UNIVERSALITY OF COPRODUCTS IN CATEGORIES OF LAX ALGEBRAS

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Abstract. Categories of lax (T, V)-algebras are shown to have pullback-stable coproducts if T preserves inverse images. The general result not only gives a common proof of this property in many topological categories but also shows that important topological categories, like the category of uniform spaces, are not presentable as a category of lax (T, V)-algebras, with T preserving inverse images. Moreover, we show that any such category of (T, V)-algebras has a concrete, coproduct preserving functor into the category of topological spaces.

Universality of coproducts is a property that distinguishes Set-based topological categories: while in many "everyday" topological categories coproducts are stable under pullback (topological spaces, preordered sets, premetric spaces, approach spaces), some others fail to enjoy the property (uniform spaces, proximity spaces, nearness spaces, merotopic spaces, see [6]). All of the topological categories in the first group happen to be presentable as categories of lax algebras, that is: they are of the form $\mathsf{Alg}(T, V)$, for a suitable extension T of a Set-monad $T_0 = (T_0, e, m)$ and a complete lattice V that comes with an associative and commutative binary operation $\otimes$ preserving suprema in each variable, and a $\otimes$-neutral element k (distinct from the bottom element $\bot$), [5]. In this note, we show that this observation is not coincidental, that is: coproducts in $\mathsf{Alg}(T, V)$ are always stable under pullback, making $\mathsf{Alg}(T, V)$ in fact an (infinitely) extensive category, provided that T_0 preserves inverse images. This condition is weaker than preservation of the Beck–Chevalley Property, as used by Clementino and Hofmann (see [4]); for an example, see [7, Example 1.3(a)].

Of special importance are *open morphisms* of lax algebras, as defined by Möbus [9] in the context of relational algebras in a category, and by Clementino and Hofmann [4] in the context used in this paper. These generalize open morphisms in the category Top of topological space. We characterize coproducts in $\mathsf{Alg}(T, V)$ by the fact that all injections of the underlying Set-coproduct are open morphisms in $\mathsf{Alg}(T, V)$. Moreover, we construct a concrete functor $\mathsf{Alg}(T, V) \to \mathsf{Top}$ which preserves open embeddings and, hence, coproducts. This result generalizes a construction of Manes [8].

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1. Preliminaries

Recall that for sets X, Y, a *V-relation* (or *V-matrix*) $r : X \nrightarrow Y$ from X to Y is a mapping $r : X \times Y \to V$; it gets composed with $s : Y \nrightarrow Z$ via

$$(sr)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

The category $\mathsf{Mat}(V)$ of sets and V-matrices is, using the pointwise order on the hom-sets, enriched over the monoidal closed category Sup of complete lattices and suprema-preserving morphisms. That is, every hom-set $\mathsf{Mat}(V)(X, Y)$ is a complete lattice, and the composition functions preserve suprema in each variable:

$$s(\bigvee r_i) = \bigvee s r_i, \qquad (\bigvee r_i)q = \bigvee r_i q.$$

Every Set-map $f : X \to Y$ can be considered as a V-relation via $f(x, y) = k$ if $f(x) = y$, and $f(x, y) = \bot$ else. We also use the converse V-relation $f^* : Y \nrightarrow X$ with $f^*(y, x) = f(x, y)$ and take note of the inequalities $1_X \le f^* f$, $f f^* \le 1_Y$, which exhibit f^* as right adjoint to f in the 2-category $\mathsf{Mat}(V)$.

Let $T_0 = (T_0, e, m)$ be a Set-monad such that T_0 *preserves inverse images*; that is: T_0 preserves every Set-pullback

$$\begin{array}{ccc} f^{-1}[B] & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle m} \\ X & \xrightarrow{\;f\;} & Y \end{array}$$

when m is a monomorphism. In particular, T_0 preserves monomorphisms.

Examples for inverse image-preserving monads abound; for instance, the powerset monad, the filter monad, the ultrafilter monad, and the free-monoid monad all preserve inverse images. But there are Set-monads (T_0, e, m) for which T_0 does not preserve inverse images: consider, for example, the trivial monad with $T_0\emptyset = \emptyset$ and $T_0 X = 1$ for all $X \neq \emptyset$.

