FACETS OF DESCENT III: MONADIC DESCENT FOR RINGS AND ALGEBRAS

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ABSTRACT. We develop an elementary approach to the classical descent problems for modules and algebras, and their generalizations, based on the theory of monads.

INTRODUCTION

In this third expository article on Descent Theory we return to the original descent problem for modules as studied by Grothendieck [7]. Formulated in the language of monads, which we will use in this paper, Grothendieck's theorem says:

> *For a homomorphism $p : R \to S$ of commutative rings, the extension-of-scalars functor $S \otimes_R (-) : R$-$\mathbf{mod} \to S$-$\mathbf{mod}$ is comonadic whenever p makes S a faithfully flat R-module.*

It is our principal goal in this paper to present what we think is the most elegant and simple proof of a much stronger result in which the sufficient condition for comonadicity is replaced by one that is also necessary:

> *For a homomorphism $p : R \to S$ of commutative rings, the extension-of-scalars functor $S \otimes_R (-) : R - \mathbf{mod} \to S - \mathbf{mod}$ is comonadic if and only if p is a pure monomorphism of R-modules.*

We actually derive this result from a more general theorem for non-commutative rings, and we also show that the principal theorem remains valid when modules get traded for various sorts of algebras, unital or not, associative or not, commutative or not, Lie, Jordan, differential, etc.

The principal theorem is not new but has a somewhat complicated history that we recall here to the best of our knowledge. It seems that it first appeared in Olivier's paper [19] where, however, no convincing indication of proof was given. Afterwards for many years people continued to use Grothendieck's original weaker theorem, thus just using the sufficient condition that S be a

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faithfully flat R-module; a standard reference for ring-theorists is the monograph [12] by Knus and Ojanguren, and for category theorists Borceux's book [2] (although we note that Borceux requires "pure plus flat" instead of "faithfully flat", see Theorem 4.7.8 in [2]). Grothendieck's theorem has an obvious monadic connection, since it follows quite easily from the fact that in the presence of coequalizers a right adjoint functor reflecting isomorphisms and preserving all coequalizers is monadic. It seems, however, that neither Grothendieck himself nor anybody of his school ever actively used the monadic approach to descent, although it was described explicitly by Bénabou and Roubaud [1] fairly soon after monads had become fashionable in the mid to late 1960s. Much later, after the publication of the important paper [11], which establishes a constructive localic analogue of the characterization of effective descent morphisms of commutative rings, the principal theorem of this paper was normally referred to (at least by topos-theorists) as an unpublished result of Joyal and Tierney. The first proof the authors actually saw was briefly shown (to the first author) by Makkai in 1997[3]. Later Mesablishvili, a former Ph. D. student of the first author, while working on the Galois theory of schemes discovered another proof which uses the "homological-algebraic" approach to purity by means of the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})$ (see [16]).

The proof we give here very much follows Mesablishvili's argumentation, the only major difference being that we explicitly state its category-theoretic backbone (see Theorem 2.3). In fact, this approach lends itself perfectly well to be used for non-commutative rings as well, a goal that we had in mind for many years and which the first author had discussed explicitly with various interested people, including Caenepeel. Hence we were glad to see that he proved the non-commutative variant of the principal theorem in [5], but the presentation appears to be a bit more cumbersome than the one we give in this paper, simply because of a less clear isolation of the categorical tools from the ring- and module-theoretic context.

In order to achieve our principal goal, at the beginning of the paper we must take the reader back to what appears to be just another twist of the many existing versions of Beck's monadicity criterion, which then gets applied to the ring- and module theoretic context. We wish to emphasize, however, that this is not a one-way street. Monads and their algebras on the one hand and rings and their modules on the other hand have a common categorical home that is most generally described in the context of a monoidal category acting on another category. In fact, they are both given by monoids and their actions in the general context. But these (very general) actions are in fact algebras for a monad, thus exhibiting monads and their algebras not just as a special instance of the general objects but in fact describing all of them, even in the most general context. Although we can allude to these facts only briefly in Section 3 of this paper (and must refer the reader to the forthcoming paper [4] for further details), we feel that these facts and our method of proof for the principal theorem show the importance of fairly simple categorical tools that, however, may become very powerful when applied in the right context.

A final comment: Unlike Nuss [18], in this paper we avoid the term "noncommutative descent" which for us, according to an observation of Caenepeel, is nothing but comonadicity of the extension-of-scalars functor. However, the way Caenepeel presents coring theory probably suggests that it would be worthwhile to develop noncommutative and in fact monoidal-categorical descent theory as such, with not only monadic but all other facets, a project left for future work.

[3] In his work on categorical logic Makkai [15] discovered a fundamental connection between descent and definability. Specifically, his theorem that every monomorphism of Boolean algebras is an effective descent morphism, implies a definability theorem for propositional (classical) logic.

We are indebted to Bachuki Mesablishvili, for detecting an error in an earlier version of Theorem 5.3, and for many other helpful comments.

