III

A Functional Approach to General Topology

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In this chapter we wish to present a categorical approach to fundamental concepts of General Topology, by providing a category $\mathcal{X}$ with an additional structure which allows us to display more directly the geometric properties of the objects of $\mathcal{X}$ regarded as *spaces*. Hence, we study topological properties for them, such as Hausdorff separation, compactness and local compactness, and we describe important topological constructions, such as the compact-open topology for function spaces and the Stone-Čech compactification. Of course, in a categorical setting, spaces are not investigated "directly" in terms of their points and neighbourhoods, as in the traditional set-theoretic setting; rather, one exploits the fact that the relations of points and parts inside a space become categorically special cases of the relation of the space to other objects in its category. It turns out that many stability properties and constructions are established more economically in the categorical rather than the set-theoretic setting, leave alone the much greater level of generality and applicability.

The idea of providing a category with some kind of topological structure is certainly not new. So-called Grothendieck topologies (see Chapter VII) and, more generally, Lawvere-Tierney topologies are fundamental for the geometrically-inspired construction of topoi. Specifically, these structures provide a notion of closure and thereby a notion of closed subobject, for every object in the category, such that all morphisms become "continuous". The notion of Dikranjan-Giuli *closure operator* [17] axiomatizes this idea and can be used to study topological properties categorically (see, for example, [9, 12]).

Here we go one step further and follow the approach first outlined in [48]. Hence, we provide the given category with a factorization structure and a special class $\mathcal{F}$ of morphisms of which we think as of the *closed morphisms*, satisfying three basic axioms; however, there is no a-priori provision of "closure" of subobjects. Depending on the parameter $\mathcal{F}$, we introduce and study basic topological properties as mentioned previously, but encounter also more advanced topics, such as exponentiability. The following features distinguish our presentation from the treatment of the same topological themes in existing topology books:

1. We emphasize the object-morphism interplay: every object notion corresponds to a morphism notion, and vice versa. For example, compact spaces "are" proper

maps, and conversely. Consequently, every theorem on compact spaces "is" a theorem on proper maps, and conversely.

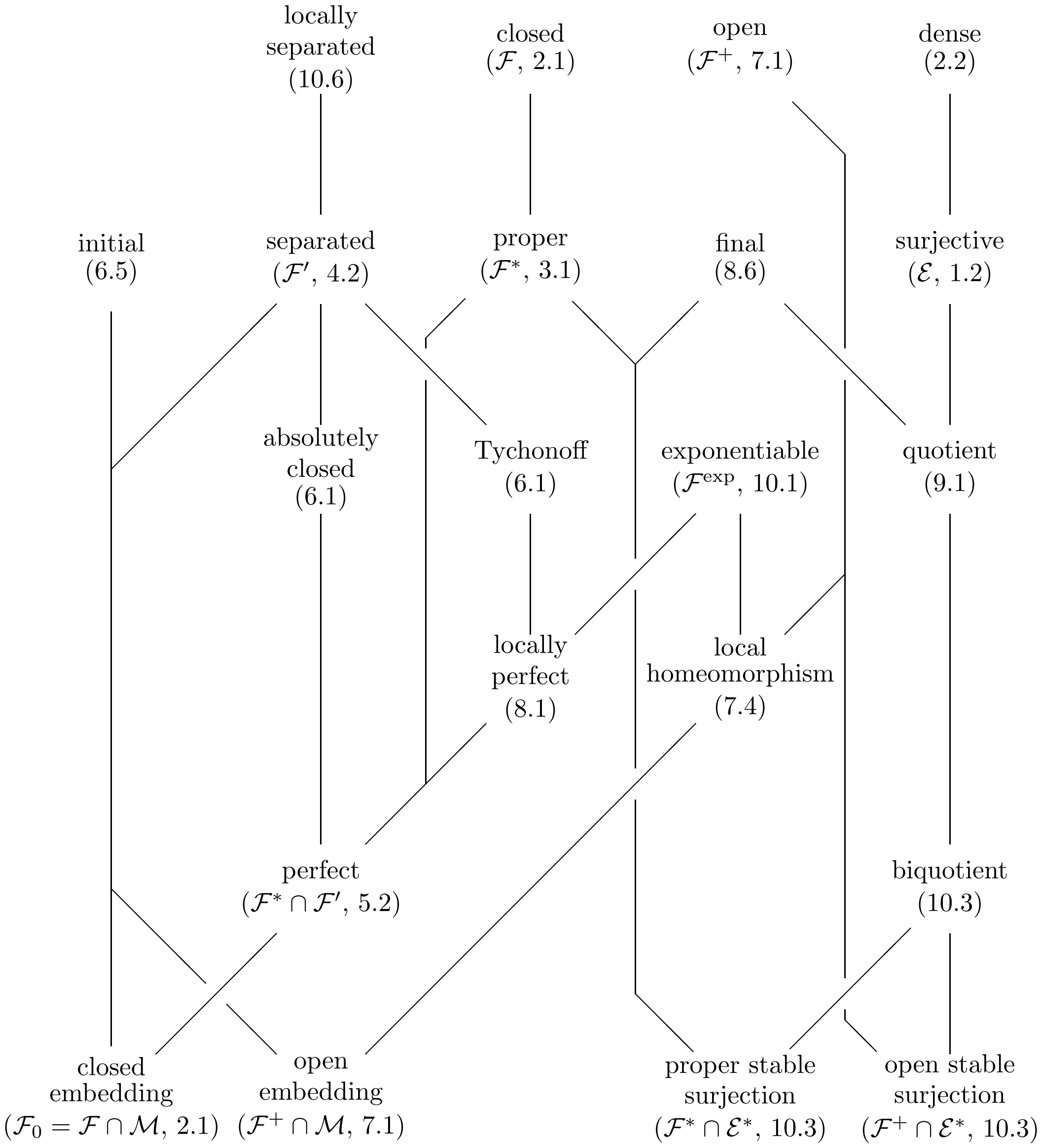


Table 1

2. We reach a wide array of applications not only by considering various categories, such as the categories of topological spaces, of locales (see Chapter II), etc., but also by varying the notion of closed morphism within the same category.

For example, in our setting, compact spaces behave just like discrete spaces, and perfect maps just like local homeomorphisms, because these notions arise from a common generalization.

3. Our categorical treatment is entirely constructive. In particular, we avoid the use of the Axiom of Choice (also when we interpret the categorical theory in the category of topological spaces), and we restrict ourselves to finitary properties (with the exception of some general remarks on the Tychonoff Theorem and the existence of the Stone-Čech compactification at the end of the chapter).

We expect the Reader to be familiar with basic categorical notions, such as adjoint functor and limit, specifically pullback, and we also assume familiarity with basic topological notions, such as topological space, neighbourhood, closure of subsets, continuous map. The Reader will find throughout the chapter mostly easy exercises; those marked with $*$ are deemed to be more demanding.

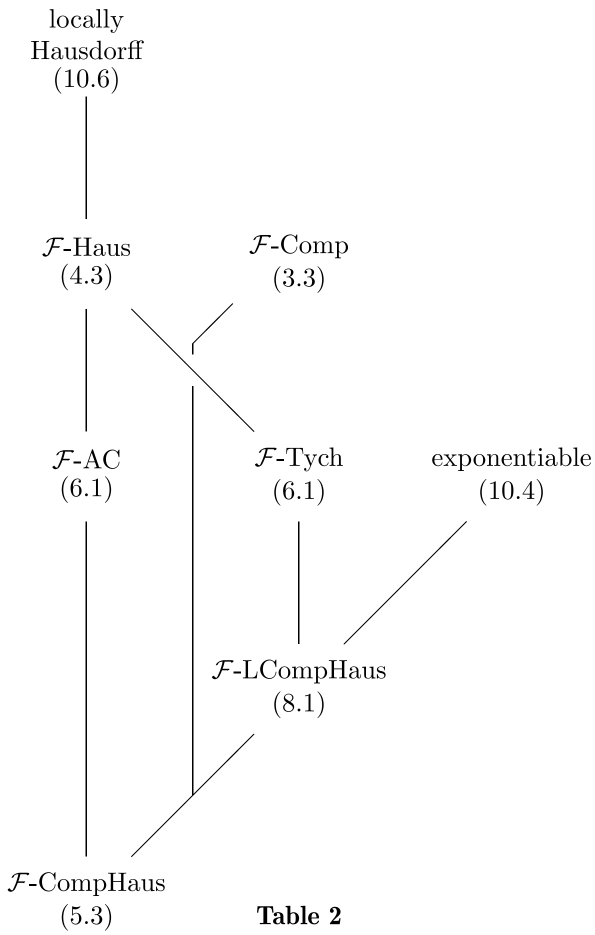


Table 2

Our basic hypotheses on the category we are working with are formulated as *axioms*

(F0)-(F2)	in 1.2	(axioms for factorization systems),
(F3)-(F5)	in 2.1	(axioms for closed maps),
(F6)-(F8)	in 11.1	(axioms giving closures),
(F9)	in 11.2	(infinite product axiom).

The morphism notions discussed in this chapter are summarized in Table 1, which also indicates the relevant subsections and the abbreviations used. Upward-directed lines indicate implications; some of these may need extra (technical) hypotheses.

Table 2 replicates part of Table 1 at the object level.

Many of these notions get discussed throughout the chapter in standard examples, as introduced in Section 2. We list some of them here in summary form, for the Reader's convenience:

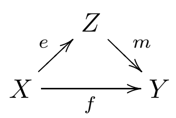
$\mathcal{T}op$:	topological spaces, $\mathcal{F}$ = closed maps (2.3),
$\mathcal{T}op_{\text{open}}$:	topological spaces, $\mathcal{F}$ = open maps (2.8),
$\mathcal{T}op_{\text{clopen}}$:	topological spaces, $\mathcal{F}$ = maps preserving clopen sets (2.6),
$\mathcal{T}op_{\text{Zariski}}$:	topological spaces, $\mathcal{F}$ = Zariski-closed sets (2.7),
$\mathcal{L}oc$:	locales, $\mathcal{F}$ = closed maps (2.4),
$\mathcal{A}bGrp$:	abelian groups, $\mathcal{F}$ = homomorphisms whose restriction to the torsion subgroups is surjective (2.9),

any topos with a universal closure operator (2.5),
any finitely-complete category with a Dikranjan-Giuli closure operator (2.5),
any lextensive category with summand-preserving morhisms (2.6),
any comma-category of any of the preceding categories (2.10).

The authors are grateful for valuable comments received from many colleagues and the anonymous referees which helped us getting this chapter into its current form. We particularly thank Jorge Picado, Aleš Pultr and Peter Johnstone for their interest in resolving the problem of characterizing open maps of locales in terms of closure (see Section 7.3), which led to the comprehensive solution given in [31].

1. Subobjects, images, preimages

1.1. Motivation. A mapping $f : X \to Y$ of sets may be decomposed as



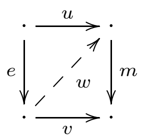
(1)

with a surjection e followed by an injection m, simply by taking $Z = f[X]$, the image of f; e maps like f, and m is an inclusion map. Hence, $f = m \cdot e$ with an epimorphism e and a monomorphism m of the category $\mathcal{S}et$ of sets. If f is a homomorphism of groups X, Y, then Z is a subgroup of Y, and e and m become epi- and monomorphisms in the category $\mathcal{G}rp$ of groups, respectively. If f is a continuous mapping of topological spaces X, Y, then Z may be endowed with the subspace topology inherited from Y, and e and m are now epi- and monomorphisms in the category $\mathcal{T}op$ of topological spaces, respectively. However, the situation in $\mathcal{T}op$ is rather different from that in $\mathcal{S}et$ and $\mathcal{G}rp$: applying the decomposition (1) to a *mono*morphism f, in $\mathcal{S}et$ and $\mathcal{G}rp$ we obtain an isomorphism e, but not so in $\mathcal{T}op$, simply because the $\mathcal{S}et$-inverse of the continuous bijective mapping e may fail to be continuous.

In what follows we therefore consider a (potentially quite special) class $\mathcal{M}$ of monomorphisms and, symmetrically, a (potentially quite special) class $\mathcal{E}$ of epimorphisms in a category $\mathcal{X}$ and assume the existence of $(\mathcal{E}, \mathcal{M})$-decompositions for all morphisms, as follows.

1.2. Axioms for factorization systems. Throughout this chapter we work in a finitely-complete category $\mathcal{X}$ with two distinguished classes of morphisms $\mathcal{E}$ and $\mathcal{M}$ such that

(F0) $\mathcal{M}$ is a class of monomorphisms and $\mathcal{E}$ is a class of epimorphisms in $\mathcal{X}$, and both are closed under composition with isomorphisms;
(F1) every morphism f decomposes as $f = m \cdot e$ with $m \in \mathcal{M}$, $e \in \mathcal{E}$;
(F2) every $e \in \mathcal{E}$ is *orthogonal* to every $m \in \mathcal{M}$ (written as $e \perp m$), that is: given any morphisms u, v with $m \cdot u = v \cdot e$, then there is a uniquely determined morphism w making the following diagram commutative:



(2)

Any pair $(\mathcal{E}, \mathcal{M})$ satisfying conditions (F0)-(F2) is referred to as a *proper* (*orthogonal*) *factorization system* of $\mathcal{X}$; "proper" gets dropped if one allows $\mathcal{M}$ and $\mathcal{E}$ to be arbitrary morphism classes, i.e., if one drops the "mono" and "epi" condition in (F0). In that case the unicity requirement for w in (F2) becomes essential; however it is redundant in our situation.

Exercises.

1. Show that every category $\mathcal{X}$ has two *trivial* but generally non-proper factorization systems: (Iso, All), (All, Iso).
2. Show that (Epi, Mono) is a proper factorization system in $\mathcal{S}et$ and $\mathcal{G}rp$, but not in $\mathcal{T}op$.
3. Find a class $\mathcal{M}$ such that (Epi, $\mathcal{M}$) is a proper factorization system in $\mathcal{T}op$.
4. Show that $f : X \to Y$ is an epimorphism in the full subcategory $\mathcal{H}aus$ of Hausdorff spaces in $\mathcal{T}op$ if and only if f is dense (i.e., its image meets every non-empty open set in Y). Find a class $\mathcal{M}$ such that (Epi, $\mathcal{M}$) is a proper factorization system in $\mathcal{H}aus$. Compare this with $\mathcal{T}op$.

(Here Iso, Epi, Mono denotes the class of all iso-, epi-, monomorphisms in the respective category, and All the class of all morphisms.)

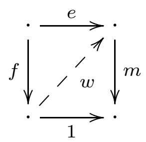
1.3. One parameter suffices. Having introduced factorization systems with two parameters we hasten to point out that one determines the other:

Lemma.

$$\mathcal{E} = \{f \in \mathrm{mor}\mathcal{X} \mid \forall m \in \mathcal{M} \,:\, f \perp m\},$$

$$\mathcal{M} = \{f \in \mathrm{mor}\mathcal{X} \mid \forall e \in \mathcal{E} \,:\, e \perp f\}.$$

Proof. By (F2), every $f \in \mathcal{E}$ satisfies $f \perp m$ for all $m \in \mathcal{M}$. Conversely, if $f \in \mathrm{mor}\mathcal{X}$ has this property, we may write $f = m \cdot e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, by (F1), and then apply the hypothesis on f as follows:



(3)

The monomorphism m with $m \cdot w = 1$ is now recognized as an isomorphism, and we obtain $f = m \cdot e \in \mathcal{E}$ with (F0). This shows the first identity, the second follows dually. □

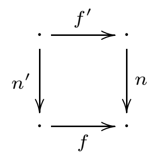
Exercises.

1. Show that the Lemma holds true without assuming properness of the factorization system.
2. Show that $(\mathcal{E}, \mathcal{M})$ is a (proper) factorization system of $\mathcal{X}$ if and only if $(\mathcal{M}, \mathcal{E})$ is a (proper) factorization system of $\mathcal{X}^{\mathrm{op}}$. Conclude that the second identity of the Lemma *follows* from the first.
3. Using properness of $(\mathcal{E}, \mathcal{M})$, show that if $g \cdot f = 1$, then $f \in \mathcal{M}$ and $g \in \mathcal{E}$. More generally: every *extremal monomorphism* (so that $m = h \cdot e$ with e epic only if e iso) lies in $\mathcal{M}$; dually, every extremal epimorphism lies in $\mathcal{E}$.

1.4. Properties of $\mathcal{M}$. We list some stability properties of $\mathcal{E}$ and $\mathcal{M}$:

Proposition.

(1) *$\mathcal{E}$ and $\mathcal{M}$ are both closed under composition, and $\mathcal{E} \cap \mathcal{M} =$* Iso.
(2) *If $n \cdot m \in \mathcal{M}$, then $m \in \mathcal{M}$.*
(3) *$\mathcal{M}$ is* stable under pullback*; hence, for every pullback diagram*



(4)

$n \in \mathcal{M}$ *implies* $n' \in \mathcal{M}$.

(4) $\mathcal{M}$ *is* stable under intersection (=multiple pullback)*; hence, if in the commutative diagram*

$$\begin{array}{ccc} & M_i & \\ {\scriptstyle j_i}\nearrow & & \searrow{\scriptstyle m_i} \\ M & \xrightarrow{\quad m \quad} & X \end{array} \tag{5}$$

$(M, m, j_i)_{i \in I}$ *is the limit of the diagram given by the morphisms* $m_i \in \mathcal{M}$ *($i \in I$), then also* $m \in \mathcal{M}$*.*

(5) *With all* $m_i : X_i \to Y_i$ *($i \in I$) in* $\mathcal{M}$*, also* $\prod_{i \in I} m_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ *is in* $\mathcal{M}$ *(if the products exist).*

Proof. We show (1), (2) and leave the rest as an exercise (see below). Clearly, if we factor an isomorphism $f = m \cdot e$ as in (F1), both e, m are isomorphisms as well, hence $f \in \mathcal{E} \cap \mathcal{M}$ with (F0). Conversely, $f \perp f$ for $f \in \mathcal{E} \cap \mathcal{M}$ shows that f must be an isomorphism.

Given a composite morphism $n \cdot m$ in $\mathcal{M}$, we may factor $m = m' \cdot e$ with $e \in \mathcal{E}$, $m' \in \mathcal{M}$ and obtain from $e \perp n \cdot m$ a morphism w with $w \cdot e = 1$. Hence, the epimorphism e is an isomorphism, and $m = m' \cdot e \in \mathcal{M}$. This shows (2).

Having again a composite $f = n \cdot m$ with both $n, m \in \mathcal{M}$, we may now factor $f = m^* \cdot e^*$ with $e^* \in \mathcal{E}$, $m^* \in \mathcal{M}$. Then $e^* \perp n$ gives a morphism w with $w \cdot e^* = m \in \mathcal{M}$, hence $e^* \in \mathcal{E} \cap \mathcal{M} = \mathrm{Iso}$ by what already has been established, and $f \in \mathcal{M}$ follows.

The assertion on $\mathcal{E}$ follows dually. □

Exercises.

1. Complete the proof of the Proposition, using the same factorization technique as in the part already established.
2. Formulate and prove the dual assertions for $\mathcal{E}$; for example: if $e \cdot d \in \mathcal{E}$, then $e \in \mathcal{E}$.
3. For any class $\mathcal{E}$ of morphisms, let $\mathcal{M} := \{f \in \mathrm{mor}\mathcal{X} \mid \forall e \in \mathcal{E} \ : \ e \perp f\}$. Show that $\mathcal{M}$ is *closed* under composition, and *under arbitrary limits*; that is: for functors $H, K : \mathcal{D} \to \mathcal{X}$ and any natural transformation $\mu : H \to K$ with $\mu_d \in \mathcal{M}$ for all $d \in \mathcal{D}$, also
$$\lim \mu : \lim H \longrightarrow \lim K$$
is in $\mathcal{M}$ (if the limits exist). Dualize this statement.
4. Prove that *any* pullback-stable class $\mathcal{M}$ satisfies: if $n \cdot m \in \mathcal{M}$ with n monic, then $m \in \mathcal{M}$.
5. * Prove that *any* class $\mathcal{M}$ closed under arbitrary limits (see 3) satisfies assertions (3)-(5) of the Proposition.

1.5. Subobjects. We usually refer to morphisms m in $\mathcal{M}$ as *embeddings* and call those with codomain X *subobjects* of X. The class of all subobjects of X is denoted by

$$\mathrm{sub}X.$$

It is preordered by

$$m \leq m' \;\Leftrightarrow\; \exists j \;:\; m' \cdot j = m;$$

$$\begin{array}{ccc} M & \xrightarrow{\;j\;} & M' \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m'} & \\ & X & \end{array} \tag{6}$$

Such j is uniquely determined and necessarily belongs to $\mathcal{M}$. Furthermore, it is easy to see that

$$m \leq m' \;\&\; m' \leq m \;\Leftrightarrow\; \exists \text{ isomorphism } j \;:\; m' \cdot j = m;$$

we write $m \cong m'$ in this case and think of m and m' as *representing the same subobject of* X.

Exercises.

1. Let $A \subseteq X$ be sets. Prove that all injective mappings $m : M \to X$ with $m[M] = A$ represent the same subobject of X in $\mathcal{S}et$ (with $\mathcal{M} = \text{Mono}$).
2. Prove that $m : M \to X$ in $\mathcal{M}$ is an isomorphism if and only if $m \cong 1_X$.

As a consequence of Proposition 1.4 we note:

Corollary. *$(\mathcal{E}, \mathcal{M})$-factorizations are essentially unique. Hence, if $f = m \cdot e = m' \cdot e'$ with $e, e' \in \mathcal{E}$, $m, m' \in \mathcal{M}$, then there is a unique isomorphism j with $j \cdot e = e'$, $m' \cdot j = m$; in particular, $m \cong m'$.*

Proof. Since $e \perp m'$ there is j with $j \cdot e = e'$, $m' \cdot j = m$, and with Prop. 1.4(1),(2),(2)$^{\text{op}}$ we have $j \in \mathcal{E} \cap \mathcal{M} = \text{Iso}$. □

1.6. Image and preimage. For $f : X \to Y$ in $\mathcal{X}$ and $m \in \text{sub}X$, one defines the *image* $f[m] \in \text{sub}Y$ *of* m *under* f by an $(\mathcal{E}, \mathcal{M})$-factorization of $f \cdot m$, as in

$$\begin{array}{ccc} M & \xrightarrow{\;e\;} & f[M] \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle f[m]} \\ X & \xrightarrow{\;f\;} & Y \end{array} \tag{7}$$

The *preimage* (or *inverse image*) $f^{-1}[n] \in \text{sub}X$ *of* $n \in \text{sub}Y$ *under* f is given by the pullback diagram (cp. 1.4(3))

$$\begin{array}{ccc} f^{-1}[N] & \xrightarrow{\;f'\;} & N \\ {\scriptstyle f^{-1}[n]}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow{\;f\;} & Y \end{array} \tag{8}$$

The pullback f' of f along n is also called a *restriction of* f. Both, image and preimage are uniquely defined, up to isomorphism, and the constructions are related, as follows:

Lemma. *For* $f : X \to Y$ *and* $m \in \mathrm{sub}X$, $n \in \mathrm{sub}Y$ *one has:*

$$f[m] \leq n \;\Leftrightarrow\; m \leq f^{-1}[n].$$

Proof. Consider diagrams (7) and (8). Then, having $j : f[M] \to N$ with $n \cdot j = f[m]$ gives $k : M \to f^{-1}[N]$ with $f^{-1}[n] \cdot k = m$, by the pullback property of (8). Viceversa, the existence of k gives the existence of j since $e \perp n$. □

Exercises.

