pre-initial objects are initial;

(iii) non-empty coproducts in \( X \) are universal and disjoint (so that the pullback of two coproduct injections with distinct labels is the initial object).

Proof. For any coproduct \((s_i : X_i \to X)_{i \in I}\) and \( j \in I \), note that

\[ X_j \xrightarrow{s_j} X \xleftarrow{\tilde{s}} \coprod_{i \neq j} X_i \]

is a binary coproduct. Therefore (ii) \implies (i) follows from 5.1.5(ii) \implies (i), and the same is true for (ii) \implies (iii); indeed, when the pullback of \( s_j \) and \( s \) is 0, so is the pullback of \( s_j \) and \( s_i \) for any \( i \neq j \) since 0 is strict by Proposition 5.1.2. \( \square \)
5. CONNECTEDNESS

We proved in Theorem III.4.3.9 that for a cartesian closed quantale \( V \), a monad \( T \) with \( T \) taut and an associative and right-whiskering lax extension \( \hat{T} \) to \( V\text{-Rel} \), the category \((\mathbb{T}, V)\text{-Cat}\) is extensive. In particular, \( \text{Ord} \), \( \text{Met} \), \( \text{Top} \), \( \text{App} \) and \((\mathbb{H}, 2)\text{-Cat} \) (for a monoid \( H \)) are extensive.

Frequently studied extensive categories which do not admit a topological functor to \( \text{Set} \) include \( \text{Cat} \) and \( \text{Rng}^{\text{op}} \) (the opposite of the category of unital rings).

5.2 Connected objects. An object \( X \) in a category \( X \) is connected if \( X(X, -) : X \to \text{Set} \) preserves all small-indexed coproducts. This definition becomes especially efficient when \( X \) is extensive.

5.2.1 Theorem. The following assertions are equivalent for an object \( X \) in an extensive category \( X \) with a terminal object \( 1 \):

(i) \( X \) is connected;

(ii) every morphism \( f : X \to \coprod_{i \in I} Y_i \) factors uniquely as \( f = t_j \cdot g \) with a uniquely determined \( j \in I \) (and \( t_j \) the corresponding coproduct injection);

(iii) every morphism \( f : X \to \coprod_{i \in I} Y_i \) factors as \( f = t_j \cdot g \) for some \( g \) and \( j \);

(iv) \( X \not\cong 0 \), and every morphism \( f : X \to 1 + 1 \) factors through one of the coproduct injections of \( 1 + 1 \);

(v) \( X \not\cong 0 \), and every extremal epimorphism \( f : X \to Y + Z \) makes one of the coproduct injections an isomorphism;

(vi) \( X \not\cong 0 \), and \( X \cong Y + Z \) implies \( Y \cong 0 \) or \( Z \cong 0 \).

Proof. The implications (i) \iff (ii) \implies (iii) are trivial, and for (iii) \implies (iv) observe that \( X \not\cong 0 \) follows since there is no coproduct injection through which the empty coproduct 0 may factor.

(iv) \implies (v): For any extremal epimorphism \( f : X \to Y + Z \), the composite

\[
X \xrightarrow{f} Y + Z \xrightarrow{1_Y + 1_Z} 1 + 1
\]

factors through one of the injections of \( 1 + 1 \), and then \( f \) factors through one of the injections of \( Y + Z \) since the latter is a pullback of the former, as \( f = t \cdot g \) with \( t : Y \to Y + Z \), say. But then the monomorphism \( t \) must be an isomorphism since the epimorphism \( f \) is extremal.

(v) \implies (vi): By hypothesis, one of the coproduct injections of \( X \cong Y + Z \) is an isomorphism, and since

\[
\begin{array}{ccc}
0 & \to & Z \\
\downarrow & & \downarrow \\
Y & \to & Y + Z
\end{array}
\]
is a pullback diagram, one of $0 \rightarrow Y$, $0 \rightarrow Z$ is an isomorphism as well.

(vi) $\implies$ (ii): For $f : X \rightarrow \coprod_{i \in I} Y_i$, one considers the pullback diagrams \[5.1.ii\] and has $X \cong \coprod_{i \in I} X_i$ by universality of coproducts. Since $X \not\cong 0$, not all $X_i$ may be initial. In fact, there is precisely one $j \in I$ with $X_j \not\cong 0$ since, by hypothesis, from $X \cong X_j + \coprod_{i \neq j} X_i$ one obtains $\coprod_{i \neq j} X_i \cong 0$ and therefore $X_i \cong 0$ for all $i \neq j$. Consequently, the coproduct injection $X_j \rightarrow X$ is an isomorphism and thus allows for the factorization $f = t_j \cdot (f_j \cdot s_j^{-1})$, which is unique since $t_j$ is a monomorphism.

