Lax Factorization Algebras

Dedicated to Max Kelly on the occasion of his seventieth birthday

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Abstract

It is shown that many weak factorization systems appearing in functorial Quillen model categories, including all those that are cofibrantly generated, come with a rich computational structure, defined by a certain lax algebra with respect to the "squaring monad" on **CAT**. This structure largely facilitates natural choices for left or right liftings once certain basic natural choices have been made. The use of homomorphisms of such lax algebras is also discussed, with focus on "lax freeness".

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1. Introduction

Weak factorization systems appear prominently in the definition of Quillen model category: for $\mathcal{C}$, $\mathcal{W}$, $\mathcal{F}$ the classes of cofibrations, weak equivalences and fibrations, respectively, one deals with two weak factorization systems, given by

$$(\mathcal{C}, \mathcal{W} \cap \mathcal{F}) \text{ and } (\mathcal{C} \cap \mathcal{W}, \mathcal{F}),$$

decomposing every morphism into a cofibration followed by a trivial fibration, and into a trivial cofibration followed by a fibration. While generally these factorizations fail to be unique (up to isomorphism), in terms of all known and interesting examples it is not restrictive to assume that they be chosen functorially, and this has indeed become a standard assumption in abstract homotopy theory (see, for example, [Ho]).

In this paper we first prove two observations which seem to have remained unnoticed so far, namely: any functorial realization of a weak factorization system $(\mathcal{L}, \mathcal{R})$ determines completely the classes $\mathcal{L}, \mathcal{R}$ and, in turn, is completely determined by its values on identity morphisms. Hence, the practice of defining a functorial weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{K}$ by the *additional* provision of a functor $F : \mathcal{K}^2 \to \mathcal{K}$ with natural transformations λ, ρ pointwise in $\mathcal{L}, \mathcal{R}$, respectively, making the diagram

$$\begin{tikzcd} & Ff \arrow[dr, "\rho_f"] & \\ A \arrow[ur, "\lambda_f"] \arrow[rr, "f"] & & B \end{tikzcd} \tag{1}$$

commute for every f, carries indeed a lot of redundant information: F, λ, ρ determine the classes $\mathcal{L}, \mathcal{R}$, and λ_f, ρ_f are determined by λ_{1_A}, ρ_{1_B}, and the functor F, see 2.3, 2.4.

For (orthogonal) factorization systems as discussed by Freyd and Kelly in [FK], so that the left/right lifting property is strengthened to the *unique* diagonalization property, this last fact is well known. In fact, in this case $\lambda_{1_A}, \rho_{1_B}$ may be chosen as identity morphisms, so that F alone carries all information about the system. The fundamental difference between an orthogonal factorization system and a functorial weak factorization system is that one may no longer assume to factor identity morphisms trivially, despite the fact that the two classes $\mathcal{L}$ and $\mathcal{R}$ contain all identity morphisms.

Orthogonal systems are known to be presentable precisely as the pseudoalgebras with respect to the "squaring monad" on **CAT**, given by the natural functors

$$E_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^2, \quad M_{\mathcal{K}} : (\mathcal{K}^2)^2 \to \mathcal{K}^2$$

for every category $\mathcal{K}$, so that the functor $F : \mathcal{K}^2 \to \mathcal{K}$ represents the algebra structure of the system (see [Co], [KT]). For a weak system, already the unity law no longer holds true; it becomes lax, but in fact *split lax*, since $\rho_{1_A} \cdot \lambda_{1_A} = 1_A$ for all objects A of $\mathcal{K}$. This leads to one of the main results of the paper which, amongst the functorial weak factorization systems, characterizes those which allow for a lax associativity law, and those which allow for a split-lax associativity law, called lax factorization algebras and symmetric lax factorization algebras, respectively (see 3.7, 3.8 below). Phrased in lay terms, one of the practical implications of this characterization is as follows: *if, for the functorial factorization* (1), *one is given a functorial choice of the liftings σ_f in the left square below, then one already has a natural choice for the liftings $\alpha_{(u,v)}$ in the right square below:*

$$\begin{tikzcd}
A \arrow[r, "\lambda_{1_A}"] \arrow[dd, "\lambda_f"'] & F1_A \arrow[r, "\lambda_{F(1_A,f)}"] & F(F(1_A,f)) \arrow[dd, "\rho_{F(1_A,f)}"] \\
& & \\
Ff \arrow[rr, "1_{Ff}"'] \arrow[uurr, "\sigma_f"] & & Ff
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\lambda_f"] \arrow[dd, "\lambda_{v \cdot f}"'] & Ff \arrow[r, "\lambda_{F(u,v)}"] & F(F(u,v)) \arrow[d, "\rho_{F(u,v)}"] \\
& & Fg \arrow[d, "\rho_g"] \\
F(v \cdot f) \arrow[rr, "\rho_{v \cdot f}"'] \arrow[uurr, "\alpha_{(u,v)}"] & & D
\end{tikzcd} \tag{2}$$

Here $(u, v) : f \to g$ and $(1_A, f) : 1_A \to f$ are the $\mathcal{K}^2$-morphisms displayed by the following commutative squares in $\mathcal{K}$:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "v"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"] & B
\end{tikzcd} \tag{3}$$

Of course, in the symmetric case, there is a similar correspondence between appropriate natural morphisms $\tau_f : F(F(f, 1_B)) \to Ff$ and $\beta_{(u,v)} : F(F(u, v)) \to F(v \cdot f)$.

In terms of examples, it is surprising to see that all important weak factorization systems used in homotopy theory automatically come with the richer structure given by such transformations σ or α, exhibiting them as lax factorization algebras. To this end we show in Section 4 that every cofibrantly generated system in a locally presentable category carries the extra structure. We also show that the (co)graph factorization in any category with finite (co)products arises from a symmetric lax factorization algebra.

While the notion of lax factorization algebra is more restrictive than that of an (op-)lax algebra w.r.t. the squaring monad (in the sense of Street [S]), the (op-)lax homomorphisms are the "right" morphisms for lax factorization algebras. They are used to show that the free factorization system on $\mathcal{K}^2$ (given by $M_{\mathcal{K}}$) is in fact "lax-free" amongst all (symmetric) lax factorization algebras (see 5.5 below). They are also shown to behave "correctly" when comparing cofibrantly generated weak factorization systems with each other (see 5.7 below).

We plan to give further applications to model categories in a subsequent paper.

2. Functorial weak factorization systems

2.1 Recall that a morphism f has the *left lifting property* w.r.t. a morphism g, and g has the *right lifting property* w.r.t. f, written as

$$f \square g$$

if every solid-arrow commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "w"] & D \end{tikzcd} \tag{4}$$

has a dotted fill-in arrow w making both triangles commutative. A pair of morphism classes $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ is a *weak factorization system* (*wfs*) (see [Be], [AHRT]) if

1. $\mathcal{K} = \mathcal{R} \cdot \mathcal{L}$, so that every morphism factors through a morphism of $\mathcal{L}$ followed by a morphism of $\mathcal{R}$,

2. $\mathcal{R} = \mathcal{L}^{\square}$, with $\mathcal{L}^{\square} := \{g \mid \forall f \in \mathcal{L} : f \square g\}$,

3. $\mathcal{L} = {}^{\square}\mathcal{R}$, with ${}^{\square}\mathcal{R} := \{f \mid \forall g \in \mathcal{R} : f \square g\}$.

Conditions 2 and 3 certainly imply:

2$'$. $\mathcal{L} \square \mathcal{R}$, i.e., $f \square g$ for all $f \in \mathcal{L}$ and $g \in \mathcal{R}$,

3$'$. $\mathcal{L}$ and $\mathcal{R}$ are both closed under retracts in the arrow category $\mathcal{K}^2$ of $\mathcal{K}$ (see 2.2 below), so that for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & C \arrow[r, "p"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "t"] \arrow[rr, bend right, "1_B"'] & D \arrow[r, "q"] & B \end{tikzcd} \tag{5}$$

$f \in \mathcal{L}$ whenever $g \in \mathcal{L}$, and $f \in \mathcal{R}$ whenever $g \in \mathcal{R}$.

Condition 3$'$ implies:

3$''$. if $t \cdot f \in \mathcal{L}$ with a split monomorphism t, then $f \in \mathcal{L}$, and if $f \cdot p \in \mathcal{R}$ with a split epimorphism p, then $f \in \mathcal{R}$.

In fact, $(\mathcal{L}, \mathcal{R})$ *is already a wfs if conditions* $1, 2', 3''$ *are satisfied*: $\mathcal{L} \subseteq {}^{\square}\mathcal{R}$ from $2'$, and for "$\supseteq$" one factors $f \in {}^{\square}\mathcal{R}$ as $f = r \cdot l$ with $l \in \mathcal{L}, r \in \mathcal{R}$; then $2'$ gives t with $t \cdot f = l \in \mathcal{L}$ and $r \cdot t = 1$, so that $3''$ yields $f \in \mathcal{L}$; dually $\mathcal{R} = \mathcal{L}^{\square}$.

2.2 A morphism $(u, v) : f \to g$ in the arrow category $\mathcal{K}^{\mathbf{2}}$ (with $\mathbf{2} = \{\cdot \to \cdot\}$) is given by the left commutative diagram of (3) in $\mathcal{K}$. One has the domain and codomain functors

$$\mathrm{dom}_{\mathcal{K}} : \mathcal{K}^2 \to \mathcal{K}, \quad \mathrm{cod}_{\mathcal{K}} : \mathcal{K}^2 \to \mathcal{K}$$

and a natural transformation $\kappa : \mathrm{dom} \to \mathrm{cod}$ with $\kappa_f = f$ for all morphism f in $\mathcal{K}$. Essentially following [Ho], a wfs $(\mathcal{L}, \mathcal{R})$ is *functorial* if there is a pair of functors $F_{\mathcal{L}}, F_{\mathcal{R}} : \mathcal{K}^2 \to \mathcal{K}^2$ with values in $\mathcal{L}$, $\mathcal{R}$ respectively, such that

$$\mathrm{dom} F_{\mathcal{L}} = \mathrm{dom}, \quad \mathrm{cod} F_{\mathcal{L}} = \mathrm{dom} F_{\mathcal{R}}, \quad \mathrm{cod} F_{\mathcal{R}} = \mathrm{cod},$$

and $F_{\mathcal{R}}(f) \cdot F_{\mathcal{L}}(f) = f$ for all morphisms f in $\mathcal{K}$. Since any functor $H : \mathcal{X} \to \mathcal{K}^2$ is equivalently described by the functors dom$H$, cod$H$ and the natural transformation κH, the pair $(F_{\mathcal{L}}, F_{\mathcal{R}})$ may be equivalently replaced by a functor $F : \mathcal{K}^2 \to \mathcal{K}$ and natural transformations λ pointwise in $\mathcal{L}$ and ρ pointwise in $\mathcal{R}$ making the diagram

$$\begin{tikzcd} & F \arrow[dr, "\rho"] & \\ \mathrm{dom}_{\mathcal{K}} \arrow[ur, "\lambda"] \arrow[rr, "\kappa"] & & \mathrm{cod}_{\mathcal{K}} \end{tikzcd} \tag{6}$$

commutative; we call such a triple (F, λ, ρ) a *functorial realization* of the wfs $(\mathcal{L}, \mathcal{R})$. The significance of the naturality of λ and ρ is the fact that the left diagram of (3) gets decomposed as:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\lambda_f"'] & C \arrow[d, "\lambda_g"] \\ Ff \arrow[r, "F(u,v)"] \arrow[d, "\rho_f"'] & Fg \arrow[d, "\rho_g"] \\ B \arrow[r, "v"'] & D \end{tikzcd} \tag{7}$$

2.3 An application of the decomposition (7) to the diagram

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[r, "f"] \arrow[d, "f"] & B \arrow[d, "1_B"] \\ A \arrow[r, "f"] & B \arrow[r, "1_B"] & B \end{tikzcd} \tag{8}$$

yields the commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\lambda_{1_A}"'] & A \arrow[r, "f"] \arrow[d, "\lambda_f"] & B \arrow[d, "\lambda_{1_B}"] \\
F1_A \arrow[r, "{F(1_A, f)}"] \arrow[d, "\rho_{1_A}"'] & Ff \arrow[r, "{F(f, 1_B)}"] \arrow[d, "\rho_f"] & F1_B \arrow[d, "\rho_{1_B}"] \\
A \arrow[r, "f"] & B \arrow[r, "1_B"] & B
\end{tikzcd}
\tag{9}
$$