1. Conclude from the Lemma:
   (a) $m \leq f^{-1}[f[m]]$, $f[f^{-1}[n]] \leq n$;
   (b) $m \leq m' \Rightarrow f[m] \leq f[m']$, $n \leq n' \Rightarrow f^{-1}[n] \leq f^{-1}[n']$.
2. Show: $f^{-1}[1_Y] \cong 1_X$, and ($f \in \mathcal{E} \Leftrightarrow f[1_X] \cong 1_Y$).

1.7. Image is left adjoint to preimage. Lemma 1.6 (with the subsequent Exercise) gives:

Proposition. *For every* $f : X \to Y$ *there is a pair of adjoint functors*

$$f[-] \dashv f^{-1}[-] : \mathrm{sub}Y \longrightarrow \mathrm{sub}X.$$

Consequently, $f^{-1}[-]$ *preserves all infima and* $f[-]$ *preserves all suprema.*

Exercises.

1. Complete the proof of the Proposition.
2. For $g : Y \to Z$, establish natural isomorphisms
$$(g \cdot f)[-] \cong g[-] \cdot f[-],\; (g \cdot f)^{-1}[-] \cong f^{-1}[-] \cdot g^{-1}[-],$$
$$1_X[-] \cong \mathrm{id}_{\mathrm{sub}X},\; 1_X^{-1}[-] \cong \mathrm{id}_{\mathrm{sub}X}.$$
3. Show that the $\cong$-classes of elements in sub$X$ carry the structure of a (possibly large) meet-semilattice, with the largest element represented by 1_X, and with the infimum $m \wedge n$ represented by $m \cdot (m^{-1}[n]) \cong n \cdot (n^{-1}[m])$.

1.8. Pullback stability. We give sufficient conditions for the image-preimage functors to be partially inverse to each other:

Proposition. *Let* $f : X \to Y$ *be a morphism in* $\mathcal{X}$.

(1) *If* $f \in \mathcal{M}$*, then* $f^{-1}[f[m]] \cong m$ *for all* $m \in \mathrm{sub}X$.
(2) *If every pullback of* f *along a morphism in* $\mathcal{M}$ *lies in* $\mathcal{E}$ *(so that in every pullback diagram* (4) *with* $n \in \mathcal{M}$ *one has* $f' \in \mathcal{E}$*), then* $f[f^{-1}[n]] \cong n$ *for all* $n \in \mathrm{sub}Y$.

Proof. (1) If $f \in \mathcal{M}$, in (7) we may take $f[m] = f \cdot m$ and $e = 1_M$, and since f is a monomorphism, (7) is now a pullback diagram, which implies the assertion.
(2) Under the given hypothesis, in (8) the morphism f' lies in $\mathcal{E}$, hence n is the image of $f^{-1}[n]$ under f. □

Exercises.

1. Prove that the sufficient condition given in (2) is also necessary.
2. Check that for the factorization systems of $\mathcal{S}et$, $\mathcal{G}rp$, $\mathcal{T}op$ given in Exercises 2 and 3 of 1.2, the class $\mathcal{E} = \text{Epi}$ is stable under pullback, but not so for the factorization system of $\mathcal{H}aus$ described in Exercise 4 of 1.2.
3. With the factorization system $(\text{Epi}, \mathcal{M})$ of $\mathcal{T}op$, find a morphism $f : X \to Y$ with $f^{-1}[f[M]] = M$ for all $M \subseteq X$ which fails to belong to $\mathcal{M}$.

2. Closed maps, dense maps, standard examples

2.1. Axioms for closed maps. A topology on a set X is traditionally defined by giving a system of open subsets of X which is stable under arbitrary joins (unions) and finite meets (intersections); equivalently, one may start with a system of closed subsets of X which is stable under arbitrary meets and finite joins, with the correspondence between the two approaches given by complementation. Continuity of $f : X \to Y$ is then characterized by preservation of closed subsets under inverse image. Among the continuous maps, these are those for which closedness of subsets is also preserved by image, called *closed maps* . Since closedness of subobjects is transitive, closed subobjects are precisely those subobjects for which the representing morphism is a closed map.

In what follows we assume that in our finitely-complete category $\mathcal{X}$ with its proper $(\mathcal{E}, \mathcal{M})$-factorization system (satisfying (F0)-(F2)) we are given a special class $\mathcal{F}$ of morphisms of which we think as of the closed maps, satisfying the following conditions:

(F3) $\mathcal{F}$ contains all isomorphisms and is closed under composition;
(F4) $\mathcal{F} \cap \mathcal{M}$ is stable under pullback;
(F5) whenever $g \cdot f \in \mathcal{F}$ with $f \in \mathcal{E}$, then $g \in \mathcal{F}$.

We often refer to the morphisms in $\mathcal{F}$ as *$\mathcal{F}$-closed maps*, and to those of

$$\mathcal{F}_0 := \mathcal{F} \cap \mathcal{M}$$

as *$\mathcal{F}$-closed embeddings* . Having fixed $\mathcal{F}$ does not prevent us from considering further classes with the same properties; hence, we call any class $\mathcal{H}$ of morphisms in $\mathcal{X}$ *$(\mathcal{E}, \mathcal{M})$-closed* if $\mathcal{H}$ satisfies (F3)-(F5) in lieu of $\mathcal{F}$.

Exercises.

1. Show that the class of closed maps in $\mathcal{T}op$ (with the factorization structure of Exercise 3 of 1.2) satisfies (F3)-(F5).
2. A continuous map of topological spaces is *open* if images of open sets are open. Prove that the class of open morphisms in $\mathcal{T}op$ is (like the system of closed maps) $(\text{Epi}, \mathcal{M})$-closed.
3. Prove that each of the following classes in $\mathcal{X}$ is $(\mathcal{E}, \mathcal{M})$-closed: Iso, $\mathcal{E}$, $\mathcal{M}$, $\mathcal{F}_0$.

4. Given a pullback-stable class $\mathcal{H}_0 \subseteq \mathcal{M}$ containing all isomorphisms and being closed under composition, define the morphism class $\mathcal{H}$ by
$$(f : X \to Y) \in \mathcal{H} \Leftrightarrow \forall m \in \mathrm{sub}X,\ m \in \mathcal{H}_0 \ :\ f[m] \in \mathcal{H}_0.$$
Show: $\mathcal{H} \cap \mathcal{M} = \mathcal{H}_0$, and $\mathcal{H}$ is $(\mathcal{E}, \mathcal{M})$-closed if every pullback of a morphism in $\mathcal{E}$ along a morphism in $\mathcal{H}_0$ lies in $\mathcal{E}$ (see Proposition 1.8(2)). Furthermore, $\mathcal{H}$ satisfies the dual of (F5): whenever $g \cdot f \in \mathcal{H}$ with $g \in \mathcal{M}$, then $f \in \mathcal{H}$.
5. Find an $(\mathcal{E}, \mathcal{M})$-closed system $\mathcal{H}$ which does not arise from a class $\mathcal{H}_0$ as described in Exercise 4.

The following statements follow immediately from the axioms (see also Exercise 3 of 1.7).

Proposition.

(1) *The $\mathcal{F}$-closed subobjects of an object X form (a possibly large) subsemilattice of the meet-semilattice* sub$X$.
(2) *Every morphism $f : X \to Y$ in $\mathcal{X}$ is* $\mathcal{F}$-continuous, *that is: $f^{-1}[-]$ preserves $\mathcal{F}$-closedness of subobjects.*
(3) *For every $\mathcal{F}$-closed morphism $f : X \to Y$ in $\mathcal{X}$, $f[-]$ preserves $\mathcal{F}$-closedness of subobjects* (but not necessarily viceversa, see Exercise 5).

□

2.2. Dense maps. A morphism $d : X \to Y$ of $\mathcal{X}$ is *$\mathcal{F}$-dense* if in any factorization $d = m \cdot h$ with an $\mathcal{F}$-closed subobject m one necessarily has $m \in$ Iso.

Lemma. *d is $\mathcal{F}$-dense if and only if $d \perp m$ for all $m \in \mathcal{F}_0$.*

Proof. "if": From $d = m \cdot h$ with $m \in \mathcal{F}_0$ and $d \perp m$ one obtains m iso as in diagram (3). "only if": If $v \cdot d = m \cdot u$, one obtains h with $d = n \cdot h$, where $n = v^{-1}[m] \in \mathcal{F}_0$ by Prop. 2.1. Consequently, n is iso, and we may put $w = v' \cdot n^{-1}$ with the pullback v' of v along m. Then $m \cdot w = m \cdot v' \cdot n^{-1} = v \cdot n \cdot n^{-1} = v$, which implies $w \cdot u = d$, as desired. □

Corollary.

(1) *Any morphism in $\mathcal{E}$ is $\mathcal{F}$-dense.*
(2) *An $\mathcal{F}$-dense and $\mathcal{F}$-closed subobject is an isomorphism.*
(3) *The class of all $\mathcal{F}$-dense morphisms is closed under composition and satisfies the duals of properties* (2)-(5) *of Proposition* 1.4.

Proof. See Lemma 1.3 and Exercises 1.4. □

Exercises.

1. Prove that $f : X \to Y$ is an $\mathcal{F}$-dense morphism if and only if $f[1_X] : f[X] \to Y$ is an $\mathcal{F}$-dense embedding.
2. Show that in $\mathcal{T}op$ (with $\mathcal{F}$ the class of closed maps, as in Exercise 1 of 2.1) a morphism $d : X \to Y$ is *$\mathcal{F}$-dense* if and only if every non-empty open set in Y meets $d[X]$.
3. Verify that the class of $\mathcal{F}$-dense maps in $\mathcal{T}op$ is not $(\mathrm{Epi}, \mathcal{M})$-closed.

In what follows we shall discuss some important examples which we shall refer to later on as *standard examples*, by just mentioning the respective category $\mathcal{X}$; the classes $\mathcal{E}$, $\mathcal{M}$, $\mathcal{F}$ are understood to be chosen as set out below.

2.3. Topological spaces with closed maps. The category $\mathcal{T}op$ will always be considered with its $(\mathrm{Epi}, \mathcal{M})$-factorization structure; here $\mathcal{M}$ is the class of *embeddings*, i.e., of those monomorphisms $m : M \to X$ for which every closed set of M has the form $m^{-1}[F]$ for a closed set F of X. (The fact that $\mathcal{M}$ is precisely the class of regular monomorphisms of the category $\mathcal{T}op$ is not being used in this chapter.) We emphasize again that here $\mathcal{E} = \mathrm{Epi}$ is stable under pullback (see Exercise 2 of 1.8). In the standard situation, $\mathcal{F}$ is always the class of closed maps. $\mathcal{F}$-closedness and $\mathcal{F}$-density for subobjects take on the usual meaning.

2.4. Locales with closed maps. The category $\mathcal{L}oc$ of *locales* has been introduced as the dual of the category $\mathcal{F}rm$ of *frames* in Chapter II. The $(\mathrm{Epi}, \mathcal{M})$-factorization structure of $\mathcal{L}oc$ coincides with the $(\mathcal{E}, \mathrm{Mono})$-factorization structure of $\mathcal{F}rm$, where $\mathcal{E}$ is the class of surjective frame homomorphisms; following the notation of Chapter II, this means:

$$m : M \to X \text{ in } \mathcal{M} \;\Leftrightarrow\; m^* : \mathcal{O}X \to \mathcal{O}M \text{ in } \mathcal{E}.$$

The subobject m is *closed* if there is $a \in \mathcal{O}X$ and an isomorphism $h : \mathcal{O}M \to \uparrow a$ of frames such that $h \cdot m^* = \check{a}$, with $\check{a} : \mathcal{O}X \to \uparrow a$ given by $\check{a}(x) = a \vee x$. The class $\mathcal{F}$ of closed maps in $\mathcal{L}oc$ can now be defined just as in $\mathcal{T}op$, by preservation of closed subobjects under image (see II.5.1).

We point out a major difference to the situation in $\mathcal{T}op$: the class Epi in $\mathcal{L}oc$ fails to be stable under pullback (see II.5.2). However, pullbacks of epimorphisms along complemented sublocales (which include both, closed sublocales and open sublocales – see II.2.7) are again epic in $\mathcal{L}oc$, since for any morphism f in $\mathcal{L}oc$ the inverse image functor $f^{-1}[-]$ preserves finite suprema, and, hence, also complements (see II.3.8).

Next we mention three (schemes of) examples of a more general nature.

2.5. A topos with closed maps with respect to a universal closure operator. Every elementary topos $\mathcal{S}$ has an $(\mathrm{Epi}, \mathrm{Mono})$-factorization system, and the class Epi is stable under pullback (see [28], 1.5). A *universal closure operator* $j = (j_X)_{X \in \mathrm{ob}\mathcal{S}}$ is given by functions $j_X : \mathrm{sub}X \to \mathrm{sub}X$ satisfying the conditions 1. $m \le j_X(m)$, 2. $m \le m' \;\Rightarrow\; j_X(m) \le j_X(m')$, 3. $j_X(j_X(m)) \cong j_X(m)$, 4. $f^{-1}[j_Y(n)] \cong j_X(f^{-1}[n])$, for all $f : X \to Y$, $m, m' \in \mathrm{sub}X$, $n \in \mathrm{sub}Y$ (see [28], 3.13, and [36], V.1). Taking for $\mathcal{F}$ now the class of those morphisms f for which not only $f^{-1}[-]$ but also $f[-]$ commutes with the closure operator, one obtains an $(\mathrm{Epi}, \mathrm{Mono})$-closed system $\mathcal{F}$ for $\mathcal{S}$; its closed maps are also described by the fact that image preserves closedness of subobjects (characterized by $m \cong j_X(m)$). A very particular feature of this structure is that the class of $\mathcal{F}$-dense maps is stable under pullback; see Chapter VII.

Although we shall not discuss this aspect any further in this chapter, we mention that the notion of universal closure operator may be generalized dramatically if we are just interested in the generation of a closed system. In fact, for any finitely-complete category $\mathcal{X}$ with a proper $(\mathcal{E}, \mathcal{M})$-factorization system and a so-called Dikranjan-Giuli closure operator one may take for $\mathcal{F}$ those morphisms for which image commutes with closure (see [18, 9] for details, as well as Section 11 below).

2.6. An extensive category with summand-preserving morphisms. A finitely-complete category $\mathcal{X}$ with binary coproducts (denoted by $X+Y$) is *(finitely) extensive* if the functor

$$\mathrm{pb}_{X,Y} : \mathcal{X}/(X+Y) \longrightarrow \mathcal{X}/X \times \mathcal{X}/Y$$

(given by pullback along the coproduct injections) is an equivalence of categories, for all objects X and Y; equivalently, one may ask its left adjoint (given by coproduct) to be an equivalence, see [6, 5, 7]. (Note that in Chapter VII *extensive* is used in the *infinitary* sense where the binary coproducts are replaced by arbitrary ones, see VII.4.1; however, in this chapter extensive always means finitely extensive.) In order to accommodate our setting we also assume that $\mathcal{X}$ has a proper $(\mathcal{E}, \mathcal{M})$-factorization structure such that every coproduct injection lies in $\mathcal{M}$. Taking for $\mathcal{F}_0$ now those morphisms $m : M \to X$ for which there is some morphism $n : N \to X$ such that $X \cong M + N$ with coproduct injections m, n, one lets $\mathcal{F}$ contain those morphisms for which image preserves the class $\mathcal{F}_0$: see Exercise 4 of 2.1.

For example, $\mathcal{X} = \mathcal{T}op$ is extensive, and the $\mathcal{F}$-closed morphisms are precisely those maps which map clopen (=closed and open) sets onto clopen sets. Whenever we refer to $\mathcal{T}op$ with this structure $\mathcal{F}$ and its $(\mathrm{Epi}, \mathcal{M})$-factorization system, we write $\mathcal{T}op_{\mathrm{clopen}}$ instead of $\mathcal{T}op$. Another prominent example of an extensive category is the dual of the category $\mathcal{CR}ng$ of commutative rings (where $\mathcal{CR}ng$ is considered with its $(\mathcal{E}, \mathrm{Mono})$-factorization system); for details we refer to [5].

Exercises.

1. Prove that the finitely complete category $\mathcal{X}$ with binary coproducts is extensive if and only if in every commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & C & \longleftarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{i_1} & X+Y & \xleftarrow{i_2} & Y \end{array} \tag{9}$$

the top row represents a coproduct (so that $C \cong A + B$) precisely when the two squares are pullback diagrams.
2. Prove that it is enough to require the existence of the coproduct $1+1$ and that $\mathrm{pb}_{1,1}$ is an equivalence to obtain extensivity, including the existence of all binary coproducts.
3. Prove that in an extensive category coproduct injections are monomorphisms, and that they are *disjoint*, i.e. the pullback of i_1, i_2 is an initial object. In particular, $\mathcal{X}$ has an initial object.

4. Prove that an extensive category is *distributive*, so that the canonical morphism $(X \times Y_1) + (X \times Y_2) \to X \times (Y_1 + Y_2)$ is an isomorphism. In addition, $X \times 0 \cong 0$, with 0 denoting the initial object (see Ex. 3).
5. Verify the extensivity of $\mathcal{T}op$ and $\mathcal{CR}ng^{\mathrm{op}}$, and of every topos.

2.7. Algebras with Zariski-closed maps. For a fixed set K let $\Omega = \{\omega_i : K^{n_i} \to K \mid i \in I\}$ be a given class of operations on K of arbitrary arities; hence, the n_i are arbitrary cardinal numbers, and there is no condition on the size of the indexing system I. The category Ω-$\mathcal{S}et$ has as objects sets X which come equipped with an Ω-subalgebra $A(X)$ of K^X; here K^X carries the pointwise structure of an Ω-algebra:

$$(K^X)^{n_i} \cong (K^{n_i})^X \to K^X.$$

A morphism $f : X \to Y$ in Ω-$\mathcal{S}et$ must satisfy $K^f[A(Y)] \subseteq A(X)$, that is: $f \cdot \beta \in A(X)$ for all $\beta \in A(Y)$. Like $\mathcal{T}op$, Ω-$\mathcal{S}et$ has a proper $(\mathrm{Epi}, \mathcal{M})$-factorization structure, with the morphisms in $\mathcal{M}$ represented by embeddings $M \hookrightarrow X$, so that $A(M) = \{\alpha|_M \mid \alpha \in A(X)\}$. Such a subobject M is called *Zariski-closed* if any $x \in X$ with

$$\forall \alpha, \beta \in A(X)\ (\alpha|_M = \beta|_M \ \Rightarrow\ \alpha(x) = \beta(x))$$

already lies in M. The class $\mathcal{F}$ of closed maps contains precisely those maps for which image preserves Zariski-closedness (see [15, 22]).

We mention in particular two special cases. First, take $K = 2 = \{0, 1\}$, with the operations given by the frame structure of $\{0 \leq 1\}$, that is: by arbitrary joins and finite meets. Equipping a set X with an Ω-subalgebra of 2^X is putting a topology (of open sets) on X, and we have Ω-$\mathcal{S}et = \mathcal{T}op$. The Zariski-closed sets on X define a new topology on X with respect to which a basic neighbourhood of a point $x \in X$ has the form $U \cap \overline{\{x\}}$, where U is a neighbourhood of x and $\overline{\{x\}}$ the closure of $\{x\}$ in the original topology. Whenever we refer to $\mathcal{T}op$ with $\mathcal{F}$ given as above, we shall write $\mathcal{T}op_{\mathrm{Zariski}}$.

Secondly, in the "classical" situation, one considers the ground field K in the category of commutative K-algebras, so the operations in Ω are

$$0, 1 : K^0 \to K, \ \ a(-) : K \to K, \ \ +, \cdot : K^2 \to K,$$

where $a(-)$ is (left) multiplication by a, for every $a \in K$. Hence, an Ω-set X comes with a subalgebra $A(X)$ of K^X, and $M \subseteq X$ is Zariski-closed if

$$M = \{x \in X \mid \forall \alpha \in A(X)\ (\alpha|_M = 0 \ \Rightarrow\ \alpha(x) = 0)\}.$$

Putting $J = \{\alpha \in A(X) \mid \alpha|_M = 0\}$ one sees that the Zariski-closed sets are all of the form

$$Z(J) = \{x \in X \mid \forall \alpha \in J\ :\ \alpha(x) = 0\}$$

for some ideal J of $A(X)$. For $X = K^n$ and $A(X)$ the algebra of polynomial functions $K^n \to K$ we have exactly the classical Zariski-closed sets considered as in commutative algebra and algebraic geometry (see [15]).

Exercises.

1. Let X be the cartesian product of the sets X_ν, with projections p_ν $(\nu \in N)$. If each X_ν is an Ω-set, let $A(X)$ be the subalgebra of K^X generated by $\{\alpha \cdot p_\nu \mid \nu \in N,\, \alpha \in A(X_\nu)\}$. Conclude that Ω-$\mathcal{S}et$ has products. More generally, show that Ω-$\mathcal{S}et$ is small-complete.
2. Show that in $\mathcal{T}op_{\text{Zariski}}$ every Zariski-closed subset of a subspace Y of X is the intersection of a Zariski-closed set in X with Y. Generalize this statement to Ω-$\mathcal{S}et$.
3. Consider $K = 2 = \{0,1\}$ and $\Omega = \emptyset$. Show that Ω-$\mathcal{S}et$ is the category $\mathcal{C}hu_2^{\text{ext}}$ of extensive Boolean Chu spaces (see [42]), whose objects are sets X with a "generalized topology" $\tau \subseteq 2^X$ (no further condition on τ), and whose morphisms are "continuous maps". Prove that a subspace M of X is Zariski-closed if for every $x \in X \setminus M$ there are ("open sets") $U, V \in \tau$ with $U \cap M = V \cap M$ and $x \in U \setminus V$.

2.8. Topological spaces with open maps. Here we mention a somewhat counter-intuitive but nevertheless important example. Again, we consider $\mathcal{T}op$ with its $(\text{Epi}, \mathcal{M})$-factorization system, but now take $\mathcal{F}$ to be the class of open maps (see Exercise 2 of 2.1). Here the $\mathcal{F}$-dense maps $f : X \to Y$ are characterized by the property that the closure $\overline{\{y\}}$ of every point $y \in Y$ meets the image $f[X]$. When equipping $\mathcal{T}op$ with this structure, we refer to $\mathcal{T}op_{\text{open}}$ rather than to $\mathcal{T}op$ (see 2.3).

2.9. Abelian groups with torsion-preserving maps. The category $\mathcal{A}b\mathcal{G}rp$ of abelian groups has, like $\mathcal{G}rp$, an $(\text{Epi}, \text{Mono})$-factorization system. Here we consider for $\mathcal{F}$ the class of homomorphisms $f : A \to B$ which map the torsion subgroup of A onto the torsion subgroup of B: $f[\text{Tor}A] = \text{Tor}B = \{b \in B \mid nb = 0 \text{ for some } n \geq 1\}$. (More generally, for any ring R, we could consider here any preradical of R-modules in lieu of Tor, or even of any finitely-complete category with a proper factorization system, in lieu of $\mathcal{A}b\mathcal{G}rp$; see [18].) The map f is $\mathcal{F}$-dense if and only if $B = \text{im}f + \text{Tor}B$.

We end our list of standard examples with an important general procedure:

2.10. Passing to comma categories. Given our standard setting with $\mathcal{X}$, $\mathcal{E}$, $\mathcal{M}$, $\mathcal{F}$ satisfying (F0)-(F5) and a fixed object B in $\mathcal{X}$, let $\mathcal{X}/B$ be the *comma category* (or *sliced category*) of *objects* (X,p) *over* B, simply given by morphisms $p : X \to B$ in $\mathcal{X}$; a morphism $f : (X,p) \to (Y,q)$ in $\mathcal{X}/B$ is a morphism $f : X \to Y$ in $\mathcal{X}$ with $q \cdot f = p$. With the forgetful functor $\Sigma_B : \mathcal{X}/B \to \mathcal{X}$, putting $\mathcal{E}_B := \Sigma_B^{-1}(\mathcal{E})$, $\mathcal{M}_B := \Sigma_B^{-1}(\mathcal{M})$, $\mathcal{F}_B := \Sigma_B^{-1}(\mathcal{F})$, it is easy to check that (F0)-(F5) hold true in $\mathcal{X}/B$.

