

APPROXIMATE INJECTIVITY

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ABSTRACT. In a locally λ -presentable category, with λ a regular cardinal, classes of objects that are injective with respect to a family of morphisms whose domains and codomains are λ -presentable, are known to be characterized by their closure under products, λ -directed colimits and λ -pure subobjects. Replacing the strict commutativity of diagrams by “commutativity up to ε ”, this paper provides an “approximate version” of this characterization for categories enriched over metric spaces. It entails a detailed discussion of the needed ε -generalizations of the notion of λ -purity. The categorical theory is being applied to the locally \aleph_1 -presentable category of Banach spaces and their linear operators of norm at most 1, culminating in a largely categorical proof for the existence of the so-called Gurarii Banach space.

1. INTRODUCTION

Recall that an object K is *injective* to a morphism $f : A \rightarrow B$ in a category \mathcal{K} if, for every morphism $g : A \rightarrow K$, there is a morphism $h : B \rightarrow K$ with $hf = g$. There is a well-developed theory of injectivity in locally presentable categories (see [2]), playing an important role in both algebra and topology. This theory applies to Banach spaces, too, because the category **Ban** of (real or complex) Banach spaces and their linear operators of norm at most 1 is locally \aleph_1 -presentable (see [2], 1.48). But in this category there is another –and probably more important– concept, of so-called *approximate injectivity* (see [14]), which is based on the fact that **Ban** is enriched over metric spaces. The basic idea is to replace the commutativity of diagrams by their “commutativity up to ε ”. The aim of our paper is to develop a theory of approximate injectivity in metric enriched categories analogously to that of injectivity in ordinary locally presentable categories, and apply it to **Ban**. In particular, we present a largely categorical existence proof for the *Gurarii space*, which attracted renewed attention in several recent papers (see, for example, [3, 6, 11]).

Let us first clarify the categorical context of this paper in as concrete terms as possible. The category **Met** of metric spaces and non-expansive maps is neither complete

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nor complete, and the tensor product $X \otimes Y$, which for $X = (X, d), Y = (Y, d')$ puts the $+$ -metric $d \otimes d'((x, y), (x', y')) = d(x, x') + d'(y, y')$ on $X \times Y$, fails to make \mathbf{Met} monoidal closed. (Note that $X \otimes Y$ must not be confused with the Cartesian product $X \times Y$ in \mathbf{Met} , which is given by the max-metric.) One therefore enlarges \mathbf{Met} to the category \mathbf{Met}_∞ of *generalized metric spaces*, by allowing distances to be ∞ while keeping all other requirements, including the type of morphisms. Then \mathbf{Met}_∞ is complete and cocomplete and monoidal closed, with the internal hom providing the hom-set $\mathbf{Met}_\infty(X, Y)$ with the sup-metric $d''(f, g) = \sup\{d'(fx, gx) \mid x \in X\}$. In what follows, we will normally denote the (generalized) metric of a space by d , using annotations or variations only for the sake of clarity.

Throughout most of this paper, we will be considering a \mathbf{Met}_∞ -enriched category \mathcal{K} . Hence, \mathcal{K} has a class $\text{ob}\mathcal{K}$ of objects, and, for all $A, B, C \in \text{ob}\mathcal{K}$, there are hom-objects $\mathcal{K}(A, B)$ in \mathbf{Met}_∞ with non-expansive composition maps $\mathcal{K}(B, C) \otimes \mathcal{K}(A, B) \rightarrow \mathcal{K}(A, C)$ and units $1 \rightarrow \mathcal{K}(A, A)$ (where 1 is a one-point metric space), satisfying the expected associativity and unity conditions. Interpretation of a \mathbf{Met}_∞ -arrow $1 \rightarrow \mathcal{K}(A, B)$ as a morphism $A \rightarrow B$ defines the underlying ordinary category \mathcal{K}_0 of \mathcal{K} , which must be carefully distinguished from \mathcal{K} (see [7] for details). Often we will nevertheless call a morphism in \mathcal{K}_0 a morphism in \mathcal{K} . Should all hom-objects $\mathcal{K}(A, B)$ of the \mathbf{Met}_∞ -enriched category \mathcal{K} happen to be ordinary metric spaces, we will allow ourselves to briefly call \mathcal{K} a \mathbf{Met} -enriched category.

Our principal examples of \mathbf{Met} -enriched categories arise from concrete categories \mathcal{K} over \mathbf{Met} , so that one has a faithful functor $U : \mathcal{K} \rightarrow \mathbf{Met}$. Then the hom-set $\mathcal{K}(A, B)$ can be considered a subspace of $\mathbf{Met}_\infty(UA, UB)$, and the composition maps of \mathcal{K} remain non-expansive. The example of primary interest in this context is the category \mathbf{Ban} , to be considered as a concrete category over \mathbf{Met} via the unit ball functor $U : \mathbf{Ban} \rightarrow \mathbf{Met}$, given by $\mathbf{Ban}(l_1(1), -)$, where l_1 is a left adjoint to the unit ball functor $\mathbf{Ban} \rightarrow \mathbf{Set}$.

We have to carefully distinguish between limits in \mathcal{K}_0 and (*conical*) *limits* in \mathcal{K} , the latter being limits of the former type (*i.e.*, limits in \mathcal{K}_0) that are preserved by all representables $\mathcal{K}(K, -) : \mathcal{K}_0 \rightarrow \mathbf{Met}_\infty$ (see [7], section 3.8). For concrete \mathbf{Met} -categories, these are limits preserved by $U : \mathcal{K} \rightarrow \mathbf{Met}$. Likewise, *conical colimits* in \mathcal{K} are colimits in \mathcal{K}_0 preserved by the representables $\mathcal{K}(-, K) : \mathcal{K}_0^{\text{op}} \rightarrow \mathbf{Met}$, a property which, for concrete \mathbf{Met} -categories, reduces to the preservation of colimits by U .

Recall that, for a regular cardinal λ , an object K of the ordinary category \mathcal{K}_0 is λ -presentable if its representable functor $\mathcal{K}_0(K, -) : \mathcal{K}_0 \rightarrow \mathbf{Set}$ preserves λ -directed colimits. \mathcal{K}_0 is λ -accessible if it has all λ -directed colimits and a set \mathcal{A} of λ -presentable objects such that every object in \mathcal{K}_0 is a λ -directed colimit of objects in \mathcal{A} ; if all small colimits exist, \mathcal{K}_0 is *locally λ -presentable*. Accessible (locally presentable) means λ -accessible (locally λ -presentable, respectively) for some λ . The underlying ordinary category of \mathbf{Met}_∞ is locally \aleph_1 -presentable (see [13], 4.5(3)), and for λ an uncountable regular cardinal, a generalized metric space X is λ -presentable if, and only if, $|X| < \lambda$.