We now consider a lax extension $T : \mathsf{Mat}(V) \to \mathsf{Mat}(V)$ of T_0 to V-relations, which respects the order of V-relations and satisfies the following conditions

(0) $T_0 f \le T f$ and $(T_0 f)^* \le T f^*$,
(1) $e_Y r \le (Tr) e_X$,
(2) $m_Y (T^2 r) \le (Tr) m_X$,
(3) $(Ts)(Tr) \le T(sr)$,

for all $f : X \to Y$, $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$. On objects T coincides with T_0. As noted in [10], these conditions suffice to derive the identities

(4) $(Ts)(Tf) = T(sf) = (Ts)(T_0 f)$,
(5) $(Tg^*)(Tr) = T(g^* r) = (T_0 g)^*(Tr)$

with $g : Y \to Z$, which play a crucial role in [5]. Moreover, we assume that

(6) $T(rg^*) = (Tr)(T_0 g^*)$

whenever g is a *monomorphism*.

2. Graphs and lax algebras

The category $\mathsf{Gph}(T,V)$ of (T,V)-*graphs* has as objects sets X equipped with a V-relation $a : TX \nrightarrow X$ satisfying $1_X \le ae_X$; a morphism $f : (X,a) \to (Y,b)$ is a Set-map $f : X \to Y$ satisfying $fa \le bT_0 f$. The forgetful functor $\mathsf{Gph}(T,V) \to \mathsf{Set}$ is topological: for a (possibly large) family $(Y_i, b_i)_{i\in I}$ of (T,V)-graphs and mappings $f_i : X \to Y_i$, the so-called *initial* structure a on X (see [1]) is given by

$$a = \bigwedge_{i\in I} f_i^* b_i (T_0 f_i),$$

(see [5]). Hence $\mathsf{Gph}(T,V)$ is complete and cocomplete. In particular, $\mathsf{Gph}(T,V)$ has pullbacks, and they are formed by putting the initial structure with respect to the two projections on the Set-pullback.

Colimits in $\mathsf{Gph}(T,V)$ are formed as follows: given a diagram with vertices $(X_i, a_i)_I$, first form the colimit $(g_i : X_i \to Y)$ of the underlying diagram in Set, and then equip Y with the *final structure*

$$b = \bigvee_{i\in I} g_i a_i (T_0 g_i)^*. \tag{1}$$

It is easy to see that $1_Y \le be_Y$ is satisfied by virtue of $(g_i)_I$ being jointly epic; equivalently, since $\bigvee_I g_i g_i^* = 1_Y$.

A (T,V)-graph (X,a) satisfies the inequality $a \le a(Ta)m_X^*$; it is called a *(lax)* (T,V)-*algebra* if it satisfies the transitivity condition $a(Ta)m_X^* \le a$ or, equivalently, $a(Ta) \le am_X$. The full subcategory of $\mathsf{Gph}(T,V)$ formed by the algebras is denoted by $\mathsf{Alg}(T,V)$. It is concretely reflective, with the reflector defined as follows (see [3]): given a graph (X,a), form the transfinite sequence of V-relations

$$a_0 = a, \qquad a_{\alpha+1} = a_\alpha (Ta_\alpha) m_X^*, \qquad a_\gamma = \bigvee_{\beta<\gamma} a_\beta \tag{2}$$

for all ordinals α and limit ordinals γ. One obtains an ascending chain of V-relations, which has to become stationary at some α_0, we write $L(X,a)$ for (X, a_{α_0}). Then $1_X : (X,a) \to L(X,a)$ is the sought universal arrow.

Thus also $\mathsf{Alg}(T,V)$ is topological, limits in $\mathsf{Alg}(T,V)$ are formed like in $\mathsf{Gph}(T,V)$, and colimits are formed by applying the reflector to the colimit in $\mathsf{Gph}(T,V)$.

3. Coproducts

For any coproduct $(t_i : X_i \to X)$ in Set, we easily obtain $t_i^* t_i = 1_{X_i}$ and $t_i^* t_j = \bot$ if $i \neq j$ as well as $\bigvee t_i t_i^* = 1_X$. In fact, these equations make $(t_i : X_i \to X)$ a *biproduct* in $\mathsf{Mat}\, V$.

Consequently, for any coproduct $\big(t_i : (X_i, a_i) \to (X, c)\big)$ of graphs,

$$t_i^* c = t_i^* \bigvee t_j a_j (T_0 t_j)^* = \bigvee t_i^* t_j a_j (T_0 t_j)^* = a_i (T_0 t_i)^*. \tag{3}$$

In other words, each injection $t_i : (X_i, a_i) \to (X, c)$ is open in the sense of [4]: a morphism $f : (X, a) \to (Y, b)$ is *open* if $f^* b = a(T_0 f)^*$. Moreover, $t_i^* c T_0 t_i = a_i (T_0 t_i)^* T_0 t_i = a_i$ holds since T_0 since preserves monomorphisms. Therefore, each open injection is an *embedding*, i.e. its domain carries the initial structure.