1. Forks and contractible pairs

1.1. A fork in a given category is a diagram of the form

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z \tag{1.1}$$

with $hf = hg$; such a fork is said to be *split* by a pair (i, j) of morphisms $i : Z \to Y$ and $j : Y \to X$ if $hi = 1_Z$, $fj = 1_Y$, and $gj = ih$. One also says that the diagram of five morphisms f, g, h, i, j above is a *split fork*, or that (i, j) is a *splitting* for the fork (1.1). Let us recall:

PROPOSITION. *(a) If f, g, h, i, j form a split fork, then diagram (1.1) is an* absolute coequalizer diagram *(i.e. a coequalizer diagram preserved by every functor).*

(b) A fork (1.1) has a splitting if and only if it is a coequalizer diagram and the pair (f, g) is contractible, *i.e. there exists a morphism $j : Y \to X$ with $fj = 1_Y$ and $gjf = gjg$.*

Accordingly, split forks are also called *split coequalizer diagrams.*

1.2. **OBSERVATION.** Every retract of a contractible pair is itself contractible. That is, if

$$\begin{array}{ccc} X & \underset{g}{\overset{f}{\rightrightarrows}} & Y \\ s\uparrow\downarrow t & & u\uparrow\downarrow v \\ X' & \underset{g'}{\overset{f'}{\rightrightarrows}} & Y' \end{array} \tag{1.2}$$

is a diagram with $vf = f't$, $vg = g't$, $fs = uf'$, $gs = ug'$, $ts = 1_{X'}$, $vu = 1_{Y'}$, then contractibility of (f, g) implies the same for (f', g'). Indeed, given $j : Y \to X$ as in Proposition 1.1(b), just take $j' = tju : Y \to X$.

1.3. **COROLLARY.** *Let $\mathbf{C}$ and $\mathbf{D}$ be arbitrary categories, $\tau : \Phi \to \Psi$ a split epimorphism of functors $\Phi, \Psi : \mathbf{C} \to \mathbf{D}$, and (f, g) a pair of parallel morphisms in $\mathbf{C}$. Then with $(\Phi(f), \Phi(g))$ also $(\Psi(f), \Psi(g))$ is contractible.*

2. General monadicity in the presence of coequalizers

2.1. Let us first recall the ordinary form of Beck's monadicity criterion (see, for exmple, [2] or [14]):

MONADICITY THEOREM IN THE PRESENCE OF COEQUALIZERS. *Let $\mathbf{A}$ and $\mathbf{X}$ be categories with coequalizers. A functor $U : \mathbf{A} \to \mathbf{X}$ is monadic if and only if the following conditions hold:*

(a) U has a left adjoint;
(b) U reflects isomorphisms;
(c) U preserves coequalizers of those pairs (f, g) for which $(U(f), U(g))$ is contractible.

2.2. Let $U : \mathbf{A} \to \mathbf{X}$ be a functor with left adjoint F and counit $\varepsilon : FU \to 1_{\mathbf{A}}$. If ε is a split epimorphism, then condition 2.1(c) can be omitted. Indeed, if $(U(f), U(g))$ is contractible, then so is $(FU(f), FU(g))$, and then Corollary 1.3 applied to $\varepsilon = \tau$ tells us that also (f, g) is contractible, and therefore its coequalizer is preserved by every functor (see Proposition 1.1). In other words, condition 2.1(c) becomes *trivial* simply because U *reflects contractibility.* Moreover, condition 2.1(b) also becomes trivial since having $\zeta : 1_{\mathbf{A}} \to FU$ with $\varepsilon\zeta = 1$ yields

$$(U(f : A \to B) \text{ invertible}) \Longrightarrow (f \text{ invertible with } f^{-1} = \varepsilon_A F(U(f)^{-1})\zeta_B)$$

Thus we obtain:

SPLIT MONADICITY THEOREM IN THE PRESENCE OF COEQUALIZERS. *Let* $\mathbf{A}$ *and* $\mathbf{X}$ *be categories with coequalizers. A functor* $U : \mathbf{A} \to \mathbf{X}$ *is monadic if the following conditions hold:*

(a) U *has a left adjoint* F*;*

(b) the counit $FU \to 1_{\mathbf{A}}$ *is a split epimorphism.*

The Split Monadicity Theorem follows also from a more general theorem proved by Pare [20] which asserts not just monadicity of the functor U under the given sufficient conditions, but even of UV for every composable monadic functor V. But for the application given in 5.1 below, we need a different type of generalization of 2.2 in which the sufficient conditions are again replaced by conditions that are also necessary for monadicity of U, as follows.

2.3. Let us consider a more general situation, where not just the counit $\varepsilon : FU \to 1_{\mathbf{A}}$ itself, but its (horizontal) composite $H\varepsilon : HFU \to H$ with some functor H is a split epimorphism. Then instead of having (f, g) contractible, we will only require $(H(f), H(g))$ to be contractible. Now, in order to conclude that U preserves the coequalizer of (f, g) we have to "connect" U and H. The following theorem gives such a "connection":

THEOREM. *Let* $\mathbf{A}$ *and* $\mathbf{X}$ *be categories with coequalizers. A functor* $U : \mathbf{A} \to \mathbf{X}$ *is monadic if and only if the following conditions hold:*

(a) U *has a left adjoint* F*;*

(b) U *reflects isomorphisms;*

(c) there exists a commutative diagram (possibly only up to an isomorphism)

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{U} & \mathbf{X} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H'} \\ \mathbf{B} & \xrightarrow[U']{} & \mathbf{Y} \end{array} \qquad (2.1)$$

of functors such that

(c₁) $H\varepsilon : HFU \to H$ *is a split epimorphism;*

(c₂) H *preserves coequalizers of those pairs* (f, g) *for which* $(U(f), U(g))$ *is contractible;*

