Mac Lane and Factorization

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Dear Walter,

Just saw your note and was astonished.

JPAA 85 (1993) 57

What do you mean?

First got up in Isbell 1957

They were in Mac Lane

Duality for Groups

BULLAMS

1957

Lose it up!!

Sincerely,
Saunders Mac Lane

Duality for groups

Bulletin for the American Mathematical Society 56 (1950) 485-516

Saunders Mac Lane

Groups, categories and duality

Bulletin of the National Academy of Sciences USA 34 (1948) 263-267
tion, we axiomatize the terms "injection homomorphism of a subgroup into a larger group" and "projection homomorphism of a group onto a quotient group." We can then define homomorphisms onto and isomorphisms into as "supermaps" and "submaps," respectively.

**Definition.** A bicategory $\mathcal{C}$ is a category with two given subclasses of mappings, the classes of "injections" ($\kappa$) and "projections" ($\pi$) subject to the axioms BC-0 to BC-6 below.²

BC-0. A mapping equal to an injection (projection) is itself an injection (projection).

BC-1. Every identity of $\mathcal{C}$ is both an injection and a projection.

BC-2. If the product of two injections (projections) is defined, it is an injection (projection).

BC-3. (Canonical decomposition). Every mapping $\alpha$ of the bicategory can be represented uniquely as a product $\alpha = \kappa \theta \pi$, in which $\kappa$ is an injection, $\theta$ an equivalence, and $\pi$ a projection.

Any mapping of the form $\lambda = \kappa \theta$ (that is, any mapping with $\pi$ equal to an identity in the canonical decomposition) is called a submap; any mapping of the form $\rho = \theta \pi$ is called a supermap.

BC-4. If the product of two submaps (supermaps) is defined, it is a submap (supermap).

Any product $\kappa_1 \pi_1 \cdots \kappa_n \pi_n$ of injections $\kappa_i$ and projections $\pi_i$ is called an idemmap.

BC-5. If two idemmaps have the same range and the same domain, they are equal.

BC-6. For each object $A$, the class of all injections with range $A$ is a set, and the class of all projections with domain $A$ is a set.

The inclusion relations between the various classes of mappings can be represented by the following Hasse diagram.

![Hasse diagram](image_of_diagram)

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¹ The term "bicategory" was suggested by Professor Grace Rose.
² In the preliminary announcement [16], axiom BC-6 did not appear, and axiom BC-5 was present only in weaker form.
11. **Partial order in a bicategory.** The axioms (especially axiom BC-5) suffice to introduce a relation of partial order (under "inclusion") in the objects of a bicategory. We define a mapping $\beta$ to be left cancellable in a category if $\beta \alpha_1 = \beta \alpha_2$ always implies $\alpha_1 = \alpha_2$, and left invertible if $\beta$ has a left inverse $\gamma$, with $\gamma \beta = I_{\beta}D$. One may readily prove, in succession, the following results.

**Lemma 11.1.** Two injections $\kappa_1$ and $\kappa_2$ such that $\kappa_1 \kappa_2$ is an identity are themselves identities.

**Lemma 11.2.** Every right factor of a submappings is a submappings.

**Lemma 11.3.** If $\alpha \beta$ is an identity, $\alpha$ is a supermap and $\beta$ a submap.

**Lemma 11.4.** Every left invertible mapping is a submap, and every submap is left cancellable.

**Theorem 11.5.** The class of objects in a bicategory is partially ordered by either of the relations

\[(11.1) \quad S \subseteq B \quad \text{if and only if there is an injection} \quad \kappa: S \to B; \]
\[(11.1^\prime) \quad Q \subseteq A \quad \text{if and only if there is a projection} \quad \pi: A \to Q.\]

If $S \subseteq B$, we call $S$ a **subobject** of $B$, while if $Q \subseteq A$, $Q$ is a **quotient-object** of $A$, the terms corresponding to those in group theory. By axiom BC-5 the mappings $\kappa$ and $\pi$ which appear in the dual definitions (11.1) and (11.1') are unique; it is more suggestive to denote them as

\[(11.2) \quad \kappa = [B \supset S]: S \to B; \quad \pi = [Q \subseteq A]: A \to Q.\]

Thus $[B \supset S]$ is a mapping, defined precisely when $S \subseteq B$ and is then an injection; every injection has this form. The notation is so chosen that

\[(11.3) \quad [B \supset S][S \supset T] = [B \supset T], \quad [R \subseteq Q][Q \subseteq A] = [R \subseteq A],\]

by BC-5, whenever the terms on the left are defined.