Proposition. *For every object B of $\mathcal{X}$, $(\mathcal{E}_B, \mathcal{M}_B)$ is a proper factorization system of $\mathcal{X}/B$ and $\mathcal{F}_B$ is an $(\mathcal{E}_B, \mathcal{M}_B)$-closed class.* □

In what follows, whenever we refer to $\mathcal{X}/B$, we think of it as being structured according to the Proposition. This applies particularly to $\mathcal{T}op/B$, $\mathcal{T}op_{\text{open}}/B$, etc. We normally drop the subscript B when referring to $\mathcal{E}_B$, $\mathcal{M}_B$, $\mathcal{F}_B$.

3. Proper maps, compact spaces

3.1. Compact objects. The pullback of a closed map in $\mathcal{T}op$ need not be a closed map. In fact, the only map from the real line $\mathbb{R}$ to a one-point space 1 is closed, but the pullback of this map along itself is not:

$$\begin{array}{ccc} \mathbb{R}\times\mathbb{R} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & 1 \end{array} \tag{10}$$

neither projection of $\mathbb{R}\times\mathbb{R}$ preserves closedness of the subspace $\{(x,y)\,|\,x\cdot y = 1\}$. Hence, we should pay attention to those closed maps which *are* stable under pullback.

In our finitely-complete category $\mathcal{X}$ with its proper $(\mathcal{E},\mathcal{M})$-factorization system and the distinguished class $\mathcal{F}$ of closed maps (satisfying (F0)-(F5)) we call a morphism f $\mathcal{F}$*-proper* if f belongs *stably* to $\mathcal{F}$, so that in every pullback diagram

$$\begin{array}{ccc} W & \xrightarrow{f'} & Z \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \tag{11}$$

$f' \in \mathcal{F}$; laxly we speak of an $\mathcal{F}$-proper map in $\mathcal{X}$ and denote by $\mathcal{F}^*$ the class of all $\mathcal{F}$-proper maps. Note that since g and g' may be chosen as identity morphisms, $\mathcal{F}^* \subseteq \mathcal{F}$.

Checking $\mathcal{F}$-properness of a morphism may be facilitated by the following criterion:

Proposition. *A morphism $f : X \to Y$ is $\mathcal{F}$-proper if and only if every restriction of $f \times 1_Z : X \times Z \to Y \times Z$ (see* (8)*) is $\mathcal{F}$-closed, for every object Z.*

Proof. Since with $f \times 1_Z$ also every of its restrictions is a pullback of f, the necessity of the condition is clear. That it is sufficient for $\mathcal{F}$-properness follows from a factorization of the pullback diagram (11), as follows:

$$\begin{array}{ccc} W & \xrightarrow{\quad f' \quad} & Z \\ {\scriptstyle <g',f'>}\downarrow & & \downarrow{\scriptstyle <g,1_Z>} \\ X\times Z & \xrightarrow{f\times 1_Z} & Y\times Z \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow{\quad f \quad} & Y \end{array} \tag{12}$$

Since both the total diagram and its lower rectangle are pullback diagrams, so is its upper rectangle. Moreover, $n := < g, 1_Z > \in \mathcal{M}$ (see Exercise 3 of 1.3), so that the condition of the Proposition applies here. $\square$

Often, but not always, the condition of the Proposition can be simplified. Let us say that $\mathcal{F}$ *is stable under restriction* if every restriction of an $\mathcal{F}$-closed morphism is $\mathcal{F}$-closed, i.e., if $\mathcal{F}$ is stable under pullback along morphisms in $\mathcal{M}$. In this case f is $\mathcal{F}$-proper if and only if $f \times 1_Z$ is $\mathcal{F}$-closed for all Z.

Exercises.

1. Show that in $\mathcal{T}op$, $\mathcal{T}op_{\mathrm{open}}$, $\mathcal{T}op_{\mathrm{clopen}}$, $\mathcal{T}op_{\mathrm{Zariski}}$, the class $\mathcal{F}$ is stable under restriction.
2. Using $\mathcal{F} = \mathcal{E}$, show that generally $\mathcal{F}$ fails to be stable under restriction.
3. Let $\mathcal{F} = \mathcal{H}$ with $\mathcal{H}$ as in Exercise 4 of 2.1, and assume that $\mathcal{E}$ is stable under pullback along $\mathcal{M}$, and that every $\mathcal{F}$-closed subobject of M with $m : M \to X$ in $\mathcal{M}$ is the inverse image of an $\mathcal{F}$-closed subobject of X under m. Show that $\mathcal{F}$ is stable under restriction. Revisit Exercise 1.

3.2. Stability properties. We show some important stability properties for $\mathcal{F}$-proper maps:

Proposition.

(1) *The class $\mathcal{F}^*$ contains $\mathcal{F} \cap \mathcal{M}$ and is closed under composition.*
(2) *$\mathcal{F}^*$ is the largest pullback-stable subclass of $\mathcal{F}$.*
(3) *If $g \cdot f \in \mathcal{F}^*$ with g monic, then $f \in \mathcal{F}^*$.*
(4) *If $g \cdot f \in \mathcal{F}^*$ with $f \in \mathcal{E}^*$, then $g \in \mathcal{F}^*$; here $\mathcal{E}^*$ is the class of the morphisms stably in $\mathcal{E}$.*

Proof. (1) follows from (F4) and the fact that a pullback of $g \cdot f$ can be obtained as a composite of a pullback of g preceded by a pullback of f. Likewise, (2) and (4) follow from the composability of adjacent pullback diagrams. (3) is a consequence of Exercise 4 of 1.4. □

Exercises.

1. Make sure that you understand every detail of the proof of the Proposition.
2. Show that with $f : X \to Y$ also the following morphisms are $\mathcal{F}$-proper:
   (a) $f \cdot m : M \to Y$ for every $\mathcal{F}$-closed embedding $m : M \to X$;
   (b) $f' : f^{-1}[N] \to N$ for every $n \in \mathrm{sub}Y$.
3. Using only closure of $\mathcal{F}^*$ under composition and pullback stability, show that with $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ also $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is $\mathcal{F}$-proper. (We say that $\mathcal{F}^*$ is closed under finite products. Hint: $f_1 \times f_2 = (f_1 \times 1) \cdot (1 \times f_2)$.)
4. Show that in the standard example $\mathcal{T}op$ (see 2.3) one has $\mathcal{E}^* = \mathcal{E}$; likewise for $\mathcal{T}op_{\mathrm{open}}$, $\mathcal{T}op_{\mathrm{clopen}}$, $\mathcal{T}op_{\mathrm{Zariski}}$, and for every topos, but not for $\mathcal{H}aus$ and $\mathcal{L}oc$.

3.3. Compact objects. An object X of $\mathcal{X}$ is called *$\mathcal{F}$-compact* if the unique morphism $!_X : X \to 1$ to the terminal object is $\mathcal{F}$-proper. Since the pullbacks of $!_X$ are the projections $X \times Y \to Y$, $Y \in \mathcal{X}$, to say that X is $\mathcal{F}$-compact means precisely that any such projection must be in $\mathcal{F}$. Before exhibiting this notion in terms of examples, let us draw some immediate conclusions from Prop. 3.2:

Theorem.

(1) *For any $\mathcal{F}$-proper $f : X \to Y$ with Y $\mathcal{F}$-compact, X is $\mathcal{F}$-compact.*
(2) *For any $f : X \to Y$ in $\mathcal{E}^*$ with X $\mathcal{F}$-compact, Y is $\mathcal{F}$-compact.*
(3) *The full subcategory $\mathcal{F}$-Comp of $\mathcal{F}$-compact objects in $\mathcal{X}$ is closed under finite products and under $\mathcal{F}$-closed subobjects.*

Proof. (1) Since

$$!_X = (X \xrightarrow{f} Y \xrightarrow{!_Y} 1),$$

the assertion follows from the compositivity of $\mathcal{F}^*$ (Prop. 3.2(1)).
(2) For the same reason one may apply Prop. 3.2(4) here.
(3) Since $\mathcal{F}^*$ contains all isomorphisms one has $1 \in \mathcal{F}$-Comp, and for $X, Y \in \mathcal{F}$-Comp one can apply (1) to the projection $X \times Y \to Y$. Likewise, one can apply (1) to $f \in \mathcal{F}_0$ to obtain closure under $\mathcal{F}$-closed subobjects. □

Exercises.

1. Show that every $\mathcal{F}$-closed subobject of an $\mathcal{F}$-compact object is $\mathcal{F}$-compact.
2. Find an example in $\mathcal{T}op$ of a morphism $f : X \to Y$ with $X, Y \in \mathcal{F}$-Comp which fails to be $\mathcal{F}$-proper.
3. Show that $\mathcal{F}$-Comp in $\mathcal{T}op$ fails to be closed under finite limits.

3.4. Categorical compactness in $\mathcal{T}op$. We show that in $\mathcal{T}op$ (with $\mathcal{F} = \{$closed maps$\}$) our notion of $\mathcal{F}$-compactness coincides with the usual notion of compactness given by the Heine-Borel open-cover property:

Theorem (Kuratowski and Mrówka). *For a topological space X, the following are equivalent:*

(i) *X is $\mathcal{F}$-compact, i.e., the projection $X \times Y \to Y$ is closed for all $Y \in \mathcal{T}op$;*
(ii) *for every family of open sets $U_i \subseteq X$ $(i \in I)$ with $X = \bigcup_{i \in I} U_i$, there is a finite set $F \subseteq I$ with $X = \bigcup_{i \in F} U_i$.*

Proof. (ii) ⇒ (i): This part is the well-known Kuratowski Theorem, we only sketch a possible proof here (see also Exercise 1 below). To see that $B = p_2(A)$ is closed for $A \subseteq X \times Y$ closed and $p_2 : X \times Y \to Y$ the second projection, assume $y \in \overline{B} \setminus B$. Then $\{A \cap p_2^{-1}[V] \mid V$ neighbourhood of $y\}$ is a base of a filter $\mathbb{F}$ on $X \times Y$. The filter $p_1(\mathbb{F})$ on X must have a cluster point x, hence

$$x \in \bigcap \{\overline{p_1[A \cap p_2^{-1}[V]]} \mid V \text{ neighbourhood of } y\},$$

which implies $(x, y) \in \overline{A} \setminus A$: contradiction.
(i) ⇒ (ii): With the given open cover $\{U_i \mid i \in I\}$ of X, define a topological space Y with underlying set $X \cup \{\infty\}$ (with $\infty \notin X$) by

$$K \subseteq Y \text{ closed } :\Leftrightarrow \ \infty \in K \text{ or } K \subseteq \bigcup_{i \in F} U_i \text{ for some finite } F \subseteq I.$$

It now suffices to show that $X \subseteq Y$ is closed in Y, and for that it suffices to prove $p_2(\overline{\Delta_X}) = X$, with the closure of $\Delta_X = \{(x, x) \mid x \in X\}$ formed in $X \times Y$. But the assumption $\infty \in p_2(\overline{\Delta_X})$ would give an $x \in X$ with $(x, \infty) \in \overline{\Delta_X}$. Since x lies in some U_i we would then have $(x, \infty) \in U_i \times (Y \setminus U_i)$, an open set in $X \times Y$ which does not meet Δ_X, a contradiction! □

We note that the proof of the Theorem does not require the Axiom of Choice. This applies also to the following exercise.

Exercise. Extending and refining the argumentation in the proof of the Theorem, prove the equivalence of the following statements for a topological space X:

(i) X is $\mathcal{F}$-compact;
(ii) $p_2 : X \times Y \to Y$ is closed for every zerodimensional, normal Hausdorff space Y;
(iii) X has the Heine-Borel open-cover property;
(iv) every filter $\mathbb{F}$ on X has a cluster point, i.e. $\bigcap\{\overline{F} \mid F \in \mathbb{F}\} \neq \emptyset$.

3.5. Fibres of proper maps. In 3.3 we defined $\mathcal{F}$-compactness via $\mathcal{F}$-properness. We could have proceeded conversely, using the following fact:

Proposition. *$f : X \to Y$ is $\mathcal{F}$-proper if and only if (X, f) is an $\mathcal{F}$-compact object of $\mathcal{X}/Y$* (see 2.10).

Proof. The terminal object in $\mathcal{X}/Y$ is $(Y, 1_Y)$, and the unique morphism $(X, f) \to (Y, 1_Y)$ is f itself. □

The question remains whether $\mathcal{F}$-properness can be characterized by $\mathcal{F}$-compactness within the category $\mathcal{X}$. Towards this we first note:

Corollary. *For the following statements on $f : X \to Y$ in $\mathcal{F}$, one has* (i) ⇒ (ii) ⇒ (iii)*:*

(i) *f is $\mathcal{F}$-proper;*
(ii) *for every pullback diagram* (11)*, if Z is $\mathcal{F}$-compact, so is W;*
(iii) *all fibres of f are $\mathcal{F}$-compact, where a fibre F of f occurs in any pullback diagram*

$$\begin{array}{ccc} F & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \qquad (13)$$

Proof. (i) ⇒ (ii): Since $\mathcal{F}^*$ is pullback-stable, in (11) with f also f' is $\mathcal{F}$-proper. Hence, the assertion of (ii) follows from Theorem 3.3(1).
(ii) ⇒ (iii) follows by putting $Z = 1$, which is $\mathcal{F}$-compact by Theorem 3.3(3). □

We now show (choice free) that in $\mathcal{T}op$ condition (iii) is already sufficient for properness:

Theorem. *In $\mathcal{T}op$ a map is $\mathcal{F}$-proper if and only if it is closed and has compact fibres.*

Proof. Let $f : X \to Y$ be closed with compact fibres, and for any space Z let $A \subseteq X \times Z$ be closed; we must show that $B := (f \times 1_Z)[A] \subseteq Y \times Z$ is closed.

Hence, for any point (y, z) in the complement $(Y \times Z) \setminus B$ we must find open sets $V_0 \subseteq Y$, $W_0 \subseteq Z$ with $(y, z) \in V_0 \times W_0 \subseteq (Y \times Z) \setminus B$. The system of all pairs (U, W) where U is an open neighbourhood in X of some $x \in f^{-1}y$ and W is an open neighbourhood in Z of z with $U \times W \subseteq (X \times Z) \setminus A$ has the property that its first components cover the compact fibre $f^{-1}y$. Hence, we obtain finitely many open sets $U_1, \cdots, U_n$, $W_1, \cdots, W_n$ $(n \geq 0)$ with $f^{-1}y \subseteq U_0 := \bigcup_{i=1}^n U_i$, $z \in W_0 := \bigcap_{i=1}^n W_i$ and $U_0 \times W_0 \subseteq (X \times Z) \setminus A$. Since f is closed, the set $V_0 := Y \setminus f(X \setminus U_0)$ is open and has the required properties. □

However, in general, in $\mathcal{T}op/B$ not even condition (ii) is sufficient for $\mathcal{F}$-properness, as the following example shows:

Example. Let B be any indiscrete topological space with at least 2 points, let X be any non-compact topological space, and let $q : 1 \to B$ be any map. Then, with $p = q \cdot f$, the unique map $f : X \to 1$ becomes a morphism $f : (X, p) \to (1, q)$ in $\mathcal{T}op/B$ which is $\mathcal{F}$-closed but not $\mathcal{F}$-proper. However, condition (ii) of the Corollary is vacuously satisfied since, given any pullback diagram (11) (with $Y = 1$) in $\mathcal{X}$ which then represents a pullback diagram also in $\mathcal{X}/B$, we observe that the object $(Z, q \cdot g)$ is never $\mathcal{F}$-compact in $\mathcal{T}op/B$ unless Z is empty; indeed, for $Z \neq \emptyset$ the constant map $q \cdot g$ is not proper, since its image is not closed in B.

Exercises.

1. In $\mathcal{T}op_{\text{open}}$, prove that every map in $\mathcal{F} = \{f \mid f \text{ open}\}$ is $\mathcal{F}$-proper. Conclude that every object is $\mathcal{F}$-compact.
2. * Prove: a space X in $\mathcal{T}op_{\text{Zariski}}$ is $\mathcal{F}$-compact if and only if if it is compact with respect to its Zariski topology (see 2.7). Conclude that this is the case precisely when (a) every subspace of X is compact (in $\mathcal{T}op$) and (b) every closed subspace of X has the form $\overline{\{x_1\}} \cup \cdots \cup \overline{\{x_n\}}$ for finitely many points $x_1, \cdots, x_n$ in X, $n \geq 0$.

3.6. Proper maps of locales. In $\mathcal{L}oc$, $\mathcal{F}$-proper maps are characterized by:

Theorem (Vermeulen [49, 50]). *For $f : X \to Y$ in $\mathcal{L}oc$, the following are equivalent:*

(i) *f is $\mathcal{F}$-proper;*
(ii) *$f \times 1_Z : X \times Z \to Y \times Z$ is $\mathcal{F}$-closed for all locales Z;*
(iii) *the restriction $f[-] : \mathcal{C}X \to \mathcal{C}Y$ of the direct-image map to closed sublocales is well defined and preserves filtered infima.*

For the proof of this theorem we must refer to Vermeulen's papers [49, 50]. However, we may point out that the equivalence proof for (i), (ii) given in [50] makes use of a refined version of Exercise 3 of 3.1. We leave it as an exercise to show that when exploiting property (iii) in case $Y = 1$, one obtains (see Theorem II.6.5):

Corollary (Pultr-Tozzi [43]). *A locale X is $\mathcal{F}$-compact if and only if every open cover of X contains a finite subcover.*

3.7. Sums of proper maps and compact objects. Next we want to examine the behaviour of $\mathcal{F}$-compact objects and $\mathcal{F}$-proper maps under the formation of finite coproducts. Clearly, some compatibility between pullbacks and finite coproducts has to be in place in order to establish any properties, in addition to closedness of $\mathcal{F}$ under finite coproducts, so that with $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ also $f_1 + f_2 : X_1 + X_2 \to Y_1 + Y_2$ is in $\mathcal{F}$. The notion of extensive category (as given in 2.6) fits our requirements:

Proposition. *Assume that the finitely-complete category $\mathcal{X}$ has finite coproducts and is extensive, and that $\mathcal{F}$ is closed under finite coproducts. Then:*

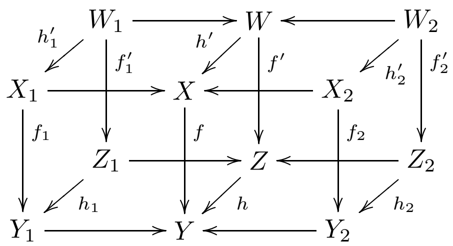
(1) *the morphism class $\mathcal{F}^*$ is closed under finite coproducts;*

(2) *the subcategory $\mathcal{F}$-Comp is closed under finite coproducts in $\mathcal{X}$ if and only if the canonical morphisms*

$$0 \longrightarrow X \text{ and } X + X \longrightarrow X$$

are $\mathcal{F}$-closed, for all objects X.

Proof. (1) Diagram (14) shows how one may obtain the pullback f' of $f = f_1 + f_2$ along h: pulling back h along the injections of $Y = Y_1 + Y_2$



(14)

one obtains h_1 and h_2, along which one pulls back f_1 and f_2 to obtain f_1' and f_2' and the induced arrows $W_1 \to W \leftarrow W_2$ giving the commutative back faces. These are pullback diagrams since all other vertical faces are; by extensivity then, $f' = f_1' + f_2'$. Hence with $f_1, f_2 \in \mathcal{F}^*$ we have $f_1', f_2' \in \mathcal{F}$ and then $f' \in \mathcal{F}$, so that $f \in \mathcal{F}^*$.

(2) Once we have $1 + 1 \in \mathcal{F}$-Comp

$$\begin{array}{ccccc} X & \longrightarrow & X + Y & \longleftarrow & Y \\ \downarrow & & \downarrow {\scriptstyle !_X + !_Y} & & \downarrow \\ 1 & \longrightarrow & 1 + 1 & \longleftarrow & 1 \end{array} \qquad (15)$$

we obtain $X+Y \in \mathcal{F}$-Comp whenever $X, Y \in \mathcal{F}$-Comp since $!_{X+Y} = !_{1+1} \cdot (!_X + !_Y)$, by an application of (1). Now,

$$\begin{aligned} 1 + 1 \in \mathcal{F}\text{-Comp} \;\; &\Leftrightarrow \;\; \forall X : ((1+1) \times X \to X) \in \mathcal{F} \\ &\Leftrightarrow \;\; \forall X : (X + X \to X) \in \mathcal{F}, \end{aligned}$$

since $(1 + 1) \times X \cong X + X$ (see Exercise 4 of 2.6). Furthermore,

$$\begin{aligned} 0 \in \mathcal{F}\text{-Comp} \;\; &\Leftrightarrow \;\; \forall X : (0 \times X \to X) \in \mathcal{F} \\ &\Leftrightarrow \;\; \forall X : (0 \to X) \in \mathcal{F}, \end{aligned}$$

since $0 \times X \cong 0$ (see Exercise 4 of 2.6). □

Exercises.

1. Check that the hypotheses of the Theorem are satisfied in $\mathcal{T}op$, $\mathcal{T}op_{\text{open}}$, $\mathcal{T}op_{\text{clopen}}$, $\mathcal{T}op_{\text{Zariski}}$, and in any extensive category with $\mathcal{F}$ as in 2.6. Conclude that in these categories $\mathcal{F}$-Comp is closed under finite coproducts.
2. Show that in $\mathcal{X} = (\mathcal{CR}ng)^{\text{op}}$ with $\mathcal{F}$ as in 2.6, the only $\mathcal{F}$-compact rings (up to isomorphism) are $\mathbb{Z}$ and $\{0\}$, and that $\mathcal{F}$-Comp fails to be closed under finite coproducts in $\mathcal{X}$.

4. Separated maps, Hausdorff spaces

4.1. Separated morphisms. With every morphism f in our category $\mathcal{X}$, structured by $\mathcal{E}$, $\mathcal{M}$, $\mathcal{F}$ as in 2.1, we may associate the morphism

$$\delta_f :< 1_X, 1_X >: X \longrightarrow X \times_Y X$$

where $X \times_Y X$ belongs to the pullback diagram

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{f_2} & X \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array} \tag{16}$$

representing the kernel pair of f. It is easy to see that

$$X \xrightarrow{\delta_f} X \times_Y X \overset{f_1}{\underset{f_2}{\rightrightarrows}} X \tag{17}$$

is an equalizer diagram. Instead of asking whether $f \in \mathcal{F}^*$, in this section we investigate those f with $\delta_f \in \mathcal{F}^*$. Actually, since $\delta_f \in \mathcal{M}$ and $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{M}$ is stable under pullback, it is enough to require $\delta_f \in \mathcal{F}$.

We call a morphism f in $\mathcal{X}$ *$\mathcal{F}$-separated* if $\delta_f \in \mathcal{F}$; laxly we speak of an $\mathcal{F}$-separated map and denote by $\mathcal{F}'$ the class of all $\mathcal{F}$-separated maps.