Let the extensive category $X$ now come with a factorization system $(\mathcal{E}, \mathcal{M})$ and an $\mathcal{E}$-topology $\mathcal{P}$. Then one easily obtains stability of connectedness under $\mathcal{E}$-images and $\mathcal{P}$-dense extensions, as follows.

5.2.2 Proposition. For a morphism $h : Z \rightarrow X$ in $X$ with $Z$ connected, under each of the following conditions $X$ is also connected.

(a) $h \in \mathcal{E}$, and coproduct injections in $X$ lie in $\mathcal{M}$;
(b) $h \in \mathcal{P}^d$, and coproduct injections in $X$ lie in $\mathcal{M} \cap \mathcal{P}$.

Proof. Since $Z$ is connected, every morphism $f : X \rightarrow \coprod_{i \in I} Y_i$ yields a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & \coprod_{i \in I} Y_i \\
\downarrow h & & \downarrow \coprod_{i \in I} t_j \\
X & \xrightarrow{f} & \coprod_{i \in I} Y_i \\
\end{array}
\]

with some $g$ and coproduct injection $t_j$. Each of the two conditions (a), (b) produces a “diagonal” $X \rightarrow Y_j$ making $f$ factor through $t_j$. \qed

Let now $X = (\mathbb{T}, \mathcal{V})$-Cat, with $\mathcal{V}$ cartesian closed, $\mathbb{T}$ taut, and $\tilde{\cdot}$ associative and right-whiskering, so that $X$ is extensive (Theorem III.4.3.9), and let $\mathcal{P} = \text{Prop}(\mathbb{T}, \mathcal{V})$ and $\mathcal{E}$ be the class of all epimorphisms. It is easy to see that, when $\tilde{T}$ preserves bottom $\mathcal{V}$-relations (so that $\tilde{T} \perp_{X,Y} = \perp_{T_X,T_Y}$, for the least $\mathcal{V}$-relation $\perp_{X,Y} : X \rightarrow Y$), then coproduct injections are closed embeddings, that is, lie in $\mathcal{M} \cap \mathcal{P}$: see Exercise 5.B). With Corollary 4.4.4 and Proposition 5.2.2 one obtains:

5.2.3 Corollary. Let $\mathcal{V}$ be cartesian closed, $\mathbb{T}$ taut, and $\tilde{\cdot}$ associative and right-whiskering. Suppose moreover that $\tilde{T}$ preserves bottom $\mathcal{V}$-relations. For a $(\mathbb{T}, \mathcal{V})$-continuous $f : X \rightarrow Y$ with $\tilde{f}(X) = Y$, if $X$ is connected, then $Y$ is connected.
It follows trivially from Theorem 5.2.1 that \((\mathbb{T}, \mathcal{V})\)-spaces \(X\) with \(|X| = 1\) are connected. For \(\mathbb{T} = \mathbb{I}\) the identity monad one also obtains easily the following characterization of all connected spaces:

5.2.4 Corollary. Let \(\mathcal{V}\) be cartesian closed. Then \((X, a)\) is connected in \(\mathcal{V}\)-\text{Cat} if and only if \(X \neq \emptyset\) and for all \(x, y \in X\) there are \(x = x_0, x_1, \ldots, x_n = y\) in \(X\) with

\[
a(x_i, x_{i+1}) \lor a(x_{i+1}, x_i) > \bot \quad (i = 0, \ldots, n - 1).
\]

Proof. The criterion is sufficient for connectedness of \((X, a)\) since continuity of any map \(f : X \to 1 + 1\) means equivalently

\[
a(x, y) > \bot \implies f(x) = f(y)
\]

for all \(x, y \in X\). Conversely, considering the least equivalence relation on \(X\) that identifies all \(x, y\) with \(a(x, y) > \bot\), for \(Z\) the subspace formed by the equivalence class of some \(z \in X \neq \emptyset\) and \(Y = X \setminus Z\), one has \(X \cong Y + Z\) in \(\mathcal{V}\)-\text{Cat}. Connectedness of \(X\) gives \(Y = \emptyset\), so that the criterion holds.