(c₃) H' *reflects isomorphisms.*

Proof. "If": By Theorem 2.1, we have to check only condition 2.1(c) on U. Let us write the coequalizer diagram of (f, g) as the fork (1.1), and consider its images under H, $U'H = H'U$, and U. We assume that (f, g) is as in 2.1(c), i.e. that $(U(f), U(g))$ is contractible, and so the H-image of (1.1) is a coequalizer diagram by (c_2). Moreover, this H-image is a split coequalizer diagram, since $(H(f), H(g))$ is contractible, being a retract of $(HFU(f), HFU(g))$ (that is, we are applying Corollary 1.3 to $\tau = H\varepsilon$). Since split coequalizer diagrams are preserved by every functor, we conclude that the $U'H$-image of (1.1) is a coequalizer diagram, and the same is true for the $H'U$-image since $U'H = H'U$. Now consider the coequalizer diagram

$$U(X) \rightrightarrows U(Y) \longrightarrow C \tag{2.2}$$

of $(U(f), U(g))$. Note that since $(U(f), U(g))$ is contractible, the H'-image of (2.2) is a coequalizer diagram. Now, since also the $H'U$-image of (1.1) is a coequalizer diagram, we can conclude that the H'-image of the canonical morphism $C \to U(Z)$ is an isomorphism. Since H' reflects isomorphisms, this implies that the U-image of (1.1) is a coequalizer diagram, as desired.

"Only if": Just take $H = U$ and $H' = U' = 1_{\mathbf{X}}$ and apply Theorem 2.1. □

2.4. **REMARK.**

(a) As we see from the "only if" part of its proof, Theorem 2.3 *contains* Theorem 2.1. On the other hand it also contains the Split Monadicity Theorem 2.2: in fact, just consider $H = 1_{\mathbf{A}}$, $H' = 1_{\mathbf{X}}$, and $U' = U$. Note also that one could present Theorem 2.3 as a corollary of the fact that conditions 2.3(a) and 2.3(c) imply condition 2.1(c).

(b) We do not really need to require the existence of all coequalizers, of course; for instance, we could only require the existence of reflexive ones since all the needed coequalizers have that property.

2.5. Theorem 2.3 uses diagram (2.1) to describe a necessary and sufficient condition for monadicity of the functor U. The following easy lemma gives another such condition:

LEMMA. *Let* $\mathbf{A}$ *and* $\mathbf{X}$ *be categories with coequalizers. A functor* $U : \mathbf{A} \to \mathbf{X}$ *is monadic if and only if the following conditions hold:*

(a) U *has a left adjoint* F*;*

(b) there exists a commutative diagram (possibly only up to an isomorphism)

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{U} & \mathbf{X} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H'} \\ \mathbf{B} & \xrightarrow[U']{} & \mathbf{Y} \end{array} \tag{2.3}$$

of functors such that

(b_1) U' *is monadic (or at least reflects isomorphisms and preserves coequalizers of those pairs* (f, g) *for which* $(U'(f), U'(g))$ *is contractible);*

(b_2) H *preserves all coequalizers;*

(b_3) H' *reflects isomorphisms;*

Proof. "If": Again, by Theorem 2.1, we have to check only condition 2.1(c) on U. Let (f, g) be a pair of parallel arrows in $\mathbf{A}$, for which $(U(f), U(g))$ is contractible. Then:

(i) $(H'U(f), H'U(g))$ is contractible, and therefore so is $(U'H(f), U'H(g))$;

(ii) therefore U' preserves the coequalizer of $(H(f), H(g))$;

(iii) since H preserves the coequalizer of (f, g), we conclude that $U'H$ preserves the coequalizer of (f, g), and hence the same is true for $H'U$;

(iv) since $(U(f), U(g))$ is contractible, H' preserves its coequalizer (in the case that H' is comonadic, of course we would not need contractibility here);

(v) since H' reflect isomorphisms, (iii) and (iv) imply that U preserves the coequalizer of (f, g), which completes the proof.

"Only if": Just consider $H = 1_{\mathbf{A}}$, $H' = 1_{\mathbf{X}}$, and $U = U'$. □

3. Monads as generalized rings

3.1. We begin this section with a brief discussion involving abstract monoid actions, where a monoid M in a monoidal category $\mathbf{C}$ acts on objects in a category $\mathbf{X}$ equipped with a "hyperaction" of $\mathbf{C}$. However, the only purpose of that discussion is to explain our intention to consider rings and modules as a special case of monads and algebras. Hence the readers not familiar with monoidal categories can safely ignore 3.1 and 3.2 and continue reading beginning from subsection 3.3.

What we mean by a hyperaction is actually a *lax* action, i.e. a functor $\bullet : \mathbf{C} \times \mathbf{X} \to \mathbf{X}$ satisfying the usual action axioms up to specified natural morphisms $\gamma : A \bullet (B \bullet X) \to (A \otimes B) \bullet X$ and $\theta : X \to I \bullet X$ (where I is the identity object for the tensor product $\otimes$ in $\mathbf{C}$) satisfying suitable coherence conditions. Such an action is called *strong* or *strict* if the aforementioned morphisms are isomorphisms or identity morphisms, respectively. Given a lax action $\bullet : \mathbf{C} \times \mathbf{X} \to \mathbf{X}$ and a monoid $R = (R, e, m)$ in $\mathbf{C}$, the category $\mathbf{X}^R$ of R-actions in $\mathbf{X}$ is defined as the category of pairs (X, h), where $h : R \bullet X \to X$ makes the diagram

$$\begin{array}{ccccccc} R \bullet (R \bullet X) & \xrightarrow{\gamma} & (R \otimes R) \bullet X & \xrightarrow{m \bullet 1} & R \bullet X & \xleftarrow{(e \bullet 1)\theta} & X \\ \downarrow{\scriptstyle 1 \bullet h} & & & & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle 1} & \\ R \bullet X & & \xrightarrow[h]{\hspace{4em}} & & X & & \end{array} \tag{3.1}$$

commute (see [4] for more details; the general idea to use actions of a monoidal category has been around since the 1960s [6] and appears explicitly in the form of strong (right) actions in Pareigis' paper [21]).