Example. In $\mathcal{T}op$, to say that $\Delta_X = \{(x,x) \mid x \in X\}$ is closed in the subspace $X \times_Y X = \{(x_1,x_2) \mid f(x_1) = f(x_2)\}$ of $X \times X$ is to say that $(X \times_Y X) \setminus \Delta_X$ is open, and this means that for all $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$, $x_1 \neq x_2$ there are open neighbourhoods $U_1 \ni x_1$, $U_2 \ni x_2$ in X with $U_1 \times_Y U_2 \subseteq (X \times_Y X) \setminus \Delta_X$, i.e., $U_1 \cap U_2 \neq \emptyset$. Hence, in $\mathcal{T}op$, a map $f : X \to Y$ is $\mathcal{F}$-separated if and only if distinct points in the same fibre of f may be separated by disjoint neighbourhoods in X. Such maps are called *separated* in fibred topology.

Exercises. Prove:

1. $f : X \to Y$ is $\mathcal{F}$-separated in $\mathcal{T}op_{\text{open}}$ if and only if f is *locally injective*, so that for every point x in X there is a neighbourhood U of x such that $f|_U : U \to Y$ is an injective map.

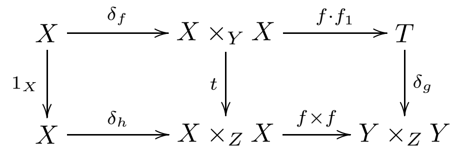
2. The $\mathcal{F}$-separated maps in $\mathcal{T}op_{\mathrm{clopen}}$ are precisely the separated and locally injective maps. Show that neither of the latter two properties implies the other.
3. $f : X \to Y$ is $\mathcal{F}$-separated in $\mathcal{T}op_{\mathrm{Zariski}}$ if and only if distinct points in the same fibre have distinct neighbourhood filters.
4. $f : A \to B$ is $\mathcal{F}$-separated in $\mathcal{A}b\mathcal{G}rp$ if and only if the restriction of f to $\mathrm{Tor}A$ is injective, i.e., $\ker f|_{\mathrm{Tor}A} = 0$.

4.2. Properties of separated maps. When establishing stability properties for $\mathcal{F}'$, it is important to keep in mind that $\mathcal{F}'$ depends only on $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{M}$; hence, only the behaviour of $\mathcal{F}_0$ plays a role in what follows.

Proposition.

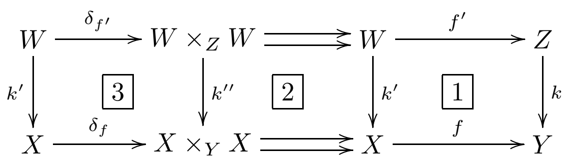
(1) *The class $\mathcal{F}'$ contains all monomorphisms of $\mathcal{X}$ and is closed under composition.*
(2) *$\mathcal{F}'$ is stable under pullback.*
(3) *Whenever $g \cdot f \in \mathcal{F}'$, then $f \in \mathcal{F}'$.*
(4) *Whenever $g \cdot f \in \mathcal{F}'$ with $f \in \mathcal{E} \cap \mathcal{F}^*$, then $g \in \mathcal{F}'$.*

Proof. (1) Monomorphisms are characterized as those morphisms f with δ_f iso. Let us now consider $f : X \to Y$, $g : Y \to Z$ and their composite $h = g \cdot f$. Then there is a unique morphism $t : X \times_Y X \to X \times_Z X$ with $h_1 \cdot t = f_1$, $h_2 \cdot t = f_2$. It makes the following diagram commutative:



(18)

The right square is in fact a pullback diagram since t is the equalizer of $f \cdot h_1 = f \cdot h_2$ (with h_1, h_2 the kernelpair of h). Consequently, with δ_g also t is in $\mathcal{F}$, and then with δ_f also $\delta_h = t \cdot \delta_f$ is in $\mathcal{F}$.
(2)



(19)

Since with $\boxed{1}$ also $\boxed{2}\&\boxed{1}$ and then $\boxed{3}$ are pullback diagrams, the assertion follows from the pullback stability of $\mathcal{F} \cap \mathcal{M}$.
(3) Going back to (18), if $\delta_h = t \cdot \delta_f$ is in the pullback-stable class $\mathcal{F}_0$, so is δ_f since t is monic (see Exercise 4 of 1.4).
(4) Since $f = f \cdot f_1 \cdot \delta_f$, from (18) one has $\delta_g \cdot f = (f \times f) \cdot \delta_h$. Now, if $h \in \mathcal{F}'$ and $f \in \mathcal{F}^*$, then $(f \times f) \cdot \delta_h = \delta_g \cdot f \in \mathcal{F}$ (see Exercise 1). Hence, if also $f \in \mathcal{E}$, $\delta_f \in \mathcal{F}$ follows with (F5). □

Exercises.

1. Show that with $f_1, f_2 \in \mathcal{F}'$ also $f_1 \times f_2 \in \mathcal{F}'$ (see Exercise 3 of 3.2).
2. In generalization of 1, prove that $\mathcal{F}'$ is closed under those limits under which $\mathcal{F} \cap \mathcal{M}$ is closed. Hint: In the setting of Exercise 3 of 1.4, prove the formula $\delta_{\lim \mu} = \lim_d \delta_{\mu_d}$.

4.3. Separated objects. An object X of $\mathcal{X}$ is called *$\mathcal{F}$-separated* (or *$\mathcal{F}$-Hausdorff*) if the unique morphism $!_X : X \to 1$ is $\mathcal{F}$-separated; this simply means that $\delta_X = < 1_X, 1_X >: X \to X \times X$ must be in $\mathcal{F}$.

Theorem. *The following conditions are equivalent for an object X:*

(i) *X is $\mathcal{F}$-separated;*
(ii) *every morphism $f : X \to Y$ is $\mathcal{F}$-separated;*
(iii) *there is an $\mathcal{F}$-separated morphism $f : X \to Y$ with Y $\mathcal{F}$-separated;*
(iv) *for every object Y the projection $X \times Y \to Y$ is $\mathcal{F}$-separated;*
(v) *for every $\mathcal{F}$-separated object Y, $X \times Y$ is $\mathcal{F}$-separated;*
(vi) *for every $\mathcal{F}$-proper morphism $f : X \to Y$ in $\mathcal{E}$, Y is $\mathcal{F}$-separated;*
(vii) *in every equalizer diagram*

$$E \xrightarrow{u} Z \rightrightarrows X,$$

u is $\mathcal{F}$-closed.

Proof. Since $!_X = !_Y \cdot f$, the equivalence of (i), (ii), (iii) follows from compositivity and left cancellation of $\mathcal{F}'$ (Prop. 4.2(1),(3)). Since the projection $X \times Y \to Y$ is a pullback of $X \to 1$, (i) $\Rightarrow$ (iv) follows from pullback stability of $\mathcal{F}'$ (Prop. 4.2(2)), and (iv) $\Rightarrow$ (v) from its compositivity again. For (v) $\Rightarrow$ (i) consider $Y = 1$, which is trivially $\mathcal{F}$-separated. For (i) $\Rightarrow$ (vi) apply Prop. 4.2(4) with $g = !_Y$, and for (vi) $\Rightarrow$ (i) let $f = 1_X$. Finally (i) $\Leftrightarrow$ (vii) follows since such equalizers u are precisely the pullbacks of δ_X. □

Corollary. *The full subcategory* $\mathcal{F}$-Haus *of $\mathcal{F}$-separated objects is closed under finite limits and under subobjects in $\mathcal{X}$. In fact, for every monomorphism $m : X \to Y$, with Y also X is $\mathcal{F}$-separated.*

Proof. Since every monomorphism is $\mathcal{F}$-separated, consider (iii), (iv) of the Theorem. □

Exercises.

1. In $\mathcal{T}op$ and $\mathcal{L}oc$, $\mathcal{F}$-separation yields the usual notion of Hausdorff separation of these categories.
2. Show that in $\mathcal{T}op_{\text{open}}$ and in $\mathcal{T}op_{\text{clopen}}$ the $\mathcal{F}$-separated objects are the discrete spaces. Conclude that, in general, $\mathcal{F}$-Haus fails to be closed under (infinite) products in $\mathcal{X}$.
3. Prove that in $\mathcal{T}op_{\text{Zariski}}$ $\mathcal{F}$-Haus is the category of T0-spaces (i.e., those spaces in which distinct points have distinct neighbourhood filters).

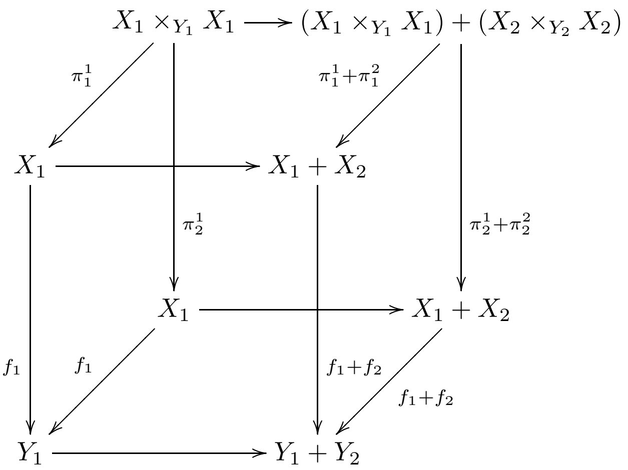
4. The quasi-component $q_X(M)$ of a subset M of a topological space X is the intersection of all clopen subsets of X containing M. Let $\mathcal{T}op_{\text{quasicomp}}$ denote the category $\mathcal{T}op$ with its (Epi, $\mathcal{M}$)-factorization structure and take for $\mathcal{F}$ the class all q-preserving maps $f : X \to Y$, i.e., $f[q_X(M)] = q_Y(f[M])$. Show that $\mathcal{F}$-Haus in $\mathcal{T}op_{\text{quasicomp}}$ is the category of totally-disconnected spaces (i.e., spaces in which all components are singletons).
5. Prove that in $\mathcal{A}b\mathcal{G}rp$ $\mathcal{F}$-Haus is the category of torsion-free abelian groups (i.e., groups in which $na = 0$ implies $n = 0$ or $a = 0$).

4.4. Sums of separated maps and objects. Closure of $\mathcal{F}'$ and of $\mathcal{F}$-Haus under finite coproducts requires not only a stability property of $\mathcal{F}_0$ under finite coproducts but, like in 3.7, also extensivity of $\mathcal{X}$:

Proposition. *Assume that the finitely-complete category $\mathcal{X}$ has finite coproducts and is extensive, and that the coproduct of two $\mathcal{F}$-closed subobjects is $\mathcal{F}$-closed. Then:*

(1) *the morphism class $\mathcal{F}'$ is closed under finite coproducts;*
(2) *the subcategory $\mathcal{F}$-*Haus *is closed under finite coproducts in $\mathcal{X}$ if and only if* $1 + 1 \in \mathcal{F}$-Haus.

Proof. (1) For $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$, by extensivity, as a coproduct of two pullback squares also the right vertical face of (20) is a pullback square.



(20)

Hence,

$$(X_1 \times_{Y_1} X_1) + (X_2 \times_{Y_2} X_2) \cong (X_1 + X_2) \times_{Y_1+Y_2} (X_1 + X_2),$$

and the formula $\delta_{f_1+f_2} \cong \delta_{f_1} + \delta_{f_2}$ follows immediately. Consequently, $f_1, f_2 \in \mathcal{F}'$ implies $f_1 + f_2 \in \mathcal{F}'$.
(2) We saw in (1) that $(\pi_1^1+\pi_1^2, \pi_2^1+\pi_2^2)$ is the kernelpair of f_1+f_2. Exploiting this fact in case $Y_1 = Y_2 = 1$, we see that the canonical morphism u, making diagram

(21) commutative, is actually an equalizer of $(f_1 + f_2) \cdot \pi_1$, $(f_1 + f_2) \cdot \pi_2$.

$$
\begin{array}{ccc}
X_1 + X_2 & \xrightarrow{\quad 1 \quad} & X_1 + X_2 \\
{\scriptstyle \delta_{X_1} + \delta_{X_2}} \downarrow & & \downarrow {\scriptstyle \delta_{X_1 + X_2}} \\
(X_1 \times X_1) + (X_2 \times X_2) & \xrightarrow{\ u\ } & (X_1 + X_2) \times (X_1 + X_2) \\
{\scriptstyle \pi_1^1 + \pi_1^2} \downarrow\downarrow {\scriptstyle \pi_2^1 + \pi_2^2} & & {\scriptstyle \pi_1} \downarrow\downarrow {\scriptstyle \pi_2} \\
X_1 + X_2 & \xrightarrow{\quad 1 \quad} & X_1 + X_2
\end{array}
\qquad (21)
$$

Hence, if $1 + 1$ is $\mathcal{F}$-separated, then u is $\mathcal{F}$-closed (see (vii) of Theorem 4.3), and with $\delta_{X_1}, \delta_{X_2} \in \mathcal{F}$ we obtain $\delta_{X_1 + X_2} \in \mathcal{F}$. Since $0 \times 0 \cong 0$, trivially $\delta_0 \in \mathcal{F}$. □

Exercises.

1. Show that, in $\mathcal{T}op$, $\mathcal{T}op_{\text{open}}$, $\mathcal{T}op_{\text{clopen}}$ and $\mathcal{T}op_{\text{Zariski}}$, $\mathcal{F}'$ and $\mathcal{F}$-Haus are closed under finite coproducts. Likewise in $\mathcal{A}b\mathcal{G}rp$.
2. Show that, in general, $1+1$ fails to be $\mathcal{F}$-separated. (Consider, for example, $\mathcal{F} = \text{Iso}$.)

4.5. Separated objects in the slices. We have presented $\mathcal{F}$-separation for objects as a special case of the morphism notion. We could have proceeded conversely, using:

Proposition. *$f : X \to Y$ is $\mathcal{F}$-separated if and only if (X, f) is an $\mathcal{F}$-separated object of $\mathcal{X}/Y$* (see 2.10)*.*

Proof. $\delta_f : X \to X \times_Y X$ in $\mathcal{X}$ serves as the morphism $\delta_{(X,f)} : (X, f) \to (X, f) \times (X, f)$ in $\mathcal{X}/Y$. □

Exercises.

1. Prove Corollary 4.3 without recourse to the previous Theorem, just using the definition of $\mathcal{F}$-Hausdorff object. Then apply the assertions of the Corollary to $\mathcal{X}/Y$ in lieu of $\mathcal{X}$ and express the assertions in terms of properties of $\mathcal{F}^*$.
2. Prove that with $\mathcal{X}$ also each slice $\mathcal{X}/Y$ is extensive. (Hint: Slices of slices of $\mathcal{X}$ are slices of $\mathcal{X}$.)

We finally mention three important "classical" examples, without proofs.

Examples.

(1) Since commutative unital R-algebras A are completely described by morphisms $R \to A$ in $\mathcal{CR}ng$, the opposite of the category $\mathcal{CA}lg_R$ is extensive, by Exercise 2:

$$\mathcal{CA}lg_R^{\text{op}} \cong \mathcal{CR}ng^{\text{op}}/R.$$

An $\mathcal{F}$-separated object in this extensive category is precisely a *separable* R-algebra; see [3].

(2) In a topos, considered as an extensive category (see Exercise 5 of 2.6), $\mathcal{F}$-separated means *decidable* (see [2], p.444). To say that every object is decidable is equivalent to the topos being Boolean.

(3) An object in a topos provided with a universal closure operator j (see 2.5) is $\mathcal{F}$-separated precisely when it is j*-separated* in the usual sense (see [36], p.227).

5. Perfect maps, compact Hausdorff spaces

5.1. Maps with compact domain and separated codomain. There is an easy but fundamental property which links compactness and separation, in the general setting provided by 2.1:

Proposition. *If* X *is* $\mathcal{F}$*-compact and* Y $\mathcal{F}$*-separated, then every morphism* $f : X \to Y$ *is* $\mathcal{F}$*-proper.*

Proof. One factors f through its "graph":

$$\begin{array}{ccccc} & & X \times Y & & \\ & {\scriptstyle <1_X, f>} \nearrow & & \searrow {\scriptstyle p_2} & \\ X & & \xrightarrow[f]{\hspace{2em}} & & Y \end{array} \tag{22}$$

Since $< 1_X, f > \cong f^{-1}[\delta_Y]$, this morphism is $\mathcal{F}$-closed, in fact $\mathcal{F}$-proper, when Y is $\mathcal{F}$-separated. Likewise p_2 is $\mathcal{F}$-closed, even $\mathcal{F}$-proper, when X is $\mathcal{F}$-compact. Closure of $\mathcal{F}^*$ under composition shows that f is $\mathcal{F}$-proper. □

Corollary. *If* Y *is* $\mathcal{F}$*-separated and* $\mathcal{F}$*-compact, then* $f : X \to Y$ *is* $\mathcal{F}$*-proper if and only if* X *is* $\mathcal{F}$*-compact.*

Proof. Combine the Theorem with Theorem 3.3(1). □

Exercises.

1. Show that a subobject of an $\mathcal{F}$-separated object with $\mathcal{F}$-compact domain is $\mathcal{F}$-closed.
2. Show that the assumption of $\mathcal{F}$-separation for Y is essential in the Theorem.

5.2. Perfect maps. Theorem 5.1 leads to a strengthening of Proposition 3.2(3), as follows:

Proposition. *If* $g \cdot f \in \mathcal{F}^*$ *with* $g \in \mathcal{F}'$*, then* $f \in \mathcal{F}^*$*.*

Proof. With $f : X \to Y$, $g : Y \to Z$, if $g \cdot f \in \mathcal{F}^*$ and $g \in \mathcal{F}'$, then $f : (X, g \cdot f) \to (Y, g)$ is a morphism in $\mathcal{X}/Z$ with $\mathcal{F}$-compact domain and $\mathcal{F}$-separated codomain (see Prop. 3.5 and Prop. 4.5). Hence, f is $\mathcal{F}$-proper in $\mathcal{X}/Z$ and also in $\mathcal{X}$. □

We call a morphism $\mathcal{F}$*-perfect* if it is both $\mathcal{F}$-proper and $\mathcal{F}$-separated. With Propositions 3.2, 3.7, 4.2 and 4.4 we then obtain:

Theorem. *The class of* $\mathcal{F}$*-perfect morphisms contains all* $\mathcal{F}$*-closed subobjects, is closed under composition and stable under pullback. Furthermore, if a composite morphism* $g \cdot f$ *is* $\mathcal{F}$*-perfect, then* f *is* $\mathcal{F}$*-perfect whenever* g *is* $\mathcal{F}$*-separated, and* g *is* $\mathcal{F}$*-perfect whenever* f *is* $\mathcal{F}$*-perfect and stably in* $\mathcal{E}$*. Finally, if the category* $\mathcal{X}$

has finite coproducts, is extensive, and if the coproduct of two $\mathcal{F}$-closed morphisms is $\mathcal{F}$-closed, also the coproduct of two $\mathcal{F}$-perfect morphisms is $\mathcal{F}$-perfect. □

Example. In $\mathcal{T}op_{\text{open}}$ a map $f : X \to Y$ is $\mathcal{F}$-perfect if and only if f is open and locally injective (see Exercise 1 of 4.1). This means precisely that f is a *local homeomorphism*, so that every point in X has an open neighbourhood U such that the restriction $U \to f[U]$ of f is a homeomorphism. In particular, local homeomorphisms enjoy all stability properties described by the Theorem.

5.3. Compact Hausdorff objects. Let $\mathcal{F}$-CompHaus denote the full subcategory of $\mathcal{X}$ containing the objects that are both $\mathcal{F}$-compact and $\mathcal{F}$-Hausdorff.

Theorem. $\mathcal{F}$-CompHaus *is closed under finite limits in $\mathcal{X}$ and under $\mathcal{F}$-closed subobjects. If $\mathcal{E}$ is stable under pullback, then the $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathcal{X}$ restricts to an $(\mathcal{E}, \mathcal{M})$-factorization system of* $\mathcal{F}$-CompHaus. *If the category $\mathcal{X}$ has finite coproducts, is extensive, and if the coproduct of two $\mathcal{F}$-closed morphisms is $\mathcal{F}$-closed, then* $\mathcal{F}$-CompHaus *is closed under finite coproducts in $\mathcal{X}$ precisely when* $1 + 1 \in \mathcal{F}$-CompHaus.

Proof. From Theorem 3.3(1) and item (vii) of Theorem 4.3 one sees that $\mathcal{F}$-CompHaus is closed under equalizers in $\mathcal{X}$; likewise for closed subobjects. For closure under finite products, use Theorem 3.3(3) and Corollary 4.3. In the $(\mathcal{E}, \mathcal{M})$-factorization (1) of f, Z is $\mathcal{F}$-compact by Theorem 3.3(2) in case $\mathcal{E} = \mathcal{E}^*$, and $\mathcal{F}$-separated by Corollary 4.3. For the last statement of the Theorem one uses Propositions 3.7 and 4.4, but in order to do so we have to make sure that the morphisms $0 \to X$ and $X + X \to X$ are $\mathcal{F}$-closed if $1 + 1 \in \mathcal{F}$-CompHaus. For $X + X \to X$, this follows directly from $1 + 1 \in \mathcal{F}$-Comp (see the proof of Prop. 3.7), and for $0 \to X$ observe that this is the equalizer of the injections $X \rightrightarrows X + X$ in the extensive category $\mathcal{X}$. Hence, $\mathcal{F}$-closedness follows again with (vii) of Theorem 4.3. □

Exercises.

1. Using previous exercises, check the correctness of the given characterization of $X \in \mathcal{F}$-CompHaus for each of the following categories: $\mathcal{T}op$: compact Hausdorff, $\mathcal{L}oc$: compact Hausdorff, $\mathcal{T}op_{\text{open}}$: discrete, $\mathcal{T}op_{\text{clopen}}$: finite discrete, $\mathcal{T}op_{\text{Zariski}}$: T0 plus the property of Exercise 2 of 3.5.
2. Find sufficient conditions for $\mathcal{F}$-CompHaus to be closed under finite coproducts.

5.4. Non-extendability of proper maps. An important property of $\mathcal{F}$-proper maps is described by:

Proposition. *An $\mathcal{F}$-proper map $f : M \to Y$ in $\mathcal{X}$ cannot be extended along an $\mathcal{F}$-dense subobject $m : M \to X$ with X $\mathcal{F}$-Hausdorff unless m is an isomorphism.*

Proof. Suppose we had a factorization $f = g \cdot m$ with $g : X \to Y$, X $\mathcal{F}$-Hausdorff and m in $\mathcal{M}$ $\mathcal{F}$-dense. Then $m : (M, f) \to (X, g)$ is an $\mathcal{F}$-dense embedding in $\mathcal{X}/Y$, with $\mathcal{F}$-compact domain and $\mathcal{F}$-separated codomain, by (ii) of Theorem 4.3

and Propositions 3.5, 4.5. Hence, an application of Proposition 5.1 (to $\mathcal{X}/Y$ in lieu of $\mathcal{X}$) gives that m is $\mathcal{F}$-closed, in fact an isomorphism by Corollary 2.2(2). □

We shall see in 6.6 below under which circumstances the property described by the Proposition turns out to be characteristic for $\mathcal{F}$-properness.

6. Tychonoff spaces, absolutely closed spaces, compactification

6.1. Embeddability and absolute closedness. In our standard setting of 2.1 we consider two important subcategories of $\mathcal{F}$-Haus, both containing $\mathcal{F}$-CompHaus. An object X of $\mathcal{X}$ is called

- *$\mathcal{F}$-Tychonoff* if it is embeddable into an $\mathcal{F}$-compact $\mathcal{F}$-Hausdorff object, so that there is $m : X \to K$ in $\mathcal{M}$ with $K \in \mathcal{F}$-CompHaus;
- *absolutely $\mathcal{F}$-closed* if it is $\mathcal{F}$-separated and $\mathcal{F}$-closed in every $\mathcal{F}$-separated extension object, so that every $m : X \to K$ in $\mathcal{M}$ with $K \in \mathcal{F}$-Haus is $\mathcal{F}$-closed.