An object K in a \mathbf{Met}_∞ -enriched category \mathcal{K} is λ -presentable (in the enriched sense) if $\mathcal{K}(K, -) : \mathcal{K} \rightarrow \mathbf{Met}_\infty$ preserves λ -directed colimits. Again, the enriched notion must be distinguished from the ordinary notion of K being λ -presentable in \mathcal{K}_0 , which postulates only that the \mathbf{Set} -valued functor $\mathcal{K}(K, -) : \mathcal{K}_0 \rightarrow \mathbf{Set}$ preserve λ -directed colimits. Since, for λ uncountable, the forgetful functor $V : \mathbf{Met}_\infty \rightarrow \mathbf{Set}$ preserves λ -directed colimits, in this case λ -presentability of an object in \mathcal{K} implies its λ -presentability in \mathcal{K}_0 . But since V does not create λ -directed colimits, the converse statement generally fails. However, following [8] 5.5 and 7.4, it does hold for $\mathcal{K} = \mathbf{Met}_\infty$, as well as in the case $\mathcal{K} = \mathbf{Ban}$ when λ is uncountable, as we confirm now with the following Lemma.

Lemma 1.1. *Let λ be an uncountable regular cardinal. Then any Banach space λ -presentable in \mathbf{Ban}_0 is λ -presentable in \mathbf{Ban} .*

Proof. For an uncountable regular cardinal λ , a Banach space A is λ -presentable in \mathbf{Ban}_0 if and only if it has a dense subset of cardinality $< \lambda$. Consider a λ -directed colimit $(b_i : B_i \rightarrow B)_{i \in I}$ in \mathbf{Ban} and a Banach space A which is λ -presentable in \mathbf{Ban}_0 . We have to show that $\mathbf{Ban}(A, b_i) : \mathbf{Ban}(A, B_i) \rightarrow \mathbf{Ban}(A, B)$ is a λ -directed colimit in \mathbf{Met} . Clearly, V sends this cocone to a λ -directed colimit in \mathbf{Set} . Consider $f, g : A \rightarrow B$. It remains to be shown that $d(f, g) = \inf d(f_i, g_i)$, where $f = b_i f_i$ and $g = b_i g_i$. Since λ is uncountable, for each $a \in A$ there is $i \in I$ such that $d(fa, ga) = d(f_i a, g_i a)$. Since A has a dense subset of cardinality $< \lambda$, there is $i \in I$ such that $d(f, g) = d(f_i, g_i)$. \square

In the following section we briefly introduce the framework of ε -commutativity (= “commutativity up to ε ”) in \mathbf{Met}_∞ -enriched categories, as well as the ensuing notion of ε -(co)limit. Having presented ε -versions of the notion of pure subobject in Section 3, we proceed to give sufficient conditions for a class of objects in a locally presentable \mathbf{Met}_∞ -enriched category to be an ε -injectivity class (Theorem 4.8), which leads us to a full characterization of approximate injectivity (= ε -injectivity, for all $\varepsilon > 0$) classes, in terms of their closure under products, directed colimits and appropriately generalized pure subobjects (Theorem 5.5). The last section is devoted to presenting a categorical framework for constructing the Urysohn metric space and the Gurarii Banach space.

2. ε -HOMOTOPY

For any $\varepsilon \in [0, \infty]$ and morphisms $f, g : A \rightarrow B$ in a \mathbf{Met}_∞ -enriched category \mathcal{K} , we say that f is ε -homotopic (or ε -close [9]) to g , if $d(f, g) \leq \varepsilon$ in the generalized metric d of $\mathcal{K}(A, B)$; we write

$$f \sim_\varepsilon g \Leftrightarrow d(f, g) \leq \varepsilon.$$

$f : A \rightarrow B$ is an ε -homotopy equivalence if there exists $f' : B \rightarrow A$ with $f'f \sim_\varepsilon \text{id}_A$ and $ff' \sim_\varepsilon \text{id}_B$. (This concept is related to the Gromov-Hausdorff distance of A and B ; see [14], 2.4.)

The relation \sim_ε is reflexive and symmetric, but generally not transitive; it is preserved by composition from either side. ε -commutativity of diagrams in \mathcal{K} has the obvious meaning. For example, to say that

$$\begin{array}{ccc} B & \xrightarrow{\bar{g}} & D \\ f \uparrow & & \uparrow \bar{f} \\ A & \xrightarrow{g} & C \end{array}$$

is an ε -commutative square simply means $\bar{f}g \sim_\varepsilon \bar{g}f$.

Remark 2.1. Our motivation for using the homotopic terminology arises from the case $\mathcal{K} = \mathbf{Met}_\infty$, as follows. For $\varepsilon > 0$, let 2_ε be the space with two points whose distance is ε , and we put $2_0 = 1$. Then 2_ε is not finitely presentable because it is a colimit of the chain formed by the spaces $2_{\varepsilon+\frac{1}{n}}$. With the injection

$$i_\varepsilon : 1 + 1 \rightarrow 2_\varepsilon$$

we get the weak factorization system $(\text{cof}(i_\varepsilon), i_\varepsilon^\square)$ (see [1]). Clearly, for any morphism h in \mathbf{Met}_∞ , one has $i_\varepsilon^\square h$ if, and only if, $d(hx, hy) \leq \varepsilon$ implies that $d(x, y) \leq \varepsilon$ for all x, y in the domain of h , and since h is non-expansive, we have the converse implication too. Consequently, 2_ε is the induced cylinder object, *i.e.*, it is given by a weak factorization of the codiagonal

$$\nabla : 1 + 1 \xrightarrow{c_\varepsilon} 2_\varepsilon \xrightarrow{s_\varepsilon} 1$$

(see [12]). In general, for a space K , the cylinder object C_K is given by a weak factorization

$$\nabla : K + K \xrightarrow{c_K} C_K \xrightarrow{s_K} K.$$

Then, for morphisms $f, g : K \rightarrow L$ to admit a morphism $h : C_K \rightarrow L$ such that

$$\begin{array}{ccc} K + K & \xrightarrow{(f,g)} & L \\ & \searrow c_K & \nearrow h \\ & & C_K \end{array}$$

commutes means precisely that f and g are ε -homotopic. Indeed, $K + K$ is obtained from K by duplicating each $x \in K$ to x' and x'' and putting $d(x', x'') = \infty$ while $d(x', x'') = \varepsilon$ in C_K .

We note that

$$d(f, g) = \inf\{\varepsilon \geq 0 \mid f \sim_\varepsilon g\}$$

(where $d(f, g) = \sup\{d(fx, gx) \mid x \in K\}$ is the metric of $\mathbf{Met}(K, L)$).