The following table recalls a standard collection of examples:

	$\mathbf{C} = (\mathbf{C}, \otimes)$	$\mathbf{X}$	$A \bullet X$	monoids in $\mathbf{C}$	R-actions in $\mathbf{X}$
(a)	(Sets,$\times$)	$\mathbf{X} = \mathbf{C}$	$A \times X$	ordinary monoids	R-sets
(b)	(Topological Spaces,$\times$)	$\mathbf{X} = \mathbf{C}$	$A \times X$	topological monoids	topological spaces equipped with a continuous R-action
(c)	(Abelian Groups,$\otimes$)	$\mathbf{X} = \mathbf{C}$	$A \otimes X$	rings[4]	R-modules[5]
(d)	(K-modules, $\otimes_K$), where K is a commutative ring (with 1)	$\mathbf{X} = \mathbf{C}$	$A \otimes_K X$	K-algebras (associative and with 1)	R-modules
(e)	(Abelian Monoids,$\otimes$)	$\mathbf{X} = \mathbf{C}$	$A \otimes X$	semirings	R-semimodules
(f)	(Complete Semilattices,$\otimes$)	$\mathbf{X} = \mathbf{C}$	$A \otimes X$	quantales	R-modules
(g)	(O-Graphs,$\times_O$) (see [14], p. 171)	(Sets$\downarrow O$)	$A \times_O X$	categories whose set of objects is O	functors from R to sets
(h)	$(\mathrm{End}(\mathbf{X}), \circ)$, the category of endofunctors of an arbitrary category $\mathbf{X}$; $\circ$ is the composition of endofunctors	the same $\mathbf{X}$	$A(X)$	monads on $\mathbf{X}$	R-algebras
(i)	$(\mathbf{C}, +)$, where $\mathbf{C}$ is an arbitrary category with finite coproducts	$\mathbf{X} = \mathbf{C}$	$A + X$	every object in $\mathbf{C}$ has a unique monoid structure	pairs (X, h), where $h : R \to X$ is a morphism in $\mathbf{C}$

Table 1

Note also that the situation (a) has two well-known generalizations: we could keep $\mathbf{C}$ = Sets but take $\mathbf{X}$ to be an arbitrary category with coproducts, and then take $A \bullet X$ to be the A-indexed coproduct of X with itself (which of course is the same as $A \times X$ when $\mathbf{X}$ = Sets); or, we could take $\mathbf{C} = \mathbf{X}$ to be an arbitrary category with finite products, and then keep $A \bullet X = A \times X$ (as in the situation (b)). In the first case, an R-action on an object X in $\mathbf{X}$ is a monoid homomorphism $R \to \mathrm{End}(X)$. In the second case R becomes an internal monoid in $\mathbf{C} = \mathbf{X}$, and, unless $\mathbf{X}$ was cartesian closed, an R-action can only be presented as a morphism $R \times X \to X$ making diagram (3.1) (with $\times$ instead of $\otimes$ and $\bullet$) commute. Similar generalizations of the situation (g) would take us from ordinary categories to internal and enriched ones.

3.2. We see that algebras over monads are (very special) monoid actions (see row (g) in Table 1 in 3.1). However the converse proposition is also true since:

(a) a lax action $\mathbf{C} \times \mathbf{X} \to \mathbf{X}$ of a monoidal category $\mathbf{C}$ can be presented as a monoidal functor $\mathbf{C} \to \mathrm{End}(\mathbf{X})$;
(b) a monoid R in $\mathbf{C}$ can be presented as a monoidal functor $\mathbf{1} \to \mathbf{C}$;
(c) the composite $\mathbf{1} \to \mathbf{C} \to \mathrm{End}(\mathbf{X})$ determines a monoid in $\mathrm{End}(\mathbf{X})$ and hence a monad on $\mathbf{X}$;
(d) the algebras over that monad are the same as R-actions in $\mathbf{X}$ with respect to the lax action $\mathbf{C} \times \mathbf{X} \to \mathbf{X}$ above.

Having this "logical equivalence" in mind, we could equally well present our generalized descent theory of modules in any of the two languages, "monoidal" or "monadic". We will choose the

[4] "Ring" always means "ring with 1" (unless we say "without 1").
[5] "Module" always means "left module" (unless we say "right module").

second one, which is more convenient for our purposes since we have already used monads in [8] and [9].

3.3. We briefly recall one of the most fundamental examples of a monad (see, for example [14]). Every ring R determines a monad on the category **Ab** of abelian groups, whose algebras are all R-modules. This can be deduced from 3.1(c) and 3.2, or directly as follows:

(a) The monad determined by R is $(R\otimes(-),\eta^R,\mu^R)$, where $(\eta^R)_X : X\to R\otimes X$ and $(\mu^R)_X : R\otimes R\otimes X \to R\otimes X$ are defined by $(\eta^R)_X(x) = 1\otimes x$ and $(\mu^R)_X(r\otimes r'\otimes x) = rr'\otimes x$ respectively (for every abelian group X).
(b) An $(R\otimes(-),\eta^R,\mu^R)$-algebra is a pair (X,h), where X is an abelian group and $h : R\otimes X\to X$ a group homomorphism making the diagram

$$\begin{array}{ccccc} R\otimes R\otimes X & \xrightarrow{(\mu^R)_X} & R\otimes X & \xleftarrow{(\eta^R)_X} & X \\ {\scriptstyle 1\otimes h}\downarrow & & {\scriptstyle h}\downarrow & \swarrow{\scriptstyle 1} & \\ R\otimes X & \xrightarrow[h]{} & X & & \end{array} \tag{3.2}$$

commute, and it is easy to see that such an h is the same as an R-module structure on X.