In particular, $\lambda_f = F(1_A, f) \cdot \lambda_{1_A}$ and $\rho_f = \rho_{1_B} F(f, 1_B)$, so that λ *and* ρ *are completely determined by their values on identity morphisms.* Expressed in functorial terms, with the full embedding

$$E = E_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^2$$

with $\mathrm{dom}E = \mathrm{cod}E = \mathrm{Id}_{\mathcal{K}}$ and $\kappa E = 1_{\mathrm{Id}_{\mathcal{K}}}$, the left and right squares of (8) represent the natural transformations $\nu = \nu^{\mathcal{K}}$ and $\mu = \mu^{\mathcal{K}}$ respectively, making the diagram

$$
\begin{tikzcd}
& \mathrm{Id}_{\mathcal{K}^2} \arrow[dr, "\mu"] & \\
E_{\mathcal{K}}\mathrm{dom}_{\mathcal{K}} \arrow[ur, "\nu"] \arrow[rr, "E_{\mathcal{K}}\kappa"'] & & E_{\mathcal{K}}\mathrm{cod}_{\mathcal{K}}
\end{tikzcd}
\tag{10}
$$

commute; they actually serve as counit and unit in the double adjunction

$$\mathrm{cod}_{\mathcal{K}} \underset{1}{\overset{\mu}{\dashv}} E_{\mathcal{K}} \underset{\nu}{\overset{1}{\dashv}} \mathrm{dom}_{\mathcal{K}} \tag{11}$$

Now, given any triple (F, λ, ρ) making (6) commutative, putting

$$\lambda_1 := \lambda E_{\mathcal{K}} : \mathrm{Id}_{\mathcal{K}} \to FE_{\mathcal{K}} \ \text{ and } \ \rho_1 := \rho E_{\mathcal{K}} : FE_{\mathcal{K}} \to \mathrm{Id}_{\mathcal{K}},$$

from (9) we obtain $\rho_1 \cdot \lambda_1 = 1_{\mathrm{Id}_{\mathcal{K}}}$ and the commutative diagram

$$
\begin{tikzcd}
\mathrm{dom} \arrow[r, "1"] \arrow[d, "\lambda_1\mathrm{dom}"'] & \mathrm{dom} \arrow[r, "\kappa"] \arrow[d, "\lambda"] & \mathrm{cod} \arrow[d, "\lambda_1\mathrm{cod}"] \\
FE\mathrm{dom} \arrow[r, "F\nu"] \arrow[d, "\rho_1\mathrm{dom}"'] & F \arrow[r, "F\mu"] \arrow[d, "\rho"] & FE\mathrm{cod} \arrow[d, "\rho_1\mathrm{cod}"] \\
\mathrm{dom} \arrow[r, "\kappa"] & \mathrm{cod} \arrow[r, "1"] & \mathrm{cod}
\end{tikzcd}
\tag{12}
$$

Conversely, given any triple (F, λ_1, ρ_1) with $\rho_1 \cdot \lambda_1 = 1_{\mathrm{Id}_{\mathcal{K}^2}}$, one puts

$$\lambda := F\nu \cdot \lambda_1 \mathrm{dom}_{\mathcal{K}} \quad \text{and} \quad \rho := \rho_1 \mathrm{cod}_{\mathcal{K}} \cdot F\mu$$

to produce the commutative diagram (6). This proves in particular:

Proposition *For a wfs $(\mathcal{L}, \mathcal{R})$, the following statements are equivalent:*

(i) $(\mathcal{L}, \mathcal{R})$ *is functorial*;

(ii) $(\mathcal{L}, \mathcal{R})$ *has a functorial realization* (F, λ, ρ);

(iii) *there is a triple* (F, λ_1, ρ_1) *with* $\rho_1 \cdot \lambda_1 = 1$ *and* $F(1_A, f) \cdot \lambda_{1_A} \in \mathcal{L}$ *and* $\rho_{1_B} \cdot F(f, 1_B) \in \mathcal{R}$ *for all* $f : A \to B$ *in* $\mathcal{K}$.

2.4 Next we show that a functorial realization of a wfs determines the system itself. In fact, for any functor $F : \mathcal{K}^2 \to \mathcal{K}$ and natural transformations λ, ρ with $\kappa = \rho \cdot \lambda$, we put

$$\mathcal{L}_F := \{f \mid \exists s : \lambda_f = s \cdot f,\ \rho_f \cdot s = 1\},$$

$$\mathcal{R}_F := \{f \mid \exists t : \rho_f = f \cdot t,\ t \cdot \lambda_f = 1\}.$$

Theorem (1) *For every wfs $(\mathcal{L}, \mathcal{R})$ with functorial realization (F, λ, ρ), one has $\mathcal{L} = \mathcal{L}_F, \mathcal{R} = \mathcal{R}_F$.*

(2) *For any triple (F, λ, ρ) with $\kappa = \rho \cdot \lambda$, such that $\lambda_f \in \mathcal{L}_F$ and $\rho_f \in \mathcal{R}_F$ for all morphisms f, $(\mathcal{L}_F, \mathcal{R}_F)$ is a wfs with functorial realization (F, λ, ρ).*

Proof (1) Since $f = \rho_f \cdot \lambda_f$ with $\lambda_f \in \mathcal{L}, \rho_f \in \mathcal{R}$ for all f, one sees $\mathcal{L} \subseteq \mathcal{L}_F$ exactly as in the argument given at the end of 1.1, while $\mathcal{L}_F \subseteq \mathcal{L}$ follows from $3''$ of 1.1. Dually, $\mathcal{R} = \mathcal{R}_F$.

(2) $\mathcal{L}_F \Box \mathcal{R}_F$ follows immediately using (7): for $f \in \mathcal{L}_F$ and $g \in \mathcal{R}_F$ one constructs the "diagonal" w of (4) as a morphism of the form $t \cdot F(u, v) \cdot s$.

Next we check closure of $\mathcal{L}_F$ under retracts. Applying the decomposition (7) to (5) one obtains

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "\lambda_f"'] & C \arrow[r, "p"] \arrow[d, "\lambda_g"] & A \arrow[d, "\lambda_f"] \\ Ff \arrow[r, "F(s,t)"] \arrow[d, "\rho_f"'] & Fg \arrow[r, "F(p,q)"] \arrow[d, "\rho_g"] & Ff \arrow[d, "\rho_f"] \\ B \arrow[r, "t"] & D \arrow[r, "q"] & B \end{tikzcd} \tag{13}$$

For $g \in \mathcal{L}_F$ there is $j : D \to Fg$ with $\lambda_g = j \cdot g,\ \rho_g \cdot j = 1$. Then $s := F(p, q) \cdot j \cdot t$ satisfies $s \cdot f = \lambda_f,\ \rho_f \cdot s = 1$, whence $f \in \mathcal{L}_F$. Dually for $\mathcal{R}_F$.

2.5 If, instead of (F, λ, ρ), we are just given (F, λ_1, ρ_1) with $\lambda_1 \cdot \rho_1 = 1_{\mathrm{Id}_\mathcal{K}}$, we may define λ, ρ as in 2.3 and put

$$\mathcal{L}_F^1 := \{f : A \to B \mid \exists s_1 : \lambda_f = s_1 \cdot \lambda_{1_B} \cdot f,\ \rho_f \cdot s_1 = \rho_{1_B}\},$$

$$\mathcal{R}_F^1 := \{f : A \to B \mid \exists t_1 : \rho_f = f \cdot \rho_{1_A} \cdot t_1,\ t_1 \cdot \lambda_f = \lambda_{1_B}\}.$$

One easily sees that always $\mathcal{L}_F^1 \subseteq \mathcal{L}_F$ and $\mathcal{R}_F^1 \subseteq \mathcal{R}_F$, and following the same argumentation as before, one shows that *Theorem* 2.4 *can be refined and remains valid if* $\mathcal{L}_F, \mathcal{R}_F$ *get traded for* $\mathcal{L}_F^1, \mathcal{R}_F^1$.

2.6 An (*orthogonal*) *factorization system* $(\mathcal{L}, \mathcal{R})$ can be defined exactly like a wfs $(\mathcal{L}, \mathcal{R})$, except that the relation $f \Box g$ in 2.1 should be replaced by $f \perp g$ (also denoted by $f \downarrow g$), meaning that every solid-arrow commutative diagram (4) has a *unique* dotted fill-in arrow w (see, for example, [FK], [T1]). It is a nice exercise to show that such a system is in fact a wfs (see [AHRT], [T2]), with a functorial realization (F, λ, ρ) such that $\lambda_1 = \lambda E, \rho_1 = \rho E$

are isomorphisms. Conversely, *given a functorial realization* (F, λ, ρ) *of a wfs* $(\mathcal{L}, \mathcal{R})$ *such that* λ_1, ρ_1 *are isomorphisms, the following assertions are equivalent* (see [KT], [JT]):

(i) *for all morphisms* f, λ_{ρ_f} *and* ρ_{λ_f} *are isomorphisms*;

(ii) *for all* f, λ_{ρ_f} *is monic and* ρ_{λ_f} *is epic*;

(iii) $(\mathcal{L}, \mathcal{R})$ *is an orthogonal factorization system.*

Systems (F, λ, ρ) with λ_1, ρ_1 iso have been studied intensively in [KT], [JT] (and were called "wfs" in those papers!). The absence of this restrictive condition on λ_1, ρ_1 is essential in what follows.

2.7 The wfs (Mono, Epi) of **Set** has a functorial realization, given by the diagram

$$\begin{tikzcd} & A+B \arrow[dr, "\rho_f"] & \\ A \arrow[ur, "\lambda_f"] \arrow[rr, "f"] & & B \end{tikzcd} \tag{14}$$

with $\lambda_f = i_A$ the injection and $\rho_f = [f, 1_B]$. Of course, the factorization (14) makes sense in any category $\mathcal{K}$ with finite coproducts, giving rise to the functorial wfs $(\mathcal{L}_F, \mathcal{R}_F)$, by Theorem 2.4(2). It is worth mentioning that the proof of this fact is entirely constructive. For $\mathcal{K} = \mathbf{Set}$, only when one insists to verify $\mathcal{L}_F =$ Mono and $\mathcal{R}_F =$ Epi the Axiom of Choice needs to be invoked.

Of course, the construction dualizes, giving a functorial wfs

$$\begin{tikzcd} & A \times B \arrow[dr, "p_B"] & \\ A \arrow[ur, "< 1_A, f >"] \arrow[rr, "f"] & & B \end{tikzcd} \tag{15}$$

in every category $\mathcal{K}$ with finite products. For $\mathcal{K} = \mathbf{Set}$, now one has $\mathcal{L}_F = \text{Mono} \setminus \mathcal{M}_0$ and $\mathcal{R}_F = \text{Epi} \cup \mathcal{M}_0$, with $\mathcal{M}_0$ the class of inclusion maps from $\emptyset$ into non-empty sets.