5.2.5 Examples.

(1) In \(\text{Ord}\) the categorical notion of connectedness retains the usual notion of a non-empty connected ordered set \((X, \leq)\): for all \(x, y \in X\) one finds a “zigzag”

\[
x = x_0 \leq x_1 \geq x_2 \leq x_3 \ldots x_{n-1} \leq x_n = y.
\]

Connectedness in \(\text{Met}\) is less interesting: every non-empty metric space \((X, d)\) with \(d\) finite is connected.

(2) In \(\text{Top}\) connected objects \(X\) are also characterized as expected: \(X = Y \cup Z\) with \(Y, Z \in \mathcal{O}X\) and \(Y \cap Z = \emptyset\) only if \(Y = \emptyset\) or \(Z = \emptyset\); but note again that the categorical notion entails \(X \neq \emptyset\).

(3) For a monoid \(H\), a connected object \((X, \rightarrow)\) in \((\mathbb{H}, 2)\)-\text{Cat} is characterized by \(X \neq \emptyset\) and the property that for all \(x, y \in X\) one has

\[
x = x_0 \xrightarrow{\alpha_0} x_1 \xleftarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} x_3 \cdots x_{n-1} \xrightarrow{\alpha_{n-1}} x_n = y
\]

for some \(x_i \in X\) and \(\alpha_i \in H\) \((i = 1, 2, \ldots, n - 1)\).

5.3 Topological connectedness governs. Throughout this subsection we assume that

- \(\mathcal{V}\) is cartesian closed;
- \(T\) is taut;
\[ \hat{\top} \text{ is associative and right-whiskering.} \]

For the extensive category \((\mathcal{T}, \mathcal{V})\text{-Cat}\) we then have the functor
\[
\Omega : (\mathcal{T}, \mathcal{V})\text{-Cat} \longrightarrow \text{Top}
\]
that provides a \((\mathcal{T}, \mathcal{V})\)-space with the topology of its \((\mathcal{T}, \mathcal{V})\)-open subsets; it preserves open maps and coproducts and also reflects coproducts: see Corollary 3.6.2. Its principal purpose arises from the following consequence of Theorem 5.2.1:

**5.3.1 Theorem.** A \((\mathcal{T}, \mathcal{V})\)-space \(X\) is connected if and only if the topological space \(\Omega X\) is connected.

**Proof.** Let \(X\) be connected in \((\mathcal{T}, \mathcal{V})\text{-Cat}\) and assume that \(X\) is the disjoint union of \((\mathcal{T}, \mathcal{V})\)-open sets \(A, B\). Then \(X \cong A + B\) in \((\mathcal{T}, \mathcal{V})\text{-Cat}\) (with \(A, B\) considered as subspaces of \(X\)); see Exercise III.4.B and Theorem III.4.3.3. Consequently, \(A = \emptyset\) or \(B = \emptyset\).

Conversely, assuming \(X \cong A + B\) in \((\mathcal{T}, \mathcal{V})\text{-Cat}\) and \(\Omega X\) connected, coproduct preservation by \(\Omega\) gives immediately \(A = \emptyset\) or \(B = \emptyset\), so that \(X\) must be connected in \((\mathcal{T}, \mathcal{V})\text{-Cat}\). \(\square\)

A connected component of a \((\mathcal{T}, \mathcal{V})\)-space \(X\) is a maximal (with respect to \(\subseteq\)) connected subspace. By Corollary 5.2.3, a connected component is always \((\mathcal{T}, \mathcal{V})\)-closed. From Theorem 5.3.1 one obtains immediately:

**5.3.2 Corollary.** The following statements on a \((\mathcal{T}, \mathcal{V})\)-space \(X\) are equivalent:

(i) \(X\) is a coproduct of connected \((\mathcal{T}, \mathcal{V})\)-spaces;

(ii) \(X\) is the coproduct of its connected components;

(iii) the connected components of \(X\) are \((\mathcal{T}, \mathcal{V})\)-open;

(iv) \(\Omega X\) is the coproduct of its connected components.

**5.3.3 Examples.**

1. Even in \(\text{Top} \cong (\mathcal{B}, 2)\text{-Cat}\), a space \(X\) may fail to satisfy the equivalent conditions of Corollary 5.3.2 (consider the subspace \(\{0\} \cup \{n \frac{1}{n} \mid n = 1, 2, \ldots\}\) of \(\mathbb{R}\), for example). Every open subspace of \(X\) satisfies them precisely when \(X\) is locally connected, that is: when every neighborhood of a point \(x\) contains a connected neighborhood of \(x\).