3.4. We will need some details of the change-of-base constructions for a morphism $p : R\to S$ of monads. Instead of checking them with long routine calculations, we display, on the table below, how the monadic situation imitates the classical one for rings:

For a ring homomorphism $p : R \to S$:	For a morphism $p : R = (R, \eta^R, \mu^R) \to S = (S, \eta^S, \mu^S)$ of monads on a category $\mathbf{X}$ with reflexive coequalizers preserved by S and reflexive equalizers:
The restriction-of-scalars functor $(-)_p : S\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$ has a left adjoint, namely the extension-of-scalars $S \otimes_R (-)$, which can be defined via the reflexive coequalizer diagram $$S \otimes R \otimes X \underset{s\otimes r\otimes x \mapsto s\otimes rx}{\overset{s\otimes r\otimes x \mapsto sp(r)\otimes x}{\rightrightarrows}} S \otimes X \longrightarrow S \otimes_R X$$ for each R-module X.	The induced functor $\mathbf{X}^p : \mathbf{X}^S \to \mathbf{X}^R$ has a left adjoint L^p, called the *change-of-base functor*, which can be defined via the reflexive coequalizer diagram (in $\mathbf{X}$ and in $\mathbf{X}^S$) $$SR(X) \underset{S(h)}{\overset{(\mu_S(Sp))_X}{\rightrightarrows}} SX \longrightarrow L^p(X,h)$$ for each R-algebra (X,h); the canonical morphism $S(X) \to L^p(X,h)$ will be denoted by $\pi_{(X,h)}$.
The unit η^p of the adjunction has its components $(\eta^p)_X$ defined as composites $$X \xrightarrow{x \mapsto 1\otimes x} S \otimes X \longrightarrow S \otimes_R X$$	The unit η^p of the adjunction has its components $(\eta^p)_{(X,h)}$ defined as composites $$X \xrightarrow{(\eta^S)_X} S(X) \xrightarrow{\pi_{(X,h)}} L^p(X,h)$$
The unit η^p is a split monomorphism if and only if there exists an (additive) group homomorphism $q : S \to$ R with $qp = 1$, $q(p(r)s) = rq(s)$ and $q(sp(r)) = q(s)r$, i.e. if and only if p is a split monomorphism of (R,R)-bimodules.	The unit η^p is a split monomorphism if and only if there exists a natural transformation $q : S \to R$ with $qp = 1$, $q(\mu^S(pS)) = \mu^R(qR)$ and $q(\mu^S(Sp)) = \mu^R(Rq)$.
Moreover, there is a natural bijection between all such q's and all splittings θ for η^p: given a $q : S \to R$, the X-component θ_X of the corresponding θ is the composite $$S \otimes_R X \xrightarrow{q\otimes_R X} R \otimes_R X \cong X$$ or, equivalently, the unique map making the right-hand square in the diagram $$\begin{array}{ccccc} S\otimes R\otimes X & \rightrightarrows & S\otimes X & \longrightarrow & S\otimes_R X \\ \downarrow{\scriptstyle q\otimes R\otimes X} & & \downarrow{\scriptstyle q\otimes X} & & \downarrow{\scriptstyle \theta_X} \\ R\otimes R\otimes X & \rightrightarrows & R\otimes X & \longrightarrow & X \end{array}$$ commute;	Moreover, there is a natural bijection between all such q's and all splittings θ for η^p: given a $q : S \to R$, the (X,h)-component $\theta_{(X,h)}$ of the corresponding θ is the unique map making the right-hand square in the diagram $$\begin{array}{ccccc} SR(X) & \underset{S(h)}{\overset{(\mu^S(Sp))_X}{\rightrightarrows}} & S(X) & \longrightarrow & L^p(X,h) \\ \downarrow{\scriptstyle q_{R(X)}} & & \downarrow{\scriptstyle q_X} & & \downarrow{\scriptstyle \theta_{(X,h)}} \\ RR(X) & \underset{R(h)}{\overset{(\mu^R)_X}{\rightrightarrows}} & R(X) & \longrightarrow & X \end{array}$$ commute;
conversely, given a splitting θ for η^p, the corresponding q is the composite $$S \cong S \otimes_R R \xrightarrow{\theta_R} R\ .$$	conversely, given a splitting θ for η^p, the X-component q_X of the corresponding q is the composite $$S(X) \cong L^p(R(X), (\mu^R)_X) \xrightarrow{\theta_{(R(X),(\mu^R)_X)}} R(X).$$

Table 2

4. Split comonadicity

4.1. Combining the Split Monadicity Theorem 2.2 and Remark 2.4(b) with the characterization of splitting(s) for η^p given in the right-hand column of Table 2, we obtain:

THEOREM. *The change-of-base functor $\mathbf{X}^R \to \mathbf{X}^S$ induced by a morphism $p : R = (R, \eta^R, \mu^R) \to S = (S, \eta^S, \mu^S)$ of monads on $\mathbf{X}$ is comonadic whenever*

(a) $\mathbf{X}$ has reflexive equalizers and reflexive coequalizers, and S preserves reflexive coequalizers;

(b) *there exists a natural transformation* $q : S \to R$ *with* $qp = 1$, $q(\mu^S(pS)) = \mu^R(qR)$ *and* $q(\mu^S(Sp)) = \mu^R(Rq)$.