2.8 *Any cofibrantly generated wfs* $(\mathcal{L}, \mathcal{R})$ *in a locally presentable category* $\mathcal{K}$ *is functorial.* Indeed, such a system is by definition of the form $(\mathrm{cof}(\mathcal{H}), \mathcal{H}^{\square})$, where $\mathcal{H}$ is a (small) set of morphisms and where $\mathrm{cof}(\mathcal{H})$ denotes the class of $\mathcal{H}$-cofibrations, i.e., of retracts of colimits of chains of pushouts of morphisms in $\mathcal{H}$ (see [AHRT]). The construction of the factorization proceeds by ordinal induction (see [FK], [Bo], [T1], [Be]), creating a diagram

$$\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, "p_i"] \arrow[drr, "p_{i+1}"] \arrow[drrr, "\lambda_f"] & & & \\
& A_i \arrow[r, "s_i"] \arrow[dl, "f_i"] & A_{i+1} \arrow[r, "k_{i+1}"] \arrow[dll, "f_{i+1}"] & Ff \arrow[dlll, "\rho_f"] \\
B & & &
\end{tikzcd} \tag{16}$$

for all ordinals i, with $Ff = \mathrm{colim}_i A_i$. Here $A_0 = A$, $f_0 = f$, and in order to construct f_{i+1} from f_i one collects all triples (x, h, y) with $h \in \mathcal{H}$ which form a commutative square

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "h"'] & A_i \arrow[d, "f_i"] \\
Y \arrow[r, "y"'] & B
\end{tikzcd} \tag{17}$$

in $\mathcal{K}$, and lets (A_{i+1}, f_{i+1}) be the colimit of all "spans" in $\mathcal{K}/B$ given by x and h; for j a limit ordinal, $A_j = \mathrm{colim}_{i<j} A_i$. Now, for a morphism $(u, v) : f \to g$ in $\mathcal{K}^2$ as in (4), and each ordinal i, one has a commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "p_i"'] & C \arrow[d, "q_i"] \\
A_i \arrow[r, "u_i"] \arrow[d, "s_i"'] & C_i \arrow[d, "t_i"] \\
A_{i+1} \arrow[r, "u_{i+1}"] \arrow[d, "f_{i+1}"'] & C_{i+1} \arrow[d, "g_{i+1}"] \\
B \arrow[r, "v"'] & D
\end{tikzcd}
\tag{18}
$$

In fact, each (x, h, y) as in (17) contributing to A_{i+1} makes the contribution $(u_i x, h, vy)$ to the construction of C_{i+1}, therefore inducing the arrow u_{i+1} which makes (18) commute. Going over to the colimits Ff and Fg, the u_i's create the needed arrow $F(u, v)$ making (7) commutative. Clearly, F is a functor, and $\lambda,\ \rho$ are natural.

2.9 As we have seen in Theorem 2.4, a functorial wfs is equivalently described by a triple (F, λ, ρ) with $\lambda \cdot \rho = \kappa$ and $\lambda_f \in \mathcal{L}_F,\ \rho_f \in \mathcal{R}_F$; this latter condition needs to be analyzed further. For $f : A \to B$ in $\mathcal{K}$, the commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "\lambda_{1_A}"] \arrow[d, "\lambda_f"'] & F1_A \arrow[d, "F(1_A, f)"] \\
Ff \arrow[r, "1_{Ff}"'] & Ff
\end{tikzcd}
\tag{19}
$$

gets decomposed as

$$
\begin{tikzcd}
A \arrow[r, "\lambda_{1_A}"] \arrow[d, "\lambda_{\lambda_f}"'] & F1_A \arrow[d, "\lambda_{F(1_A,f)}"] \\
F\lambda_f \arrow[r, "F(\lambda_{1_A}, 1_{Ff})"] \arrow[d, "\rho_{\lambda_f}"'] & F(F(1_A,f)) \arrow[d, "\rho_{F(1_A,f)}"] \\
Ff \arrow[r, "1_{Ff}"'] & Ff
\end{tikzcd}
\tag{20}
$$

Now, $\lambda_f \in \mathcal{L}_F$ means that there is $s : Ff \to F\lambda_f$ with

$$(*) \qquad \lambda_{\lambda_f} = s \cdot \lambda_f \;\text{ and }\; \rho_{\lambda_f} \cdot s = 1,$$

whereas $F(1_A, f) \in \mathcal{L}_F$ means that there is $\sigma : Ff \to F(F(1_A, f))$ with

$$\lambda_{F(1_A,f)} = \sigma \cdot F(1_A, f) \;\text{ and }\; \rho_{F(1_A,f)} \cdot \sigma = 1,$$

which implies

$$(**) \qquad \lambda_{F(1_A,f)} \cdot \lambda_{1_A} = \sigma \cdot \lambda_f \;\text{ and }\; \rho_{F(1_A,f)} \cdot \sigma = 1.$$

Clearly, $(*)$ implies $(**)$ (just put $\sigma := F(\lambda_{1_A}, 1_{Ff}) \cdot s$), but generally not vice versa, unless λ_{1_A} is an isomorphism (as is the case for orthogonal factorization systems). It is this "technicality" which makes functorial wfs a lot harder to deal with than orthogonal factorization systems. Fortunately, as we shall see in the next section, our principal examples 2.7 and 2.8 still come with a *natural* choice for the morphisms $\sigma = \sigma_f$, and in the case of 2.7 also with a natural choice for the dual morphisms $\tau = \tau_f : F(F(f, 1_B)) \to Ff$, which facilitate an equational presentation as lax algebras.

3. Lax factorization algebras

3.1 The functor $E = E_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^2$ of 2.3 belongs, of course, to the monad $((-)^2, E, M)$ on **CAT** (see, for example, [KT]). Here the functor

$$M = M_{\mathcal{K}} : (\mathcal{K}^2)^2 \to \mathcal{K}^2$$

maps $(u,v): f \to g$ in $\mathcal{K}^2$, considered as an object of $(\mathcal{K}^2)^2$, to the $\mathcal{K}$-morphism $vf = gu$, considered as an object of $\mathcal{K}^2$; morphisms of $(\mathcal{K}^2)^2$ get mapped as follows:

$$\begin{tikzcd} f \arrow[r, "{(a,b)}"] \arrow[d, "{(u,v)}"'] & f' \arrow[d, "{(u',v')}"] \\ g \arrow[r, "{(c,d)}"] & g' \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "v \cdot f"'] & A' \arrow[d, "v' \cdot f'"] \\ D \arrow[r, "d"] & D' \end{tikzcd} \tag{21}$$

From the first of the monad identities

$$M_{\mathcal{K}} E_{\mathcal{K}^2} = \mathrm{Id}_{\mathcal{K}^2} = M_{\mathcal{K}}(E_{\mathcal{K}})^2, \;\; M_{\mathcal{K}} M_{\mathcal{K}^2} = M_{\mathcal{K}}(M_{\mathcal{K}})^2$$

we see that $M_{\mathcal{K}}$ belongs to the functorial realization $(M_{\mathcal{K}}, \tilde{\lambda}, \tilde{\rho})$ of a wfs on $\mathcal{K}^2$, since one may put $\tilde{\lambda}_1 = 1_{\mathrm{Id}_{\mathcal{K}^2}} = \tilde{\rho}_1$. This is actually an orthogonal factorization system, with

$$\mathcal{L}_M = \{(u,v) \mid u \;\text{ iso}\}, \;\; \mathcal{R}_M = \{(u,v) \mid v \;\text{ iso}\},$$

which decomposes $(u,v): f \to g$ in $\mathcal{K}^2$ as $(u,v) = \tilde{\rho}_{(u,v)} \cdot \tilde{\lambda}_{(u,v)}$, as in:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "f"'] & A \arrow[r, "u"] \arrow[d, "v \cdot f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"] & D \arrow[r, "1_D"] & D \end{tikzcd} \tag{22}$$

3.2 For future reference we note that for any functor $H: \mathcal{K} \to \mathcal{K}'$, the functor $H^2 : \mathcal{K}^2 \to (\mathcal{K}')^2$ satisfies

$$\mathrm{dom}_{\mathcal{K}'} H^2 = H \mathrm{dom}_{\mathcal{K}}, \;\; \mathrm{cod}_{\mathcal{K}'} H^2 = H \mathrm{cod}_{\mathcal{K}}, \;\; \kappa' H^2 = H\kappa.$$

For a natural transformation $\gamma : H \to K$, the transformation $\gamma^2 : H^2 \to K^2$ is defined by $\gamma^2_f = (\gamma_A, \gamma_B) : Hf \to Kf$ for all $f: A \to B$ in $\mathcal{K}$; hence,

$$\mathrm{dom}_{\mathcal{K}'} \gamma^2 = \gamma \mathrm{dom}_{\mathcal{K}}, \;\; \mathrm{cod}_{\mathcal{K}'} \gamma^2 = \gamma \mathrm{cod}_{\mathcal{K}}.$$

In particular, $\mathrm{dom}_{\mathcal{K}}$ and $\mathrm{cod}_{\mathcal{K}}$ are, like $E_{\mathcal{K}}$ and $M_{\mathcal{K}}$, natural in $\mathcal{K}$.

3.3 Definition (1) A *lax factorization algebra* (*lfa*) is a quadruple $(F, \lambda_1, \rho_1, \alpha)$ with a functor $F : \mathcal{K}^2 \to \mathcal{K}$, natural transformations

$$\lambda_1 : \mathrm{Id}_{\mathcal{K}} \to FE_{\mathcal{K}}, \quad \rho_1 : FE_{\mathcal{K}} \to \mathrm{Id}_{\mathcal{K}}, \quad \alpha : FM_{\mathcal{K}} \to FF^2,$$

such that

1. $\rho_1 \cdot \lambda_1 = 1_{\mathrm{Id}_{\mathcal{K}}}$,

2a. $\alpha E_{\mathcal{K}^2} E_{\mathcal{K}} \cdot \lambda_1 = \lambda_1 F E_{\mathcal{K}} \cdot \lambda_1 = \alpha (E_{\mathcal{K}})^2 E_{\mathcal{K}} \cdot \lambda_1$,

2b. $\rho_1 F \cdot \alpha E_{\mathcal{K}^2} = 1_F = F\rho_1^2 \cdot \alpha (E_{\mathcal{K}})^2$,

3. $\alpha (F^2)^2 \cdot \alpha M_{\mathcal{K}^2} = F\alpha^2 \cdot \alpha (M_{\mathcal{K}})^2$.

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "E_{\mathcal{K}}"] \arrow[dr, "\mathrm{Id}_{\mathcal{K}}"'] & \mathcal{K}^2 \arrow[d, "F"] \arrow[d, phantom, shift right=3ex, "\overset{\lambda_1}{\Rightarrow}\,\underset{\rho_1}{\Leftarrow}"] \\
& \mathcal{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{K}^2)^2 \arrow[r, "F^2"] \arrow[d, "M_{\mathcal{K}}"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathcal{K}^2 \arrow[d, "F"] \\
\mathcal{K}^2 \arrow[r, "F"'] & \mathcal{K}
\end{tikzcd}
\qquad (23)$$

$$\begin{tikzcd}
\mathrm{Id}_{\mathcal{K}} \arrow[r, "\lambda_1"] \arrow[d, "\lambda_1"'] & FE_{\mathcal{K}} \arrow[d, "\lambda_1 F E_{\mathcal{K}}"] \\
FE_{\mathcal{K}} \arrow[r, shift left, "\alpha E_{\mathcal{K}^2} E_{\mathcal{K}}"] \arrow[r, shift right, "\alpha (E_{\mathcal{K}})^2 E_{\mathcal{K}}"'] & FE_{\mathcal{K}}FE_{\mathcal{K}}
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, "\alpha E_{\mathcal{K}^2}"] \arrow[d, "\alpha (E_{\mathcal{K}})^2"'] \arrow[dr, "1_F"] & FF^2E_{\mathcal{K}^2} \arrow[d, "\rho_1 F"] \\
FF^2(E_{\mathcal{K}})^2 \arrow[r, "F\rho_1^2"'] & F
\end{tikzcd}
\qquad (24)$$

$$\begin{tikzcd}
FM_{\mathcal{K}}M_{\mathcal{K}^2} \arrow[r, "\alpha (M_{\mathcal{K}})^2"] \arrow[d, "\alpha M_{\mathcal{K}^2}"'] & FF^2(M_{\mathcal{K}})^2 \arrow[d, "F\alpha^2"] \\
FF^2M_{\mathcal{K}^2} \arrow[r, "\alpha (F^2)^2"'] & FF^2(F^2)^2
\end{tikzcd}
\qquad (25)$$

Here we use the naturality of E and M, so that

$$E_{\mathcal{K}}F = F^2 E_{\mathcal{K}^2}, \quad F^2 M_{\mathcal{K}^2} = M_{\mathcal{K}}(F^2)^2.$$

We also note that $E_{\mathcal{K}^2}E_{\mathcal{K}} = (E_{\mathcal{K}})^2E_{\mathcal{K}}$, so that *only one of the two identities in* 2*a is needed.* An lfa is *special* if instead of condition 2 the following (stronger) condition is satisfied:

2*. $\alpha E_{\mathcal{K}^2} = \lambda_1 F, \;\; \alpha(E_{\mathcal{K}})^2 = F\lambda_1^2.$

(2) A *symmetric lax factorization algebra* (*slfa*) is an lfa which comes with an additional natural transformation

$$\beta : FF^2 \to FM_{\mathcal{K}},$$

such that

4. $\beta \cdot \alpha = 1_{FM_{\mathcal{K}}}$,

5a. $\rho_1 \cdot \beta E_{\mathcal{K}^2}E_{\mathcal{K}} = \rho_1 \cdot FE_{\mathcal{K}}\rho_1 = \rho_1 \cdot \beta(E_{\mathcal{K}})^2E_{\mathcal{K}}$,

5b. $\beta E_{\mathcal{K}^2} \cdot \lambda_1 F = 1_F = \beta(E_{\mathcal{K}})^2 \cdot F\lambda_1^2$,

6. $\beta M_{\mathcal{K}^2} \cdot \beta(F^2)^2 = \beta(M_{\mathcal{K}})^2 \cdot F\beta^2$.