Denoting the respective full subcategories of $\mathcal{X}$ by $\mathcal{F}$-Tych and $\mathcal{F}$-AC, with Proposition 5.1, Theorem 3.3(3), Corollary 4.3 we obtain:

Proposition. $\mathcal{F}$-CompHaus $= \mathcal{F}$-Tych $\cap\, \mathcal{F}$-AC.

Examples.

(1) In $\mathcal{T}op$, the $\mathcal{F}$-Tychonoff spaces are precisely the completely regular Hausdorff spaces X, characterized by the property that for every closed set A in X and $x \in X \setminus A$, there is a continuous mapping $g : X \to [0,1]$ into the unit interval with $g[A] \subseteq \{1\}$ and $g(x) = 0$. Using the Stone-Čech compactification $\beta_X : X \to \beta X$ one can prove this assertion quite easily.

(2) In $\mathcal{L}oc$, the $\mathcal{F}$-Tychonoff locales are exactly the completely regular locales (see II.6 and [29], IV.1.7).

(3) In $\mathcal{T}op$, the absolutely $\mathcal{F}$-closed spaces are (by definition) the so-called *H-closed* spaces: see [20], p.223.

(4) In Ω-$\mathcal{S}et$ (see 2.7) the absolutely $\mathcal{F}$-closed objects have been characterized in [15] as the so-called *algebraic Ω-sets*. In the case that K is given by the ground field in the category of commutative K-algebras, all such objects are subobjects of K^I for some set I and are called *K-algebraic* . Here the K-algebraic set K^I is equipped with the K-algebra of polynomial functions on K^I, and its K-algebraic subsets are precisely the zero sets of sets of polynomials in $K[X_i]_{i\in I}$.

Exercises.

1. Prove that in $\mathcal{T}op_{\text{open}}$ one has $\mathcal{F}$-Haus $= \mathcal{F}$-CompHaus $= \mathcal{F}$-Tych $= \mathcal{F}$-AC, given by the subcategory of discrete spaces. Prove a similar result for $\mathcal{A}b\mathcal{G}rp$.

2. Prove that in $\mathcal{T}op_{\text{Zariski}}$ $\mathcal{F}$-Haus is the category of $T0$-spaces, whereas $\mathcal{F}$-AC is the category of sober $T0$-spaces (see Chapter II).
3. Find examples in $\mathcal{T}op$ of a Tychonoff space which is not absolutely closed, and of an absolutely closed space which is not Tychonoff. (Following standard praxis we left off the prefix $\mathcal{F}$ here.)
4. Consider the category $\mathcal{C}hu_2^{\text{ext}}$ of Exercise 3 of 2.7. For an object (X,τ) of this category, let $N : X \to 2^X$, $x \mapsto \{U \in \tau \mid x \in U\}$ the "neighbourhood filter map". Prove that (X,τ) is $\mathcal{F}$-Hausdorff in $\mathcal{C}hu_2^{\text{ext}}$ if and only if N is injective, and absolutely $\mathcal{F}$-closed if and only if N is bijective. (*Hint*: For (X,τ) absolutely $\mathcal{F}$-closed, assume that N fails to be surjective, witnessed by $\alpha \subseteq \tau$; then consider the structure σ on $Y = X + \{\infty\}$, given by $\sigma = (\tau \setminus \alpha) \cup \{U \cup \{\infty\} \mid U \in \alpha\}$, and verify that (Y,σ) is $\mathcal{F}$-Hausdorff.)

6.2. Stability properties of Tychonoff objects. $\mathcal{F}$-Tych has the expected stability properties:

Proposition. $\mathcal{F}$-Tych *is closed under finite limits and under subobjects in* $\mathcal{X}$*.*

Proof. With $m_i : X_i \to K_i$ in $\mathcal{M}$ $(i = 1, 2)$, also $m_1 \times m_2 : X_1 \times X_2 \to K_1 \times K_2$ is in $\mathcal{M}$ (see 1.4(5)), and we can use 3.3(3) to obtain closure of $\mathcal{F}$-Tych under finite products. Closure under equalizers follows from closure under subobjects, and the latter property is trivial. □

Remark. $\mathcal{F}$-AC generally fails to be closed under finite products or under equalizers. For example, in $\mathcal{T}op$ $\mathcal{F}$-AC fails to be closed under equalizers, although it is closed under arbitrary products, by a result of Chevalley and Frink [8]. In $\mathcal{T}op_{\text{open}}$ $\mathcal{F}$-AC is still closed under finite products, but not under infinite ones: see Exercise 1 of 6.1.

6.3. Extending the notions to morphisms. Using comma categories, it is natural to extend the notions introduced in 6.1 from objects to morphisms of $\mathcal{X}$. Hence, a morphism $f : X \to Y$ in $\mathcal{X}$ is called

- $\mathcal{F}$*-Tychonoff* if (X, f) is an $\mathcal{F}$-Tychonoff object in $\mathcal{X}/Y$, which means that there is a factorization $f = p \cdot m$ with $m \in \mathcal{M}$ and p $\mathcal{F}$-perfect (see Prop. 3.5, 4.5);
- *absolutely* $\mathcal{F}$*-closed* if (X, f) is an absolutely $\mathcal{F}$-closed object in $\mathcal{X}/Y$, which means that f is $\mathcal{F}$-separated and, whenever there is a factorization $f = p \cdot m$ with $m \in \mathcal{M}$ and p $\mathcal{F}$-separated, then m is $\mathcal{F}$-closed.

There is some interaction between the object and morphism notions, similarly to what we have seen for compactness and separation.

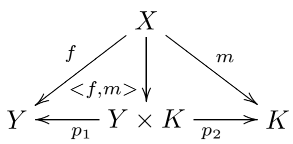
Proposition. *In each* (1) *and* (2)*, the assertions* (i)-(iii) *are equivalent:*

(1) i. $X \in \mathcal{F}$-Tych*;*
 ii. *every morphism* $f : X \to Y$ *is* $\mathcal{F}$*-Tychonoff;*
 iii. *there is an* $\mathcal{F}$*-Tychonoff map* $f : X \to Y$ *with* $Y \in \mathcal{F}$-CompHaus*;*

(2) i. $X \in \mathcal{F}$-AC*;*
 ii. *every morphism* $f : X \to Y$ *with* $Y \in \mathcal{F}$-Haus *is absolutely* $\mathcal{F}$*-closed;*

iii. *there is an absolutely $\mathcal{F}$-closed morphism $f : X \to Y$ with $Y \in \mathcal{F}$-*CompHaus.

Proof. For morphisms $f : X \to Y$, $m : M \to K$ in $\mathcal{X}$ consider the diagram



(23)

If $K \in \mathcal{F}$-CompHaus, then p_1 is $\mathcal{F}$-perfect, and $m = p_2\cdot < f, m > \in \mathcal{M}$ implies $< f, m > \in \mathcal{M}$; this proves (1) i $\Rightarrow$ ii. Similarly, for (2) iii $\Rightarrow$ i, the hypotheses $m \in \mathcal{M}$ and $K \in \mathcal{F}$-Haus give $< f, m > \in \mathcal{M}$ and $p_1 \in \mathcal{F}'$, so that $< f, m >$ must be $\mathcal{F}$-closed since f is absolutely $\mathcal{F}$-closed; furthermore, p_2 is $\mathcal{F}$-closed since Y is $\mathcal{F}$-compact, so that $m = p_2\cdot < f, m >$ is $\mathcal{F}$-closed.

Let us now assume $f = p \cdot m$ with $m \in \mathcal{M}$. If $p : K \to Y$ is $\mathcal{F}$-perfect and $Y \in \mathcal{F}$-CompHaus, then also $K \in \mathcal{F}$-CompHaus, by Theorems 3.3(1), 4.3(iii). This proves (1) iii $\Rightarrow$ i. For (2) i $\Rightarrow$ ii, assume both p and Y to be $\mathcal{F}$-separated, so that also K is $\mathcal{F}$-separated; now the hypothesis $X \in \mathcal{F}$-AC gives that m is $\mathcal{F}$-closed, as desired.

The implications ii $\Rightarrow$ iii are trivial in both cases: take $Y = 1$. □

Of course, the Proposition just proved may be applied to the slices of $\mathcal{X}$ rather than to $\mathcal{X}$ itself and then leads to stability properties for the classes of $\mathcal{F}$-Tychonoff and of absolutely $\mathcal{F}$-closed maps, the proof of which we leave as exercises.

Exercises.

1. The class of $\mathcal{F}$-Tychonoff maps is stable under pullback. If $g \cdot f$ is $\mathcal{F}$-Tychonoff, so is f; conversely, f $\mathcal{F}$-Tychonoff and g $\mathcal{F}$-perfect imply $g \cdot f$ $\mathcal{F}$-Tychonoff.
2. If $g \cdot f$ is absolutely $\mathcal{F}$-closed and g $\mathcal{F}$-separated, then f is absolutely $\mathcal{F}$-closed; conversely, if f is absolutely $\mathcal{F}$-closed and g $\mathcal{F}$-perfect, then $g \cdot f$ is absolutely $\mathcal{F}$-closed.

Remark. Unlike spaces, $\mathcal{F}$-Tychonoff maps in $\mathcal{T}op$ do not seem to allow for an easy characterization in terms of mapping properties into the unit interval. Other authors (see [16, 41, 33]) studied separated maps $f : X \to Y$ with the property that for every closed set $A \subseteq X$ and every $x \in X \setminus A$ there is an open neighbourhood U of $f(x)$ in Y and a continuous map $g : f^{-1}[U] \to [0, 1]$ with $g(x) = 0$ and $g[A \cap f^{-1}[U]] \subseteq \{1\}$. In case $Y = 1$ this amounts to saying that X is completely regular Hausdorff, hence Tychonoff. For general Y, maps f with this property are $\mathcal{F}$-Tychonoff, i.e. restrictions of perfect maps, but the converse is generally false: see [51].

6.4. Compactification of objects. An *$\mathcal{F}$-compactification* of an object X is given by an $\mathcal{F}$-dense embedding $X \to K$ with $K \in \mathcal{F}$-CompHaus. Of course, only objects in $\mathcal{F}$-Tych can have $\mathcal{F}$-compactifications. For our purposes it is important to be provided with a *functorial* choice of a compactification for every $X \in \mathcal{F}$-Tych.

Hence, we call an endofunctor $\kappa : \mathcal{F}\text{-Tych} \to \mathcal{F}\text{-Tych}$ which comes with a natural transformation

$$\kappa_X : X \longrightarrow \kappa X \ (X \in \mathcal{F}\text{-Tych})$$

a *functorial $\mathcal{F}$-compactification* if

- each κ_X is an $\mathcal{F}$-dense embedding
- each κX is $\mathcal{F}$-compact and $\mathcal{F}$-separated.

By illegitimally denoting the endofunctor and the natural transformation by the same letter we follow standard praxis in topology. In $\mathcal{T}op$, the prime example for a functorial $\mathcal{F}$-compactification is provided by the Stone-Čech compactification

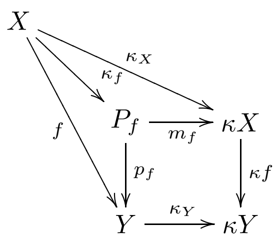
$$\beta_X : X \longrightarrow \beta X.$$

These morphisms serve as reflexions, showing the reflexivity of $\mathcal{F}$-CompHaus in $\mathcal{F}$-Tych. We shall revisit this theme in Section 11 below; here we just note that the universal property makes the first of the two requirements for a functorial $\mathcal{F}$-compactification redundant, also in our general setting:

Proposition. *If $\mathcal{F}$-CompHaus is reflective in $\mathcal{F}$-Tych, with reflexions $\beta_X : X \to \beta X$, then these provide a functorial $\mathcal{F}$-compactification.*

Proof. We just need to show that each β_X is an $\mathcal{F}$-dense embedding. But for $X \in \mathcal{F}\text{-Tych}$ we have some $m : X \to K$ in $\mathcal{M}$ with $K \in \mathcal{F}\text{-CompHaus}$, which factors as $m = f \cdot \beta_X$, by the universal property of β_X. Hence $\beta_X \in \mathcal{M}$. If $\beta_X = n \cdot g$ with $n : N \to \beta X$ $\mathcal{F}$-closed, then $N \in \mathcal{F}\text{-CompHaus}$, and we can apply the universal property again to obtain $h : \beta X \to N$ with $h \cdot \beta_X = g$. Then $n \cdot h \cdot \beta_X = \beta_X$ shows $n \cdot h = 1_{\beta X}$, so that n is an isomorphism. Consequently, β_X is $\mathcal{F}$-dense. □

6.5. Compactification of morphisms. In order to take full advantage of the presence of a functorial $\mathcal{F}$-compactification κ, we should extend this gadget from objects to morphisms, as follows: for $f : X \to Y$ in $\mathcal{F}$-Tych we form the pullback $P_f = Y \times_{\kappa Y} \kappa X$ and the induced morphism κ_f making the following diagram commutative:



(24)

Calling a morphism $f : X \to Y$ *$\mathcal{F}$-initial* if every $\mathcal{F}$-closed subobject of X is the inverse image of an $\mathcal{F}$-closed subobject of Y under f, we observe:

Proposition.

(1) *$\kappa_f \in \mathcal{M}$, and p_f is $\mathcal{F}$-perfect, with $P_f \in \mathcal{F}$-Tych.*
(2) *If every morphism in $\mathcal{M}$ is $\mathcal{F}$-initial, then κ_f is $\mathcal{F}$-dense.*

Proof. (1) Since $\kappa_X = m_f \cdot \kappa_f \in \mathcal{M}$ we have $\kappa_f \in \mathcal{M}$. As a morphism in $\mathcal{F}$-CompHaus, κf is $\mathcal{F}$-perfect (by Prop. 5.1 and Theorem 4.3), and so is its pullback p_f. Since $\mathcal{F}$-Tych is closed under finite limits in $\mathcal{X}$, $P_f \in \mathcal{F}$-Tych.
(2) The given hypothesis implies the following cancellation property for $\mathcal{F}$-dense subobjects: if $n \cdot k$ is $\mathcal{F}$-dense with $n, k \in \mathcal{M}$, then k is $\mathcal{F}$-dense. This may then be applied to $\kappa_X = m_f \cdot \kappa_f$ since $m_f \in \mathcal{M}$, as a pullback of κ_Y. □

Hence, we should think of $\kappa_f : (X, f) \to (P_f, p_f)$ as of an $\mathcal{F}$-compactification of (X, f) in $\mathcal{F}$-Tych$/Y$, especially under the hypothesis given in (2).

Exercises.

1. Show $P_f \cong \kappa X$ in case $Y = 1$ and $\kappa_1 \cong 1$.
2. Show that by putting $\kappa(X, f) := (P_f, p_f)$ one obtains a functorial $\mathcal{F}$-compactification for $\mathcal{F}$-Tych$/Y$, under the same hypothesis as in (2) of the Proposition. (Here we think of $\mathcal{F}$-Tych$/Y$ as a full subcategory of $\mathcal{X}/Y$ that has inherited the factorization system and the closed system from there.)

6.6. Isbell-Henriksen characterization of perfect maps. In case $\mathcal{X} = \mathcal{T}op$ the following theorem goes back to Isbell and Henriksen (see [23]). Categorical versions of it can be found in [9] and [48].

Theorem. *Let κ be a functorial $\mathcal{F}$-compactification and let every morphism in $\mathcal{M}$ be $\mathcal{F}$-initial. Then the following assertions are equivalent for $f : X \to Y$ in $\mathcal{F}$-*Tych*:*

(i) *f is $\mathcal{F}$-perfect;*
(ii) *f cannot be extended along an $\mathcal{F}$-dense embedding $m : X \to Z$ with Z $\mathcal{F}$-Hausdorff unless m is an isomorphism;*
(iii) *the naturality diagram*

$$\begin{array}{ccc} X & \xrightarrow{\kappa_X} & \kappa X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \kappa f} \\ Y & \xrightarrow{\kappa_Y} & \kappa Y \end{array} \qquad (25)$$

is a pullback diagram.

Proof. (i) ⇒ (ii) was shown in 5.4.
(ii) ⇒ (iii): By Prop. 6.5, κ_f of (24) is an $\mathcal{F}$-dense embedding, hence an isomorphism by hypothesis. Consequently, (25) coincides with the pullback square of (24), up to isomorphism.
(iii) ⇒ (i): By hypothesis $f \cong p_f$, hence f is $\mathcal{F}$-perfect by 6.5. □

Example. For $\mathcal{X} = \mathcal{T}op$ and $\kappa = \beta$ the Stone-Čech compactification, by item (iii) of the Theorem perfect maps $f : X \to Y$ between Tychonoff spaces are characterized by the property that their extension $\beta f : \beta X \to \beta Y$ maps the remainder $\beta X \setminus X$ into the remainder $\beta Y \setminus Y$.

6.7. Antiperfect-perfect factorization. Suppose that, in our general setting of 2.1, $\kappa = \beta$ is given by reflexivity of $\mathcal{F}$-CompHaus in $\mathcal{F}$-Tych. Then there is another way of thinking of the factorization $f = p_f \cdot \beta_f$ established in diagram (24). With $\mathcal{P}_\mathcal{F}$ denoting the class of $\mathcal{F}$-perfect morphisms in $\mathcal{F}$-Tych, and with

$$\mathcal{A}_\mathcal{F} := \{m : X \to Y \mid m \in \mathcal{M},\ X, Y \in \mathcal{F}\text{-Tych},\ \beta m : \beta X \to \beta Y \text{ iso}\},$$

whose morphisms are also called *$\mathcal{F}$-antiperfect*, one obtains:

Theorem. *$(\mathcal{A}_\mathcal{F}, \mathcal{P}_\mathcal{F})$ is a (generally non-proper) factorization system of* $\mathcal{F}$-Tych*, provided that* $\mathcal{F}$-CompHaus *is reflective in* $\mathcal{F}$-Tych *and every morphism in $\mathcal{M}$ is $\mathcal{F}$-initial.*

Proof. The outer square and the upper triangle of

$$\begin{array}{ccc} X & \xrightarrow{\beta_X} & \beta X \\ {\scriptstyle \beta_f}\downarrow & \nearrow{\scriptstyle m_f} & \downarrow{\scriptstyle \overline{\beta_f}} \\ P_f & \xrightarrow[\beta_{P_f}]{} & \beta P_f \end{array} \qquad (26)$$

are commutative, with $\overline{\beta_f} := \beta(\beta_f)$. According to Theorem 2.7 of [27], the assertion of the Theorem follows entirely from the universal property of β once we have established commutativity of the lower triangle of (26). For that, let $u : E \to P_f$ be the equalizer of $\beta_{P_f}, \overline{\beta_f} \cdot m_f$, which is $\mathcal{F}$-closed since βP_f is $\mathcal{F}$-separated (see Theorem 4.3). Since β_f factors through u by a unique morphism $l : X \to E$, and since β_f is $\mathcal{F}$-dense by Prop. 6.5, u is an isomorphism, and we have $\overline{\beta_f} \cdot m_f = \beta_{P_f}$. □

An important consequence of the Theorem is that *essentially* it allows us to carry the hypothesis of reflectivity of $\mathcal{F}$-CompHaus in $\mathcal{F}$-Tych from $\mathcal{X}$ to its slices, as follows. Consider again a morphism $f : X \to Y$ with $X, Y \in \mathcal{F}$-Tych. Then Prop. 6.5(1) shows that the object (X, f) in $\mathcal{X}/Y$ (in fact: $\mathcal{F}$-Tych$/Y$) is $\mathcal{F}$-Tychonoff. Now, with the Theorem, one easily shows:

Corollary. *Under the provisions of the Theorem, $\beta_f : (X, f) \to (P_f, p_f)$ is a reflexion of the $\mathcal{F}$-Tychonoff object (X, f) of* $\mathcal{F}$-Tych$/Y$ *into the full subcategory of $\mathcal{F}$-compact Hausdorff objects of $\mathcal{X}/Y$.* □

Hence, the construction given by diagram (24) with $\kappa = \beta$ *is* really the "Stone-Čech $\mathcal{F}$-compactification" of $\mathcal{F}$-Tychonoff objects in $\mathcal{X}/Y$, provided that we restrict ourselves to those objects $(X, f : X \to Y)$ for which X, Y are $\mathcal{F}$-Tychonoff objects in $\mathcal{X}$.

Exercise. Work out the details of the proof of the Theorem and the Corollary.

7. Open maps, open subspaces

7.1. Open maps. In the standard setting of 2.1, a morphism $f : X \to Y$ is said to *reflect $\mathcal{F}$-density* if $f^{-1}[-]$ maps $\mathcal{F}$-dense subobjects of Y to $\mathcal{F}$-dense subobjects of X. The *morphism f* is *$\mathcal{F}$-open* if every pullback f' of f (see diagram (11)) reflects $\mathcal{F}$-density. A *subobject* is $\mathcal{F}$-open if its representing morphism is $\mathcal{F}$-open. By definition one has

$$\mathcal{F}^+ := \{f \mid f \ \mathcal{F}\text{-open}\ \} = \{f \mid f \text{ reflects } \mathcal{F}\text{-density}\}^*$$

in the notation of Section 3.

Since reflection of $\mathcal{F}$-density is obviously closed under composition, one obtains immediately some stability properties for $\mathcal{F}$-open maps, just as for $\mathcal{F}$-proper maps.

Proposition.

(1) *The class $\mathcal{F}^+$ contains all isomorphisms, is closed under composition and stable under pullback.*
(2) *If $g \cdot f \in \mathcal{F}^+$ with g monic, then $f \in \mathcal{F}^+$.*
(3) *If $g \cdot f \in \mathcal{F}^+$ with $f \in \mathcal{E}^*$, then $g \in \mathcal{F}^+$.*

Proof. For (1), (2), proceed as in Prop. 3.2. (3): A pullback of $g \cdot f$ with $f \in \mathcal{E}^*$ has the form $g' \cdot f'$ with $f' \in \mathcal{E}$ a pullback of f and g' a pullback of g. So, it suffices to check that g reflects $\mathcal{F}$-density if $g \cdot f$ does and $f \in \mathcal{E}^*$. But for every $\mathcal{F}$-dense subobject d of the codomain of g one obtains from Prop. 1.8 that

$$g^{-1}[d] = f[f^{-1}[g^{-1}[d]]] = f[(g \cdot f)^{-1}[d]]$$

is an image of an $\mathcal{F}$-dense subobject under f. But since $f \in \mathcal{E}$, trivially $f[-]$ preserves $\mathcal{F}$-density (by Cor. 2.2(1),(3)). □

Analogously to Prop. 2.1 we can state:

Corollary.