4.2. We could also use the left-hand column of Table 2 to obtain a sufficient condition for comonadicity of the functor $S \otimes_R (-) : R\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$; or, we could deduce it from Theorem 4.1, or from the following intermediate result:

COROLLARY. *Let* $\mathbf{X} = (\mathbf{X}, \otimes)$ *be a monoidal category and* $p : R \to S$ *a morphism of monoids in* X. *The extension-of-scalars functor* $S \otimes_R (-) : \mathbf{X}^R \to \mathbf{X}^S$ *is comonadic whenever*

(a) $\mathbf{X}$ *has reflexive equalizers and reflexive coequalizers, and the functor* $S \otimes (-) : \mathbf{X} \to \mathbf{X}$ *preserves reflexive coequalizers (which is always the case when* $(\mathbf{X}, \otimes)$ *is a monoidal-closed category with reflexive equalizers),*
(b) *there exists a morphism* $q : S \to R$ *in* $\mathbf{X}$ *with* $qp = 1$, *which is a morphism of left and right* R*-actions at the same time.*

Note that this fact was independently observed, in a (essentially) more general situation by Mesablishvili (unpublished), and in the case $\mathbf{X}$ = Sets by Laan [13], together with several related results.

4.3. Every *commutative algebraic theory* (see e.g. [2, Section 3.10]) has its category of models monoidal closed, and hence yields an example for Corollary 4.2, where condition 4.2(a) holds automatically. Examples of such monoidal closed categories are:

(a) Sets; the monoids here are ordinary monoids.
(b) Pointed sets; the monoids are monoids with zero.
(c) Commutative monoids; the monoids are semirings (with 1).
(d) Abelian groups; the monoids are rings, and this is the "classical" case of row (c) in Table 1 and of the left-hand column of Table 2. We will return to this case again; at the moment let us just point out that what we get from Corollary 4.2 here follows also from what we get for semirings, simply because every semimodule over a ring is a module.

4.4. Another interesting example is provided by the monoidal-closed category of complete semilattices. Its monoids are called *quantales*, and among these there are the *locales*, a fundamental notion of "pointless" topology (see, for example, [3]). Recall that a locale can be defined as a complete Heyting algebra, and a morphism of locales is the same as a $\bigvee$-complete homomorphism of lattices in the opposite direction. For a morphism $p : S \to R$ (which is in fact a map from R to S as above!) of locales, the conditions required on q in 4.2 become

$$qp = 1 \quad \text{and} \quad q(p(r) \wedge s) = r \wedge q(s). \tag{4.1}$$

Recall that p is an *open surjection* in the sense of the theory of locales if and only if it satisfies (4.1), with q being the left adjoint of p (see [3, Section 1.6]). Since condition (4.1) is also satisfied whenever p is a split monomorphism of complete lattices (=split epimorphism of locales), it is certainly weaker than the open-surjection condition. On the other hand (4.1) is stronger than what Plewe [22] requires for p to be a *triquotient* map of locales.

5. Comonadicity for ordinary modules

5.1. According to Corollary 4.2 (see also 4.3(d)), a ring homomorphism $p : R \to S$ renders the extension-of-scalars functor $S \otimes_R (-) : R\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$ comonadic whenever p is a split monomorphism of (R, R)-bimodules. In this section we will use Theorem 2.3 to obtain a stronger result, which however cannot be extended to abstract monoidal categories.

For a ring homomorphism $p : R \to S$, let us choose the data involved in Theorem 2.3 as follows:

- $\mathbf{A} = (R\text{-}\mathbf{mod})^{\mathrm{op}}$, the opposite category of R-modules;
- $\mathbf{X} = (S\text{-}\mathbf{mod})^{\mathrm{op}}$;
- $U : \mathbf{A} \to \mathbf{X}$ the dual of the extension-of-scalars functor $S \otimes_R (-) : R\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$;
- $F : \mathbf{X} \to \mathbf{A}$ is thus the dual of the restriction-of-scalars functor;
- $\mathbf{B} = \mathbf{mod}\text{-}R$, the category of right R-modules;
- $\mathbf{Y} = \mathbf{Ab}$, the category of abelian groups;
- $H : \mathbf{A} \to \mathbf{B}$ defined by $H(A) = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ with the right R-module structure on $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ defined by $(hr)(a) = h(ra)$; of course the only reason for using $\mathbb{Q}/\mathbb{Z}$ here is that it is an injective cogenerator in $\mathbf{Ab}$;
- $U' : \mathbf{B} \to \mathbf{Y}$ defined by $U'(A) = \mathrm{Hom}_R(S, A)$ (considering $\mathrm{Hom}_R(S, A)$ just as an abelian group of course);
- $H' : \mathbf{X} \to \mathbf{Y}$ defined by $H(X) = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$.