An slfa is *special* if instead of conditions 2 and 5 the (stronger) conditions 2* and

5*. $\beta E_{\mathcal{K}^2} = \rho_1 F, \;\; \beta(E_{\mathcal{K}})^2 = F\lambda_1^2$

are satisfied.

3.4 (1) The transformations $\rho_1,\;\; \alpha$ satisfying 2.b and 3 of 3.3 present an lfa as an op-lax $(-)^2$-algebra (as defined by Street [S]). Hence, *an lfa is precisely an op-lax* $(-)^2$*-algebra for which the "unit transformation"* ρ_1 *has a specified section* λ_1 *satisfying* 2a *of* 3.3. Likewise, *an slfa is simultaneously a lax and an op-lax* $(-)^2$*-algebra in which both, the two unit transformations and the two associativity transformations, are partially inverse to each other* (*as in* 1 *and* 4 *of* 3.3) *and satisfy* 2a *and* 5a *of* 3.3.

(2) As follows from the results of [KT], *orthogonal factorization systems correspond precisely to those* (*symmetric*) *lfa's for which both the unit and the associativity transformation are isomorphisms, i.e., to pseudo* $(-)^2$*-algebras.* In fact, mere existence of those isomorphic transformations forces all equations 1-5* of 3.3 to hold true, since all of them arise from the *unique* diagonalization property. Orthogonal factorization systems are in particular special, a rare property amongst (s)lfa's, as we shall see next.

3.5 In a category with finite coproducts, we consider the *cograph factorization* (14) with

$F(f : A \to B) = A + B$. For every $(u, v) : f \to g$ in $\mathcal{K}^2$ (as in (4)) one may define natural morphisms

$$F(v \cdot f) \xrightarrow{\alpha_{(u,v)}} F(F(u,v)) \xrightarrow{\beta_{(u,v)}} F(v \cdot f)$$

in $\mathcal{K}$ which fit the diagram

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[d, "\lambda_f = i_A"'] & A + D \arrow[r, "{[v \cdot f, 1_D]}"] \arrow[d, shift right, "\alpha_{(u,v)}"'] & D \\
A + B \arrow[r, "i_{A+B}"] & (A+B) + (C+D) \arrow[r, "{[u+v, 1_{C+D}]}"] \arrow[u, shift right, "\beta_{(u,v)}"'] & C + D \arrow[u, "\rho_g = {[g, 1_D]}"']
\end{tikzcd} \tag{26}$$

where $\alpha_{(u,v)} = i_A + i_D$ and $\beta_{(u,v)} = [1_A + v, [i_D \cdot g, 1_D]]$; the top row represents the factorization of $v \cdot f$, and the bottom row that of $F(u, v)$. The verifications that α, β be natural and satisfy the identities 1-6 of 3.3 are lengthy and, at times, tedious but nevertheless straightforward. Hence:

Proposition *The (co)graph factorization has the structure of a symmetric lax factorization algebra.*

We emphasize that this algebra fails (badly) to be special: already for $\mathcal{K} = \mathbf{Set}$, neither of the four identities listed in 2^* and 5^* of 3.3 holds true.

Before discussing further examples of lfa's, we should take advantage of a simplified setting for lfa's which, after some preparation, we discuss next.

3.6 Like $E_{\mathcal{K}}$, also $M_{\mathcal{K}}$ has both adjoints,

$$L_{\mathcal{K}} \underset{\psi}{\overset{1}{\dashv}} M_{\mathcal{K}} \underset{1}{\overset{\varphi}{\dashv}} R_{\mathcal{K}} \tag{27}$$

with $L_{\mathcal{K}}$, $R_{\mathcal{K}}$ being defined by the counit ν, unit μ of (11), respectively; hence, on $\mathcal{K}^2$-objects one has

$$L_{\mathcal{K}} f = \nu_f = (1_A, f), \quad R_{\mathcal{K}} f = \mu_f = (f, 1_B).$$

The counit $\psi : L_{\mathcal{K}} M_{\mathcal{K}} \to \mathrm{Id}_{(\mathcal{K}^2)^2}$ and unit $\varphi : \mathrm{Id}_{(\mathcal{K}^2)^2} \to R_{\mathcal{K}} M_{\mathcal{K}}$ of (25) are given by the left and right square, respectively, of the diagram

$$
\begin{tikzcd}
1_A \arrow[r, "{(1_A, f)}"] \arrow[d, "{(1_A, v\cdot f)}"'] & f \arrow[r, "{(1_A, v)}"] \arrow[d, "{(u,v)}"] & v\cdot f \arrow[d, "{(g\cdot u, 1_D)}"] \\
v\cdot f \arrow[r, "{(u, 1_D)}"] & g \arrow[r, "{(g, 1_D)}"] & 1_D
\end{tikzcd}
\tag{28}
$$

for every morphism $(u, v) : f \to g$ in $\mathcal{K}^2$ (considered as an object of $(\mathcal{K}^2)^2$). With $\chi : L_{\mathcal{K}} \to R_{\mathcal{K}}$ defined by

$$\chi_f := ((1_A, f), (f, 1_B)) : L_{\mathcal{K}} f \to R_{\mathcal{K}} f,$$

diagram (28) represents the factorization

$$
\begin{tikzcd}
 & \mathrm{Id}_{(\mathcal{K}^2)^2} \arrow[dr, "\varphi"] & \\
L_{\mathcal{K}} M_{\mathcal{K}} \arrow[ur, "\psi"] \arrow[rr, "\chi M_{\mathcal{K}}"'] & & R_{\mathcal{K}} M_{\mathcal{K}}
\end{tikzcd}
\tag{29}
$$

which renders an analogous situation as depicted by (10).

For future reference, we list a number of identities without proof (see also [KT], but observe change of notation, in particular for L and R):

$$\mathrm{dom}_{\mathcal{K}^2} L_{\mathcal{K}} = E_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}}, \;\; \mathrm{cod}_{\mathcal{K}^2} R_{\mathcal{K}} = E_{\mathcal{K}} \mathrm{cod}_{\mathcal{K}},$$

$$\mathrm{cod}_{\mathcal{K}^2} L_{\mathcal{K}} = \mathrm{Id}_{\mathcal{K}^2} = \mathrm{dom}_{\mathcal{K}^2} R_{\mathcal{K}}, \;\; (\mathrm{cod}_{\mathcal{K}})^2 L_{\mathcal{K}} = \mathrm{Id}_{\mathcal{K}^2} = (\mathrm{dom}_{\mathcal{K}})^2 R_{\mathcal{K}},$$

$$L_{\mathcal{K}} E_{\mathcal{K}} = E_{\mathcal{K}^2} E_{\mathcal{K}} = (E_{\mathcal{K}})^2 E_{\mathcal{K}} = R_{\mathcal{K}} E_{\mathcal{K}},$$

$$L_{\mathcal{K}^2} L_{\mathcal{K}} = (L_{\mathcal{K}})^2 L_{\mathcal{K}}, \;\; R_{\mathcal{K}^2} R_{\mathcal{K}} = (R_{\mathcal{K}})^2 R_{\mathcal{K}},$$

$$\bar{\kappa} L_{\mathcal{K}} = \nu, \;\; \bar{\kappa} R_{\mathcal{K}} = \mu \;\; (\text{with } \; \bar{\kappa} = \kappa^{\mathcal{K}^2}),$$

$$L_{\mathcal{K}} \nu = \bar{\nu} L_{\mathcal{K}}, \;\; R_{\mathcal{K}} \mu = \bar{\mu} R_{\mathcal{K}} \;\; (\text{with } \; \bar{\mu} = \mu^{\mathcal{K}^2}, \; \bar{\nu} = \nu^{\mathcal{K}^2}),$$

$$\mathrm{dom}_{\mathcal{K}} M_{\mathcal{K}} = \mathrm{dom}_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}^2}, \;\; \mathrm{cod}_{\mathcal{K}} M_{\mathcal{K}} = \mathrm{cod}_{\mathcal{K}} \mathrm{cod}_{\mathcal{K}^2},$$

$$M_{\mathcal{K}} L_{\mathcal{K}} = \mathrm{Id}_{\mathcal{K}^2} = M_{\mathcal{K}} R_{\mathcal{K}}, \;\; M_{\mathcal{K}} \psi = 1_{\mathrm{Id}_{(\mathcal{K}^2)^2}} = M_{\mathcal{K}} \varphi,$$

$\psi L_{\mathcal{K}} = 1_{\mathrm{Id}_{\mathcal{K}^2}} = \varphi R_{\mathcal{K}}, \;\; \psi R_{\mathcal{K}} = \chi = \varphi L_{\mathcal{K}},$

$\psi E_{\mathcal{K}^2} = \bar{\mu} L_{\mathcal{K}}, \;\; \varphi E_{\mathcal{K}^2} = \bar{\nu} R_{\mathcal{K}},$

$\psi (E_{\mathcal{K}})^2 = \mu^2 L_{\mathcal{K}}, \;\; \varphi (E_{\mathcal{K}})^2 = \mu^2 R_{\mathcal{K}},$

$\mathrm{dom}_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}^2} \psi = 1_{\mathrm{dom}_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}^2}}, \;\; \mathrm{cod}_{\mathcal{K}} \mathrm{cod}_{\mathcal{K}^2} \varphi = 1_{\mathrm{cod}_{\mathcal{K}} \mathrm{cod}_{\mathcal{K}^2}},$

$\mathrm{cod}_{\mathcal{K}} \mathrm{cod}_{\mathcal{K}^2} \psi = 1_{\mathrm{cod}_{\mathcal{K}} \mathrm{cod}_{\mathcal{K}^2}}, \;\; \mathrm{dom}_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}^2} \varphi = 1_{\mathrm{dom}_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}^2}},$

$L_{\mathcal{K}^2} \psi M_{\mathcal{K}^2} = (L_{\mathcal{K}})^2 \psi (M_{\mathcal{K}})^2, \;\; R_{\mathcal{K}^2} \varphi M_{\mathcal{K}^2} = (R_{\mathcal{K}})^2 \varphi (M_{\mathcal{K}})^2.$

As discussed in [KT], most of these identities arise from the comonoid structure of $\mathbf{2} = \{0 \to 1\}$, via the internal-hom of the cartesian closed 2-category **CAT**. We omit all details.

3.7 For any functor $F : \mathcal{K}^2 \to \mathcal{K}$ and any natural transformation $\alpha : FM_{\mathcal{K}} \to FF^2$ we obtain (with the identities listed in 3.6):

$$FF^2\psi \cdot \alpha L_{\mathcal{K}} M_{\mathcal{K}} = \alpha \mathrm{Id}_{(\mathcal{K}^2)^2} \cdot FM_{\mathcal{K}} \psi = \alpha,$$

$$FF^2\varphi \cdot \alpha \mathrm{Id}_{(\mathcal{K}^2)^2} = \alpha R_{\mathcal{K}} M_{\mathcal{K}} \cdot FM_{\mathcal{K}} \varphi = \alpha R_{\mathcal{K}} M_{\mathcal{K}}.$$

which imply

$$FF^2\psi E_{\mathcal{K}^2} \cdot \alpha L_{\mathcal{K}} = \alpha E_{\mathcal{K}^2}, \;\; FF^2 \psi (E_{\mathcal{K}})^2 \cdot \alpha L_{\mathcal{K}} = \alpha (E_{\mathcal{K}})^2,$$

$$FF^2\varphi E_{\mathcal{K}^2} \cdot \alpha E_{\mathcal{K}^2} = \alpha R_{\mathcal{K}} = FF^2 \varphi (E_{\mathcal{K}})^2 \cdot \alpha (E_{\mathcal{K}})^2.$$

It is worth mentioning that the first of these identities implies that α is completely determined by $\alpha L_{\mathcal{K}}$, i.e., by its values on $\nu_f = (1_A, f)$ for all f in $\mathcal{K}$. We must therefore pay attention to the transformation

$$\sigma := \alpha L_{\mathcal{K}} : F \to FF^2 L_{\mathcal{K}}$$

which, for every $f : A \to B$ in $\mathcal{K}$, gives a morphism

$$\sigma_f : Ff \to F(F(1_A, f)).$$

If α belongs to an lfa, then σ_f satisfies $(**)$ of 2.9, and in turn, these morphisms determine α. More precisely:

Theorem *Given any triple* $(F : \mathcal{K}^2 \to \mathcal{K},\ \lambda : \mathrm{dom}_{\mathcal{K}} \to F,\ \rho : F \to \mathrm{cod}_{\mathcal{K}})$ *with* $\rho \cdot \lambda = \kappa$, *the natural transformations* $\alpha : FM_{\mathcal{K}} \to FF^2$ *making* $(F, \lambda_1, \rho_1, \alpha)$ *a lax factorization algebra correspond bijectively to the natural transformations* $\sigma : F \to FF^2L_{\mathcal{K}}$ *satisfying the following identities:*

2′a. $\sigma \cdot \lambda = \lambda F^2 L_{\mathcal{K}} \cdot \lambda E_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}}$,

2′b. $\rho F^2 L_{\mathcal{K}} \cdot \sigma = 1_F = F\rho^2 L_{\mathcal{K}} \cdot \sigma$,

3′ $\sigma F^2 L_{\mathcal{K}} \cdot \sigma = F\sigma^2 L_{\mathcal{K}} \cdot \sigma$.