(1) *The $\mathcal{F}$-open subobjects of an object X form a (possibly large) subsemilattice of the meet-semilattice* sub$X$.
(2) *For every morphism f, $f^{-1}[-]$ preserves $\mathcal{F}$-openness of subobjects.*
(3) *For every $\mathcal{F}$-open morphism f, $f[-]$ preserves $\mathcal{F}$-openness of subobjects provided that $\mathcal{E}$ is stable under pullback.*

Proof. (1) follows from the Proposition, and for (2) remember that $\mathcal{F}$-openness of subobjects is, by definition, pullback stable. (3) If f and $m \in \mathcal{M}$ are both open, then $\mathcal{F}$-openness of $f \cdot m = f[m] \cdot e$ with $e \in \mathcal{E}^*$ yields $\mathcal{F}$-openness of $f[m]$, by the Proposition. □

7.2. Open maps of topological spaces. It is time for a "reality check" in terms of our standard examples.

Proposition. *In $\mathcal{T}op$* (with $\mathcal{F} = \{$closed maps$\}$), *the following assertions are equivalent for a continuous map $f : X \to Y$:*

(i) *f is $\mathcal{F}$-open;*
(ii) *for all subspaces N of Y, $f^{-1}[\overline{N}] = \overline{f^{-1}[N]}$;*
(iii) *f is open, i.e. $f[-]$ preserves openness of subspaces.*

Proof. (i) ⇒ (ii): Given $N \subseteq Y$, by hypothesis, the restriction $f' : f^{-1}[\overline{N}] \to \overline{N}$ of f reflects the density of N in $\overline{N}$, hence $\overline{f^{-1}[N]} = f^{-1}[\overline{N}]$.
(ii) ⇒ (iii): Given $O \subseteq X$ open, $f^{-1}[\overline{Y \setminus f[O]}] = \overline{f^{-1}[Y \setminus f[O]]} \subseteq \overline{X \setminus O} = X \setminus O$, hence $\overline{Y \setminus f[O]} \subseteq Y \setminus f[O]$, that is $f[O]$ is open.
(iii) ⇒ (i): Openness of maps (in the sense of preservation of openness of subspaces) is easily shown to be stable under pullback. Hence, it suffices to show that an open map $f : X \to Y$ reflects density of subspaces. But if $D \subseteq Y$ is dense, then $f[X \setminus \overline{f^{-1}[D]}]$ is open, by hypothesis, and this set would have to meet D under the assumption that $X \setminus \overline{f^{-1}[D]}$ is not empty, which is impossible. Hence, $f^{-1}[D]$ is dense in X. □

We emphasize that a map in $\mathcal{T}op$ which reflects $\mathcal{F}$-density need not be open (see Exercise 1 below); hence, the stability requirement in Definition 7.1 of $\mathcal{F}$-openness is essential.

Exercises.

1. Show that in $\mathcal{T}op$ the embedding of the closed unit interval into $\mathbb{R}$ reflects density.
2. In $\mathcal{T}op_{\text{open}}$, $D \subseteq X$ is $\mathcal{F}$-dense if and only if D meets every non-empty closed set in X. Hence, if X is a $T1$-space (so that all singleton sets are closed), only X itself is a dense subobject of X. Show that every proper (=stably-closed) map is $\mathcal{F}$-open, but not conversely.
3. In $\mathcal{T}op_{\text{clopen}}$, $D \subseteq X$ is $\mathcal{F}$-dense if and only if D meets every non-empty clopen set in X. Prove that f is $\mathcal{F}$-open if and only if f is $\mathcal{F}$-proper (i.e., every pullback of f maps clopen subsets onto clopen subsets).
4. Show that in a topos with a universal closure operator (see 2.5), every morphism is $\mathcal{F}$-open.

7.3. Open maps of locales. In $\mathcal{L}oc$, $\mathcal{F}$-open morphisms $f : X \to Y$ are characterized like in $\mathcal{T}op$ as those that have stably the property that $f^{-1}[-]$ commutes with the usual closure, just as in (ii) of Prop. 7.2; this follows formally from the fact that the closure in $\mathcal{L}oc$ is given by an idempotent and hereditary closure operator (see [18]). But more importantly we need to compare this notion with the usual notion of open map of locales (see Chapter II):

Proposition. *If $f[-]$ maps open sublocales to open sublocales (i.e., if f is open), then f is $\mathcal{F}$-open in $\mathcal{L}oc$.*

Proof. Since openness of localic maps is stable under pullback (see Theorem II 5.2), it suffices to show

$$f^{-1}[\overline{n}] = \overline{f^{-1}[n]}$$

for $f : X \to Y$ open and any sublocale $n : N \rightarrowtail Y$. For this it suffices to show

$$f^*[c(n)] = c(f^{-1}[n]),$$

where $c(n) = \bigvee\{b \in OY \mid n^*(b) = 0\}$ (see II, 2.9 and 3.6), which means:

$$\bigvee\{f^*(b) \mid b \in OY,\, n^*(b) = 0\} = \bigvee\{a \in OX \mid m^*(a) = 0\},$$

with $m = f^{-1}[n] : M \rightarrowtail X$. Now, for this last identity, "$\leq$" is trivial, while "$\geq$" follows when we put $b = f_!(a)$ for every a with $m^*(a) = 0$ and use

$$n^*(f_!(a))) = (f')_!(m^*(a)),$$

where $f_!$ is the left adjoint of f^* and $f' : M \to N$ is the restriction of f. $\square$

After the authors posed the converse statement of the Proposition as an open problem, in April 2002 P.T. Johnstone proved its validity:

Theorem. *The $\mathcal{F}$-open maps in $\mathcal{L}oc$ are precisely the (usual) open maps of locales.*

For its rather intricate proof we must refer the Reader to [31]. The paper also exhibits various subtypes of openness. For example, like in $\mathcal{T}op$, also in $\mathcal{L}oc$ there are examples of morphisms reflecting $\mathcal{F}$-density which fail to be $\mathcal{F}$-open, as also shown in II.5.2.

7.4. Local homeomorphisms. Proposition 7.1 shows that, if $\mathcal{E}$ is stable under pullback in $\mathcal{X}$, then $\mathcal{F}^+$ is a new $(\mathcal{E}, \mathcal{M})$-closed class in $\mathcal{X}$ (see 2.1), and in view of Exercise 1 of 4.3 and Example 5.2, it would make sense to define:

$$\begin{aligned} X\ \mathcal{F}\textit{-discrete} \quad &:\Leftrightarrow \quad X\ \mathcal{F}^+\text{-separated} \\ &\Leftrightarrow \quad \delta_X : X \to X \times X\ \mathcal{F}\text{-open} \\ f : X \to Y\ \textit{local}\ \mathcal{F}\textit{-homeomorphism} \quad &:\Leftrightarrow \quad X\ \mathcal{F}^+\text{-perfect} \\ &\Leftrightarrow \quad f \text{ and } \delta_f : X \to X \times_Y X\ \mathcal{F}\text{-open} \end{aligned}$$

Here we cannot explore these notions further, but must leave the Reader with:

Exercises.

1. Collect stability properties for $\mathcal{F}$-discrete objects and local $\mathcal{F}$-homeomorphisms, from the properties already shown for $\mathcal{F}$-separated objects and $\mathcal{F}$-proper morphisms.
2. In a topos $\mathcal{S}$ with a universal closure operator (see 2.5) every object is $\mathcal{F}$-discrete and every morphism is a local $\mathcal{F}$-homeomorphism.

7.5. Sums of open maps. The proof of the following Proposition is left as an exercise as well:

Proposition. *Assume that the finitely-complete category $\mathcal{X}$ has finite coproducts and is extensive, and that the classes $\mathcal{M}$ and $\mathcal{F}_0$ are closed under finite coproducts. Then:*

$$\begin{aligned} f_1 + f_2\ \ \mathcal{F}\textit{-dense} \quad &\Leftrightarrow \quad f_1,\ f_2\ \ \mathcal{F}\textit{-dense}, \\ f_1 + f_2\ \ \mathcal{F}\textit{-open} \quad &\Leftrightarrow \quad f_1,\ f_2\ \ \mathcal{F}\textit{-open}. \end{aligned}$$

$\square$

8. Locally perfect maps, locally compact Hausdorff spaces

8.1. Locally perfect maps. In the setting of 2.1, a morphism $f : X \to Y$ is *locally $\mathcal{F}$-perfect* if it is a restriction of an $\mathcal{F}$-perfect morphism $p : K \to Y$ to an $\mathcal{F}$-open subobject $u : X \to K$:

$$\begin{array}{ccc} & K & \\ {\scriptstyle u}\nearrow & & \searrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{27}$$

Such morphisms are in particular $\mathcal{F}$-Tychonoff. An object X is *locally $\mathcal{F}$-compact Hausdorff* if $X \to 1$ is locally $\mathcal{F}$-perfect, that is: if there is an $\mathcal{F}$-open embedding $u : X \to K$ with $K \in \mathcal{F}$-CompHaus. If it is clear that X is $\mathcal{F}$-Hausdorff, we may simply call X *locally $\mathcal{F}$-compact.*

Examples.

(1) In $\mathcal{T}op$, X is locally $\mathcal{F}$-compact Hausdorff if and only if X is Hausdorff and locally compact, in the sense that every point in X has a base of compact neighbourhoods. By constructing the Alexandroff one-point compactification for such spaces, one sees that they are locally $\mathcal{F}$-compact. Conversely, compact Hausdorff spaces are locally compact, and local compactness is open-hereditary.

(2) Every locally compact Hausdorff locale (in the sense of Chapter II, 7) is an open sublocale of its Stone-Čech compactification (see II.6.7), hence the embedding is $\mathcal{F}$-open by Prop. 7.3 and X is a locally $\mathcal{F}$-compact Hausdorff object in $\mathcal{L}oc$. We conjecture that also the converse proposition is true.

8.2. First stability properties. The following properties are easy to prove:

Proposition.

(1) *Every morphism representing an $\mathcal{F}$-closed or an $\mathcal{F}$-open subobject is a locally $\mathcal{F}$-perfect morphism, and so is every $\mathcal{F}$-perfect morphism.*

(2) *The class of locally $\mathcal{F}$-perfect morphisms is stable under pullback; moreover, if in the pullback diagram* (11) *both f and g are locally $\mathcal{F}$-perfect, so is $g \cdot f' = f \cdot g'$.*

Proof. (1) $\mathcal{F}$-closed subobjects give $\mathcal{F}$-perfect morphisms. Hence, the assertion follows by choosing u or p in (27) to be an identity morphism.
(2) Both $\mathcal{F}$-open subobjects and $\mathcal{F}$-perfect morphisms are stable under pullback. Furthermore, if $f = p \cdot u$, $g = q \cdot v$ are both locally $\mathcal{F}$-perfect, the pullback diagram

(11) is decomposed into four pullback diagrams, as follows:

$$\begin{array}{ccccc} \cdot & \xrightarrow{u'} & \cdot & \xrightarrow{p'} & \cdot \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle v''} & & \downarrow{\scriptstyle v} \\ \cdot & \xrightarrow{u''} & \cdot & \xrightarrow{p''} & \cdot \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q''} & & \downarrow{\scriptstyle q} \\ \cdot & \xrightarrow{u} & \cdot & \xrightarrow{p} & \cdot \end{array} \tag{28}$$

Hence $g \cdot f = (q \cdot v) \cdot (p' \cdot u') = (q \cdot p'') \cdot (v'' \cdot u')$, with $q \cdot p''$ $\mathcal{F}$-perfect and $v'' \cdot u'$ an $\mathcal{F}$-open subobject. □

Corollary. *The full subcategory* $\mathcal{F}$-LCompHaus *of locally $\mathcal{F}$-compact Hausdorff objects in $\mathcal{X}$ is closed under finite products and under $\mathcal{F}$-open subobjects in $\mathcal{X}$.*

Proof. For the first statement, choose $Y = 1$ in diagram (11) and apply assertion (2) of the Proposition. The second statement is trivial. □

Exercise. Prove: if $U \to X$ in sub$X$ is $\mathcal{F}$-open and $A \to X$ in sub$X$ $\mathcal{F}$-closed, then $U \wedge A \to X$ is locally $\mathcal{F}$-perfect.

8.3. Local compactness via Stone-Čech. For the remainder of this Section, *we assume that* $\mathcal{F}$-CompHaus *is reflective in* $\mathcal{F}$-Tych, so that in particular we have a functorial $\mathcal{F}$-compactification

$$\beta_X : X \longrightarrow \beta X \quad (X \in \mathcal{F}\text{-Tych})$$

at our disposal. We shall also use its extension to morphisms, as described in diagram (24), with $\kappa = \beta$.

Theorem (Clementino-Tholen [13]). *An object X is locally $\mathcal{F}$-compact Hausdorff if and only if X is $\mathcal{F}$-Tychonoff and β_X is $\mathcal{F}$-open.*

Proof. Sufficiency of the condition is trivial. For its necessity, let $u : X \to K$ be an $\mathcal{F}$-open subobject with $K \in \mathcal{F}$-CompHaus. With the unique morphism $f : \beta X \to K$ with $f \cdot \beta_X = u$ we can then form the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{u'} & \beta X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{u} & K \end{array} \tag{29}$$

and have an induced morphism $d : X \to P$ with $f' \cdot d = 1_X$ and $u' \cdot d = \beta X$. Since u' is $\mathcal{F}$-open and β_X $\mathcal{F}$-dense (by Cor. 7.1(2) and Prop. 6.4), the pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle \beta_X} \\ P & \xrightarrow{u'} & \beta X \end{array} \tag{30}$$

shows that d is $\mathcal{F}$-dense. On the other hand, the equalizer diagram

$$X \xrightarrow{d} P \underset{d \cdot f'}{\overset{1_P}{\rightrightarrows}} P \tag{31}$$

shows that d is $\mathcal{F}$-closed (since $P \in \mathcal{F}$-Haus, by Cor. 4.3). Therefore, d is an isomorphism (see Cor. 2.2(2)), and $\beta_X \cong u'$ is $\mathcal{F}$-open. □

With Corollary 6.7 we obtain immediately from the Theorem:

Corollary. *If every morphism in $\mathcal{M}$ is $\mathcal{F}$-initial, then a morphism $f : X \to Y$ with $X, Y \in \mathcal{F}$*-Tych *is locally $\mathcal{F}$-perfect if and only if the morphism $\beta_f : X \to P_f$ making diagram* (24) *(with $\beta = \kappa$) commutative is $\mathcal{F}$-open.* □

8.4. Composites of locally perfect maps. We can now embark on improving some of the properties given in 8.2.

Proposition. *Let every morphism in $\mathcal{M}$ be $\mathcal{F}$-initial. Then the class of locally $\mathcal{F}$-perfect morphisms with $\mathcal{F}$-Tychonoff domain and codomain is closed under composition.*

Proof. It obviously suffices to show that any composite morphism $h = v \cdot p$ with $p : K \to Y$ $\mathcal{F}$-perfect and $v : Y \to L$ an $\mathcal{F}$-open subobject is locally $\mathcal{F}$-perfect. For that we must show that its $\mathcal{F}$-antiperfect factor $\beta_h : K \to P_h$ is $\mathcal{F}$-open. But the β-naturality diagram for h decomposes as

$$\begin{array}{ccc} K & \xrightarrow{\beta_K} & \beta K \\ {\scriptstyle p}\big\downarrow & \boxed{1} & \big\downarrow{\scriptstyle \beta p} \\ Y & \xrightarrow{\beta_Y} & \beta Y \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle \beta v} \\ L & \xrightarrow{\beta_L} & \beta L \end{array} \tag{32}$$

with the upper square a pullback, since p is $\mathcal{F}$-perfect (see Theorem 6.6). Now (32) decomposes further, as

$$\begin{array}{ccccc} K & \xrightarrow{\beta_h} & P_h & \xrightarrow{m_h} & \beta K \\ {\scriptstyle p}\downarrow & \boxed{2} & \downarrow & \boxed{3} & \downarrow{\scriptstyle \beta p} \\ Y & \xrightarrow{\beta_v} & P_v & \xrightarrow{m_v} & \beta Y \\ {\scriptstyle v}\downarrow & & {\scriptstyle p_v}\downarrow & \boxed{4} & \downarrow{\scriptstyle \beta v} \\ L & \xrightarrow{1_L} & L & \xrightarrow{\beta_L} & \beta L \end{array} \tag{33}$$

Here $\boxed{3}$ is a pullback diagram, since $\boxed{4}$ and the concatenation $\boxed{3}\&\boxed{4}$ are pullback diagrams. Now, with $\boxed{1} = \boxed{2}\&\boxed{3}$ also $\boxed{2}$ is a pullback diagram. Since the $\mathcal{F}$-antiperfect factor β_v of the $\mathcal{F}$-open (and therefore locally $\mathcal{F}$-perfect) morphism v is $\mathcal{F}$-open, also its pullback β_h must be $\mathcal{F}$-open. □

Corollary. *If every morphism in $\mathcal{M}$ is $\mathcal{F}$-initial, then the following assertions are equivalent for $X \in \mathcal{F}$-*Tych*:*

(i) *X is locally $\mathcal{F}$-compact;*
(ii) *every morphism $f : X \to Y$ with $Y \in \mathcal{F}$-*Tych *is locally $\mathcal{F}$-perfect;*
(iii) *there is a locally $\mathcal{F}$-perfect map $f : X \to Y$ with $Y \in \mathcal{F}$-*LCompHaus*.*

Proof. (i) ⇒ (ii): In diagram (24) with $\kappa = \beta$, β_f is open since β_X is, and since m_f is monic, by Prop. 7.1(2). (ii) ⇒ (iii) is trivial. (iii) ⇒ (ii): $!_X = !_Y \cdot f$ is locally $\mathcal{F}$-perfect, by the Proposition. □

8.5. Further stability properties. We can now improve the assertions made in 8.2:

Theorem. *If every morphism in $\mathcal{M}$ is $\mathcal{F}$-initial, then the full subcategory $\mathcal{F}$-*LCompHaus *of locally $\mathcal{F}$-compact Hausdorff objects is closed under finite limits and under $\mathcal{F}$-closed or $\mathcal{F}$-open subobjects in $\mathcal{X}$.*

Proof. We already stated closure under finite products and $\mathcal{F}$-open subobjects in Cor. 8.2. Closure under $\mathcal{F}$-closed subobjects follows from Prop. 8.4, and this fact then implies closure under equalizers, by Thm. 4.3(viii). □

8.6. Invariance theorem for local compactness. We finally turn to invariance of local $\mathcal{F}$-compactness under $\mathcal{F}$-perfect maps. For that purpose let us call a morphism $f : X \to Y$ *$\mathcal{F}$-final* if a subobject of Y is $\mathcal{F}$-closed whenever its inverse image under f is $\mathcal{F}$-closed. Clearly, with Prop. 1.8(2) one obtains:

Lemma. *If every pullback of $f \in \mathcal{F}$ along a morphism in $\mathcal{M}$ lies in $\mathcal{E}$, then f is $\mathcal{F}$-final.*

Translated into standard topological terms, the Lemma asserts that in $\mathcal{T}op$ closed surjective maps are quotient maps. Now, in $\mathcal{T}op$, to say that the map

$f : X \to Y$ has the property that $N \subseteq Y$ is closed as soon as $f^{-1}[N] \subseteq X$ is closed is trivially equivalent to saying that N is open whenever $f^{-1}[N]$ is open. In general, however, we may not assume such equivalence.

Example. In $\mathcal{T}op_{\text{open}}$, $\mathcal{F}$-finality takes on the usual meaning: $N \subseteq Y$ is open (= $\mathcal{F}$-closed) as soon as $f^{-1}[N]$ is open in X. However, every map $f : X \to Y$ in $\mathcal{T}op_{\text{open}}$ has the property that $N \subseteq Y$ is $\mathcal{F}$-open as soon as $f^{-1}[N]$ is $\mathcal{F}$-open, provided that X and Y are $T1$-spaces. Indeed, in this case all subobjects are $\mathcal{F}$-open, because there are only isomorphic $\mathcal{F}$-dense subobjects: see Exercise 2 of 7.2.

Theorem. *Let every morphism in $\mathcal{M}$ be $\mathcal{F}$-initial, let $\mathcal{E}$ be stable under pullback along $\mathcal{M}$-morphisms and let every $\mathcal{F}$-final morphism in $\mathcal{E}$ have the property that a subobject of its codomain is $\mathcal{F}$-open whenever its inverse image is $\mathcal{F}$-open. Then, for every $\mathcal{F}$-perfect morphism $f : X \to Y$ with $X, Y \in \mathcal{F}$-*Tych*, one has:*

(1) *if Y is locally $\mathcal{F}$-compact, so is X;*
(2) *if $f \in \mathcal{E}$ and X is locally $\mathcal{F}$-compact, so is Y.*

Proof. (1) is a special case of Cor. 8.4 (iii) $\Rightarrow$ (i). For (2), first observe that $\mathcal{F}$-density of f and β_Y gives the same first for $\beta f \cdot \beta_X = \beta_Y \cdot f$ and then for βf. But $\beta f : \beta X \to \beta Y$ is also $\mathcal{F}$-closed (see Prop. 5.1), hence $\beta f \in \mathcal{E}$ (see Exercise 1 of 2.2). Furthermore, by the Lemma, βf is $\mathcal{F}$-final. Since the $\mathcal{F}$-perfect morphism f makes

$$\begin{array}{ccc} X & \xrightarrow{\beta_X} & \beta X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \beta f} \\ Y & \xrightarrow{\beta_Y} & \beta Y \end{array} \tag{34}$$

a pullback diagram with

$$\beta_X = (\beta f)^{-1}[\beta_Y]$$

$\mathcal{F}$-open, by hypothesis also β_Y must be $\mathcal{F}$-open. $\square$

Remark. Analyzing the condition of pullback stability of $\mathcal{E}$ along embeddings one observes that this stability property is needed only along $\mathcal{F}$-closed embeddings.

Exercise. Using Prop. 7.5, give sufficient conditions for $\mathcal{F}$-LCompHaus to be closed under finite coproducts in $\mathcal{X}$.

9. Pullback stability of quotient maps, Whitehead's Theorem

9.1. Quotient maps. The formation of quotient spaces is an important tool in topology when constructing new spaces from old: given a space X and a surjective mapping $f : X \to Y$, one provides the set Y with a topology such that any $B \subseteq Y$ is open (closed) if $f^{-1}[B] \subseteq X$ is open (closed). In our general setting of 2.1, we have called a morphism $f : X \to Y$ in $\mathcal{X}$ *$\mathcal{F}$-final* (see 8.6) if any $b \in \mathrm{sub}Y$ is $\mathcal{F}$-closed whenever $f^{-1}[b] \in \mathrm{sub}X$ is $\mathcal{F}$-closed, and we refer to an $\mathcal{F}$-final morphism in $\mathcal{E}$ as an *$\mathcal{F}$-quotient map.* Certain pullbacks of $\mathcal{F}$-final morphisms are $\mathcal{F}$-final:

Proposition. *The restriction $f' : f^{-1}[B] \to B$ of an $\mathcal{F}$-final morphism $f : X \to Y$ to an $\mathcal{F}$-closed subobject $b : B \to Y$ is $\mathcal{F}$-final.*

Proof. For a subobject $c : C \to B$ one has

$$f^{-1}[b \cdot c] \cong f^{-1}[b] \cdot (f')^{-1}[c].$$

Hence, if $(f')^{-1}[c]$ is $\mathcal{F}$-closed, so is $f^{-1}[b \cdot c]$ and also $b \cdot c$, by hypothesis. But $\mathcal{F}$-closedness of $b \cdot c$ implies the same for c, by Prop. 3.2(3). □

In general, $\mathcal{F}$-finality fails badly to be stable under pullback, already for $\mathcal{X} = \mathcal{T}op$ and in very elementary situations.

Example. We consider the quotient map $f : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, so that $f(x) = f(y)$ if and only if $x - y \in \mathbb{Z}$, and the subspace $S := \mathbb{R} \setminus \{\frac{1}{n} \,|\, n \in \mathbb{N},\ n \geq 2\}$ of $\mathbb{R}$. Then the map

$$g := \mathrm{id}_S \times f : S \times \mathbb{R} \to S \times \mathbb{R}/\mathbb{Z}$$

is the pullback of f along the projection $S \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, but fails to be a quotient map. Indeed, the set

$$B := \{\frac{1}{i} + \frac{\pi}{j}, \frac{i+1}{j}) \,|\, i, j \geq 2\}$$

is closed in $S \times \mathbb{R}$ and $g^{-1}[g[B]] = B$, but $g[B]$ is not closed in $S \times \mathbb{R}/\mathbb{Z}$ (the point $(0, f(1))$ lies in $\overline{g[B]} \setminus g[B]$).

Let us also note that the map f is closed; hence: the product of two closed quotient maps needs neither to be closed nor a quotient map.

Exercise. Verify the claims made in the Example and show that the space S fails to be locally compact.