To study coproducts of (T, V)-algebras, we first note:

Lemma 1. *If $f : (X, a) \to (Y, b)$ is an open injection from a (T, V)-algebra into a (T, V)-graph, then $f : (X, a) \to L(Y, b)$ is also open.*

Proof. It suffices to show that $f^* b_\alpha = a(T_0 f)^*$ holds for all ordinals α. Indeed, the initial step is trivial, and for a successor ordinal we have:

$$\begin{aligned} f^* b_{\alpha+1} &= f^* b_\alpha (T b_\alpha) m_Y^* \\ &= a(T_0 f)^* (T b_\alpha) m_Y^* \\ &= a T(f^* b_\alpha) m_Y^* \\ &= a T(a(T_0 f)^*) m_Y^* \\ &= a(Ta)(T_0 T_0 f)^* m_Y^* \\ &= a(Ta) m_X^* (T_0 f)^* \\ &= a(T_0 f)^*. \end{aligned}$$

In case β is a limit ordinal, we have $f^* b_\beta = f^* \bigvee_{\alpha<\beta} b_\alpha = \bigvee_{\alpha<\beta} f^* b_\alpha = \bigvee_{\alpha<\beta} a(T_0 f)^* = a(T_0 f)^*$. □

Theorem 2. *Let $(X_i, a_i)_I$ be a family of (T, V)-algebras and $(t_i : X_i \to X)_I$ be a Set-coproduct. The following statements are equivalent for a (T, V)-graph (X, a):*

1. *$\big(t_i : (X_i, a_i) \to (X, a)\big)_I$ is a coproduct in $\mathsf{Gph}(T, V)$;*
2. *$(X, a) \in \mathsf{Alg}(T, V)$, and $\big(t_i : (X_i, a_i) \to (X, a)\big)_I$ is a coproduct in $\mathsf{Alg}(T, V)$;*
3. *each $t_i : (X_i, a_i) \to (X, a)$ is open.*

Proof. (1) implies (2): Let (X, c) denote the coproduct of the (X_i, a_i) in $\mathsf{Alg}(T, V)$. By Lemma 1, each $t_i : (X_i, a_i) \to (X, c)$ is open. Thus $c = \bigvee t_i t_i^* c = \bigvee t_i a_i (T_0 t_i)^*$, and this is, by hypothesis, precisely the final structure a with respect to the coproduct injections.

(2) implies (3): Lemma 1.

(3) implies (1): From (3) one obtains $a = \bigvee t_i t_i^* a = \bigvee t_i a_i (T_0 t_i)^*$, so that, again, a is the final structure with respect to the coproduct injections. □

Lemma 3. *The following statements are equivalent for a set-indexed family* $\big(f_i : (X_i, a_i) \to (Y, b)\big)_I$ *of morphisms in* $\mathsf{Alg}(T, V)$*:*

(1) *each* f_i *is open;*
(2) *the induced morphism* $f : \coprod_I (X_i, a_i) \to (Y, b)$ *is open.*

Proof. (1) implies (2): Let $\big(t_i : (X_i, a_i) \to (X, a)\big)_I$ be a coproduct in $\mathsf{Alg}(T, V)$. Using the easily verified formula $f^* = \bigvee t_i f_i^*$, we obtain:

$$a(T_0 f)^* = \bigvee t_i a_i (T_0 t_i)^* (T_0 f)^* = \bigvee t_i a_i (T_0 f_i)^* = \bigvee t_i f_i^* b = f^* b$$

and thus f is open.

(2) implies (1): Follows from the closedness of open morphisms under composition. □

In particular, open morphisms are closed under coproducts.

Lemma 4. *Openness of embeddings is stable under pullback in* $\mathsf{Alg}(T, V)$*.*

For the proof we recall two well-known properties of $\mathsf{Mat}\, V$:

Remark 5. Set*-pullbacks are* Beck–Chevalley squares *in* $\mathsf{Mat}\, V$*. That is, given a set pullback*

$$\begin{array}{ccc} P & \xrightarrow{p} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

$f^* g = q p^*$ *holds in* $\mathsf{Mat}\, V$*.*

Proof. Straightforward. □

Remark 6. $\mathsf{Mat}\, V$ *satisfies the following weak form of* Freyd's Modular Law*: Given* V*-matrices* $r : X \nrightarrow Z$*,* $s : Y \nrightarrow Z$*, and an injective function* $m : X \to Y$*, we have*

$$(r \wedge sm)m^* = rm^* \wedge s. \tag{4}$$

Proof. We assume m to be a subset-inclusion, and obtain:

$$rm^*(y, z) = \begin{cases} r(y, z) & \text{if } y \in X, \\ \bot & \text{else.} \end{cases}$$