Proof Having α with 2a, 2b, 3 of 3.3 and putting $\sigma := \alpha L_{\mathcal{K}}$, we first check the identities 2′a, 2′b, 3′:

$$\begin{aligned}
\sigma \cdot \lambda &= \alpha L_{\mathcal{K}} \cdot F\nu \cdot \lambda_1 \mathrm{dom}_{\mathcal{K}} && \text{(by (12))}\\
&= FF^2 L_{\mathcal{K}} \nu \cdot \alpha L_{\mathcal{K}} E_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}} \cdot \lambda_1 \mathrm{dom}_{\mathcal{K}} && (\alpha, \nu \text{ nat.})\\
&= FF^2 \bar{\nu} L_{\mathcal{K}} \cdot \alpha E_{\mathcal{K}^2} E_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}} \cdot \lambda_1 \mathrm{dom}_{\mathcal{K}} && \text{(by (3.6))}\\
&= F\nu F^2 L_{\mathcal{K}} \cdot \lambda_1 F E_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}} \cdot \lambda_1 \mathrm{dom}_{\mathcal{K}} && \text{(by 3.3, 2a)}\\
&= F\nu F^2 L_{\mathcal{K}} \cdot \lambda_1 \mathrm{dom}_{\mathcal{K}} F^2 L_{\mathcal{K}} \cdot \lambda_1 \mathrm{dom}_{\mathcal{K}} && \text{(by 3.6)}\\
&= \lambda F^2 L_{\mathcal{K}} \cdot \lambda E_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}}, && \text{(by (12))}\\
\rho F^2 L_{\mathcal{K}} \cdot \sigma &= \rho_1 \mathrm{cod}_{\mathcal{K}} F^2 L_{\mathcal{K}} \cdot F\mu F^2 L_{\mathcal{K}} \cdot \alpha L_{\mathcal{K}} && \text{(by (12))}\\
&= \rho_1 F \cdot FF^2 \psi E_{\mathcal{K}^2} \cdot \alpha L_{\mathcal{K}} && \text{(by 3.6)}\\
&= \rho_1 F \cdot \alpha E_{\mathcal{K}^2} && \text{(see 3.7 above)}\\
&= 1_F, && \text{(by 3.3, 2.b)}\\
F\rho^2 L_{\mathcal{K}} \cdot \sigma &= F\rho_1^2 (\mathrm{cod}_{\mathcal{K}})^2 L_{\mathcal{K}} \cdot FF^2 \mu^2 L_{\mathcal{K}} \cdot \alpha L_{\mathcal{K}} && \text{(by (12))}\\
&= F\rho_1^2 \cdot FF^2 \psi (E_{\mathcal{K}})^2 \cdot \alpha L_{\mathcal{K}} && \text{(by 3.6)}\\
&= F\rho_1^2 \cdot \alpha (E_{\mathcal{K}})^2 && \text{(see 3.7 above)}\\
&= 1_F, && \text{(by 2.b)}\\
\sigma F^2 L_{\mathcal{K}} \cdot \sigma &= \alpha L_{\mathcal{K}} F^2 L_{\mathcal{K}} \cdot \alpha L_{\mathcal{K}}
\end{aligned}$$

$$= \alpha(F^2)^2 L_{\mathcal{K}^2} L_{\mathcal{K}} \cdot \alpha M_{\mathcal{K}^2} L_{\mathcal{K}^2} L_{\mathcal{K}} \qquad \text{(by 3.6)}$$

$$= F\alpha^2 (L_{\mathcal{K}})^2 L_{\mathcal{K}} \cdot \alpha (M_{\mathcal{K}})^2 (L_{\mathcal{K}})^2 L_{\mathcal{K}} \qquad \text{(by 3.3, 3)}$$

$$= F\sigma^2 L_{\mathcal{K}} \cdot \sigma. \qquad \text{(by 3.6)}$$

Conversely, given σ satisfying 2′a, 2′b, 3′, we define α by

$$\alpha = (FM_{\mathcal{K}} \xrightarrow{\sigma M_{\mathcal{K}}} FF^2 L_{\mathcal{K}} M_{\mathcal{K}} \xrightarrow{FF^2\psi} FF^2)$$

and verify 2a, 2b, 3 of 3.3, by first showing that the following diagram commutes (the pointwise version of which appaers in (2); see also (26)):

$$\begin{tikzcd}
\mathrm{dom}_{\mathcal{K}} M_{\mathcal{K}} \arrow[r, "\lambda M_{\mathcal{K}}"] \arrow[d, "\lambda \mathrm{dom}_{\mathcal{K}^2}"'] & FM_{\mathcal{K}} \arrow[r, "\rho M_{\mathcal{K}}"] \arrow[d, "\alpha"] & \mathrm{cod}_{\mathcal{K}} M_{\mathcal{K}} \\
\mathrm{dom}_{\mathcal{K}} F^2 \arrow[r, "\lambda F^2"] & FF^2 \arrow[r, "\rho F^2"] & \mathrm{cod}_{\mathcal{K}} F^2 \arrow[u, "\rho \mathrm{cod}_{\mathcal{K}^2}"']
\end{tikzcd} \qquad (30)$$

$$\begin{aligned}
\alpha \cdot \lambda M_{\mathcal{K}} &= FF^2\psi \cdot \sigma M_{\mathcal{K}} \cdot \lambda M_{\mathcal{K}} \\
&= FF^2\psi \cdot \lambda F^2 L_{\mathcal{K}} M_{\mathcal{K}} \cdot \lambda E_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}} M_{\mathcal{K}} && \text{(by 2′a)} \\
&= \lambda F^2 \cdot \mathrm{dom}_{\mathcal{K}} F^2 \psi \cdot \lambda \mathrm{dom}_{\mathcal{K}^2} L_{\mathcal{K}} M_{\mathcal{K}} && (\psi, \lambda \text{ nat., 3.6}) \\
&= \lambda F^2 \cdot \lambda \mathrm{dom}_{\mathcal{K}^2} \cdot \mathrm{dom}_{\mathcal{K}} \mathrm{dom}_{\mathcal{K}^2} \psi && (\psi, \lambda \text{ nat.}) \\
&= \lambda F^2 \cdot \lambda \mathrm{dom}_{\mathcal{K}^2}, && \text{(by 3.6)}
\end{aligned}$$

$$\begin{aligned}
\rho \mathrm{cod}_{\mathcal{K}^2} \cdot \rho F^2 \cdot \alpha &= \rho \mathrm{cod}_{\mathcal{K}^2} \cdot \rho F^2 \cdot FF^2\psi \cdot \sigma M_{\mathcal{K}} \\
&= \rho \mathrm{cod}_{\mathcal{K}^2} \cdot \mathrm{cod}_{\mathcal{K}} F^2 \psi \cdot \rho F^2 L_{\mathcal{K}} M_{\mathcal{K}} \cdot \sigma M_{\mathcal{K}} && (\rho, \lambda \text{ nat.}) \\
&= \mathrm{cod}_{\mathcal{K}} \mathrm{cod}_{\mathcal{K}^2} \psi \cdot \rho \mathrm{cod}_{\mathcal{K}^2} L_{\mathcal{K}} M_{\mathcal{K}} && \text{(by 2′b)} \\
&= \rho M_{\mathcal{K}}. && \text{(by 3.6)}
\end{aligned}$$

The first of these identities implies 2a:

$$\begin{aligned}
\alpha E_{\mathcal{K}^2} E_{\mathcal{K}} \cdot \lambda_1 &= \alpha E_{\mathcal{K}^2} E_{\mathcal{K}} \cdot \lambda M_{\mathcal{K}} E_{\mathcal{K}^2} E_{\mathcal{K}} \\
&= \lambda F^2 E_{\mathcal{K}^2} E_{\mathcal{K}} \cdot \lambda \mathrm{dom}_{\mathcal{K}^2} E_{\mathcal{K}^2} E_{\mathcal{K}} && \text{(by (30))}
\end{aligned}$$

$$= \lambda E_{\mathcal{K}} F E_{\mathcal{K}} \cdot \lambda E_{\mathcal{K}}$$

$$= \lambda_1 F E_{\mathcal{K}} \cdot \lambda_1.$$

For 2b and 3, we have:

$$\begin{aligned}
\rho_1 F \cdot \alpha E_{\mathcal{K}^2} &= \rho_1 F \cdot F F^2 \psi E_{\mathcal{K}^2} \cdot \sigma M_{\mathcal{K}} E_{\mathcal{K}^2} \\
&= \rho_1 \mathrm{cod}_{\mathcal{K}} F^2 L_{\mathcal{K}} \cdot F \mu F^2 L_{\mathcal{K}} \cdot \sigma && \text{(by 3.6)} \\
&= \rho F^2 L_{\mathcal{K}} \cdot \sigma && \text{(by(12))} \\
&= 1_F, && \text{(by 2}'\text{b)}
\end{aligned}$$

$$\begin{aligned}
F \rho_1^2 \cdot \alpha (E_{\mathcal{K}})^2 &= F \rho_1^2 \cdot F F^2 \psi (E_{\mathcal{K}})^2 \cdot \sigma M_{\mathcal{K}} (E_{\mathcal{K}})^2 \\
&= F \rho_1^2 (\mathrm{cod}_{\mathcal{K}})^2 L_{\mathcal{K}} \cdot F F^2 \mu^2 L_{\mathcal{K}} \cdot \sigma && \text{(by 3.6)} \\
&= F \rho^2 L_{\mathcal{K}} \cdot \sigma && \text{(by (12))} \\
&= 1_F, && \text{(by 2'b)}
\end{aligned}$$

$$\begin{aligned}
&\alpha (F^2)^2 \cdot \alpha M_{\mathcal{K}^2} \\
&= F F^2 \psi (F^2)^2 \cdot \sigma M_{\mathcal{K}} (F^2)^2 \cdot F F^2 \psi M_{\mathcal{K}^2} \cdot \sigma M_{\mathcal{K}} M_{\mathcal{K}^2} \\
&= F F^2 (F^2)^2 \psi^2 \cdot F F^2 L_{\mathcal{K}} F^2 \psi M_{\mathcal{K}^2} \cdot \sigma F^2 L_{\mathcal{K}} M_{\mathcal{K}} M_{\mathcal{K}^2} \cdot \sigma M_{\mathcal{K}} M_{\mathcal{K}^2} && (\sigma, \psi \ \text{nat.}) \\
&= F F^2 (F^2)^2 \psi^2 \cdot F F^2 (F^2)^2 (L_{\mathcal{K}})^2 \psi (M_{\mathcal{K}})^2 \cdot F \sigma^2 L_{\mathcal{K}} M_{\mathcal{K}} (M_{\mathcal{K}})^2 \cdot \sigma M_{\mathcal{K}} (M_{\mathcal{K}})^2 && \text{(by 3.6, 3}'\text{)} \\
&= F F^2 (F^2)^2 \psi^2 \cdot F \sigma^2 (M_{\mathcal{K}})^2 \cdot F F^2 \psi (M_{\mathcal{K}})^2 \cdot \sigma M_{\mathcal{K}} (M_{\mathcal{K}})^2 && (\sigma, \psi \ \text{nat.}) \\
&= F \alpha^2 \cdot \alpha (M_{\mathcal{K}})^2.
\end{aligned}$$

For the bijectivity assertion, note that if $\alpha = F F^2 \psi \cdot \sigma M_{\mathcal{K}}$, then

$$\alpha L_{\mathcal{K}} = F F^2 \psi L_{\mathcal{K}} \cdot \sigma M_{\mathcal{K}} L_{\mathcal{K}} = \sigma$$

by 3.6, and if $\sigma = \alpha L_{\mathcal{K}}$, then

$$F F^2 \psi \cdot \sigma M_{\mathcal{K}} = F F^2 \psi \cdot \alpha L_{\mathcal{K}} M_{\mathcal{K}} = \alpha \cdot F M_{\mathcal{K}} \psi = \alpha$$

by naturality of ψ, α, and by 3.6. $\square$

We call $(F, \lambda, \rho, \sigma)$ the *reduced presentation* of the lfa $(F, \lambda_1, \rho, \alpha)$, and we note that the "pointwise display" of the passage from σ to α was given in the Introduction by diagram (2).