In examining prospective examples of bicategories, it is easier to formulate the axioms directly in terms of these constructions on the objects.
A brief history of factorization systems

- Mac Lane 1948/1950
- Isbell 1957/1964
- Quillen 1967
- Kennison 1968
- Kelly 1969
- Ringel 1970/1971
- Freyd-Kelly 1972
- Pumplün 1972
(Orthogonal) factorization system \((\mathcal{E}, \mathcal{M})\) in \(\mathcal{C}\)

\[
\begin{array}{c}
\text{e} \perp \text{m} \\
\text{e} \downarrow \\
\text{m} \\
\end{array}
\]

1. \((FS*1\&2)\) \(\mathcal{E} = \perp \mathcal{M}, \mathcal{M} = \mathcal{E}^{\perp}\)
2. \((FS*3)\) \(\mathcal{C} = \mathcal{M} \cdot \mathcal{E}\)

3. \((FS*1)\) \(\text{Iso} \cdot \mathcal{E} \subseteq \mathcal{E}, \mathcal{M} \cdot \text{Iso} \subseteq \mathcal{M}\)
4. \((FS*2)\) \(\mathcal{E} \perp \mathcal{M}\)
5. \((FS*3)\) \(\mathcal{C} = \mathcal{M} \cdot \mathcal{E}\)
Alternative characterization

(FS1) $\text{Iso} \subseteq \mathcal{E} \cap \mathcal{M}$

(FS2) $\mathcal{E} \cdot \mathcal{E} \subseteq \mathcal{E}, \mathcal{M} \cdot \mathcal{M} \subseteq \mathcal{M}$

(FS3) $\mathcal{C} = \mathcal{M} \cdot \mathcal{E}$

(FS3!) $\sim \rightsquigarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow m \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow m' \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow m' \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow e \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow e'$

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Strict factorization system \((\mathcal{E}_0, \mathcal{M}_0)\) in \(\mathcal{C}\) (M. Grandis)

(SFS1) \(\text{Id} \subseteq \mathcal{E}_0 \cap \mathcal{M}_0\)
(SFS2) \(\mathcal{E}_0 \cdot \mathcal{E}_0 \subseteq \mathcal{E}_0, \mathcal{M}_0 \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0\)
(SFS3) \(\mathcal{C} = \mathcal{M}_0 \cdot \mathcal{E}_0\)
(SFS3!) \(\mathcal{C} = \mathcal{M}_0 \cdot \mathcal{E}_0\)
“Higher” Justification:

- $F : C^2 \rightarrow C \iff$ Eilenberg-Moore structure w.r.t. $\square^2$
- $fs \iff$ normal pseudo-algebras (Coppey, Korostenski-Tholen)
- $sfs \iff$ strict algebras (Rosebrugh-Wood)
Free structure on $\mathcal{C}^2$

\[
\begin{align*}
& \begin{array}{c}
\bullet \\
\downarrow f \\
\bullet \\
\end{array} \\
\begin{array}{c}
\bullet \\
\downarrow g \\
\bullet \\
\end{array} \\
\begin{array}{c}
\bullet \\
\downarrow v \\
\bullet \\
\end{array} \\
& = \\
\begin{array}{c}
\bullet \\
\downarrow f \\
\bullet \\
\end{array} \\
\begin{array}{c}
\bullet \\
\downarrow \downarrow d \\
\bullet \\
\downarrow g \\
\bullet \\
\end{array} \\
\begin{array}{c}
\bullet \\
\downarrow v \\
\bullet \\
\end{array} \\
\begin{array}{c}
\bullet \\
\downarrow 1 \\
\bullet \\
\end{array}
\end{align*}
\]
Mac Lane again:

(BC1) \( \text{Id} \subseteq \mathcal{E}_0 \cap \mathcal{M}_0 \)

(BC2) \( \mathcal{E}_0 \cdot \mathcal{E}_0 \subseteq \mathcal{E}_0, \mathcal{M}_0 \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0 \)

(BC3) \( \mathcal{C} = \mathcal{M}_0 \cdot \text{Iso} \cdot \mathcal{E}_0 \)

(BC3!)\[\begin{array}{ccc}
  \text{e} & \downarrow & \text{j} \\
  \downarrow & & \downarrow \\
  \text{1} & \rightarrow & \text{m} \\
 \end{array}\]

(BC4) \( \mathcal{E}_0 \cdot \text{Iso} \subseteq \text{Iso} \cdot \mathcal{E}_0, \text{Iso} \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0 \cdot \text{Iso} \)

(BC5) \( |\mathcal{M}_0 \cdot \mathcal{E}_0 \cap \mathcal{C}(A, B)| \leq 1 \)
Epimorphisms from $G$ $\iff$ congruences on $G$
Set

objects: sets $X$ with equivalence relation $\sim_X$
morphisms: $[f] : X \to Y$

$x \sim_X x' \implies f(x) \sim_Y f(x')$
$f \sim g \iff \forall x \in X : f(x) \sim_Y g(x)$
closure: $Z \subseteq X, Z^\sim = \{x \in X \mid \exists z \in Z : x \sim_X z\}$
compare: Freyd completion!
\[ x \sim_f x' \iff f(x) \sim_Y f(x') \]

\[ \mathcal{E}_0 = \{ [1_X : X \to X' | \sim_X \subseteq \sim_{X'} \} \]

\[ \mathcal{M}_0 = \{ [Z \hookrightarrow Y] | Z \sim = Z \} \]

\[ [f] \text{ mono} \iff \sim_X = \sim_f \]

\[ [f] \text{ epi} \iff f(X) \sim = Y \]

\[ \text{Epi} \cap \text{Mono} = \text{Iso} \iff AC \]

\[ \iff \text{Epi} = \text{SplitEpi} \]
Grp\home ~ = Grp(Set\home ~)

- groups with a congruence relation
- homomorphisms “up to congruence”
- Grp\home ~ \rightarrow Set\home ~ reflects isos
Mac Lane: \( U \subseteq X_f \text{ open} \iff \exists V \subseteq Y \text{ open} : U = f^{-1}(V) \)

Better: \( U \subseteq X_f \text{ open} \iff \exists V = V^\sim \subseteq Y : U = f^{-1}(V) \text{ open} \)
Double factorization system \((\mathcal{E}_0, \mathcal{J}, \mathcal{M}_0)\) in \(\mathcal{C}\)

\[ (e, j) \perp (k, m) \]

\[ \begin{array}{c}
\bullet \quad u \\
\downarrow e \\
\downarrow j \\
\downarrow v \\
\bullet \quad k \\
\downarrow !w \\
\downarrow !z \\
\downarrow m \\
\bullet
\end{array} \]

\textbf{(DFS*1)} \quad \text{Iso} \cdot \mathcal{E}_0 \subseteq \mathcal{E}_0, \text{Iso} \cdot \mathcal{J} \cdot \text{Iso} \subseteq \mathcal{J}, \mathcal{M}_0 \cdot \text{Iso} \subseteq \mathcal{M}_0

\textbf{(DFS*2)} \quad (\mathcal{E}_0, \mathcal{J}) \perp (\mathcal{J}, \mathcal{M}_0)

\textbf{(DFS*3)} \quad \mathcal{C} = \mathcal{M}_0 \cdot \mathcal{J} \cdot \mathcal{E}_0

\text{(\(\mathcal{E}, \mathcal{M}\)) fs} \quad \iff \quad \text{(\(\mathcal{E}, \text{Iso}, \mathcal{M}\)) dfs}
(DFS1) $\text{Iso} \subseteq \mathcal{E}_0 \cap \mathcal{I} \cap \mathcal{M}_0$