2. Let \(M\) be a multiplicative monoid, considered as a compact Hausdorff \((\mathcal{L}, 2)\text{-space}: see Corollary 1.4.4. A subset \(A \subseteq M\) is closed when it is closed under the binary operation of \(M\), and \(A\) is open in \(M\) if
\[
\forall x, y \in M \ (x \cdot y \in A \implies x \in A \land y \in A) .
\]
Hence, non-empty open sets in $M$ must contain the neutral element and, consequently, $M$ is connected in $(\mathbb{L}, 2)$-$\text{Cat}$.

When $M$ is commutative, for $A \subseteq M$ to be open means precisely that $A$ is down-closed with respect to the divisibility order:

$$x | z \iff \exists y \in M \ (x \cdot y = z).$$

Hence, the topology of the Alexandroff space $\Omega M$ is induced by this order.

Recall that a topological space is Alexandroff if intersections of open sets are open: see Example 5.10.5. The connected component $C_x$ of a point $x$ in an Alexandroff space $X$ is not only closed but also open, since

$$C_x = \bigcap_{y \in X \setminus C_x} (X \setminus C_y).$$

We can now show that when $T$ preserves intersections (that is, multiple pullbacks of sinks of monomorphisms), the values of the functor $\Omega$ are always Alexandroff.

5.3.4 Proposition. Let $T : \mathcal{S}et \to \mathcal{S}et$ preserve intersections. Then, for every $(\mathbb{T}, \mathcal{V})$-space $X = (X, a)$, the topological space $\Omega X$ is Alexandroff and $X$ is the coproduct of its connected components.

Proof. The second assertion follows from the first. To prove the first, consider open sets $U_i$ ($i \in I$) in $\Omega X$ and $x \in U$, $y \in TY$ with $a(y, x) > \bot$. Then $y \in TU_i$ by hypothesis on $U_i$ for all $i \in I$, hence $y \in TU$ by hypothesis on $T$. Consequently $U \hookrightarrow X$ is $(\mathbb{T}, \mathcal{V})$-open.

5.4 Products of connected spaces. Throughout this subsection, we assume

- $\mathcal{V}$ is cartesian closed and integral;
- $T$ is taut and $T1 \cong 1$;
- $\mathbb{\hat{T}}$ is associative and right-whiskering.

5.4.1 Proposition. Finite products of connected $(\mathbb{T}, \mathcal{V})$-spaces are connected.

Proof. The terminal $(\mathbb{T}, \mathcal{V})$-space $1 = (1, \top)$ is certainly connected. For $X, Y$ connected, it suffices to show that any $(x_0, y_0), (x_1, y_1) \in X \times Y$ lie in the same connected component of $X \times Y$. The additional assumptions on $T$ and $\mathcal{V}$ make $x_0 : 1 \to X$ a morphism, and the split monomorphism $x_0 \times 1_Y : Y \cong 1 \times Y \to X \times Y$ is preserved by $\Omega$, making $\{x_0\} \times Y$ a connected subspace of $\Omega(X \times Y)$, by Theorem 5.3.1 likewise for $X \times \{y_1\}$. Since $\{(x_0) \times Y\} \cap (X \times \{y_1\}) \neq \varnothing$, both $(x_0, y_0), (x_1, y_1)$ lie in the connected subspace $(\{x_0\} \times Y) \cup (X \times \{y_1\})$ of $\Omega(X \times Y)$, showing that $\Omega(X \times Y)$ is connected. By 5.3.1 again, $X \times Y$ is connected in $(\mathbb{T}, \mathcal{V})$-$\text{Cat}$.

The hypothesis $T1 \cong 1$ is essential for this Proposition to hold, and so is the restriction to finite products, as the following examples show.
5.4.2 Examples.

(1) For the list monad $\mathbb{L}$, consider the $(\mathbb{L}, 2)$-space $(X, \vdash)$ with $X = \{x, y\}$ and $\vdash$ the least $(\mathbb{L}, 2)$-structure on $X$ with $(x, y) \vdash x$, and $(Y, \vdash)$ with $Y = \{\ast\}$ discrete. Since $X \not\sim \{x\} + \{y\}$ and $|Y| = 1$, the spaces $X$ and $Y$ are connected, but $X \times Y \not\sim (\{x\} \times Y) + (\{y\} \times Y)$ is not.