We observe:

(a) For each R-module A, there are canonical isomorphisms

$$\mathrm{Hom}_R(S, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})) \approx \mathrm{Hom}_{\mathbb{Z}}(S \otimes_R A, \mathbb{Q}/\mathbb{Z}) \approx \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})), \tag{5.1}$$

where however the first Hom_R is used for the right R-module homomorphisms, while the second is used for the left ones, assuming that $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is considered as an R-module via $(rh)(s) = h(sr)$. The first isomorphism tells us that in this case diagram (2.1) indeed commutes up to an isomorphism.

(b) Since the forgetful functor from the category of modules (over any ring) to $\mathbf{Ab}$ is exact and reflects isomorphisms, and since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in $\mathbf{Ab}$, the functors H and H' preserve all coequalizers (in fact all limits and all finite colimits!) and reflect isomorphisms.

(c) If

$$\mathrm{Hom}_{\mathbb{Z}}(p, \mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}) \tag{5.2}$$

is a split epimorphism of(R, R)-bimodules ($= R \otimes R^{op}$-modules), then each

$$\mathrm{Hom}_{\mathbb{Z}}(\varepsilon_A, \mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}_{\mathbb{Z}}(S \otimes_R A, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \tag{5.3}$$

is a split epimorphism of right R-modules and, moreover, the splitting is natural in A. This follows from the fact that (up to a natural isomorphism) (5.3) can be rewritten as

$$\mathrm{Hom}_{\mathbb{Z}}(p \otimes_R A, \mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}_{\mathbb{Z}}(S \otimes_R A, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R \otimes_R A, \mathbb{Q}/\mathbb{Z}), \tag{5.4}$$

and then as

$$\mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(p, \mathbb{Q}/\mathbb{Z}))) : \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})) \to \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})). \tag{5.5}$$

Thus, whenever $S \otimes_R (-)$ reflects isomorphisms and (5.2) is a split epimorphism of (R, R)-bimodules, all assumptions of Theorem 2.3 are indeed satisfied.

5.2. Together with the observations 5.1(a)–(c), Theorem 2.3 yields:

LEMMA. *For a ring homomorphism $p : R \to S$, the extension-of-scalars functor $S \otimes_R (-) : R$-$\mathbf{mod} \to S$-$\mathbf{mod}$ is comonadic whenever it reflects isomorphisms and* $\mathrm{Hom}_{\mathbb{Z}}(p, \mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ *is a split epimorphism of (R, R)-bimodules.*

5.3. Let us see what it means for the homomorphism $\mathrm{Hom}_{\mathbb{Z}}(p, \mathbb{Q}/\mathbb{Z})$ involved in Lemma 5.2 to be a split epimorphism of (R, R)-bimodules. By the Yoneda Lemma this property holds if and only if

$$\mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(p, \mathbb{Q}/\mathbb{Z})) : \mathrm{Hom}_{R\otimes R^{\mathrm{op}}}(A, \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})) \to \mathrm{Hom}_{R\otimes R^{\mathrm{op}}}(A, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))$$

is surjective for every (R, R)-bimodule A. Now, the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(A \otimes_{R\otimes R^{\mathrm{op}}} S, \mathbb{Q}/\mathbb{Z}) \approx \mathrm{Hom}_{R\otimes R^{\mathrm{op}}}(A, \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})), \tag{5.6}$$

formed similarly to the second isomorphism in (5.1), tells us that the desired split property holds if and only if $A \otimes p : A \otimes_{R\otimes R^{\mathrm{op}}} R \to A \otimes_{R\otimes R^{\mathrm{op}}} S$ is a monomorphism for every R-module A, i.e. if and only if $p : R \to S$ is a *pure monomorphism* of (R, R)-bimodules.

On the other hand, it is a well-known general fact that a right adjoint functor reflects isomorphisms if and only if all components of the counit of adjunction are extremal epimorphisms. Since the extremal epimorphisms in $(R\text{-}\mathbf{mod})^{\mathrm{op}}$ are precisely the monomorphisms in R-**mod**, and since in our case these components can be presented as $p \otimes_R A : R \otimes_R A \to S \otimes_R A$, we conclude that the reflection of isomorphisms in Lemma 5.2 is equivalent to the purity of p as a right R-module homomorphism implies the comonadicity of $S \otimes_R (-)$. Moreover, since comonadicity always implies reflection of isomorphisms, we obtain:

THEOREM. *For a ring homomorphism $p : R \to S$, we have (a)⇒(b)⇒(c), with*

(a) p is a pure monomorphism of (R, R)-bimodules;
*(b) the extension-of-scalars functor $S \otimes_R (-) : R$-**mod** $\to S$-**mod** is comonadic;*
(c) p is a pure monomorphism of right R-modules.

This is a (corrected version of a) theorem originally proved by Caenepeel [5]; he describes his arguments as an adoption of Mesablishvili's arguments [16] to the noncommutative case. A simplified version of Theorem 5.3 is obtained from its proof in the commutative case, as follows.

5.4. **COROLLARY.** *For a homomorphism $p : R \to S$ of commutative rings, the following conditions are equivalent:*

(a) p is a pure monomorphism of R-modules;
*(b) the extension-of-scalars functor $S \otimes_R (-) : R$-**mod** $\to S$-**mod** is comonadic.*

6. From modules to algebras

6.1. Lemma 2.5 tells us that, for commutative R and S, the comonadicity of the extension-of-scalars functor for modules implies the same property for algebras. Indeed, we can take diagram (2.3) to be

$$\begin{array}{ccc} (R\text{-}\mathbf{alg})^{\mathrm{op}} & \xrightarrow{(S\otimes_R(-))^{\mathrm{op}}} & (S\text{-}\mathbf{alg})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ (R\text{-}\mathbf{mod})^{\mathrm{op}} & \xrightarrow[(S\otimes_R(-))^{\mathrm{op}}]{} & (S\text{-}\mathbf{mod})^{\mathrm{op}}, \end{array} \tag{6.1}$$

where the vertical arrows are the (duals of the) forgetful functors from algebras to modules. Note that this observation actually applies to *many kinds of algebras*, and in particular to the following ones:

(a) arbitrary (not necessarily associative or commutative) algebras, with or without 1;
(b) associative algebras, with or without 1;
(c) (associative and) commutative algebras, with or without 1;

(d) Lie algebras;
(e) Jordan algebras;
(f) differential algebras.