Definition 2.2. Given a small category \mathcal{D} , an ε -*diagram* in the (underlying ordinary category of the) \mathbf{Met}_∞ -enriched category \mathcal{K} is given by maps $D_{i,j} : \mathcal{D}(d, d') \rightarrow \mathcal{K}(Di, Dj)$ ($i, j \in \mathcal{D}$), such that $D\text{id}_i = \text{id}_{Di}$ and $D(gf) \sim_\varepsilon D(g)D(f)$, for all composable morphisms f, g in \mathcal{D} . An ε -*cocone* $\delta : D \rightarrow K$ in \mathcal{K} with vertex K is a family $(\delta_i : Di \rightarrow K)_{i \in \mathcal{D}}$ of morphisms in \mathcal{K} such that $\delta_j Df \sim_\varepsilon \delta_i$ for every $f : i \rightarrow j$ in \mathcal{D} . An ε -*colimit* of D is an ε -cocone $\delta : D \rightarrow K$ such that, for every ε -cocone $\gamma : D \rightarrow L$, there is a unique morphism $t : K \rightarrow L$ in \mathcal{K} such that $t\delta_i = \gamma_i$ for all $i \in \mathcal{D}$.

Up to isomorphism, ε -colimits are uniquely determined. We denote the ε -colimit of D by $\text{colim}_\varepsilon D$. In case of a discrete diagram, the ε -notion of colimit coincides with the ordinary one: ε -coproducts are precisely coproducts. An ε -pushout is an ε -commutative square

$$\begin{array}{ccc} B & \xrightarrow{\bar{g}} & D \\ f \uparrow & & \uparrow \bar{f} \\ A & \xrightarrow{g} & C \end{array}$$

such that for any ε -commutative square

$$\begin{array}{ccc} B & \xrightarrow{g'} & D' \\ f \uparrow & & \uparrow f' \\ A & \xrightarrow{g} & C \end{array}$$

there is a unique $t : D \rightarrow D'$ such that $t\bar{f} = f'$ and $t\bar{g} = g'$.

Lemma 2.3. \mathbf{Met}_∞ has ε -pushouts.

Proof. Consider $f : A \rightarrow B$ and $g : A \rightarrow C$. In the coproduct $B + C$ we have $d(fx, gx) = \infty$ for all $x \in A$. Changing all distances $d(fx, gx)$ to ε gives a distance function that satisfies all axioms of a generalized metric but the triangle inequality. Such structures are called semimetrics and, following [13] 4.5(3), \mathbf{Met}_∞ is reflective in the corresponding category \mathbf{SMet}_∞ : the reflector provides a semimetric space (X, d) with the metric \bar{d} given by

$$\bar{d}(x, z) = \inf \left\{ \sum_{i=0}^{n-1} d(y_i, y_{i+1}) \mid n \geq 1, y_i \in X, y_0 = x, y_n = z \right\}.$$

It now suffices to take the reflection D of the resulting semimetric space to \mathbf{Met}_∞ to obtain an ε -pushout in \mathbf{Met}_∞ . \square

Corollary 2.4. \mathbf{Met}_∞ has ε -colimits.

Proof. An ε -coequalizer

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \xrightarrow{h} D$$

may be given by an ε -pushout

$$\begin{array}{ccc} B & \xrightarrow{h} & D \\ \uparrow (f, \text{id}_B) & & \uparrow h \\ A + B & \xrightarrow{(g, \text{id}_B)} & B, \end{array}$$

and ε -colimits are constructed with the help of coproducts and ε -coequalizers in the same way as colimits are constructed from coproducts and coequalizers. \square

Corollary 2.5. \mathbf{Ban} has ε -colimits.

Proof. Let \mathcal{S} be a representative full subcategory of separable Banach spaces. Consider the functor

$$E : \mathbf{Ban} \rightarrow \mathbf{Met}^{\mathcal{S}\text{op}}$$

given by $(EK)(A) = \mathbf{Ban}(K, A)$. The functor E preserves limits and, following 1.1, it preserves \aleph_1 -directed colimits. Since E is a full embedding, it makes \mathbf{Ban} a reflective full subcategory of $\mathbf{Met}^{\mathcal{S}\text{op}}$. Following 2.4, $\mathbf{Met}^{\mathcal{S}\text{op}}$ has ε -colimits calculated pointwise. Given an ε -diagram $D : \mathcal{D} \rightarrow \mathbf{Ban}$, its ε -colimits is given by a reflection of the ε -colimit of ED . \square

Remark 2.6. For a diagram D in a \mathbf{Met}_∞ -enriched category \mathcal{K} , existence of the needed colimits granted, the induced morphism

$$q : \text{colim}_\varepsilon D \rightarrow \text{colim } D$$

is an ε -monomorphism; that is: whenever defined, $qf = qg$ always implies $f \sim_\varepsilon g$.

3. ε -PURITY

The notion of λ -pure morphism in a locally λ -presentable category as given in [2] allows for an obvious generalization in the case of a \mathbf{Met}_∞ -enriched category, as follows. The latter notion entails the former when one puts $\varepsilon = 0$.

Definition 3.1. Let \mathcal{K} be a \mathbf{Met}_∞ -enriched category and λ a regular cardinal. We say that a morphism $f : K \rightarrow L$ is λ - ε -pure if for any ε -commutative square

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ \uparrow u & & \uparrow v \\ A & \xrightarrow{g} & B \end{array}$$

with A and B λ -presentable in \mathcal{K} there exists $t : B \rightarrow K$ such that $tg \sim_\varepsilon u$.

Remark 3.2. (1) A composite of λ - ε -pure morphisms is λ - ε -pure.

(2) If $f_2 f_1$ is λ - ε -pure, then f_1 is λ - ε -pure.

(3) Every split monomorphism is λ - ε -pure, for any λ . (Indeed, when $pf = \text{id}_A$, consider the ε -commutative square of 3.1. Then $fu \sim_\varepsilon vfg$ implies $u = pfu \sim_\varepsilon pvg$.)

(4) Every λ - ε -pure morphism is λ' - ε -pure, for all $\lambda' \leq \lambda$.

(5) The λ -0-pure morphisms are precisely the λ -pure morphisms (as defined in [2]).

Before discussing λ - ε -purity further, let us also consider some variations of the notion.

Definition 3.3. Let $f : K \rightarrow L$ be a morphism in the \mathbf{Met}_∞ -enriched category \mathcal{K} . We say that a morphism $f : K \rightarrow L$ is *weakly (barely) λ - ε -pure* if for every ε -commutative (commutative) square as in 3.1, with A and B λ -presentable in \mathcal{K} , there exists $t : B \rightarrow K$ such that $tg \sim_{2\varepsilon} u$ ($tg \sim_\varepsilon u$, respectively). We also say that f is ε -split if there is $p : L \rightarrow K$ in \mathcal{K} with $pf \sim_\varepsilon \text{id}_K$.

The following easily verified statements all rely on the transitivity rule for ε -homotopy, which just rephrases the triangle inequality:

$$f \sim_\varepsilon g \quad \text{and} \quad g \sim_\delta h \implies f \sim_{\varepsilon+\delta} h.$$

Lemma 3.4. (1) *Every λ - ε -pure morphism is weakly λ - ε -pure and barely λ - ε -pure.*

(2) *Every weakly λ - ε -pure morphism is barely λ - 2ε -pure.*

(3) *Every split monomorphism is ε -split, and every ε -split morphism is both, weakly and barely λ - ε -pure, for any λ , and it is a 2ε -monomorphism.*

Remark 3.5. (1) Note that an ε -split morphism does not need to be a monomorphism, not even an ε -monomorphism: in \mathbf{Met}_∞ , for $0 < \varepsilon < \infty$, consider $\{a, b, c\} \rightarrow 1$ with $d(a, b) = d(b, c) = \varepsilon$ and $d(a, c) = 2\varepsilon$.