Thus the left-hand side as well as the right-hand side of (4) evaluate to $\bot$ for every (y, z) with $y \notin X$. Obviously both evaluate to $r(y, z) \wedge s(y, z)$ for all (y, z) such that $y \in X$, and hence (4) holds. □

Proof of Lemma 4. Suppose

$$\begin{array}{ccc} (P,d) & \xrightarrow{p} & (Y,b) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Z,c) \end{array}$$

is a pullback diagram in $\mathsf{Alg}(T,V)$ and $f:(X,a)\to(Z,c)$ is an open embedding. We have to show that also $p:(P,d)\to(Y,b)$ is open. Observe that with f also T_0p is injective, and that the T_0 image of the underlying Set-pullback is still a pullback, hence a Beck–Chevalley square. Thus we obtain $q^*a(T_0q)(T_0p)^* = q^*a(T_0f)^*(T_0g) = q^*f^*c(T_0g) = p^*g^*c(T_0g)$, hence

$$\begin{aligned} d(T_0p)^* &= (p^*b(T_0p) \wedge q^*a(T_0q))(T_0p)^* \\ &= p^*b \wedge q^*a(T_0q)(T_0p)^* \\ &= p^*b \wedge p^*g^*c(T_0g) \\ &= p^*\big(b \wedge g^*c(T_0g)\big) \\ &= p^*b. \end{aligned}$$

□

Lemma 4, together with Theorem 2 and the fact that coproducts in Set are universal, immediately yields:

Theorem 7. *Coproducts in* $\mathsf{Alg}(T,V)$ *are universal; that is: coproducts are stable under pullback.*

Corollary 8. $\mathsf{Alg}(T,V)$ *is (infinitely) extensive.*

Proof. Coproducts in $\mathsf{Alg}(T,V)$ are trivially disjoint, and this property in conjunction with universality gives extensitivity (see [2]). □

Since coproducts fail to be universal in Unif (see [6]), we also obtain:

Corollary 9. *The category of uniform spaces is not presentable in the form* $\mathsf{Alg}(T,V)$ *with* T_0 *preserving inverse images.*

Note, however, that Unif is a coreflective subcategory of a category of lax (T,V)-proalgebras (see [5]).

4. The topology of a (T,V)-algebra

Lemma 10. *gf open and f surjective implies g open.*

Proof. Let $f:(X,a)\to(Y,b)$ and $g:(Y,b)\to(Z,c)$ be morphisms such that gf is open and f is surjective; equivalently, $ff^*=1_Y$. It suffices to show $g^*c \le b(T_0g)^*$: $g^*c = ff^*g^*c = fa(T_0f)^*(T_0g)^*$ holds since gf is open, and thus, since f is a morphism, $g^*c \le b(T_0g)^*$. □

For a (T, V)-algebra (X, a), we write $\mathcal{O}(X, a)$ for the set of all open subalgebras of (X, a), that is,

$$\mathcal{O}(X, a) = \{\, \iota_A : A \hookrightarrow X \mid \iota_A \text{ is open w.r.t. the initial structure} \,\}.$$

Lemma 11. *$\mathcal{O}(X, a)$ is closed under arbitrary suprema and finite infima in the powerset PX of X.*

Proof. Suprema in PX can be described as follows: given a family $(\iota_{A_i} : A_i \hookrightarrow X)_I$ of subset-inclusions, first form the induced morphism $h = [\iota_{A_i}] : \coprod A_i \to X$, and then factor $h = \iota p$ with ι a subset-inclusion and p surjective. ι is the sought supremum.

Now if each ι_{A_i} is open, so is h by Lemma 3, and thus ι is open by Lemma 10.

By Lemma 4, the set $\mathcal{O}(X, a)$ is closed under binary intersections, and since every isomorphism is open, this implies closedness under finite intersections. □

Hence $\mathcal{O}(X, a)$ is a topology on X. Moreover, since open embeddings are stable under pullback in $\mathsf{Alg}(T, V)$, every lax morphism $f : (X, a) \to (Y, b)$ gives rise to a continuous function $f : \mathcal{O}(X, a) \to \mathcal{O}(Y, b)$. Thus:

Theorem 12. *There exists a concrete functor $O : \mathsf{Alg}(T, V) \to \mathsf{Top}$, defined by*

$$(X, a) \mapsto (X, \mathcal{O}(X, a)).$$

O preserves open embeddings and coproducts.

Proof. Clearly O is a concrete functor which preserves open embeddings. Preservation of coproducts follows thus from the characterization of coproducts in $\mathsf{Alg}(T, V)$ (and hence in Top) given in Theorem 2. □

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