3.8 Corollary *For the reduced presentation* $(F, \lambda, \rho, \sigma)$ *of an lfa* $(F, \lambda_1, \rho_1, \alpha)$, *the natural transformations* $\beta : FF^2 \to FM_{\mathcal{K}}$ *making it symmetric correspond bijectively to the natural transformations* $\tau : FF^2R_{\mathcal{K}} \to F$ *satisfying the following identities*:

$4'$. $\tau \cdot FF^2\chi \cdot \sigma = 1_F$,

$5'$a. $\rho \cdot \tau = \rho E_{\mathcal{K}}\mathrm{cod}_{\mathcal{K}} \cdot \rho F^2 R_{\mathcal{K}}$,

$5'$b. $\tau \cdot \lambda F^2 R_{\mathcal{K}} = 1_F = \tau \cdot F\lambda^2 R_{\mathcal{K}}$,

$6'$. $\tau \cdot \tau F^2 R_{\mathcal{K}} = \tau \cdot F\tau^2 R_{\mathcal{K}}$.

The transformation α *of an lfa satisfies*

$2''$a. $\alpha \cdot \lambda M_{\mathcal{K}} = \lambda F^2 \cdot \lambda \mathrm{dom}_{\mathcal{K}^2}$,

$2''$b. $\rho \mathrm{cod}_{\mathcal{K}^2} \cdot \rho F^2 \cdot \alpha = \rho M_{\mathcal{K}}$,

and the transformation β *of an slfa satisfies*

$5''$a. $\rho M_{\mathcal{K}} \cdot \beta = \rho \mathrm{cod}_{\mathcal{K}^2} \cdot \rho F^2$,

$5''$b. $\beta \cdot \lambda F^2 \cdot \lambda \mathrm{dom}_{\mathcal{K}^2} = \lambda M_{\mathcal{K}}$.

Proof Dually to the correspondence $\alpha \longleftrightarrow \sigma$, the correspondence $\beta \longleftrightarrow \tau$ is facilitated by the equations

$$\tau = \beta R_{\mathcal{K}}, \ \beta = \tau M_{\mathcal{K}} \cdot FF^2\varphi,$$

under which 4 of 3.4 corresponds exactly to $4'$, as one easily verifies using (29). That $5'$, $6'$ correspond to 5, 6 of 3.4 follows dually from $2'$, $3' \longleftrightarrow 2$, 3 as in Theorem 2.7. For $2''$, see (30); the equation $5''$ follow dually.

We call $(F, \lambda, \rho, \sigma, \tau)$ the *reduced presentation* of the slfa $(F, \lambda_1, \rho_1, \alpha, \beta)$.

3.9 We note that $2''$b, $5''$b tell us how to compute the factorization of a composite gf in an slfa: with $(f, g) : f \to g$ in $\mathcal{K}^2$, the following diagram commutes:

$$\begin{tikzcd}
Ff \arrow[r, "\lambda_{F(f,g)}"] & F(F(f,g)) \arrow[r, "\beta_{(f,g)}"] & F(f\cdot g) \arrow[r, "\alpha_{(f,g)}"] \arrow[drr, "\rho_{g\cdot f}"] & F(F(f,g)) \arrow[r, "\rho_{F(f,g)}"] & Fg \arrow[d, "\rho_g"] \\
A \arrow[u, "\lambda_f"] \arrow[urr, "\lambda_{g\cdot f}"'] \arrow[rr, "f"'] & & B \arrow[rr, "g"'] & & C
\end{tikzcd} \tag{31}$$

3.10 We give an *example of an lfa* $(F, \lambda, \rho, \sigma)$ (in reduced form) *for which* $(\mathcal{L}_F, \mathcal{R}_F)$ *fails to be a wfs.* Simply consider the poset $\mathbb{N}$ of natural numbers as a category, and put

$$F(n \to m) = \begin{cases} n & \text{if } n = m, \\ n+1 & \text{if } n < m, \end{cases}$$

which is clearly functorial. To obtain

$$\sigma_{n\to m} : F(n \to m) \to FF^2L(n \to m),$$

first observe that λ_1, ρ_1 are identity morphisms, so that

$$F^2L(n \to m) = \lambda_{n\to m} = \begin{cases} (n \to n) & \text{if } n = m, \\ (n \to n+1) & \text{if } n < m; \end{cases}$$

hence, $F(n \to m) = FF^2L(n \to m)$ in both cases, and $\sigma_{n\to m}$ is simply the identity morphism. Hence, F has the structure of an lfa (which fails to be symmetric). Although

$$\lambda_{n\to m} \in \mathcal{L}_F = \{k \to k,\ k \to k+1 \mid k \in \mathbb{N}\},$$

in general

$$\rho_{n\to m} \notin \mathcal{R}_F = \mathcal{L}_F \neq \mathcal{L}_F^{\square} = \{k \to k \mid k \in \mathbb{N}\}.$$

3.11 A functorial wfs may carry very distinct structures as an slfa. For example, in any category with finite products and coproducts, we may consider the factorization

$$\begin{tikzcd}
& A + (A \times B) \arrow[dr, "{[f, p_B]}"] & \\
A \arrow[ur, "i_A"] \arrow[rr, "f"'] & & B
\end{tikzcd} \tag{32}$$

which can be made into an slfa via

$$\alpha_{(u,v)} := i_A + < i_A p_A, i_{C\times D}(u \times 1_D) >:$$
$$A + (A\times D) \longrightarrow (A + (A\times B)) + ((A + (A\times B)) \times (C + (C\times D))),$$

$$\beta_{(u,v)} := [1_A + (1_A \times v), i_{A\times D} \cdot ([1_A, p_A] \times [g, p_D])].$$

As a wfs, in **Set** we obtain, as in (15), $\mathcal{L}_F = \text{Mono} \setminus \mathcal{M}_0$, $\mathcal{R}_F = \text{Epi} \cup \mathcal{M}_0$, with $\mathcal{M}_0$ the class of inclusion maps from $\emptyset$ into non-empty sets.

3.12 *Any left factorization system* (*as defined in* [JT]) *gives an example of an lfa which is not an slfa.* For example, let (F, λ, ρ) be given by factoring a morphism f through the coequalizer of its kernelpair, in any category that admits these (co)limits. Then $\mathcal{L}_F$ is the class of regular epimorphisms, and $\mathcal{R}_F$ is the class of monomorphisms. Since λ_1, ρ_1 are isomorphisms, and since $\lambda_f \in \mathcal{L}_F$ for all f, there is a natural isomorphism σ exhibiting $(F, \lambda, \rho, \sigma)$ as an lfa (see 2.9). However, the existence of a natural transformation τ making it symmetric would mean that $\rho_f \in \mathcal{R}_F$ for all f, which would make $\mathcal{L}_F$ closed under composition. However, in **cat**, or in the opposite of the category of semigroups, regular epimorphisms do not compose. Like in 3.10, $(\mathcal{L}_F, \mathcal{R}_F)$ is not a wfs.

4. Cofibrantly generated lax factorization algebras

4.1 The purpose of this section is to show:

Theorem *Every cofibrantly generated wfs in a locally presentable category has the structure of a lax factorization algebra.*

In fact, a functorial realization (F, λ, ρ) of the system $(\text{cof}(\mathcal{H}), \mathcal{H}^{\square})$ (with $\mathcal{H}$ a small set of morphisms in the locally presentable category $\mathcal{K}$) was established in 2.8. Hence, all that is needed is to construct a natural transformation $\sigma : F \to FF^2L_{\mathcal{K}}$ satisfying 2′a, 2′b, 3′ of 3.7.

For the construction of $\sigma_f : Ff \to F(F(1_A, f))$, for every $f : A \to B$ in $\mathcal{K}$, putting $l := \lambda_{1_A} : A \to F1_A =: \tilde{A}$ and $\tilde{f} := F(1_A, f) : \tilde{A} \to Ff$, one constructs l_i making the following diagram commutative, for every ordinal i (cp. (18)):

$$\begin{tikzcd}
A \arrow[r, "l"] \arrow[d, "p_i"'] & \tilde{A} \arrow[d, "\tilde{p}_i"] \\
A_i \arrow[r, "l_i"] \arrow[d, "s_i"'] & \tilde{A}_i \arrow[d, "\tilde{s}_{i+1}"] \\
A_{i+1} \arrow[r, "l_{i+1}"] \arrow[d, "k_{i+1}"'] & \tilde{A}_{i+1} \arrow[d, "\tilde{k}_{i+1}"] \\
Ff \arrow[r, "1"'] & Ff
\end{tikzcd} \tag{33}$$

In fact, each contribution (x, h, y) as in (17) to the colimit (A_{i+1}, f_{i+1}) makes, with its colimit injection $w : Y \to Ff$, the contribution $(l_i x, h, w)$ to the colimit $(\tilde{A}_{i+1}, \tilde{f}_{i+1})$, which then induces the arrow l_{i+1}. Taking the colimits of the respective chains, one obtains the desired arrow σ_f making the following diagram commutative:

$$\begin{tikzcd}
A \arrow[r, "\lambda_{1_A}"] \arrow[d, "\lambda_f"'] & F1_A \arrow[d, "\lambda_{F(1_A,f)}"] \\
Ff \arrow[r, "\sigma_f"] \arrow[d, "\rho_f"'] \arrow[dr, "1_{Ff}"'] & F(F(1_A, f)) \arrow[d, "\rho_{F(1_A,f)}"] \\
B & Ff \arrow[l, "\rho_f"]
\end{tikzcd} \tag{34}$$

4.2 Lemma σ_f *is natural in* f.

Proof We must show that, for $(u, v) : f \to g$ in $\mathcal{K}^2$, the bottom face of

$$
\begin{tikzcd}
& F1_A \arrow[rr, "F(u,u)"] \arrow[dd, "\lambda_{F(1_A,f)}" near start] & & F1_C \arrow[dd, "\lambda_{F(1_A,g)}"] \\
A \arrow[ur, "\lambda_{1_A}"] \arrow[rr, "u"' near start] \arrow[dd, "\lambda_f"'] & & C \arrow[ur, "\lambda_{1_C}"] \arrow[dd, "\lambda_g" near start] & \\
& F(F(1_A,f)) \arrow[rr, "F(F(u,u),F(u,v))"' near start] & & F(F(1_C,g)) \\
Ff \arrow[ur, "\sigma_f"] \arrow[rr, "F(u,v)"'] & & Fg \arrow[ur, "\sigma_g"'] &
\end{tikzcd}
\tag{35}
$$

commutes. This face is the colimit of a chain of faces above it, each of which needs to be shown to be commutative. We perform only the first step of this ordinal induction assuming, for ease of notation, that the first step is already the final one, i.e., $A_1 = Ff$, $p_1 = \lambda_f$, etc. Now, consider a contribution (x, h, y) to the colimit (Ff, ρ_f); then, according to their definitions, the arrows of the bottom face of (35) transform this contribution, as follows:

$$(x,h,y) \xmapsto{\sigma_f} (\lambda_{1_A} \cdot x, h, w) \xmapsto{F(F(u,u),F(u,v))} (F(u,u) \cdot \lambda_{1_A} \cdot x, h, F(u,v) \cdot w),$$

where $w : Y \to Ff$ is the colimit injection belonging to (x, h, y);

$$(x,h,y) \xmapsto{F(u,v)} (u \cdot x, h, v \cdot y) \xmapsto{\sigma_g} (\lambda_{1_C} \cdot u \cdot x, h, w'),$$

where $w' : Y \to Fg$ is the colimit injection belonging to $(u \cdot x, h, v \cdot y)$. But $F(u,u) \cdot \lambda_{1_A} = \lambda_{1_C} \cdot u$ and $F(u,v) \cdot w = w'$. Hence, the two transforms coincide, which shows that the two composite arrows themselves coincide. $\square$

4.3 ¿From diagram (34) we conclude directly:

$$\sigma \cdot \lambda = \lambda F^2 L_{\mathcal{K}} \cdot \lambda_1 \mathrm{dom}_{\mathcal{K}}, \quad \rho F^2 L_{\mathcal{K}} \cdot \sigma = 1_F.$$

We now show:

Lemma $F\rho^2 L_{\mathcal{K}} \cdot \sigma = 1_F$.