(2) *For λ uncountable, every barely λ - ε -pure morphism in \mathbf{Met}_∞ is 2ε -monomorphic.* Indeed, for $f : K \rightarrow L$ barely λ - ε -pure, consider $a, b \in K$ with $fa = fb$. With

$\delta = d(a, b)$ we exploit the commutative square

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ u \uparrow & & \uparrow v \\ 2_\delta & \xrightarrow{g} & 1, \end{array}$$

with u mapping 2_δ onto $\{a, b\}$. Since 2_δ and 1 are λ -presentable, there is $t : 1 \rightarrow K$ such that $tg \sim_\varepsilon u$. With c the image of t , this forces $d(a, b) \leq d(a, c) + d(b, c) \leq 2\varepsilon$.

Let us also record to which extent λ - ε -purity gets transported along ε -homotopy:

Lemma 3.6. *Let $f \sim_\varepsilon f'$.*

- (1) *If f is λ - 2ε -pure, then f' is weakly λ - ε -pure.*
- (2) *If f is λ - ε -pure, then f' is barely λ - ε -pure.*

In conjunction with Remark 3.2 (2), Lemma 3.6 gives:

Corollary 3.7. *Let $gf \sim_\varepsilon h$.*

- (1) *If h is λ - 2ε -pure, then f is weakly λ - ε -pure.*
- (2) *If h is λ - ε -pure, then f is barely λ - ε -pure.*

We are now ready to prove an important stability property of λ - ε -pure morphisms:

Proposition 3.8. *Let \mathcal{K} be a \mathbf{Met}_∞ -enriched category and λ be an uncountable regular cardinal. Then λ - ε -pure morphisms in \mathcal{K} are closed under λ -directed colimits in \mathcal{K} .*

Proof. Let $E : \mathcal{E} \rightarrow \mathcal{K}^\rightarrow$ be a λ -directed diagram in the morphism category of \mathcal{K} with $Ee : K_e \rightarrow L_e$ λ - ε -pure for all e in \mathcal{E} . For $f = \operatorname{colim} E : K \rightarrow L$ in \mathcal{K} with a colimit cocone $(k_e, l_e) : Ee \rightarrow f$ we have that $k_e : K_e \rightarrow K$ and $l_e : L_e \rightarrow L$ are colimits in \mathcal{K} . Consider an ε -commutative square

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ u \uparrow & & \uparrow v \\ A & \xrightarrow{g} & B \end{array}$$

with A and B λ -presentable in \mathcal{K} .

We will show that there are e_0 in \mathcal{E} and $u_{e_0} : A \rightarrow K_{e_0}$, $v_{e_0} : B \rightarrow L_{e_0}$ in \mathcal{K} , such that $u = k_{e_0}u_{e_0}$, $v = l_{e_0}v_{e_0}$ and $(Ee_0)u_{e_0} \sim_\varepsilon v_{e_0}g$. As Ee_0 is λ - ε -pure, there is then $t : B \rightarrow K_{e_0}$ in \mathcal{K} with $tg \sim_\varepsilon u_{e_0}$. Hence, $u = k_{e_0}u_{e_0} \sim_\varepsilon k_{e_0}tg$, and the proof for f to be λ - ε -pure will be complete.

Indeed, since A and B are λ -presentable in \mathcal{K}_0 , first one finds e in \mathcal{E} and $u_e : A \rightarrow K_e$ and $v_e : B \rightarrow L_e$ with $u = k_e u_e$ and $v = l_e v_e$. Since $l_e(Ee)u_e = f k_e u_e = f u$ and $l_e v_e g = v g$,

$$d(l_e(Ee)u_e, l_e v_e g) \leq \varepsilon,$$

follows. Since A is λ -presentable in \mathcal{K} , $\mathcal{K}(A, l_e) : \mathcal{K}(A, L_e) \rightarrow \mathcal{K}(A, L)$ is a colimit in \mathbf{Met}_∞ . By the construction of directed colimits in $\tilde{\mathbf{Met}}$, whereby

$$d(l_e(Ee)u_e, l_e v_e g) = \inf_{e' \geq e} d(l_{e,e'}(Ee)u_e, l_{e,e'} v_e g),$$

with $(k_{e,e'}, l_{e,e'}) : Ee \rightarrow Ee'$ given by the diagram E , for all $n = 1, 2, \dots$, there are then $e_n \geq e$ in \mathcal{E} with $d(l_{e,e_n}(Ee)u_e, l_{e,e_n} v_e g) \leq \varepsilon + \frac{1}{n}$. Finally, since λ is uncountable, we can find $e_0 \geq e_n$ for all n and obtain $u = k_{e_0} u_{e_0}$, $v = l_{e_0} v_{e_0}$ and $(Ee_0)u_{e_0} \sim_\varepsilon v_{e_0} g$. \square

Remark 3.9. As in Proposition 3.8 one proves that the classes of weakly and barely λ - ε -pure morphisms are both closed under λ -directed colimits in \mathcal{K} , for λ uncountable.

Corollary 3.10. *Let λ be a regular uncountable cardinal and \mathcal{K} be a \mathbf{Met}_∞ -enriched category with λ -directed colimits such that \mathcal{K}_0 is locally λ -presentable. Then every λ -pure morphism is λ - ε -pure, for all $\varepsilon \geq 0$.*

Proof. Since λ -pure morphisms are λ -directed colimits of split morphisms (see [2] 2.30), the result follows from Remark 3.2(3) and Proposition 3.8. \square

We can finally give the following characterization of barely λ - ε -pure morphisms in a large class of categories.

Proposition 3.11. *Let λ be an uncountable regular cardinal and \mathcal{K} be a $\tilde{\mathbf{Met}}$ -enriched category with λ -directed colimits and ε -pushouts such that \mathcal{K}_0 is locally λ -presentable. Then the following assertions are equivalent for a morphism f in \mathcal{K} :*

- (i) f is barely λ - ε -pure;
- (ii) there are g, h such that $gf \sim_\varepsilon h$, with h is λ -pure;
- (iii) there are g, h such that $gf \sim_\varepsilon h$, with h is λ - ε -pure.