(2) In $\text{Ord} \cong (\mathbb{L}, 2)$-$\text{Cat}$, the product of a family of objects in which any two elements have a lower bound is connected (by Corollary 5.2.4). However, if we let $X_n = \{x_1 \leq x_2 \geq x_3 \leq \ldots x_n\}$ be a “zigzag” of length $n$, then $X_n$ is connected, but $X = \prod_n X_n$ is not since the sequences $(x_1)_n$ and $(x_n)_n$ lie in distinct components of $X$. Note that $\Omega : \text{Ord} \to \text{Top}$ is the coreflective embedding that provides an ordered set with its Alexandroff topology, and that the topological space $\prod_n(\Omega X_n)$ is connected (see Corollary 5.4.4 below) while $\Omega X$ is not. In particular, $\Omega$ does not preserve infinite products, but it does preserve finite products.

Here is a criterion for infinite products of connected $(\mathbb{T}, \mathcal{V})$-spaces to be connected. We assume $\hat{T}$ to preserve bottom $\mathcal{V}$-relations.

5.4.3 Theorem. For a family $(X_i)_{i \in I}$ of connected $(\mathbb{T}, \mathcal{V})$-spaces, $X = \prod_{i \in I} X_i$ is connected if and only if there is a connected subspace $A$ of $X$ such that

$$\hat{A} = \{(x_i) \in X \mid \exists (z_i) \in A \ (x_i = z_i \text{ for all but finitely many } i \in I)\}$$

is $\text{Prop}(\mathbb{T}, \mathcal{V})$-dense in $X$.

Proof. The condition is trivially necessary since we may choose $A = X$. Conversely, for $z = (z_i)_{i \in I} \in A$ and $F \subseteq I$ finite, there is a morphism $1 \to \prod_{i \in I \setminus F} X_i$ with constant value $(z_i)_{i \in I \setminus F}$ and then a split monomorphism

$$\prod_{i \in F} X_i \to \prod_{i \in F} X_i \times \prod_{i \in I \setminus F} X_i \cong X .$$

Since the domain is connected, as in Proposition 5.4.1 its image

$$F_z = \{(x_i)_{i \in F} \mid \forall i \in I \setminus F \ (x_i = z_i)\}$$

is connected as well, and so is

$$\hat{A} = \bigcup_{z \in A, F \subseteq I \text{ finite}} F_z .$$

Indeed, for $x \in F_z$, $y \in G_w$ with $z, w \in A$ and $F, G \subseteq I$ finite, the connected components of $x$ and $z$ coincide since $F_z$ is connected, and so do the connected components of $y$ and $w$, but also of $z$ and $w$ since $A$ is connected. With Corollary 5.2.3 connectedness of $X$ follows. $\square$
Choosing for \( A \) any singleton subset of \( X \), Theorem 5.4.3 shows in particular:

5.4.4 Corollary. The product of connected spaces in \( \text{Top} \cong (\beta, 2)\text{-Cat} \) is connected.

5.4.5 Remark. In general, the connected subspace \( A \) in Theorem 5.4.3 may not be chosen to be a singleton set, not even finite. Indeed, in \( \text{Ord} \cong (1, 2)\text{-Cat} \), the product of countably many copies of the set \( \mathbb{Z} \) of integers is connected (see Example 5.4.22). For any finite set \( A \subseteq \mathbb{Z}^N \) consider a point \( x = (x_n)_{n \in \mathbb{N}} \) with \( x_m < z_m \) for all \( z = (z_n)_{n \in \mathbb{N}} \in A \) and \( m \in \mathbb{N} \); since the idempotent grand closure is given by the up-closure, such \( x \) cannot lie in the grand closure of \( \hat{A} \).

Exercises

5.A Connected categories. Confirm that connected objects in \( \text{Cat} \) are characterized as in Exercise 2.7.Q

5.B Coproduct injections are closed embeddings. Let \( \hat{T} \) and \( V \) be such that the lax extension \( \hat{T} : V\text{-Rel} \to V\text{-Rel} \) preserves bottom \( V \)-relations (see Corollary 5.2.3). Then coproduct injections in \( (T, V)\text{-RGph} \) are \( O \)-initial (for \( O : (T, V)\text{-RGph} \to \text{Set} \)). Moreover, when \( \hat{T} \) is associative the corresponding statement holds for \( (T, V)\text{-Cat} \), and every coproduct injection is proper.