Proof For $f : A \to B$ in $\mathcal{K}$,

$$\rho^2_{L_{\mathcal{K}} f} = (\rho_{1_A}, \rho_f) : F(1_A, f) \to f.$$

Hence, we must show that the bottom composite arrow in

$$\begin{tikzcd}
A \arrow[r, "\lambda_{1_A}"] \arrow[d, "\lambda_f"'] & F1_A \arrow[r, "\rho_{1_A}"] \arrow[d, "\lambda_{F(1_A,f)}"] & A \arrow[d, "\lambda_f"] \\
Ff \arrow[r, "\sigma_f"'] & F(F(1_A, f)) \arrow[r, "F(\rho_{1_A}, \rho_f)"'] & Ff
\end{tikzcd} \tag{36}$$

is the identity morphism. But, in the symbolic setting of 4.2 one indeed has:

$$(x, h, y) \overset{\sigma_f}{\longmapsto} (\lambda_{1_A} \cdot x, h, w) \xmapsto{F(\rho_{1_A}, \rho_f)} (\rho_{1_A} \cdot \lambda_{1_A} \cdot x, h, \rho_f \cdot w) = (x, h, y).$$

$\square$

4.4 Lemma $\sigma F^2 L_{\mathcal{K}} \cdot \sigma = F\sigma^2 L_{\mathcal{K}} \cdot \sigma$

Proof For $f : A \to B$ in $\mathcal{K}$ and $\tilde{f} := F(1_A, f) : \tilde{A} = F1_A \to Ff$, we must show that the bottom face of

$$\begin{tikzcd}
& \tilde{A} \arrow[rr, "\lambda_{1_{\tilde{A}}}"] \arrow[dd, "\lambda_{\tilde{f}}"] & & F1_{\tilde{A}} \arrow[dd, "\lambda_{F(1_{\tilde{A}}, \tilde{f})}"] \\
A \arrow[ur, "\lambda_{1_A}"] \arrow[rr, "\lambda_{1_A}"'] \arrow[dd, "\lambda_f"'] & & \tilde{A} \arrow[ur, "\sigma_{1_A}"] \arrow[dd, "\lambda_{\tilde{f}}"] & \\
& F\tilde{f} \arrow[rr, "\sigma_{\tilde{f}}"'] & & F(F(1_{\tilde{A}}, \tilde{f})) \\
Ff \arrow[ur, "\sigma_f"] \arrow[rr, "\sigma_f"'] & & F\tilde{f} \arrow[ur, "F(\sigma_{1_A}, \sigma_f)"'] &
\end{tikzcd} \tag{37}$$

commutes. Note that all other faces commute, by definition of σ and functoriality of F. In the symbolic setting of 4.2 we observe:

$$(x,h,y) \xmapsto{\ \sigma_f\ } (\lambda_{1_A} \cdot x, h, w) \xmapsto{\ \sigma_{\tilde f}\ } (\lambda_{1_{\tilde A}} \cdot \lambda_{1_A} \cdot x, h, w'),$$

where $w : Y \to Ff$ is the colimit injection belonging to (x,h,y), and $w' : Y \to F\tilde{f}$ is the colimit injection belonging to $(\lambda_{1_A} \cdot x, h, w)$, which is precisely $\sigma_f \cdot w$. Hence, the above composite transformation coincides with

$$(x,h,y) \xmapsto{\ \sigma_f\ } (\lambda_{1_A} \cdot x, h, w) \xmapsto{\ F(\sigma_{1_A}, \sigma_f)\ } (\sigma_{1_A} \cdot \lambda_{1_A} \cdot x, h, \sigma_f \cdot w),$$

as desired. □

5. Lax homomorphisms

5.1 A *left morphism* from a wfs $(\mathcal{L}, \mathcal{R})$ on $\mathcal{K}$ to a wfs $(\mathcal{L}', \mathcal{R}')$ on $\mathcal{K}'$ is a functor $H : \mathcal{K} \to \mathcal{K}'$ with $H(\mathcal{L}) \subseteq \mathcal{L}'$; dually, a *right morphism* H satisfies $H(\mathcal{R}) \subseteq \mathcal{R}'$. It is easy to see that, if $H \dashv K : \mathcal{K}' \to \mathcal{K}$, then H is a left morphism if and only if K is a right morphism. In case of functorial wfs, one has:

Proposition *Let (F, λ, ρ), (F', λ', ρ') be a functorial realization of a wfs on $\mathcal{K}$, $\mathcal{K}'$, respectively. Then $H : \mathcal{K} \to \mathcal{K}'$ is a left morphism from $\mathcal{K}$ to $\mathcal{K}'$ if and only if, for all morphisms $f : A \to B$ in $\mathcal{K}$ there is a morphism $\Phi_f : HFf \to F'Hf$ in $\mathcal{K}'$ which makes*

$$\begin{tikzcd} HA \arrow[r, "\lambda'_{Hf}"] \arrow[d, "H\lambda_f"'] & F'Hf \arrow[d, "\rho'_{Hf}"] \\ HFf \arrow[ur, dashed, "\Phi_f"] \arrow[r, "H\rho_f"'] & HB \end{tikzcd} \tag{38}$$

commutative.

Proof The necessity of the condition follows from the lifting property of $(\mathcal{L}_{F'}, \mathcal{R}_{F'})$. In order to show its sufficiency, consider $f \in \mathcal{L}_F$, so that $\lambda_f = s \cdot f$, $\rho_f \cdot s = 1_B$ for some $s : B \to Ff$. But then, with $s' := \Phi_f \cdot Hs$, one obtains $\lambda'_{Hf} = s' \cdot Hf$, $\rho'_{Hf} \cdot s' = 1_{HB}$, hence $Hf \in \mathcal{L}_{F'}$.

5.2 Definition (1) A *lax homomorphism* $(H, \Phi) : (F, \lambda_1, \rho_1, \alpha) \to (F', \lambda_1', \rho_1', \alpha')$ *from an lfa on* $\mathcal{K}$ *to an lfa on* $\mathcal{K}'$ consists of a functor $H : \mathcal{K} \to \mathcal{K}'$ and a natural transformation $\Phi : HF \to F'H^2$ such that

1. $\rho_1' H \cdot \Phi E_{\mathcal{K}} = H\rho_1$,

2. $\alpha'(H^2)^2 \cdot \Phi M_{\mathcal{K}} = F'\Phi^2 \cdot \Phi F^2 \cdot H\alpha$.

$$
\begin{tikzcd}
HFE_{\mathcal{K}} \arrow[r, "\Phi E_{\mathcal{K}}"] \arrow[dr, "H\rho_1"'] & F'H^2E_{\mathcal{K}} \arrow[d, "\rho_1' H"] \\
 & H
\end{tikzcd}
\qquad
\begin{tikzcd}
HFM_{\mathcal{K}} \arrow[r, "H\alpha"] \arrow[d, "\Phi M_{\mathcal{K}}"'] & HFF^2 \arrow[r, "\Phi F^2"] & F'H^2F^2 \arrow[d, "F'\Phi^2"] \\
F'H^2M_{\mathcal{K}} \arrow[rr, "\alpha'(H^2)^2"'] & & F'(F')^2(H^2)^2
\end{tikzcd}
\tag{39}
$$

Hence, in the terminology of [S], it is an op-lax homomorphism $(H, \Phi) : (F, \rho_1, \alpha) \to (F', \rho_1', \alpha')$.

(2) A *lax homomorphism* $(H, \Phi, \Theta) : (F, \lambda_1, \rho, \alpha, \beta) \to (F', \lambda_1', \rho_1', \alpha', \beta')$ *from an slfa on* $\mathcal{K}$ *to an slfa on* $\mathcal{K}'$ involves, in addition to H, Φ as above satisfying 1, 2, a natural transformation $\Theta : F'H^2 \to HF$ such that

3. $\Theta E_{\mathcal{K}} \cdot \lambda_1' H = H\lambda_1$,

4. $\Theta M_{\mathcal{K}} \cdot \beta'(H^2)^2 = H\beta \cdot \Theta F^2 \cdot F'\Theta^2$.

Hence, in the terminology of [S], $(H, \Theta) : (F, \lambda_1, \beta) \to (F', \lambda_1', \beta')$ is a lax homomorphism.

5.3 In terms of their reduced presentations, a lax homomorphism $(H, \Phi) : (F, \lambda, \rho, \sigma) \to (F', \lambda', \rho', \sigma')$ of lfa's is characterized by the conditions

$1'$. $\rho' H^2 \cdot \Phi = H\rho$,

$2'$. $\sigma' H^2 \cdot \Phi = F'\Phi^2 L_{\mathcal{K}} \cdot \Phi F^2 L_{\mathcal{K}} \cdot H\sigma$,

and a lax homomorphism $(H, \Phi, \Theta) : (F, \lambda, \rho, \sigma, \tau) \to (F', \lambda', \rho', \sigma', \tau')$ has to satisfy the additional conditions

$3'$. $\Theta \cdot \lambda' H^2 = H\lambda$,

$4'$. $\Theta \cdot \tau' H^2 = H\tau \cdot \Theta F^2 R_{\mathcal{K}} \cdot F'\Theta^2 R_{\mathcal{K}}$.

We must leave the verifications of these assertions to the Reader.

Corollary *A lax homomorphism* $(H, \Phi) : (F, \lambda_1, \rho_1, \alpha) \to (F', \lambda_1', \rho_1', \alpha')$ *of lfa's with*

$$\Phi E_{\mathcal{K}} \cdot H\lambda_1 = \lambda_1' H$$

satisfies $H(\mathcal{L}_F) \subseteq \mathcal{L}_{F'}$*; hence,* H *is a left morphism if each lfa induces a wfs. For a lax homomorphisms* (H, Φ, Θ) *of slfa's with*

$$H\rho_1 \cdot \Theta E_{\mathcal{K}} = \rho_1' H$$

one has $H(\mathcal{R}_F) \subseteq \mathcal{R}_{F'}$*, so that* H *is a right morphism if each slfa induces a wfs.*

Proof Just like 1 of 5.2 is equivalent to 1′, the equation $\Phi E_{\mathcal{K}} \cdot H\lambda_1 = \lambda_1' H$ is equivalent to $\Phi \cdot H\lambda = \lambda' H^2$, so that the first assertion follows from 5.1. The second assertion follows dually. □

5.4 Proposition *Let* $H : \mathcal{K} \to \mathcal{K}'$ *be any functor of categories with finite coproducts, and let* $(F, \lambda_1, \rho_1, \alpha, \beta)$, $(F', \lambda_1', \rho_1', \alpha', \beta')$ *be the slfa given by the respective cograph factorization in* $\mathcal{K}$, $\mathcal{K}'$ *(see 3.5). Then* $H(\mathcal{R}_F) \subseteq \mathcal{R}_{F'}$*, and there are* Φ, Θ *such that* (H, Φ, Θ) *is a lax homomorphism of slfa's. Also,* $H(\mathcal{L}_F) \subseteq \mathcal{L}_{F'}$*, whenever* H *preserves finite coproducts.*

Proof For $f : A \to B$ in $\mathcal{K}$ one puts

$$\Phi_f := (H(A+B) \xrightarrow{H[f, 1_B]} HB \xrightarrow{i_{HB}} HA + HB), \ \Theta_f := [Hi_A, Hi_B]$$

and (patiently) shows naturality of Φ_f, Θ_f as well as the identities 1-4 of 5.2. The inclusion $H(\mathcal{R}_F) \subseteq \mathcal{R}_{F'}$ follows with 5.1 and 5.3. If H preserves finite coproducts, then Θ is a natural isomorphism, and (H, Θ^{-1}, Θ) becomes a lax homomorphism, showing in particular $H(\mathcal{L}_F) \subseteq \mathcal{L}_{F'}$, with 5.1. □

Let $\mathbf{CAT}_+$ be the category of all categories with finite coproducts, as a *full* subcategory of **CAT**. By **SLFA** we denote the category of all slfa's and their lax homomorphisms (leaving it to the Reader to show that lax homomorphisms compose!), and $\mathbf{SLFA}_+$ is its full subcategory of slfa's on categories with finite coproducts given by the cograph factorization. The Proposition shows that the forgetful functor

$$\mathbf{SLFA}_+ \to \mathbf{CAT}_+$$

is full. The Proposition also establishes a faithful functor

$$\mathbf{CAT}_+ \to \mathbf{SLFA}_+, \quad H \mapsto (H, \Phi, \Theta),$$

which, however, is *not* full. In fact, any functor $H : \mathcal{K} \to \mathcal{K}'$ gets structured differently from above when we trade Θ_f for $\tilde{\Theta}_f$ with $\tilde{\Theta}_f := Hi_B \cdot [Hf, 1_{HB}]$.