Proof. (i) \Rightarrow (ii): Assume that $f : A \rightarrow B$ is barely λ - ε -pure. We proceed as in the proof of [2] 2.30(ii) and express f as a λ -directed colimit of morphisms $f_i : A_i \rightarrow B_i$ ($i \in I$), with A_i and B_i λ -presentable. Since f is barely λ - ε -pure, for every i there is $t_i : B_i \rightarrow A$ such that $t_i f_i \sim_\varepsilon u_i$. Therefore, in the ε -pushouts

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}_i} & \bar{B}_i \\ \uparrow u_i & & \uparrow \bar{u}_i \\ A_i & \xrightarrow{f_i} & B_i \end{array}$$

every \bar{f}_i is a split monomorphism. We get a λ -directed diagram $(\text{id}_A, \bar{b}_{ij}) : \bar{f}_i \rightarrow \bar{f}_j$. Its colimit $\bar{f} : A \rightarrow \bar{B}$ is λ -pure (as a λ -directed colimit of split monomorphisms), and it may be realized as the ε -pushout

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & \bar{B} \\ \text{id}_A \uparrow & & \uparrow g \\ A & \xrightarrow{f} & B, \end{array}$$

so that we have $gf \sim_\varepsilon \bar{f}$.

(ii) \Rightarrow (iii): Corollary 3.10.

(iii) \Rightarrow (i): Corollary 3.7(2). □

4. ε -INJECTIVITY

Definition 4.1. Let \mathcal{K} be a \mathbf{Met}_∞ -enriched category. Given a morphism $f : A \rightarrow B$, we say that an object K is ε -injective to f if, for every $g : A \rightarrow K$, there exists $h : B \rightarrow K$ in \mathcal{K} with $hf \sim_\varepsilon g$.

Remark 4.2. (1) 0-injectivity coincides with the ordinary injectivity notion.

(2) If K is ε -injective to f , then K is also ε' -injective to f , for all $\varepsilon' \geq \varepsilon$.

(3) K is ∞ -injective to $f : A \rightarrow B$ precisely when $\mathcal{K}(B, K) = \emptyset$ only if $\mathcal{K}(A, K) = \emptyset$.

For a class \mathcal{F} of morphisms in \mathcal{K} , we denote by

$$\text{Inj}_\varepsilon \mathcal{F}$$

the class of objects ε -injective to every $f \in \mathcal{F}$. Trivially, following 4.2(2), $\text{Inj}_\varepsilon \mathcal{F} \subseteq \text{Inj}_{\varepsilon'} \mathcal{F}$ whenever $\varepsilon \leq \varepsilon'$. A class of objects in \mathcal{K} is an ε -injectivity class if, for some \mathcal{F} , it is of the form $\text{Inj}_\varepsilon \mathcal{F}$, and if \mathcal{F} is a set, then $\text{Inj}_\varepsilon \mathcal{F}$ is called a *small ε -injectivity class*. If the domains and codomains of morphisms in \mathcal{F} are all λ -presentable in \mathcal{K} , $\text{Inj}_\varepsilon \mathcal{F}$ is called a *λ - ε -injectivity class*. If \mathcal{K}_0 is locally λ -presentable then every λ - ε -injectivity class is a small ε -injectivity class.

Compatibility of \sim_ε with the category composition immediately gives the expected closure properties of ε -injectivity classes, as follows.

Lemma 4.3. *Let \mathcal{L} be an ε -injectivity class in the \mathbf{Met}_∞ -enriched category \mathcal{K} with products. Then \mathcal{L} is closed under retracts, and \mathcal{L} is also closed under products in \mathcal{K} .*

Proof. Closure under retracts is obvious. For the product of a family of ε -injective objects K_i , since the canonical

$$\mathcal{K}(A, \prod_{i \in I} K_i) \rightarrow \prod_{i \in I} \mathcal{K}(A, K_i), \quad g \mapsto (gp_i)_{i \in I},$$

(with product projections p_i) is an isomorphism, one has

$$\forall i \in I (p_i g \sim_\varepsilon p_i g') \implies g \sim_\varepsilon g'$$

whenever $g, g' : A \rightarrow \prod_{i \in I} K_i$ in \mathcal{K} , a property which is immediately seen to guarantee product stability of the ε -injectivity class. \square

Lemma 4.4. *Let \mathcal{K} be a \mathbf{Met}_∞ -enriched category such that \mathcal{K}_0 is locally presentable. Then every small ε -injectivity class in \mathcal{K} is closed under λ -directed colimits, for some regular cardinal λ .*

Proof. For any given set \mathcal{F} of morphisms in \mathcal{K} we can find λ such that the domains of morphisms from \mathcal{F} are all λ -presentable. The proof that $\text{Inj}_\varepsilon \mathcal{F}$ is closed under λ -directed colimits is then straightforward. \square

We say that a class \mathcal{L} of objects is *closed under (weakly) λ - ε -pure morphisms* in \mathcal{K} if, for every (weakly) λ - ε -pure morphism $K \rightarrow L$, with L in \mathcal{L} one has also K in \mathcal{L} .

Lemma 4.5. *Let \mathcal{K} be a \mathbf{Met}_∞ -enriched category having all objects presentable. Then every small ε -injectivity class in \mathcal{K} is closed under λ - ε -pure morphisms, for some regular cardinal λ .*

Proof. Take λ such that the domains and the codomains of morphisms in \mathcal{F} are all λ -presentable in \mathcal{K} . Let $p : K \rightarrow L$ be λ - ε -pure with L in $\text{Inj}_\varepsilon \mathcal{F}$, and consider $f : A \rightarrow B$ in \mathcal{F} and any $g : A \rightarrow K$. ε -injectivity of L gives $h : B \rightarrow L$ such that $gf \sim_\varepsilon pg$, and then λ - ε -purity of p gives a morphism $t : B \rightarrow K$ with $tf \sim_\varepsilon g$. \square

We now have the tools enabling us to state:

Proposition 4.6. *Let \mathcal{K} be a \mathbf{Met}_∞ -enriched category with products, such that all objects in \mathcal{K} are presentable and the ordinary category \mathcal{K}_0 is locally presentable. Then every small ε -injectivity class in \mathcal{K} is a small injectivity class.*

Proof. According to Theorem 4.8 in [2], it suffices to show that a small ε -injectivity class is accessible and accessibly embedded into the ambient locally presentable category, as well as closed under products. While the latter condition is satisfied by Lemma 4.3, the former two conditions are guaranteed by Lemmas 4.4 and 4.5, in conjunction with Corollary 2.36 in [2]. \square

Remark 4.7. By Theorem 2.2 in [15], in a locally λ -presentable category, λ -injectivity (= λ -0-injectivity) classes are characterized by closure under products, λ -directed colimits and λ -pure subobjects. Consequently, every λ - ε -injectivity class in a category satisfying the hypotheses of Proposition 4.6 is a λ -injectivity class.