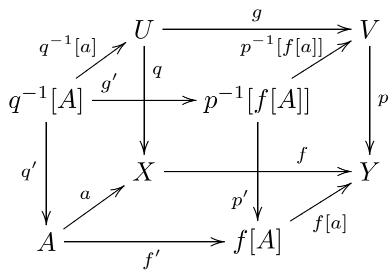
9.2. Beck-Chevalley Property. In what follows we would like to extend Proposition 9.1 greatly by showing that $\mathcal{F}$-quotient maps are stable under pullback along $\mathcal{F}$-perfect maps. But this needs some preparations and extra conditions. The condition given in the following proposition is known as (an instance of) the Beck-Chevalley Property.

Proposition. *The class $\mathcal{E}$ of the factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathcal{X}$ is stable under pullback if and only if, for every pullback diagram*

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad (35)$$

and every $a \in \mathrm{sub}X$, $g[q^{-1}[a]] \cong p^{-1}[f[a]]$.

Proof. Considering $a \cong 1_X$ one sees that the condition is sufficient for pullback stability: with Exercise 2 of 1.6, $f \in \mathcal{E}$ implies $g \in \mathcal{E}$. For its necessity we consider the commutative diagram



(36)

It suffices to show that the morphism g' (induced by the pullback property of $p^{-1}[f[A]]$) is in $\mathcal{E}$. But since all other vertical faces are pullback diagrams, also the front face of (36) is a pullback diagram; consequently, with $f' \in \mathcal{E}$ we obtain $q' \in \mathcal{E}$, by hypothesis on $\mathcal{E}$. □

9.3. Fibre-determined categories. We showed in 3.5 that in $\mathcal{T}op$ proper maps are characterized as closed maps with compact fibres, but that this characterization fails in general. Since in what follows we would like to make crucial use of it, we say that our category $\mathcal{X}$ is *fibre-determined* (with respect to its $(\mathcal{E}, \mathcal{M})$-closed system $\mathcal{F}$) if the following two conditions are satisfied:

1. a morphism f lies in $\mathcal{E}$ if and only if all of its fibres have *points*, that is: if for every fibre F of f there is a morphism $1 \to F$;
2. a morphism f is in $\mathcal{F}^*$ if and only if $f[-]$ preserves $\mathcal{F}$-closedness of subobjects and f has $\mathcal{F}$-compact fibres.

Of course, in condition 2 "only if" comes for free (see Prop. 2.1(3) and Cor. 3.5). Condition 1 means equivalently that $\mathcal{E}$ contains exactly those morphisms with respect to which the terminal object 1 is projective, as the Reader will readily check:

Exercises.

1. Prove that for a morphism $f : X \to Y$ all fibres have points if and only if the hom-map $\mathcal{X}(1, f) : \mathcal{X}(1, X) \to \mathcal{X}(1, Y)$ is surjective.
2. Prove that the class of all morphisms f for which $\mathcal{X}(1, f)$ is surjective is stable under pullback in $\mathcal{X}$.

As a consequence we see that condition 1 forces $\mathcal{E}$ to be stable under pullback. We note that, granted pullback stability of $\mathcal{E}$, in the remainder of this section we shall use only the "only if" part of condition 1.

Observe that $\mathcal{T}op$ is fibre-determined (by Theorem 3.5), and so is $\mathcal{T}op_{\text{open}}$ (trivially, see Exercise 1 of 3.5), but that $\mathcal{T}op/B$ is not (by Example 3.5).

9.4. Pullbacks of quotient maps. Next we establish a result which, even in the case $\mathcal{X} = \mathcal{T}op$, was established only recently (see [10]):

Theorem (Richter-Tholen [46]). *If $\mathcal{X}$ is fibre-determined, then every pullback of an $\mathcal{F}$-quotient map along an $\mathcal{F}$-perfect map is an $\mathcal{F}$-quotient map.*

Proof. We consider the pullback diagram (35) with an $\mathcal{F}$-perfect map f and an $\mathcal{F}$-quotient map p. In order to show that q is an $\mathcal{F}$-quotient map as well, for $a \in \mathrm{sub}X$ we assume $q^{-1}[a]$ $\mathcal{F}$-closed and must show that a is $\mathcal{F}$-closed.

First, with Prop. 9.2 and the Exercises 9.3 we obtain

$$g[q^{-1}[a]] \cong p^{-1}[f[a]],$$

and this subobject is $\mathcal{F}$-closed since g (as a pullback of $f \in \mathcal{F}^*$) is $\mathcal{F}$-closed. Whence $f[a]$ is $\mathcal{F}$-closed, by hypothesis on p. Therefore it suffices to show that the morphism f' of diagram (36) is $\mathcal{F}$-proper, since then $\mathcal{F}$-properness of

$$f[a] \cdot f' = f \cdot a$$

and $\mathcal{F}$-separatedness of f give $\mathcal{F}$-closedness of a, with Prop. 5.2.

Now, in order to show $\mathcal{F}$-properness of f' we invoke condition 2 of 9.3 and first show that $f'[-]$ preserves $\mathcal{F}$-closedness of subobjects. Hence we consider $b \in \mathrm{sub}A$ and note

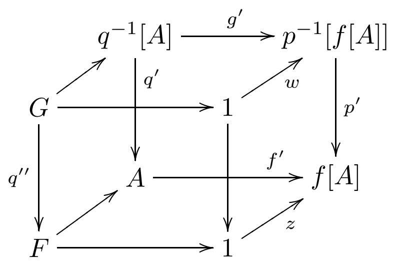
$$g'[(q')^{-1}[b]] \cong (p')^{-1}[f'[b]]$$

with the Beck-Chevalley Property again, applied to the front face of (36). This subobject is $\mathcal{F}$-closed since the morphism g' is $\mathcal{F}$-closed; in fact, since

$$g \cdot q^{-1}[a] = p^{-1}[f[a]] \cdot g'$$

is $\mathcal{F}$-proper, so is g', by Prop. 3.2(3). Furthermore, p' is an $\mathcal{F}$-quotient map, by Prop. 9.1, and we can conclude that $f'[b]$ is $\mathcal{F}$-closed.

Lastly, it remains to be shown that f' has $\mathcal{F}$-compact fibres. But by condition 1 and Exercises 9.3, any point $z : 1 \to f[A]$ factors as $z = p' \cdot w$, with $w : 1 \to p^{-1}[f[A]]$. With F, G denoting the fibres of f', g' belonging to z, w, respectively, we can form the commutative diagram



(37)

Its front face is a pullback diagram since the back-, top- and bottom faces are pullback diagrams. Hence, q'' is an isomorphism, and $F \cong G$ is $\mathcal{F}$-compact since g' is $\mathcal{F}$-proper, as shown earlier. □

Corollary. *For $\mathcal{X}$ fibre-determined and K an $\mathcal{F}$-compact $\mathcal{F}$-Hausdorff object, with $p : V \to Y$ also $1_K \times p : K \times V \to K \times Y$ is an $\mathcal{F}$-quotient map.*

Proof. Apply the Theorem to the $\mathcal{F}$-perfect projection $K \times Y \to Y$. □

Exercises.

1. By applying the Theorem to $\mathcal{T}op_{\text{open}}$, conclude that pullbacks of quotient maps along local homeomorphisms are quotient maps.
2. Verify that when $\mathcal{E}$ is stable under pullback, the class $\mathcal{F}^+$ of $\mathcal{F}$-open maps in $\mathcal{X}$ is $(\mathcal{E}, \mathcal{M})$-closed. What does the Theorem say when we exchange $\mathcal{F}$ for $\mathcal{F}^+$?

9.5. Whitehead's Theorem. An $\mathcal{F}^+$-final morphism $f : X \to Y$ has, by definition, the property that $b \in \text{sub}Y$ is $\mathcal{F}$-open whenever $f^{-1}[b] \in \text{sub}X$ is $\mathcal{F}$-open. From Prop. 9.1 applied to $\mathcal{F}^+$ in lieu of $\mathcal{F}$ one obtains immediately:

Lemma. *The restriction $f' : f^{-1}[B] \to B$ of an $\mathcal{F}^+$-final morphism $f : X \to Y$ to an $\mathcal{F}$-open subobject $b : B \to Y$ is $\mathcal{F}^+$-final.* □

In $\mathcal{T}op$, $\mathcal{F}^+$-quotient maps (=$\mathcal{F}^+$-final maps in $\mathcal{E}$) coincide with $\mathcal{F}$-quotient maps. If this happens in $\mathcal{X}$, we can use the lemma in order to upgrade Theorem 9.4 to:

Theorem. *If $\mathcal{X}$ is fibre-determined and if $\mathcal{F}^+$-quotient maps coincide with $\mathcal{F}$-quotient maps, then every pullback of an $\mathcal{F}$-quotient map along a locally $\mathcal{F}$-perfect map is an $\mathcal{F}$-quotient map.* □

Corollary. *If $\mathcal{X}$ is fibre-determined and if $\mathcal{F}^+$-quotient maps coincide with $\mathcal{F}$-quotient maps, then for every locally $\mathcal{F}$-compact $\mathcal{F}$-Hausdorff object K and every $\mathcal{F}$-quotient map p, also $1_K \times p$ is an $\mathcal{F}$-quotient map.* □

In $\mathcal{T}op$, the assertion of the Corollary is known as *Whitehead's Theorem.*

10. Exponentiable maps, exponentiable spaces

10.1. Reflection of quotient maps. In the previous section we have seen that, under certain hypotheses, locally $\mathcal{F}$-perfect maps have (stably) the property that $\mathcal{F}$-quotients pull back along them. This is a fundamental property which is worth investigating separately. Analogously to the terminology employed in the definition of $\mathcal{F}$-open maps, we say that a morphism *reflects $\mathcal{F}$-quotients* if in every pullback diagram (35) with p also q is an $\mathcal{F}$-quotient map, and f is *$\mathcal{F}$-exponentiable* if every pullback f' of f (see diagram (11)) reflects $\mathcal{F}$-quotients. Hence,

$$\mathcal{F}^{\text{exp}} := \{f \mid f\ \mathcal{F}\text{-exponentiable}\} = \{f \mid f \text{ reflects } \mathcal{F}\text{-quotients}\}^*$$

in the notation of Section 3. The reason for this terminology will become clearer at the end of 10.2 below, and fully transparent in 10.9.

In this terminology, since (locally) $\mathcal{F}$-perfect maps are pullback stable, we proved in Theorems 9.4 and 9.5:

Corollary. *Let $\mathcal{X}$ be fibre-determined. Then every $\mathcal{F}$-perfect map is $\mathcal{F}$-exponentiable. Even every locally $\mathcal{F}$-perfect map has this property, if the notions of $\mathcal{F}^+$-quotient map and $\mathcal{F}$-quotient map are equivalent in $\mathcal{X}$.* □

Exercise. Conclude that in $\mathcal{T}op$ both perfect maps and local homeomorphisms are $\mathcal{F}$-exponentiable.

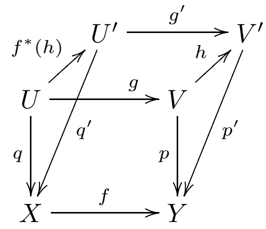
Example. For a space X, the map $X \to 1$ in $\mathcal{T}op$ always reflects $\mathcal{F}$-quotients (since the pullback of $V \to 1$ along it is the projection $X \times V \to X$), but generally not every pullback of $X \to 1$ reflects $\mathcal{F}$-quotients: see Example 9.1. Hence, in $\mathcal{T}op$, reflection of $\mathcal{F}$-quotients is a property properly weaker than $\mathcal{F}$-exponentiability.

Before developing a theory of $\mathcal{F}$-exponentiability, we should provide a first justification for the terminology.

10.2. Exponentiable maps. For every morphism $f : X \to Y$ we have the pullback functor

$$f^* : \mathcal{X}/Y \to \mathcal{X}/X$$

which sends an object (V, p) in $\mathcal{X}/Y$ to $(U, q) = (X \times_Y V, \mathrm{proj}_1)$ in $\mathcal{X}/X$ (see diagram (35)), and for a morphism $h : (V, p) \to (V', p')$, $f^*(h) : (U, q) \to (U', q')$ has underlying $\mathcal{X}$-morphism $1_X \times h : X \times_Y V \to X \times_Y V'$. Of course, the functor $f^{-1}[-]$ of 1.7 is just a restriction of f^*.



(38)

Proposition. *f is $\mathcal{F}$-exponentiable if and only if the functor f^* maps an $\mathcal{F}$-quotient morphism of $\mathcal{X}/Y$ to an $\mathcal{F}$-quotient morphism of $\mathcal{X}/X$.*

Proof. If f is $\mathcal{F}$-exponentiable and $h : (V, p) \to (V', p')$ an $\mathcal{F}$-quotient map, then g' (as a pullback of f) reflects the $\mathcal{F}$-quotient map h in $\mathcal{X}$, hence its pullback $f^*(h)$ along g' is an $\mathcal{F}$-quotient map in $\mathcal{X}$ and also in $\mathcal{X}/X$. Conversely, considering any pullback g' of f and an $\mathcal{F}$-quotient map h in $\mathcal{X}$, we can consider this as a morphism in $\mathcal{X}/Y$ and argue in the same way as before. □

Exercises.

1. Prove that if $\mathcal{X}$ has a certain type of colimits, so does $\mathcal{X}/B$, and the forgetful functor $\Sigma_B : \mathcal{X}/B \to \mathcal{X}$ preserves them.
2. For $\mathcal{X} = \mathcal{T}op$, prove that f^* preserves coproducts for every f.
3. For $\mathcal{X} = \mathcal{T}op$, prove that f^* preserves coequalizers if and only if f^* preserves $\mathcal{F}$-quotient morphisms.

4. Conclude that in $\mathcal{T}op$ (as well as in $\mathcal{T}op_{\mathrm{open}}$) the $\mathcal{F}$-exponentiable morphisms f are exactly those for which f^* preserves all small colimits.
5. Apply *Freyd's Special Adjoint Functor Theorem* (see Mac Lane [35]) to prove that for an $\mathcal{F}$-exponentiable morphism f in $\mathcal{T}op$ the functor f^* actually has a right adjoint.

Exercise 5 identifies the $\mathcal{F}$-exponentiable morphisms f in $\mathcal{T}op$ (and in $\mathcal{T}op_{\mathrm{open}}$) as those for which f^* has a right adjoint functor, a property that may be considered in any category $\mathcal{X}$ with pullbacks and that is equivalent to the absolute (="$\mathcal{F}$-free") categorical notion of *exponentiability*; see 10.8 below.

10.3. Stability properties. We shall collect some properties of the class $\mathcal{F}^{\exp}$ of $\mathcal{F}$-exponentiable morphisms. First we note:

Lemma. *If $g \cdot f$ is $\mathcal{F}$-final, so is g.*

Proof. If for a subobject c of the codomain of g $g^{-1}[c]$ is $\mathcal{F}$-closed, so is $f^{-1}[g^{-1}[c]] = (g \cdot f)^{-1}[c]$. Hence $\mathcal{F}$-closedness of c follows with the hypothesis. □

Following the terminology used in topology we call a morphism f in $\mathcal{X}$ an $\mathcal{F}$-*biquotient map* if every pullback of f is an $\mathcal{F}$-quotient map. In $\mathcal{T}op$ other names in use are *universal quotient map* and *descent map*. In order not to divert too much, we leave it to the Reader to establish their characterization in terms of points and open sets.

Exercises.

1. Show that every $\mathcal{F}$-proper map in $\mathcal{E}^*$ is an $\mathcal{F}$-biquotient map.
2. Conclude that if the notions of $\mathcal{F}^+$-quotient map and $\mathcal{F}$-quotient map are equivalent in $\mathcal{X}$, then every $\mathcal{F}$-open map in $\mathcal{E}^*$ is an $\mathcal{F}$-biquotient map.
3. * Show that in $\mathcal{T}op$ $f : X \to Y$ is an ($\mathcal{F}$-)biquotient map if and only if, for every point $y \in Y$ and every open cover $\{U_i \mid i \in I\}$ of the fibre $f^{-1}y$, the system $\{f[U_i] \mid i \in F\}$ covers some neighbourhood of y in Y, for some finite $F \subseteq I$. (See Day-Kelly [14].)

Proposition.

(1) *The class $\mathcal{F}^{\exp}$ contains all isomorphisms, is closed under composition and stable under pullback.*

(2) *If $g \cdot f \in \mathcal{F}^{\exp}$ with g monic, then $f \in \mathcal{F}^{\exp}$.*

(3) *If $g \cdot f \in \mathcal{F}^{\exp}$ with f an $\mathcal{F}$-biquotient map, then $g \in \mathcal{F}^{\exp}$.*

Proof. Statements (1) and (2) are trivial, see Prop. 3.2. A pullback of $g \cdot f$ with f an $\mathcal{F}$-biquotient map has the form $g' \cdot f'$ with f' and $\mathcal{F}$-(bi)quotient map and $g' \cdot f' \in \mathcal{F}^{\exp}$. Hence it suffices to show that g reflects $\mathcal{F}$-quotients if $g \cdot f$ does, with f an $\mathcal{F}$-quotient map. But if $r : W \to Z$ is an $\mathcal{F}$-quotient map, we can form the consecutive pullback diagram

$$\begin{array}{ccccc} U & \xrightarrow{f'} & V & \xrightarrow{g'} & W \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array} \tag{39}$$

and have that q is an $\mathcal{F}$-quotient map, by hypothesis on $g \cdot f$. But then also $f \cdot q = p \cdot f'$ is an $\mathcal{F}$-quotient map, whence p is one too, by the Lemma and Exercise 2 of 1.4. □

10.4. Exponentiable objects. An object X of $\mathcal{X}$ is $\mathcal{F}$*-exponentiable* if the unique morphism $!_X : X \to 1$ is $\mathcal{F}$-exponentiable. This means precisely that for every $\mathcal{F}$-quotient map $p : W \to V$ also

$$1_X \times p : X \times W \to X \times V$$

is an $\mathcal{F}$-quotient map. From Corollaries 9.4 and 9.5 (or from 9.1) we have:

Corollary. *Let $\mathcal{X}$ be fibre-determined. Then every $\mathcal{F}$-compact Hausdorff object is $\mathcal{F}$-exponentiable. Even every locally $\mathcal{F}$-compact Hausdorff object has this property, if the notions of $\mathcal{F}^+$-quotient map and $\mathcal{F}$-quotient map are equivalent.* □

Just as Theorem 3.3 follows from Prop. 3.2, one can conclude the following Theorem from Prop. 10.3:

Theorem.

(1) *For any $\mathcal{F}$-exponentiable $f : X \to Y$ with Y $\mathcal{F}$-exponentiable, also X is $\mathcal{F}$-exponentiable.*
(2) *For any $\mathcal{F}$-biquotient map $f : X \to Y$ with X $\mathcal{F}$-exponentiable, also Y is $\mathcal{F}$-exponentiable.*
(3) *With X and Y also $X \times Y$ is $\mathcal{F}$-exponentiable.*

□

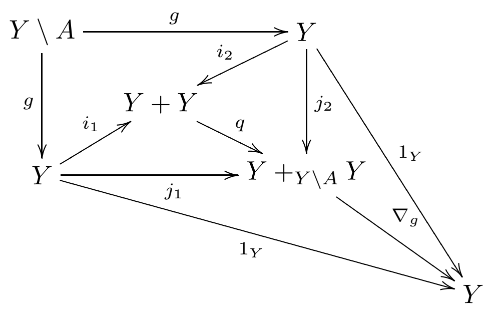
10.5. Exponentiability in $\mathcal{T}op$. It is time to shed more light on the notion of exponentiability in case $\mathcal{X} = \mathcal{T}op$. A characterization in traditional topological terms is, however, not quite obvious, and to a large extent we must refer the interested Reader to the literature. Reasonably manageable is the case of a subspace embedding.

Proposition (Niefield [38]). *For a subspace A of a topological space X, the following are equivalent:*

(i) *the inclusion map $A \hookrightarrow X$ is exponentiable;*
(ii) *A is open in its closure $\overline{A}$ in X;*
(iii) *A is locally closed in X, that is: $A = O \cap C$, for an open set O and a closed set C in X.*

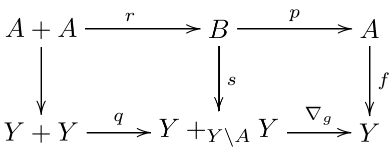
Proof (Richter [44]). The equivalence (iii) $\Leftrightarrow$ (ii) is straightforward, while (ii) $\Rightarrow$ (i) follows easily from the characterization of quotients in $\mathcal{T}op$ via open subsets.
(i) $\Rightarrow$ (ii): Let the inclusion map $A \hookrightarrow X$ be exponentiable. Then the inclusion $f : A \hookrightarrow \overline{A} := Y$ is exponentiable by Prop. 10.3(2). Consider the inclusion of the

complement $g : Y \setminus A \hookrightarrow Y$ and the pushout $(j_1, j_2 : Y \to Y +_{Y\setminus A} Y)$ of (g, g). We therefore have the following diagram



(40)

where q and ∇_g are quotient maps, since q is the coequalizer of $(i_1 \cdot g, i_2 \cdot g)$ and ∇_g is by construction a split epimorphism. Form now the pullback diagrams



(41)

where $s : B \hookrightarrow Y +_{Y\setminus A} Y$ is an inclusion and the map $r : A + A \to B$ is an identity. Since with f also s is exponentiable, then $r : A + A \to B$ is a quotient, hence an homeomorphism. The inclusion $k_1 : A \hookrightarrow A + A \cong B$ is open, and therefore there is an open subset O of $Y +_{Y\setminus A} Y$ such that $O \cap (A + A) = k_1(A)$, hence $j_1^{-1}(O) \supseteq A$. Moreover, $j_2^{-1}(O) = \emptyset = j_1^{-1}(O) \cap (Y \setminus A)$ since otherwise $j_2^{-1}(O) \cap A \neq \emptyset$ because A is dense in Y. Therefore $A = j_1^{-1}(O)$ is open in Y as claimed. □

For the characterization of exponentiable spaces we refer to [14, 26, 21]:

Theorem (Day-Kelly). *A topological space is exponentiable if and only if for every neighbourhood V of a point there is a smaller neighbourhood U such that every open cover of V contains a finite subcover of U.* □

Spaces satisfying the condition of the Theorem are called *core-compact*. These are precisely the spaces whose system of open sets forms a *continuous lattice* (see [47, 26]). We already know that locally compact Hausdorff spaces are of this type. But in fact, without any separation condition, already Brown [4] proved that local compactness (in the sense of Example 1 of 8.1) implies exponentiability. The converse proposition in case of a Hausdorff space goes back to Michael [37]; actually, the separation condition may be eased from Hausdorff to sober, see [24]. But in general, exponentiable spaces fail to be locally compact, although no constructive example is known (see Isbell [26]).

Niefield [38] established a characterization of exponentiable maps in $\mathcal{T}op$ in the Day-Kelly style and thereby greatly generalized the Theorem above. For recent accounts of this characterization we refer the Reader to Niefield [40] and Richter

[45], with the latter paper giving a smooth extension of the Day-Kelly result in terms of *fibrewise core-compactness.*

10.6. Local separatedness. We briefly look at morphisms in the class $(\mathcal{F}^{\exp})'$. Hence, we call $f : X \to Y$ *locally $\mathcal{F}$-separated* if $\delta_f : X \to X \times_Y X$ is $\mathcal{F}$-exponentiable. With Prop. 4.2 applied to $\mathcal{F}^{\exp}$ in lieu of $\mathcal{F}$ and with Prop. 10.3(3) we obtain:

Proposition.