5.5 The orthogonal factorization system on $\mathcal{K}^2$ depicted by (22) is known to be the "free system over $\mathcal{K}$" (see [G], [RV]). It maintains this role amongst lfa's and slfa's if one changes "free" to "lax-free", as we show next.

Theorem *The embedding $E_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^2$ and the lfa $(M_{\mathcal{K}}, 1, 1, 1)$ on $\mathcal{K}^2$ have the following lax-universal property: given a functor $G : \mathcal{K} \to \mathcal{K}'$ into a category with an lfa $(F', \lambda'_1, \rho'_1, \alpha')$, there are a lax homomorphism*

$$(H, \Phi) : (M_{\mathcal{K}}, 1, 1, 1) \to (F', \lambda'_1, \rho'_1, \alpha')$$

and a natural transformation $\varepsilon : HE_{\mathcal{K}} \to G$ such that, for any other lax homomorphism

$$(\tilde{H}, \tilde{\Phi}) : (M_{\mathcal{K}}, 1, 1, 1) \to (F', \lambda'_1, \rho'_1, \alpha')$$

and natural transformation $\tilde{\varepsilon} : \tilde{H}E_{\mathcal{K}} \to G$, there is a unique transformation $\delta : \tilde{H} \to H$ satisfying

$$\varepsilon \cdot \delta E_{\mathcal{K}} = \tilde{\varepsilon} \ \text{ and } \ \Phi \cdot \delta M_{\mathcal{K}} = F'\delta^2 \cdot \tilde{\Phi}$$

.

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "E_{\mathcal{K}}"] \arrow[dr, "G"'] & \mathcal{K}^2 \arrow[d, "{(H,\Phi)}"] \arrow[dl, phantom, "\overset{\varepsilon}{\Leftarrow}"] & & \mathcal{K}^2 \arrow[d, "{(\tilde{H},\tilde{\Phi})}"'] \arrow[ll, dashed, "\delta"'] & \mathcal{K} \arrow[l, "E_{\mathcal{K}}"'] \arrow[dl, "G"] \\
& \mathcal{K}' & & \mathcal{K}' \arrow[ur, phantom, "\overset{\tilde{\varepsilon}}{\Rightarrow}"] &
\end{tikzcd} \tag{40}$$

In particular, for $G = \mathrm{Id}_{\mathcal{K}}$, every lfa on $\mathcal{K}$ is exhibited as a "lax quotient" of the free system on $\mathcal{K}^2$.

Proof Putting $H := F'G^2$, $\Phi := \alpha'(G^2)^2$, $\varepsilon := \rho'_1 G$, we can leave it to the Reader to check that (H, Φ) is a lax homomorphism. We note that Φ and ε are related by

$$
\begin{aligned}
F'\varepsilon^2 \cdot \Phi(E_{\mathcal{K}})^2 &= F'(\rho_1')^2 G^2 \cdot \alpha'(G^2)^2(E_{\mathcal{K}})^2 \\
&= F'(\rho_1')^2 G^2 \cdot \alpha'(E_{\mathcal{K}'})^2 G^2 \\
&= 1_H.
\end{aligned}
$$

Now, given $\tilde{H}$, $\tilde{\Phi}$, $\tilde{\varepsilon}$ as above, we define δ as the composite

$$
\tilde{H} \xrightarrow{\tilde{\Phi}(E_{\mathcal{K}})^2} F'\tilde{H}^2(E_{\mathcal{K}})^2 \xrightarrow{F'\tilde{\varepsilon}^2} F'G^2 = H.
$$

Then

$$
\begin{aligned}
\varepsilon \cdot \delta E_{\mathcal{K}} &= \rho_1' G \cdot F'\tilde{\varepsilon}^2 E_{\mathcal{K}} \cdot \tilde{\Phi}(E_{\mathcal{K}})^2 E_{\mathcal{K}} \\
&= \tilde{\varepsilon} \cdot \rho_1' \tilde{H} E_{\mathcal{K}} \cdot \tilde{\Phi} E_{\mathcal{K}^2} E_{\mathcal{K}} && (\varepsilon, \rho_1 \text{ nat.}) \\
&= \tilde{\varepsilon} && (\text{by 5.2, 1})
\end{aligned}
$$

$$
\begin{aligned}
F'\delta^2 \cdot \tilde{\Phi} &= F'(F')^2(\tilde{\varepsilon}^2)^2 \cdot F'\tilde{\Phi}^2((E_{\mathcal{K}})^2)^2 \cdot \tilde{\Phi}(M_{\mathcal{K}})^2((E_{\mathcal{K}})^2)^2 \\
&= F'(F')^2(\tilde{\varepsilon}^2)^2 \cdot \alpha'(\tilde{H}^2)^2((E_{\mathcal{K}})^2)^2 \cdot \tilde{\Phi} M_{\mathcal{K}^2}((E_{\mathcal{K}})^2)^2 && (\text{by 5.2, 2}) \\
&= \alpha'(G^2)^2 \cdot F' M_{\mathcal{K}}(\tilde{\varepsilon}^2)^2 \cdot \tilde{\Phi}(E_{\mathcal{K}})^2 M_{\mathcal{K}} && (\tilde{\varepsilon}, \alpha' \text{ nat.}) \\
&= \Phi \cdot \delta M_{\mathcal{K}}.
\end{aligned}
$$

If $\eta : \tilde{H} \to H$ satisfies $\varepsilon \cdot \eta E_{\mathcal{K}} = \tilde{\varepsilon}$, $F'\eta^2 \cdot \tilde{\Phi} = \Phi \cdot \eta M_{\mathcal{K}}$, then

$$
\begin{aligned}
\delta &= F'\tilde{\varepsilon}^2 \cdot \tilde{\Phi}(E_{\mathcal{K}})^2 \\
&= F'\varepsilon^2 \cdot F'\eta^2(E_{\mathcal{K}})^2 \cdot \tilde{\Phi}(E_{\mathcal{K}})^2 \\
&= F'\varepsilon^2 \cdot \Phi(E_{\mathcal{K}})^2 \cdot \eta M_{\mathcal{K}}(E_{\mathcal{K}})^2 \\
&= 1_H \cdot \eta = \eta.
\end{aligned}
$$

$\square$

5.6 Corollary *For every functor $G : \mathcal{K} \to \mathcal{K}'$ into an slfa $(F', \lambda_1', \rho_1', \alpha', \beta')$, there are a lax homomorphism*

$$
(H, \Phi, \Theta) : (M_{\mathcal{K}}, 1, 1, 1, 1) \to (F', \lambda_1', \rho_1', \alpha', \beta')
$$

and natural transformations

$$
\varepsilon : HE_{\mathcal{K}} \to G, \;\; \varsigma : G \to HE_{\mathcal{K}} \;\text{ with }\; \varepsilon \cdot \varsigma = 1_G
$$

such that, for any other homomorphism

$$(\tilde{H}, \tilde{\Phi}, \tilde{\Theta}) : (M_{\mathcal{K}}, 1, 1, 1, 1) \to (F', \lambda_1', \rho_1', \alpha', \beta')$$

and transformations $\tilde{\varepsilon} : \tilde{H}E_{\mathcal{K}} \to G$, $\tilde{\varsigma} : G \to \tilde{H}E_{\mathcal{K}}$, *there are unique transformations* $\delta : \tilde{H} \to H$, $\gamma : H \to \tilde{H}$ *satisfying* $\varepsilon \cdot \delta E_{\mathcal{K}} = \tilde{\varepsilon}$, $\Phi \cdot \delta M_{\mathcal{K}} = F'\delta^2 \cdot \tilde{\Phi}$, $\gamma E_{\mathcal{K}} \cdot \varsigma = \tilde{\varsigma}$, $\gamma M_{\mathcal{K}} \cdot \Theta = \tilde{\Theta} \cdot F'\gamma^2$.

Proof Dually from 4.5, with $\varsigma := \lambda_1' G$, $\Theta := \beta'(G^2)^2$, $\gamma = \tilde{\Theta}(E_{\mathcal{K}})^2 F'\tilde{\varsigma}^2$. □

We note that, even if $\mathcal{K} = \mathcal{K}'$ and $G = \mathrm{Id}_{\mathcal{K}}$ and if the given slfa induces a wfs, *in general the functor H is not a left or right morphism of wfs.* In fact, if $(F, \lambda_1, \rho_1, \alpha, \beta)$ is the slfa given by the cograph factorization, $H = F : \mathcal{K}^2 \to \mathcal{K}$ fails badly to map $\mathcal{L}_{M_{\mathcal{K}}}$ into $\mathcal{L}_F$ or $\mathcal{R}_{M_{\mathcal{K}}}$ into $\mathcal{R}_F$.

5.7 Finally we return to cofibrantly generated lfa's and sketch the proof of:

Theorem *Let* $H : \mathcal{K} \to \mathcal{K}'$ *be a cocontinuous functor of locally presentable categories, mapping a given small set* $\mathcal{H}$ *of morphisms in* $\mathcal{K}$ *into the given set* $\mathcal{H}'$ *of morphisms in* $\mathcal{K}'$*. Then H is a left morphism of the cofibrantly generated wfs's induced by* $\mathcal{H}$ *and* $\mathcal{H}'$*; indeed, H carries the structure of a lax homomorphism of the respective lfa's, as given by 4.1.*

Proof With $(F, \lambda, \rho, \sigma)$, $(F', \lambda', \rho', \sigma')$ denoting the lfa's in question, for each ordinal i one defines a commutative diagram

$$\begin{tikzcd}
HA \arrow[r, "1"] \arrow[d, "Hp_i"'] & HA \arrow[d, "\tilde{p}_i"] \\
HA_i \arrow[r, "l_i"] \arrow[d, "Hs_i"'] & (HA)_i \arrow[d, "\tilde{s}_i"] \\
HA_{i+1} \arrow[r, "l_{i+1}"] \arrow[d, "Hf_{i+1}"'] & (HA)_{i+1} \arrow[d, "(Hf)_{i+1}"] \\
HB \arrow[r, "1"'] & HB
\end{tikzcd} \tag{41}$$

by observing that, since $H(\mathcal{H}) \subseteq \mathcal{H}'$, each contribution (x, h, y) to the colimit (A_{i+1}, f_{i+1}) gives the contribution (Hx, Hh, Hy) to the colimit $((HA)_{i+1}, (Hf)_{i+1})$, hence defining the morphism l_{i+1} since H preserves the colimit (A_{i+1}, f_{i+1}). As a colimit of the morphisms l_i one obtains $\Phi_f : HFf \to F'Hf$ rendering diagram (38) commutative. We omit the proofs that Φ_f is natural in f and that the conditions $1'$, $2'$ of 5.3 are satisfied, since the arguments are similar to those used in the proof of Theorem 4.1. Hence (H, Φ) is a lax homomorphism, and H is a left morphism by 5.1. □

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