Theorem 4.8. *Let λ be an uncountable regular cardinal and \mathcal{K} a \mathbf{Met}_∞ -enriched category with λ -directed colimits, such that \mathcal{K}_0 is locally λ -presentable and any λ -presentable object in \mathcal{K}_0 is λ -presentable in \mathcal{K} . Then every class \mathcal{L} of objects in \mathcal{K} closed under products, λ -directed colimits and weakly λ - ε -pure morphisms is a λ - ε -injectivity class and, in particular, a small injectivity class.*

Proof. Let \mathcal{L} be closed under products, λ -directed colimits and weakly λ - ε -pure morphisms. We will follow the proof of [15] 2.2. Following 3.10 and [2] 2.36 and 4.8, \mathcal{L} is weakly reflective. This means that every K in \mathcal{K} comes with a morphism $r_K : K \rightarrow K^*$, $K^* \in \mathcal{L}$ such that every object of \mathcal{L} is injective to r_K . Let \mathcal{F} consist of all morphisms $f : A \rightarrow B$ such that A and B are λ -presentable and every object of \mathcal{L} is ε -injective to f . By the definition of \mathcal{F} we have $\mathcal{L} \subseteq \text{Inj}_\varepsilon \mathcal{F}$, and the converse inclusion $\text{Inj}_\varepsilon \mathcal{F} \subseteq \mathcal{L}$ will follow from the closure of \mathcal{L} under weakly λ - ε -pure morphisms once we have shown that, for $K \in \text{Inj}_\varepsilon \mathcal{F}$, any weak reflection of K into \mathcal{L} is weakly λ - ε -pure.

Thus, given $K \in \text{Inj}_\varepsilon \mathcal{F}$ and a weak reflection $r : K \rightarrow K^*$ in \mathcal{L} , we are to prove that in any ε -commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow u & & \downarrow v \\ K & \xrightarrow{r} & K^* \end{array}$$

with A and B λ -presentable the morphism u 2ε -factors through h . We will say that $(u, v) : h \rightarrow r$ is an ε -morphism in this situation.

Claim: There is a factorization $u = u_2 \cdot u_1$ and an ε -morphism $(u_1, v_1) : h \rightarrow \bar{r}$ where $\bar{r} : \bar{K} \rightarrow \bar{K}^*$ is a weak reflection into \mathcal{L} of a λ -presentable \bar{K} .

Proof of claim. Consider all ε -morphisms $(u_1, v_1) : h \rightarrow \bar{r}$ where $\bar{r} : \bar{K} \rightarrow \bar{K}^*$ is a weak reflection of \bar{K} in \mathcal{L} and $u = u_2 \cdot u_1$ for some u_2 . Since $(u, v) : h \rightarrow r$ is such an ε -morphism, we can take the smallest α such that \bar{K} is α -presentable. We are to prove $\alpha \leq \lambda$. Assuming $\alpha > \lambda$ we will obtain a contradiction. As in [15], we express \bar{K} as a colimit of a smooth chain $k_{ij} : K_i \rightarrow K_j$ ($i \leq j < \alpha$) of objects K_i of presentability less than α . This provides weak reflections $r_i : K_i \rightarrow K_i^*$ into \mathcal{L} such that their colimit $r_\alpha : \bar{K} \rightarrow K_\alpha^*$ factorizes through \bar{r} , i.e., $r_\alpha = s\bar{r}$ for some $s : \bar{K}^* \rightarrow K_\alpha^*$. Since $\bar{r}u_1 \sim_\varepsilon v_1h$, we have $r_\alpha u_1 = s\bar{r}u_1 \sim_\varepsilon sv_1h$, so that $(u_1, sv_1) : h \rightarrow r_\alpha$ is an ε -morphism. In the same way as in the proof of 3.8, this ε -morphism ε -factors through some r_i , $i < \alpha$. This means that there is an ε -morphism $h \rightarrow r_i$, which contradicts the minimality of α and proves the claim.

We are ready to prove that u 2ε -factors through h . Let us consider a factorization $u = u_2 \cdot u_1$ and a morphism $(u_1, v_1) : h \rightarrow \bar{r}$ as in the above claim. Let us express \bar{K}^* as a λ -directed colimit of λ -presentable objects Q_t , $t \in T$, with a colimit cocone $q_t : Q_t \rightarrow \bar{K}^*$. Since both \bar{K} and B are λ -presentable, the morphisms \bar{r} and v_1 both factor through q_{t_0} for some $t_0 \in T$. Since A is λ -presentable, there then exists $t_1 \geq t_0$ in T with an ε -commutative diagram, as follows:

$$\begin{array}{ccc}
 A & \xrightarrow{h} & B \\
 \downarrow u_1 & & \swarrow \tilde{v}_1 \\
 & & Q_{t_1} \\
 \tilde{r} \nearrow & & \searrow q_{t_1} \\
 \bar{K} & \xrightarrow{\bar{r}} & \bar{K}^* \\
 & & \downarrow v_1
 \end{array}$$

Since all objects of \mathcal{L} are injective to \bar{r} , they are also injective to \tilde{r} ; moreover, \bar{K} and Q_{t_1} are both λ -presentable. Thus $\tilde{r} \in \mathcal{F}$. This implies that K is ε -injective to \tilde{r} . Choosing $d : Q_{t_1} \rightarrow K$ with $u_2 \sim_\varepsilon d\tilde{r}$ we obtain

$$u = u_2 u_1 \sim_\varepsilon d\tilde{r} u_1 \sim_\varepsilon d\tilde{v}_1 h.$$

Hence, r is weakly λ - ε -pure, and thus K lies in \mathcal{L} . \square

Remark 4.9. (1) Let

$$\begin{array}{ccc}
 B & \xrightarrow{\bar{g}} & D \\
 \uparrow f & & \uparrow \bar{f} \\
 A & \xrightarrow{g} & C
 \end{array}$$

be an ε -pushout and K be ε -injective to f . Then K is injective to \bar{f} . Indeed, considering $u : C \rightarrow K$ we obtain $v : B \rightarrow K$ such that $vf \sim_\varepsilon u$. Thus there is $w : D \rightarrow K$ such that $w\bar{f} = u$.

(2) Let

$$\begin{array}{ccc}
 A & \xrightarrow{\bar{f}} & \bar{B} \\
 \uparrow \text{id}_A & & \uparrow p \\
 A & \xrightarrow{f} & B
 \end{array}$$

be an ε -pushout as in the proof of 3.11 (which corresponds to the mapping cylinder in homotopy theory). Then an object K is ε -injective to f if and only if it injective to \bar{f} . Indeed, the “if”-part of this statement follows from (1), and the converse is evident.

(3) If λ is an uncountable regular cardinal and A, B are λ -presentable, then \bar{B} in (2) is λ -presentable. The verification is analogous to that in the proof of 3.8.

This yields a direct proof of 4.7.

Problem 4.10. Let λ be an uncountable regular cardinal and \mathcal{K} a \mathbf{Met}_∞ -enriched category with λ -directed colimits, such that \mathcal{K}_0 is locally λ -presentable and any λ -presentable object in \mathcal{K}_0 is λ -presentable in \mathcal{K} . Are λ - ε -injectivity classes in \mathcal{K} precisely classes closed under products, λ -directed colimits and λ - ε -pure morphisms?

5. APPROXIMATE INJECTIVITY

The following definition is motivated by [14] 3.2.

Definition 5.1. Let \mathcal{K} be a \mathbf{Met}_∞ -enriched category. We say that an object K is *approximately injective* to $f : A \rightarrow B$ in \mathcal{K} if it is ε -injective to f for every $\varepsilon > 0$.