(DFS2) $\mathcal{E}_0 \cdot \mathcal{E}_0 \subseteq \mathcal{E}_0$, $\mathcal{I} \cdot \mathcal{I} \subseteq \mathcal{I}$, $\mathcal{M}_0 \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0$

(DFS3) $\mathcal{C} = \mathcal{M}_0 \cdot \mathcal{I} \cdot \mathcal{E}_0$

(DFS3!) $\mathcal{J} \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0 \cdot \mathcal{J}$, $\mathcal{E}_0 \cdot \mathcal{I} \subseteq \mathcal{I} \cdot \mathcal{E}_0$

(DFS4) $\mathcal{I} \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0 \cdot \mathcal{I}$, $\mathcal{E}_0 \cdot \mathcal{I} \subseteq \mathcal{I} \cdot \mathcal{E}_0$

$(\mathcal{E}_0, \mathcal{I}, \mathcal{M}_0)$ dfs $\iff$ $(\mathcal{E}_0, \mathcal{M}_0 \cdot \mathcal{I}), (\mathcal{I} \cdot \mathcal{E}_0, \mathcal{M}_0)$ fs

$\mathcal{I} = \mathcal{I} \cdot \mathcal{E}_0 \cap \mathcal{M}_0 \cdot \mathcal{I}$
Free structure on $C^3$:
\((E_0, \mathcal{I}, M_0) \leftrightarrow (E, W, M)\)

\(E_0 = E \cap W\) \hspace{1cm} \(E = \mathcal{I} \cdot E_0\)

\(\mathcal{I} = E \cap M\) \hspace{1cm} \(W = M_0 \cdot E_0\)

\(M_0 = M \cap W\) \hspace{1cm} \(M = M_0 \cdot \mathcal{I}_0\)

- \(W\) is closed under retracts in \(C^3\).
- When does \(W\) have the 2-out-of-3 property?
Double factorization systems $(\mathcal{E}_0, \mathcal{I}, \mathcal{M}_0)$:

$(\mathcal{E}_0, \mathcal{M}_0 \cdot \mathcal{I}), (\mathcal{I} \cdot \mathcal{E}_0, \mathcal{M}_0)$ fs,
$\mathcal{E}_0 \cdot \mathcal{M}_0 \subseteq \mathcal{M}_0 \cdot \mathcal{E}_0,$
$ej \in \mathcal{E}_0, e \in \mathcal{E}_0, j \in \mathcal{I} \implies j \text{ iso},$
$jm \in \mathcal{M}_0, m \in \mathcal{M}_0, j \in \mathcal{I} \implies j \text{ iso}.$

“Quillen factorization systems” $(\mathcal{E}, \mathcal{W}, \mathcal{M})$:

$(\mathcal{E} \cap \mathcal{W}, \mathcal{M}), (\mathcal{E}, \mathcal{M} \cap \mathcal{W})$ fs,
$\mathcal{W}$ has 2-out-of-3 property.

(Pultr-Tholen 2002)
Weak factorization system \((\mathcal{E}, \mathcal{M})\) in \(\mathcal{C}\)

\[
\begin{array}{c}
\begin{tikzcd}
\circ \\
\arrow{e}{u} \\
\end{tikzcd}
\end{array}
\]
\[
\begin{array}{c}
\begin{tikzcd}
\circ \\
\arrow{e}{u} \\
\end{tikzcd}
\end{array}
\]

(WFS*1&2) \(\mathcal{E} = \square \mathcal{M}, \mathcal{M} = \mathcal{E} \square\)

(WFS*3) \(\mathcal{C} = \mathcal{M} \cdot \mathcal{E}\)

(WFS*1a) \(gf \in \mathcal{E}, g\) split mono \(\implies f \in \mathcal{E}\)

(WFS*1b) \(gf \in \mathcal{M}, f\) split epi \(\implies g \in \mathcal{M}\)

(WFS*2) \(\mathcal{E} \square \mathcal{M}\)