5.C Infinite products of connected spaces. Consider the subspaces \( X_n = \{0, n\} \ (n \in \mathbb{N}) \) of \( \mathbb{R} \) with their Euclidean metric. Show that the product \( \prod_{n \in \mathbb{N}} X_n \) fails to be connected in \( \text{Met} \) although each \( X_n \) is connected.

5.D Total disconnectedness. A \( (T, V) \)-space \( Y \) is called totally disconnected if each of its connected components has precisely one element. Show that the following statements hold under the hypotheses of Corollary 5.2.3.

(1) \( Y \) is totally disconnected if and only if every \( (T, V) \)-continuous map \( f : X \to Y \) with \( X \) connected is constant.

(2) A \( (T, V) \)-space \( X \) is connected if and only if every \( (T, V) \)-continuous map \( f : X \to Y \) with \( Y \) totally disconnected is constant.

(3) The full subcategory of totally disconnected \( (T, V) \)-spaces is strongly epireflective in \( (T, V)\text{-Cat} \).

(4) In \( \text{Top} \cong (\beta, 2)\text{-Cat} \), an extremally disconnected T0-space must be totally disconnected, but not conversely. Indiscrete spaces are both connected and extremally disconnected.

5.E Shortcomings of \( \Omega \). While the functor \( \Omega : (T, V)\text{-Cat} \to \text{Top} \) of 5.3 preserves open maps and open embeddings, it generally fails to preserve embeddings or finite products.
5.F A right adjoint to Ω. If V = 2, T is taut, and \( \hat{\mathbb{T}} \) is associative and right-whiskering, then the functor

\[
\Omega : (\mathbb{T}, 2)\text{-Cat} \to \text{Top}
\]

has a right adjoint which assigns to a topological space X the \((\mathbb{T}, 2)\)-space \((X, \rightarrow)\) with

\[
\chi \rightarrow x \iff \forall U \in \mathcal{O}X \ (x \in U \implies \chi \in TU)
\]

for all \( x \in X, \chi \in TX \).
5. CONNECTEDNESS

Notes on Chapter V

For concrete categories endowed with a notion of closed subobject, in [Manes, 1974] Manes essentially considers stably-closed maps and defines an object X to be compact and Hausdorff if, respectively, \( X \to 1 \) and \( X \to X \times X \) are stably closed, just as in the axiomatic setting of Section 4. Furthermore, for a monad \( T \) on \( Set \) endowed with its Barr extension to \( Rel \), he in essence considers the category \((T, 2)-Cat\) and the notion of closed subobject as in Remark 3.6.4 and gives a relational characterization of compactness and Hausdorff separation as in our Definition 1.1.1. He also observes that the stably-closed maps are equationally defined, in the same way as proper maps are defined in Section 3. Briefly, his paper is to be considered an eminent precursor to large parts of Chapter V. There are two remarkable (but not well known or accessible) PhD theses which greatly extended Manes’ ideas in an abstract relational setting, by Kamnitzer [1974a] (written under the direction of G.C.L. Brümmer; see also Kamnitzer [1974b]) and Möbus [1981] (written under the direction of H. Schubert; see also Möbus [1978]). Specifically, Kamnitzer considers T0, T1, and Hausdorff separation and compactness as used in this chapter, and our definition of R0, R1, regularity, normality and extremal disconnectedness follows Möbus who considers these notions in the general relational context of Klein [1970] and Meisen [1974]. Our treatment of order-separation can be traced back to Marny’s definition of T0-separation for topological categories [Marny, 1979].

In the particular case of approach or \((β, P_+)-spaces\), compactness coincides with 0-compactness as developed in [Lowen, 1988, 1997]. A study of low separation properties in that setting goes back to [Lowen and Sioen 2003], where order-separation is called T0. The R1 property was considered by Robeys [1992] where an approach space satisfying this condition is called complemented. Regularity for approach or \((β, P_+)-spaces\) coincides with the notion considered in Robeys [1992] and is further characterized in terms of the tower of the approach space in [Brock and Kent, 1998]. A notion of normality for approach spaces weaker than the one used in Section 2 was introduced in Van Olmen’s thesis [Van Olmen, 2005]; here it appears as item (iii) in Theorem 2.5.2.