The class of objects in \mathcal{K} approximately injective to a class \mathcal{F} of morphisms in \mathcal{K} will be denoted $\text{Inj}_{\text{ap}} \mathcal{F}$. If \mathcal{F} is a set, then $\text{Inj}_{\text{ap}} \mathcal{F}$ is called an *approximate small injectivity class*. If the domains and the codomains of all morphisms in \mathcal{F} are λ -presentable in \mathcal{K} , $\text{Inj}_{\text{ap}} \mathcal{F}$ is called an *approximate λ -injectivity class*. If \mathcal{K}_0 is locally presentable, then any approximate λ -injectivity class is an approximate small injectivity class.

Definition 5.2. A morphism in \mathcal{K} is *(weakly, barely) λ -ap-pure* if it is (weakly, barely) λ - ε -pure for every $\varepsilon > 0$.

Remark 5.3. (1) A composite of λ -ap-pure morphisms is λ -ap-pure.

(2) If the composite morphism $f_2 f_1$ is λ -ap-pure, then f_1 is also λ -ap-pure.

(3) Let λ be an uncountable regular cardinal and \mathcal{K} be a \mathbf{Met}_∞ -enriched category with λ -directed colimits and ε -pushouts, such that \mathcal{K}_0 is locally λ -presentable. Then every barely λ -ap-pure morphism is a monomorphism. Indeed, considering the ε -pushouts

$$\begin{array}{ccc} A & \xrightarrow{f_\varepsilon} & B_\varepsilon \\ \text{id}_A \uparrow & & \uparrow g \\ A & \xrightarrow{f} & B, \end{array}$$

and applying the characterization 3.11(ii), since λ -pure morphisms in an accessible category are monomorphisms, we see that every f_ε is monic. Consequently, as a directed colimit of these morphisms, also f is a monomorphism.

(4) It follows easily from Lemma 3.4(2) that every weakly λ -ap-pure morphism is barely λ -ap-pure and, hence, a monomorphism, by (3). Consequently, rather than referring to its closure under weakly λ -ap-pure morphisms we may say that a class \mathcal{L} of objects be *closed under weakly λ -ap-pure subobjects*.

Lemma 5.4. Let λ be an uncountable regular cardinal and \mathcal{K} a $\tilde{\mathbf{Met}}$ -enriched category with λ -directed colimits, such that \mathcal{K}_0 is locally λ -presentable. Then every λ -pure morphism is λ -ap-pure.

Proof. The statement follows from Corollary 3.10. \square

Theorem 5.5. *Let λ be an uncountable regular cardinal and \mathcal{K} a \mathbf{Met}_∞ -enriched category with λ -directed colimits, such that \mathcal{K}_0 is locally λ -presentable and every λ -presentable object in \mathcal{K}_0 is λ -presentable in \mathcal{K} . Then the approximate λ -injectivity classes in \mathcal{K} are precisely the full subcategories closed under products, λ -directed colimits and weakly λ -ap-pure subobjects.*

Proof. Since

$$\mathrm{Inj}_{\mathrm{ap}} \mathcal{F} = \bigcap_{\varepsilon > 0} \mathrm{Inj}_\varepsilon \mathcal{F},$$

every approximate λ -injectivity class is closed under products and λ -directed colimits (see 4.3 and 4.4). We will show that $\mathrm{Inj}_{\mathrm{ap}} \mathcal{F}$ is closed under weakly λ -ap-pure subobjects. Let $p : K \rightarrow L$ be λ -ap-pure and L belong to $\mathrm{Inj}_{\mathrm{ap}} \mathcal{F}$. Consider $f : A \rightarrow B$ in \mathcal{F} , $\varepsilon > 0$ and $u : A \rightarrow K$. There is $v : B \rightarrow L$ such that $pu \sim_{\frac{\varepsilon}{2}} vf$. Since p is weakly λ - $\frac{\varepsilon}{2}$ -pure, there exists $t : B \rightarrow K$ with $tf \sim_\varepsilon u$. Thus K is ε -injective to f .

Let \mathcal{L} be closed under products, λ -directed colimits and weakly λ -ap-pure subobjects. We will proceed in the same way as in 4.8. Let \mathcal{F} consist of all morphisms $f : A \rightarrow B$ such that A and B are λ -presentable and every object of \mathcal{L} is approximately injective to f . We have $\mathcal{L} \subseteq \mathrm{Inj}_{\mathrm{ap}} \mathcal{F}$, and the converse inclusion will follow from the fact that every weak reflection of $K \in \mathrm{Inj}_{\mathrm{ap}} \mathcal{F}$ into \mathcal{L} is weakly λ -ap-pure. Since $K \in \mathrm{Inj}_{\frac{\varepsilon}{2}} \mathcal{F}$, a weak reflection $r : K \rightarrow K^*$ is weakly λ - ε -pure. Hence r is λ -ap-pure. \square

Corollary 5.6. *Under the hypotheses of Theorem 5.5, every approximate λ -injectivity class in \mathcal{K} is a λ -injectivity class.*

Proof. The statement follows from 5.4, 5.5 and [15] 2.2. \square

Remark 5.7. Continuing to work under the hypotheses of Theorem 5.5, we let \mathcal{L} be an approximate λ -injectivity class in \mathcal{K} . Then, following 5.6 and [2] 4.8, \mathcal{L} is weakly reflective. We claim that every object K that is approximately injective to its weak reflection $r : K \rightarrow K^*$ must lie in \mathcal{L} . Indeed, r ε -splits for every $\varepsilon > 0$ and then, by 3.4, must be weakly λ -ap-pure, for any λ . Hence, $K \in \mathcal{L}$ follows.

Recall that *Vopěnka's Principle* is a large-cardinal principle which guarantees that injectivity classes in a locally presentable category are characterized by their closure under products and split subobjects (see [2] 6.26). We are now ready to conclude the validity of an “ap-version” of this theorem. To state it, we say that f is *ap-split* if it is ε -split for every $0 < \varepsilon$.

Theorem 5.8. *Under Vopěnka's Principle, the following conditions are equivalent for a full subcategory \mathcal{L} of a category \mathcal{K} satisfying the hypotheses of Theorem 5.5:*

- (1) \mathcal{L} is closed under products and ap-split subobjects,
- (2) \mathcal{L} is an ap-injectivity class,

(3) \mathcal{L} is weakly reflective and closed under ap-split subobjects.

Proof. (3) \Rightarrow (2): By Remark 5.7, \mathcal{L} is an ap-injectivity class w.r.t. weak reflections of \mathcal{K} -objects to \mathcal{L} .

(2) \Rightarrow (1) follows from 3.4 and 5.5.

(1) \Rightarrow (3) follows from [2] 6.26, since closure under ap-split subobjects trivially entails closure under split subobjects. \square

6. THE COUNTABLE CASE

Regular monomorphisms in \mathbf{Met}_∞ are isometries, and these are stable under pushout. Every finite generalized metric space A is \aleph_0 -generated, in the sense that $\mathbf{Met}_\infty(A, -) : \mathbf{Met}_\infty \rightarrow \mathbf{Set}$ preserves directed colimits of isometries.