6.2. It is interesting that the converse is also true, that is, the comonadicity of the extension-of-scalars functor for any kind of algebras above implies the same property for modules. This again follows from Lemma 2.5, but this time applied to the diagram

$$\begin{array}{ccc} (R\text{-}\mathbf{mod})^{\mathrm{op}} & \xrightarrow{(S\otimes_R(-))^{\mathrm{op}}} & (S\text{-}\mathbf{mod})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ (R\text{-}\mathbf{alg})^{\mathrm{op}} & \xrightarrow[(S\otimes_R(-))^{\mathrm{op}}]{} & (S\text{-}\mathbf{alg})^{\mathrm{op}}, \end{array} \tag{6.2}$$

where the vertical arrows are the functors carrying modules M to

(a) M equipped with the zero multiplication if our algebras are not required to have 1;
(b) the semidirect product of M with the ground ring (i.e. R for the left-hand vertical arrow, and S for the right-hand one).

6.3. Summarizing we obtain:

THEOREM. *For a homomorphism $p : R \to S$ of commutative rings, the extension-of-scalars functor $S \otimes_R (-) : R\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$ is comonadic if and only if the induced extension-of-scalars functor $S \otimes_R (-) : R\text{-}\mathbf{alg} \to S\text{-}\mathbf{alg}$ is comonadic, for any kind of algebras as listed in 6.1(a)–(f).*

6.4. **REMARK.** Like many other things, our previous arguments and Theorem 6.3 easily extend (under obvious additional conditions) from rings to monoids in a monoidal categories. Omitting details, let us just mention that since commutativity is involved, those monoidal categories should actually be *symmetric* (or at least *braided* in the sense of [10]). Note also that the straightforward translation of 6.1 (not 6.2!) from rings to locales would tell us that the comonadicity of the extension-of-scalars functor for modules over locales implies the comonadicity of the corresponding functor for locales themselves. In particular, the localic open-surjection descent theorem of Joyal and Tierney (for modules over locales; see [11]) implies the localic open-surjection descent theorem of Moerdijk [17]. Unlike these two theorems, the triquotient descent theorem of Plewe [22] is not covered by the results presented here (see also 4.4). We are omitting details again; however the relationship between comonadicity and descent will be recalled in the next section.

7. From comonadicity to (co)descent

7.1. It is well known (see 3.7 and 3.8 in [8]), that the basic (bi)fibration over a category $\mathbf{C}$ with pullbacks always satisfies the Beck-Chevalley condition. This simply means that, for every pullback square

$$\begin{array}{ccc} D & \xrightarrow{s} & E \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow[p]{} & B, \end{array} \tag{7.1}$$

the canonical morphism between the two composites in the diagram

$$\begin{array}{ccc} (\mathbf{C} \downarrow D) & \xrightarrow{s_!} & (\mathbf{C} \downarrow E) \\ {\scriptstyle t^*}\uparrow & & \uparrow{\scriptstyle q^*} \\ (\mathbf{C} \downarrow F) & \xrightarrow[p_!]{} & (\mathbf{C} \downarrow B) \end{array} \tag{7.2}$$

is an isomorphism[6]. We could also express this property more briefly, by saying that for every object C in $\mathbf{C}$ equipped with a morphism from C to E, we have

$$(E \times_B F) \times_F C \approx E \times_B C. \tag{7.3}$$

When $\mathbf{C}$ is the opposite category of commutative rings, (7.3) becomes

$$(E \otimes_B F) \otimes_F C \approx E \otimes_B C, \tag{7.4}$$

which in fact holds for every F-module C and has nothing to do with any multiplication on C. Hence also the (bi)fibration

$$(\text{Modules over Commutative Rings})^{\mathrm{op}} \to (\text{Commutative Rings})^{\mathrm{op}} \tag{7.5}$$

satisfies the Beck-Chevalley condition, and of course the same is true for all kinds of algebras listed in 6.1(a)–(f). From this fact and Theorem 6.3 we obtain:

THEOREM. *A homomorphism $p : R \to S$ of commutative rings is an effective descent morphism (considered as a morphism $S \to R$ in the opposite category of commutative rings) with respect to any, or to every, (bi)fibration of modules or algebras from 6.1(a)–(f) if and only if the extension-of-scalars functor $S \otimes_R (-) : R\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$ is comonadic.*

7.2. Corollary 5.4 can now be reformulated as:

THEOREM. *A homomorphism $p : R \to S$ of commutative rings is an effective descent morphism, i.e. it satisfies the equivalent conditions of Theorem 7.1, if and only if it is a pure monomorphism of R-modules.*

7.3. **REMARK.** Since the basic fibration for the opposite category of commutative rings is among those that occur in Theorem 7.1 (see 6.1(c)), we can say that the descent theory of modules over commutative rings is a special case of what we called global descent (theory) in [8].

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