(1) *The class of locally $\mathcal{F}$-separated maps contains all monomorphisms, is closed under composition and stable under pullback.*
(2) *If $g \cdot f$ is locally $\mathcal{F}$-separated, so is f.*
(3) *If $g \cdot f$ is locally $\mathcal{F}$-separated with an $\mathcal{F}$-exponentiable $\mathcal{F}$-biquotient map f, also g is locally $\mathcal{F}$-separated.*

□

Example. The terminology justifies itself in case $\mathcal{X} = \mathcal{T}op$. With Prop. 10.5 one easily verifies that $f : X \to Y$ is locally ($\mathcal{F}$-)separated if and only if every point in X has a neighbourhood U such that $f|_U : U \to Y$ is separated. Both separated and locally injective maps (see Exercise 1 of 4.1) are locally separated.

An object X of $\mathcal{X}$ is *locally $\mathcal{F}$-separated* (or *locally $\mathcal{F}$-Hausdorff*) if $X \to 1$ is locally $\mathcal{F}$-separated, i.e., if $\delta_X : X \to X \times X$ is $\mathcal{F}$-exponentiable. For $\mathcal{X} = \mathcal{T}op$, this means that every point in X has a Hausdorff neighbourhood.

Exercises.

1. Apply Theorem 4.3 and Corollary 4.3 with $\mathcal{F}^{\exp}$ in lieu of $\mathcal{F}$ in order to establish stability properties for locally $\mathcal{F}$-separated spaces. (Attention: special care needs to be given when "translating" condition (vi) of Theorem 4.3: see (3) of the Proposition above.)
2. Show that in $\mathcal{T}op$ locally Hausdorff spaces are sober T1-spaces, but not conversely.

10.7. Maps with exponentiable domain and locally separated codomain. A powerful sufficient criterion for $\mathcal{F}$-exponentiability of morphisms (a weaker version of which was established in [39]) is obtained "for free" when we apply Prop. 5.1 to $\mathcal{F}^{\exp}$ in lieu of $\mathcal{F}$:

Theorem. *Every morphism $f : X \to Y$ in $\mathcal{X}$ with X $\mathcal{F}$-exponentiable and Y locally $\mathcal{F}$-Hausdorff is $\mathcal{F}$-exponentiable.* □

Just as Prop. 5.2 follows from Prop. 5.1, we conclude from the Theorem the following strengthening of the assertion of Prop. 10.3(2):

Corollary. *If $g \cdot f$ is $\mathcal{F}$-exponentiable with g locally $\mathcal{F}$-separated, then f is $\mathcal{F}$-exponentiable.*

Exercise. Prove that the full subcategory of $\mathcal{F}$-exponentiable and locally $\mathcal{F}$-separated objects is closed in $\mathcal{X}$ under finite limits and under $\mathcal{F}$-exponentiable subobjects.

10.8. Adjoints describing exponentiability. We saw in 10.2 that $f : X \to Y$ is $\mathcal{F}$-exponentiable in $\mathcal{T}op$ if and only if $f^* : \mathcal{T}op/Y \to \mathcal{T}op/X$ has a right adjoint functor. It is a slightly tricky exercise on adjoint functors to show that the latter property may equivalently be expressed by the adjointness of two other functors (see Niefield [38]):

Proposition. *In a finitely-complete category $\mathcal{X}$, the following assertions are equivalent for a morphism $f : X \to Y$:*

(i) $f^* : \mathcal{X}/Y \to \mathcal{X}/X$ *has a right adjoint;*
(ii) *the functor* $X \times_Y (-) : \mathcal{X}/Y \to \mathcal{X}$, $(V, p) \mapsto X \times_Y V$, *has a right adjoint;*
(iii) *the functor* $(X, f) \times (-) : \mathcal{X}/Y \to \mathcal{X}/Y$, $(V, p) \mapsto (X \times_Y V, f \cdot f^*(p))$, *has a right adjoint.*

Proof. Note that the functor described by (ii) is simply $\Sigma_X f^*$, with the forgetful $\Sigma_X : \mathcal{X}/X \to \mathcal{X}$, and the functor described by (iii) may be written as $f_! f^*$, where

$$f_! : \mathcal{X}/X \to \mathcal{X}/Y, \quad (U, q) \mapsto (U, f \cdot q),$$

is left adjoint to f^*. □

Property (iii) identifies (X, f) as an exponentiable object of the category $\mathcal{X}/Y$ since, in general, one calls an object X of a category $\mathcal{X}$ with finite products *exponentiable* if $X \times (-) : \mathcal{X} \to \mathcal{X}$ has a right adjoint.

10.9. Function space topology. As important as establishing the existence of the right adjoints in question is their actual description. In case $\mathcal{X} = \mathcal{T}op$, the right adjoint to $X \times (-)$ for an exponentiable (=core-compact, see 10.5) space is described by the *function-space functor*

$$(-)^X : \mathcal{T}op \to \mathcal{T}op,$$

where the space Y^X has underlying set $C(X, Y) = \mathcal{T}op(X, Y)$ (the set of continuous functions from X to Y); its open sets are generated by the sets

$$N(U, V) = \{f \in C(X, Y) \mid U \ll f^{-1}(V)\},$$

where $U \ll f^{-1}(V)$ for open sets $U \subseteq X$, $V \subseteq Y$ means that every open cover of $f^{-1}(V)$ has a finite subcover of U (see [21]). In case X is a locally compact Hausdorff space (hence exponentiable, see Cor. 10.4) the topology of Y^X has generating open sets

$$N(C, V) = \{f \in C(X, Y) \mid f(C) \subseteq V\},$$

with $C \subseteq X$ compact and $V \subseteq Y$ open, which is known by the name *compact-open topology* (see [20]).

One of the great advantages of Scott's *way-below relation* $\ll$ (which we already encountered in the formulation 10.5 of the Day-Kelly Theorem) is that it translates smoothly from topological spaces to locales and actually leads to corresponding results. For example, a locale L is exponentiable in $\mathcal{L}oc$ if and only if it is a continuous lattice; see [25].

10.10. Partial products. How does the function-space functor look like in $\mathcal{T}op/Y$, rather than in $\mathcal{T}op$? Here we just indicate how the right adjoint of $X \times_Y (-) : \mathcal{T}op/Y \to \mathcal{T}op$ (see Prop. 10.8(ii)) looks like for $f : X \to Y$ exponentiable in $\mathcal{T}op$. Such right adjoint functor must produce, for every object Z in $\mathcal{T}op$, an object $(P, p : P \to Y)$ in $\mathcal{T}op/Y$ and a morphism $\varepsilon : X \times_Y P \to Z$ which makes

$$\begin{array}{ccccc} Z & \xleftarrow{\varepsilon} & X \times_Y P & \longrightarrow & P \\ & & \downarrow & & \downarrow{\scriptstyle p} \\ & & X & \xrightarrow{f} & Y \end{array} \tag{42}$$

terminal amongst all diagrams

$$\begin{array}{ccccc} Z & \xleftarrow{d} & X \times_Y T & \longrightarrow & T \\ & & \downarrow & & \downarrow{\scriptstyle t} \\ & & X & \xrightarrow{f} & Y \end{array} \tag{43}$$

such that there is a unique morphism $h : T \to P$ with $p \cdot h = t$ and $\varepsilon \cdot (1_X \times h) = d$ (see [19]). Exploiting the bijective correspondence

$$\frac{h : T \to P}{t : T \to Y,\ d : X \times_Y T \to Z}$$

in case $T = 1$, one sees that $P = P(f, Z)$ should have underlying set

$$P = \{(t, d) \,|\, t \in Y,\ d : f^{-1}t \to Z \text{ continuous}\},$$

with projection $p : P \to Y$, and that $\varepsilon = \varepsilon_{f,Z}$ is the evaluation map $(x, d) \mapsto d(x)$ when we realize the underlying set of the pullback $X \times_Y P$ as

$$X \times_Y P = \{(x, d) \,|\, x \in X,\ d : f^{-1}(f(x)) \to Z \text{ continuous}\}.$$

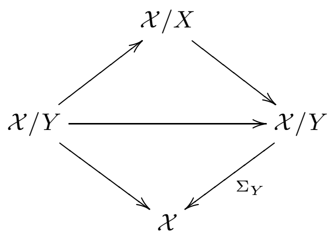
Now, for f exponentiable, there is a coarsest topology on P which makes both p and ε continuous. This topology has been described quite succinctly in terms of ultrafilter convergence in [10].

The case that $f : U \hookrightarrow Y$ is the embedding of an open subspace of Y (hence exponentiable, by Prop. 10.5) deserves special mentioning. In this case, the underlying sets of P and $U \times_Y P$ may be realized as $(U \times Z) + (Y \setminus U)$ and $U \times Z$, respectively, with $p|_{U \times Z}$ and ε projection maps and $p|_{Y \setminus U}$ the inclusion map. Here the coarsest topology on P making p and ε continuous provides the pullback $U \times_Y P = U \times Z$ with the product topology, and it was first described by Pasynkov [41] who called P the *partial product of* Y *over* U *with fibre* Z. He gave various interesting examples for such spaces; for instance, the n-dimensional sphere S^n can be obtained recursively via partial products from the n-dimensional cube I^n and the discrete doubleton D_2, as

$$S^n \cong P(I^n \setminus S^{n-1} \hookrightarrow I^n, D_2).$$

Exercises.

1. Identify the functors described by Prop. 10.8 in the commutative diagram



(44)

2. Show that $Z \mapsto (Y \times Z, Y \times Z \to Y)$ is right adjoint to Σ_Y.
3. Conclude that when we denote by $(-)^{(X,f)}$ the right adjoint of $(X,f)\times(-)$ in $\mathcal{T}op/Y$, then $P(f,Z) \cong (Y \times Z, Y \times Z \to Y)^{(X,f)}$.

11. Remarks on the Tychonoff Theorem and the Stone-Čech compactification

11.1. Axioms giving closures. If $\mathcal{X}$ is $\mathcal{T}op$ or $\mathcal{L}oc$, in addition to (F3)-(F5) the class $\mathcal{F}$ satisfies also the following condition:

(F6) arbitrary intersections of morphisms in $\mathcal{F}\cap\mathcal{M}$ exist and belong to $\mathcal{F}\cap\mathcal{M}$.

Hence, $\mathcal{F}\cap\mathcal{M}$ is *stable under intersection* (see Prop. 1.4(4); we allow the indexing system to be arbitrarily large). Equipped with this additional condition in our category $\mathcal{X}$ satisfying the conditions of 2.1, one introduces the $\mathcal{F}$*-closure*

$$c_X(m) : c_X(M) \to X$$

of a subobject $m : M \to X$ in X by

$$c_X(m) := \bigwedge\{k \in \mathrm{sub}X \mid k \geq m,\ k\ \mathcal{F}\text{-closed}\}.$$

The following properties are easily checked:

1. $m \leq c_X(m)$,
2. $m \leq n \ \Rightarrow\ c_X(m) \leq c_X(n)$,
3. $c_X(m) \cong c_X(c_X(m))$,
4. $f[c_X(m)] \leq c_Y(f[m])$,

for all $m, n \in \mathrm{sub}X$ and $f : X \to Y$ in $\mathcal{X}$. In other words, $c = (c_X)_{X\in\mathcal{X}}$ is an *idempotent closure operator* in the sense of [17]. In what follows we therefore assume that $\mathcal{F}$ is given as in 2.5, that is:

(F7) a morphism f lies in $\mathcal{F}$ if and only if $f[-]$ preserves $\mathcal{F}$-closed subobjects.

The condition for f means equivalently that image commutes with closure, i.e., in assertion 4 above we may write $\cong$ instead of $\leq$ when f is closed. Of course, (F7) implies (F5) when $\mathcal{E}$ is stable under pullback along morphisms in $\mathcal{F}_0 \cap \mathcal{M}$ (see Exercise 4 of 2.1). In addition, we assume the closure operator c to be *hereditary*, that is:

(F8) $c_Y(m) \cong y^{-1}[c_X(y \cdot m)]$ for all $m : M \to Y$, $y : Y \to X$ in $\mathcal{M}$.

Exercises.

1. Verify that $\mathcal{T}op$ and $\mathcal{L}oc$ satisfy conditions (F6)-(F8).
2. Show that (F8) implies that every morphism in $\mathcal{M}$ is $\mathcal{F}$-initial (see 6.5).
3. Prove that when $\mathcal{X}$ satisfies (F6)-(F8), so does $\mathcal{X}/B$.

11.2. Products as inverse limits of finite products. Every product of objects in $\mathcal{X}$ is an inverse limit of its "finite subproducts". This means, given a product

$$X_I = \prod_{i \in I} X_i$$

of objects in $\mathcal{X}$, one has

$$X_I \cong \lim_{F \subseteq I \text{finite}} X_F,$$

where $X_F = \prod_{i \in F} X_i$, with canonical bonding morphisms $X_G \to X_F$ for $F \subseteq G \subseteq I$. It is then natural to expect that the closure operator defined in 11.1 commutes with this limit presentation:

(F9) $c_{X_I}(M) \cong \lim_{F \subseteq I \text{finite}} c_{X_F}(\pi_F[M])$ for all $m : M \to X_I$ in $\mathcal{M}$;

here $\pi_F : X_I \to X_F$ is the canonical morphism. Equivalently, this formula may be written as:

$$c_{X_I}(m) \cong \bigwedge_{F \subseteq I \text{finite}} \pi_F^{-1}[c_{X_F}(\pi_F[m])]$$

(see [9]). Hence, by (F9) the "topological structure" of X_I is completely determined by the structure of the "finite subproducts".

Exercises.

1. Verify that $\mathcal{T}op$ and $\mathcal{L}oc$ satisfy condition (F9).
2. Prove that when $\mathcal{X}$ satisfies (F6)-(F9), so does $\mathcal{X}/B$.

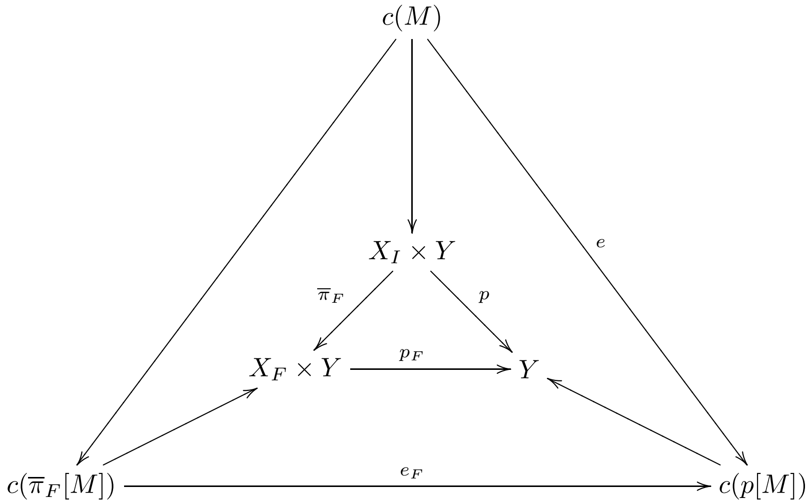
11.3. Towards proving the Tychonoff Theorem. Let us now assume that all X_i $(i \in I)$ are $\mathcal{F}$-compact. In order to show that the projection

$$p : X_I \times Y \to Y$$

is $\mathcal{F}$-closed for every object Y in $\mathcal{X}$, by (F7) we would have to establish

$$p[c(m)] \cong c(p[m]) \qquad (*)$$

for all $m : M \to X_I \times Y$ in $\mathcal{M}$. For that consider the commutative diagram



(45)

Here e, e_F are determined by the projections p, p_F, respectively, and $\overline{\pi}_F = \pi_F \times 1_Y$ represents $X_I \times Y$ as an inverse limit of $X_F \times Y$ ($F \subseteq I$ finite). Formula $(*)$ says precisely that e should be in $\mathcal{E}$. By Theorem 3.3(3), p_F is $\mathcal{F}$-closed for every finite $F \subseteq I$, hence $e_F \in \mathcal{E}$. Now, by (F9),

$$c(M) \cong \lim_{F} \, c(\overline{\pi}_F[M])$$

in $\mathcal{X}$, which implies

$$(c(M), e) \cong \lim_{F} \, (c(\overline{\pi}_F[M]), e_F)$$

in $\mathcal{X}/c(p[M])$.

11.4. The Tychonoff Theorem. We call $\mathcal{E}$ a *surjectivity class* if for some class $\mathcal{P}$ of objects in $\mathcal{X}$ one has $f : X \to Y$ in $\mathcal{E}$ exactly when all maps $\mathcal{X}(P, f) : \mathcal{X}(P, X) \to \mathcal{X}(P, Y)$ ($P \in \mathcal{P}$) are surjective. Note that if $\mathcal{E}$ in $\mathcal{X}$ is a surjectivity class, so is $\mathcal{E}_B$ in $\mathcal{X}/B$. Furthermore, when $\mathcal{X}$ is fibre-determined (see 9.3), then $\mathcal{E}$ is a surjectivity class. In $\mathcal{T}op$, $\mathcal{E}$ is a surjectivity class (take $\mathcal{P} = \{1\}$), but in $\mathcal{L}oc$ it is not (since every surjectivity class is pullback stable).

Theorem (Clementino-Tholen [11]). *In addition to* (F1)-(F5)*, let $\mathcal{X}$ satisfy conditions* (F6)-(F9) *of* 11.1/2*. Then* each *of the following two assumptions makes* $\mathcal{F}$-Comp *closed under direct products in $\mathcal{X}$:*

- (L) *if $(X, \pi_\nu : X \to X_\nu)_\nu$ is an inverse limit in $\mathcal{X}$ and $(e_\nu : X_\nu \to Y)_\nu$ a compatible family of morphisms in $\mathcal{E}$, then the induced morphism $e : X \to Y$ lies in $\mathcal{E}$;*
- (T) *$\mathcal{E}$ is a surjectivity class, and the Axiom of Choice holds true.*

Proof (sketch). By 11.3, assumption (L) makes $\mathcal{F}$-Comp closed under products in $\mathcal{X}$. Regarding assumption (T), it is sufficient to deduce that for the *particular* inverse system of diagram (43), $e_F \in \mathcal{E}$ for all finite $F \subseteq I$ implies e in $\mathcal{E}$. This is shown (using an ordinal induction) in [11] and [9]. □

Examples.

(1) Via (T) one obtains the classical *Tychonoff Theorem* in $\mathcal{T}op$. Note that the validity of the Tychonoff Theorem in $\mathcal{T}op$ is logically equivalent to the Axiom of Choice (see [32]).

(2) While (L) fails in $\mathcal{T}op$, the condition is satisfied in $\mathcal{L}oc$ (see [49], 2.3), and the Theorem gives a choice-free proof for the Tychonoff Theorem in $\mathcal{L}oc$. (See also Chapter II.)

Exercises.

1. Assuming (F1)-(F6) show that for every monic family $(p_i : X \to X_i)_{i\in I}$ (so that for $x, y : P \to X$, $p_i \cdot x = p_i \cdot y$ for all $i \in I$ always implies $x = y$) $X_i \in \mathcal{F}$-Haus implies $X \in \mathcal{F}$-Haus.
2. Show that under the assumptions of the Theorem, $\mathcal{F}$-CompHaus is closed under all small limits in $\mathcal{X}$, if $\mathcal{X}$ has them.

11.5. Products of proper maps. Since all assumptions of Theorem 11.4 are invariant under "slicing", we may apply it to the slices $\mathcal{X}/B$ rather than to $\mathcal{X}$ and obtain:

Corollary. *If $\mathcal{X}$ satisfies* (F1)-(F9) *and condition* (L) or (T) *of Theorem* 11.4*, then the direct product of $\mathcal{F}$-proper ($\mathcal{F}$-perfect) morphisms in $\mathcal{X}$ is again $\mathcal{F}$-proper ($\mathcal{F}$-perfect, respectively).*

Proof. Given $\mathcal{F}$-proper ($\mathcal{F}$-perfect) maps $f_i : X_i \to Y_i$, $i \in I$, the morphism $\prod_{i\in I} f_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i = Y$ may be regarded as an object of $\mathcal{X}/Y$ and is then a product of the objects (P_i, f_i'), where $f_i' : P_i \to Y$ is the pullback of f_i along the projection $Y \to Y_i$, $i \in I$. □

In case $\mathcal{X} = \mathcal{T}op$ the Corollary (which formally generalizes Theorem 11.4) is known as *Frolik's Theorem.*

11.6. Existence of the Stone-Čech compactification. A full subcategory $\mathcal{A}$ of $\mathcal{X}$ is said to be *$\mathcal{F}$-cowellpowered* if every object $X \in \mathcal{A}$ admits only a small set of non-isomorphic $\mathcal{F}$-dense morphisms with domain X and codomain in $\mathcal{A}$; that is: if there is a set-indexed family $d_i : X \to A_i$ $(i \in I)$ of $\mathcal{F}$-dense morphisms in $\mathcal{A}$ such that every $\mathcal{F}$-dense morphism $d : X \to A$ in $\mathcal{A}$ factors as $d = j \cdot d_i$ for some isomorphism j and $i \in I$.

An easy application of *Freyd's General Adjoint Functor Theorem* gives (see [35]):

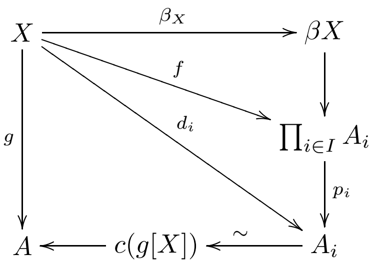
Theorem. *Let $\mathcal{X}$ be small-complete, and let $\mathcal{F}$-*Haus *be $\mathcal{F}$-cowellpowered. Then $\mathcal{F}$-*CompHaus *is reflective in $\mathcal{F}$-*Haus *with $\mathcal{F}$-dense reflexions if and only if $\mathcal{F}$-*CompHaus *is closed under products in $\mathcal{X}$.*

Proof (sketch). The necessity of the condition is clear (see also Exercise 1 of 11.4). For its sufficiency, let $X \in \mathcal{F}$-Haus and consider a representative system $d_i : X \to A_i$ $(i \in I)$ of $\mathcal{F}$-dense morphisms as in the definition of $\mathcal{F}$-cowellpoweredness, with the additional condition that $A_i \in \mathcal{F}$-CompHaus. By (F6), the induced morphism $f : X \to \prod_{i \in I} A_i$ factors through the $\mathcal{F}$-closed subobject

$$\beta X := c(f[X]) \longrightarrow \prod_{i \in I} A_i$$

by a unique morphism $\beta_X : X \to \beta X$, which is $\mathcal{F}$-dense by (F8). Note that βX is $\mathcal{F}$-compact Hausdorff, by Theorem 3.3(3), since $\prod_{i \in I} A_i$ is.

An arbitrary morphism $g : X \to A$ with $A \in \mathcal{F}$-CompHaus factors through the $\mathcal{F}$-dense morphism $g' : X \to c(g[X])$, which must be isomorphic to some d_i and must therefore factor through β_X. The resulting factorization is unique since β_X is $\mathcal{F}$-dense and A is $\mathcal{F}$-Hausdorff. $\square$



(46)

Of course, reflectivity of $\mathcal{F}$-CompHaus in $\mathcal{F}$-Haus gives in particular reflectivity of $\mathcal{F}$-CompHaus in $\mathcal{F}$-Tych and therefore the existence of the *Stone-Čech compactification*, as needed in Section 6.

Exercise. Show that the cardinal number of the underlying set of a Hausdorff space X with dense subset A cannot exceed $2^{2^{\operatorname{card} A}}$. Conclude that the full subcategory $\mathcal{F}$-Haus in $\mathcal{T}op$ is $\mathcal{F}$-cowellpowered, so that Theorem 11.6 in conjunction with the Tychonoff Theorem gives the existence of the classical Stone-Čech compactification of Tychonoff spaces.

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