By definition, a generalized metric space is \aleph_0 -saturated if it is injective with respect to isometries between finite generalized metric spaces.

Theorem 6.1. *There is a countable \aleph_0 -saturated generalized metric space.*

Proof. Isometries in \mathbf{Met}_∞ are stable under pushout, and every isometry between finite generalized metric spaces is a multiple pushout of isometries between finite rational generalized metric spaces, *i.e.*, of finite spaces whose distances are either rational or ∞ . Hence a generalized metric space is \aleph_0 -saturated if, and only if, it is injective to isometries between finite rational generalized metric spaces. Up to isomorphism, there are only countably many such spaces. Let \mathcal{S} be the set of all isometries between them; \mathcal{S} is countable again. We express \mathcal{S} as a union of a countable chain of finite subsets \mathcal{S}_n and will construct a countable chain $(k_{ij} : K_i \rightarrow K_j)_{i < j < \omega}$ of finite generalized metric spaces and isometries, as follows. Let $K_0 = \emptyset$. Having $(k_{ij} : K_i \rightarrow K_j)_{i < j \leq n}$ we take the diagram consisting of all spans (u, h) where $h \in \mathcal{S}_n$. Then $k_{n,n+1} : K_n \rightarrow K_{n+1}$ is given by the corresponding multiple pushout, *i.e.*, by the pushout

$$\begin{array}{ccc} K_n & \xrightarrow{k_{n,n+1}} & K_{n+1} \\ \uparrow \langle u \rangle & & \uparrow \\ \coprod_h X_h & \xrightarrow{\coprod_h} & \coprod_h Y_h \end{array}$$

with $h : X_h \rightarrow Y_h$ running through \mathcal{S}_n . We claim that $K = \operatorname{colim} K_n$ is \aleph_0 -saturated. Consider $h : X \rightarrow Y$ in \mathcal{S} and $u : X \rightarrow K$. Since X is \aleph_0 -small, there is $u' : X \rightarrow K_n$ such that $k_n u' = u$; here $k_n : K_n \rightarrow K$ is a colimit injection. Without any loss of generality, we may assume $h \in \mathcal{S}_n$. Thus $k_{n,n+1} u' = v h$ for some $v : Y \rightarrow K_{n,n+1}$. Hence $u = k_n u' = v h$, as desired. \square

A countable \aleph_0 -saturated generalized metric space is, in fact, uniquely determined, up to isomorphism: see, *e.g.*, [16] Theorem 2; it is the Fraïssé limit of finite generalized

metric spaces. Similarly one establishes the existence of a countable \aleph_0 -saturated metric space, called *Urysohn space* in the literature (cf. [10]). In order for us to establish a corresponding result in **Ban**, we introduce some needed definitions more generally at the level of \mathbf{Met}_∞ -enriched categories.

Definition 6.2. Let \mathcal{K} be a \mathbf{Met}_∞ -enriched category. A morphism $f : A \rightarrow B$ is called an *isometry* if, for every $\varepsilon \geq 0$ and all $u, v : C \rightarrow A$, one has

$$fu \sim_\varepsilon fv \Rightarrow u \sim_\varepsilon v.$$

Example 6.3. Both in **Met** and **Ban**, isometries have their usual meaning. In **Met**, it suffices to test them on $u, v : 1 \rightarrow A$, and in **Ban** on $u, v : l_1(1) \rightarrow A$.

Definition 6.4. Let \mathcal{K} be a \mathbf{Met}_∞ -enriched category. An object A in \mathcal{K} is λ - ε -generated if, for any λ -directed diagram of isometries $(k_{ij} : K_i \rightarrow K_j)_{i \leq j \in I}$ with colimit cocone $k_i : K_i \rightarrow K$ and every morphism $f : A \rightarrow K$, there is $i \in I$ such that

- (1) f ε -factorizes through k_i , i.e., $f \sim_\varepsilon k_i g$ for some $g : A \rightarrow K_i$,
- (2) the ε -factorization is ε -essentially unique, in the sense that, if $f \sim_\varepsilon k_i g$ and $f \sim_\varepsilon k_i g'$, then $k_{ij} g \sim_\varepsilon k_{ij} g'$ for some $j \geq i$.

We say that A is λ -ap-generated if it is λ - ε -generated for every $\varepsilon > 0$.

Example 6.5. (1) In **Ban**, every finite-dimensional space A is \aleph_0 -ap-generated.

(2) Since every Banach space is a directed colimit of finite-dimensional Banach spaces and isometries, every \aleph_0 - ε -generated Banach space admits an ε -split morphism to a finite-dimensional Banach space, for any $\varepsilon > 0$.

(3) More generally, for every isometry $f : X \rightarrow Y$ between \aleph_0 -ap-generated Banach spaces and every $\varepsilon > 0$, there is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow u & & \uparrow v \\ A & \xrightarrow{g} & B \end{array}$$

in **Ban** with an isometry g between finite-dimensional Banach spaces A, B , as well as morphisms $s : X \rightarrow A$, $t : Y \rightarrow B$, such that $us \sim_\varepsilon \text{id}_X$, $vt \sim_\varepsilon \text{id}_Y$ and $gs \sim_\varepsilon tf$.

Definition 6.6. Let \mathcal{K} be a \mathbf{Met}_∞ -enriched category. We say that an object K is \aleph_0 -ap-saturated if it is approximately injective to isometries between ap- \aleph_0 -generated objects.

Theorem 6.7. *There is a separable \aleph_0 -ap-saturated Banach space.*

Proof. By 6.5(2), a Banach space is \aleph_0 -ap-saturated if, and only if, it is approximately injective to isometries between finite-dimensional Banach spaces. Now we proceed in

the same way as in 6.1. Isometries are pushout stable in **Ban** (see [3], 2.1) and, following [6] 2.7 and [11], a Banach space is approximately injective to finite-dimensional Banach spaces if, and only if, it is approximately injective to rational isometries between finite-dimensional rational Banach spaces. Up to isomorphism, there are only countably many such isometries. We will show that K , constructed analogously to the construction in 6.1 from the countable set \mathcal{S} of relevant isometries as a colimit of finite-dimensional spaces K_n (with K_0 the null space), is \aleph_0 -ap-saturated. For that, consider $h : X \rightarrow Y$ in \mathcal{S} and $u : X \rightarrow K$, and let $\varepsilon > 0$. Since X is \aleph_0 -ap-generated, there is $u' : X \rightarrow K_n$ such that $k_n u' \sim_\varepsilon u$. Without loss of generality we may assume that $h \in \mathcal{S}_n$. Thus $k_{n,n+1} u' = v h$ for some $v : Y \rightarrow K_{n,n+1}$. Hence

$$u \sim_\varepsilon k_n u' = k_{n+1} k_{n,n+1} u' = k_{n+1} v h,$$

as desired. □

A separable \aleph_0 -ap-saturated Banach space is in fact uniquely determined, up to isomorphism; it coincides with the *Gurarii space* (see [11]).

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