(WFS*3) \(\mathcal{C} = \mathcal{M} \cdot \mathcal{E}\)
(Mono,Epi) in Set

- (Mono,Mono$\Box$) wfs in $\mathcal{C}$ with binary products and enough injectives
- (⊔, SplitEpi) wfs in every lextensive category $\mathcal{C}$
fs $\implies$ wfs

$E \Box$: closed under composition, direct products
stable under pullback, intersection

If $C$ has kernel pairs, any of the following will make a wfs $(E, M)$ an fs:

- $M$ closed under any type of limit
- $gf \in M, g \in M \implies f \in M$
- $gf = 1, g \in M \implies f \in M$

\[ \mathcal{C} \text{ finitely well-complete} \]

- reflective subcategories of \( \mathcal{C} \) (full, replete)
- factorization systems \((\mathcal{E}, \mathcal{M})\) with \( g f \in \mathcal{E}, g \in \mathcal{E} \implies f \in \mathcal{E} \)

\((\mathcal{E}, \mathcal{M}) \mapsto \mathcal{F}(\mathcal{M}) = \{ B \in \mathcal{C} \mid (B \to 1) \in \mathcal{M} \}\)
\( \mathcal{F} \) reflective in finitely complete \( \mathcal{C} \) with reflection \( \rho : 1 \to R \)

\[
(\mathcal{E}, \mathcal{M}) = (R^{-1}(\text{Iso}), \text{Cart}(R, \rho)) \quad \text{fs} \iff \forall f : A \to B : \quad \quad \quad (A \xrightarrow{(\rho_A, f)} RA \times_{RB} B) \in \mathcal{E}
\]

\( \mathcal{E} \) stable under pb along \( \mathcal{M} \) \iff \( \mathcal{F} = \mathcal{F}(\mathcal{M}) \) semilocalization

\( \mathcal{E} \) stable under pullback \iff \( \mathcal{F} = \mathcal{F}(\mathcal{M}) \) localization
\( \mathcal{C} \) with 0

\[(\mathcal{E}, \mathcal{M}) \) torsion theory \iff (\mathcal{E}, \mathcal{M}) \) fs,
\( \mathcal{E}, \mathcal{M} \) have 2-out-of-3 property

\[
\begin{align*}
\mathcal{F}(\mathcal{M}) &= \{ B \mid (B \to 0) \in \mathcal{M} \} \\
\mathcal{T}(\mathcal{E}) &= \{ A \mid (0 \to A) \in \mathcal{E} \}
\end{align*}
\]
$C$ with kernels and cokernels

\[
\begin{array}{cccccccccc}
SKC & \cong & SC & \overset{1}{\rightarrow} & SC & \rightarrow & 0 \\
\sigma_{KC} \cong & \alpha_C & \downarrow & & \downarrow & \sigma_C & \\
KC & \overset{\kappa_C}{\rightarrow} & C & \overset{\pi_C}{\rightarrow} & QC \\
\downarrow & & \downarrow \rho_C & & \downarrow \beta_C \cong & \rho_{QC} \\
0 & \rightarrow & RC & \overset{1}{\rightarrow} & RC & \cong & RQC
\end{array}
\]

$C \in \mathcal{F}(\mathcal{M}) \iff SC = 0 \iff KC = 0$

$C \in \mathcal{T}(\mathcal{E}) \iff RC = 0 \iff QC = 0$
\( \alpha_C \text{ iso} \iff \beta_C \text{ iso} \iff \pi_C \kappa_C = 0 \)

\( (\mathcal{E}, \mathcal{M}) \text{ simple} \implies (\mathcal{E}, \mathcal{M}) \text{ normal} \)

\[ \mathcal{C} \text{ homological, } \mathcal{C}^{\text{op}} \text{ homological:} \]

normal torsion theories \((\mathcal{E}, \mathcal{M}) \iff \text{ standard torsion theories } (\mathcal{T}, \mathcal{F}) \)

\[ 0 \to T \to C \to F \to 0 \]

\[ \mathcal{C}(\mathcal{T}, \mathcal{F}) = 0 \]
Example

\( \mathcal{C} \): abelian groups with \((4x = 0 \implies 2x = 0)\)

\( \mathcal{F} \): abelian groups with \(2x = 0\)