Versions of the Tychonoff Theorem and the Čech–Stone compactification appear in various contexts, including the ones already mentioned (see in particular Clementino, Giuli, and Tholen [1996] Clementino and Tholen [1996] Lowen [1997] Möbus [1981], but its treatment in the general \((T, V)-context\) as given in Proposition 1.2.1 and Theorem 1.2.3 draws heavily on Proposition 3.1.2.1 which first appeared in [Clementino and Hofmann, 2009].

The equational definition of proper morphism as presented in \((T, V)-Cat\) in Section 3 may be traced back to Manes [1974], called perfect by him and strongly closed in Kamnitzer [1974a], while we maintain Bourbaki’s terminology Bourbaki [1989]. The corresponding definition of open morphism as used here appears first in Möbus [1981]. Generalizations of the Kuratowski–Mrówka Theorem, named after Kuratowski [1931] (who proved that product projections along a compact space are closed) and Mrówka [1959] (who showed that Kuratowski’s property characterizes compactness) have been considered by various authors, notably in the context of closure operators by Dikranjan and Giuli [1989] and for approach spaces in [Colebunders, Lowen, and Wuyts, 2005]. There is also a fairly general version of the Kuratowski–Mrówka Theorem in a different setting in Hofmann’s fundamental article [Hofmann, 2007] while the construction used in Theorem 3.4.1 relies on Proposition 3.4.1 that draws on [Clementino and Hofmann, 2012]. The other crucial ingredient to the characterization of proper maps, Theorem 3.2.5 appeared only recently in Clementino and Tholen [2013], following which Solovyov [2012] proposed the notion of closed map as given in Definition 3.3.6. The characterization of proper maps of approach spaces as stably closed maps or as closed maps with 0-compact fibres appears in [Colebunders, Lowen, and Wuyts, 2005]. Open maps of convergence spaces are described in [Kent and Richardson, 1973]; for approach spaces openness is introduced in [Lowen and Verbeeck, 1998]; 2003 in terms of the associated distance, a notion coinciding with that of inversely closed maps for \((β, P_+)-spaces\).

Theorem 3.5.6 on the product stability of proper maps (which entails the Tychonoff Theorem) originates
with Schubert's thesis [Schubert, 2006] who also observed the crucial role of complete distributivity as in Proposition 3.5.4. Conditions for the openness of coproduct injections are considered in [Möbus, 1981]. Our presentation of stability of openness under coproducts relies on Mahmoudi, Schubert, and Tholen [2006].

The first axiomatic categorical treatment of compactness and Hausdorff separation depending on a parameter \( P \) as in Section 4 was given by Penon [1972]; in fact the axioms we have imposed on a topology \( P \) may be considered as a finitary version of Penon's axioms, except that he does not require closure under composition. But as emphasized in [Tholen, 1999], closure under composition is essential to fully exhibit the beautiful interplay of the two notions. A slightly different axiomatic approach, starting with a notion of closed map from which proper is derived as stably-closed, is presented in [Clementino, Giuli, and Tholen, 2004a] where further topological themes like local compactness and exponentiability are being pursued in greater depth.

Of course, there is a rich supply of articles recognizing and axiomatizing the key role of the class of proper or perfect maps, including [Herrlich, 1974; Herrlich, Salicrup, and Strecker, 1987; Manes, 1974] which, once a definite notion of categorical closure operator had been introduced by Dikranjan and Giuli [1987], led to many investigations of compactness and separation in that context; see in particular [Clementino, Giuli, and Tholen, 1996; Clementino and Tholen, 1996; Dikranjan and Giuli, 1989]. The term topology on a category as in Section 4 appears in [Tholen, 2003] and is adopted in [Schubert, 2006] and published in [Hofmann and Tholen, 2012].

Extensive categories (in the finitary sense) were studied by Carboni, Lack, and Walters [1993]. Their elegant definition as given in Section 5 is due to Steve Schanuel, and their characterization as given in Theorem 5.1.5 draws also on [Börger, 1994], an extended preprint of which appeared as [Börger, 1987]. Connected objects as defined in Section 5.2 appeared in Hoffmann's thesis [Hoffmann, 1972], and their characterization as given in Theorem 5.2.1 draws on Janelidze [2004]. Theorem 5.3.1 (for \( V = 2 \)) and its consequences are due to Clementino, Hofmann, and Montoli [2013].

Bibliography


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