LAWVERE COMPLETION AND SEPARATION VIA CLOSURE

DIRK HOFMANN AND WALTER THOLEN

Dedicated to Bill Lawvere at the occasion of his seventieth birthday

Abstract. For a quantale V, first a closure-theoretic approach to completeness and separation in V-categories is presented. This approach is then generalized to $\mathcal{T}$-categories, where $\mathcal{T}$ is a topological theory that entails a set monad $\mathbb{T}$ and a compatible $\mathbb{T}$-algebra structure on V.

Introduction

Bill Lawvere's 1973 milestone paper "Metric spaces, generalized logic, and closed categories" helped us to detect categorical structures in previously unexpected surroundings. His revolutionary idea was not only to regard individual metric spaces as categories (enriched over the monoidal-closed category given by the non-negative extended real half-line, with arrows provided by $\geq$ and tensor by $+$), but also to expose the purely categorical nature of the key concept of the theory, Cauchy completeness. The first step to this end was to disregard metric conditions that actually obscure the categorical intuition. In fact, once one has dropped the symmetry requirement it seems much more natural to regard the metric d of a space X as the categorical hom and, given a Cauchy sequence (a_n) in X, to associate with it the pair of functions

$$\varphi(x) = \lim d(a_n, x) \qquad \text{and} \qquad \psi(x) = \lim d(x, a_n).$$

Lawvere's great insight was to expose these functions as pairs of adjoint (bi)modules whose representability as

$$\varphi(x) = d(a, x) \qquad \text{and} \qquad \psi(x) = d(x, a)$$

is facilitated precisely by a limit a for (a_n). Hence, a new notion of completeness for categories enriched over any symmetric monoidal-closed category V was born. Also in the enriched category context it is often referred to as Cauchy completeness. But since Lawvere's brilliant notion entails no sequences at all, just the representability requirement for bimodules, this name seems to be far-fetched and, contrary to popular belief, was in fact not proposed in his paper. Hence, here we use *L-completeness* instead.

In the first part of this paper we give a quick introduction to V-category theory (see [Kel82]) in the special case of a commutative unital quantale V, focussing on the themes of L-completion and *L-separation*. We are not aware of an explicit prior occurrence of the latter notion, and both themes are treated with the help of a new closure operator that arises most naturally in the 2-category $\mathsf{V}\text{-}\mathsf{Cat}$, as follows. Call a V-functor $m : M \longrightarrow X$ *L-dense* if $f \cdot m = g \cdot m$ implies $f \cong g$ for all V-functors $f, g : X \longrightarrow Y$; the *L-closure* of a subobject M of X is then the largest subobject $\overline{M}$ of X for which $M \longrightarrow \overline{M}$ is L-dense. For L-separated V-categories, L-dense simply means epimorphism. The L-separated reflection of

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a V-category X is its image under the Yoneda functor $y : X \longrightarrow \mathsf{V}^{X^{\mathrm{op}}} = \hat{X}$, and its L-completion is the L-closure of that image in $\hat{X}$.

The main part of the paper is devoted to a substantial generalization of the first part which, however, without the reader's recalling of the more familiar V-category context, may be hard to motivate, especially in view of the considerable additional "technical" difficulties. The quantale V gets augmented by a *topological theory* $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ which now entails also a Set-monad $\mathbb{T}$ and a $\mathbb{T}$-algebra structure ξ on V, with suitable compatibility conditions (see [Hof07]). While a V-category X comes with a V-relation $a : X \relbar\!\!\!+\!\!\!\rightarrow X$ (given by a function $a : X \times X \longrightarrow \mathsf{V}$), $\mathcal{T}$-categories come with a V-relation $a : TX \relbar\!\!\!+\!\!\!\rightarrow X$ making X a lax $\mathbb{T}$-algebra. For $\mathbb{T}$ the ultrafilter monad and $\mathsf{V} = 2$, $\mathcal{T}$-Cat provides Barr's [Bar70] relational description of the category of topological spaces (which, in turn, was based on Manes' [Man69] description of compact Hausdorff spaces); for the same monad but with V the Lawvere half-line, one obtains Lowen's approach spaces [Low89], as shown by Clementino and Hofmann [CH03].

The V-to-$\mathcal{T}$ generalization must necessarily entail the provision of a Yoneda functor for a $\mathcal{T}$-category X. But what is X^{op} supposed to be in this highly asymmetric context? Fortunately, this problem was solved in [CH08]: the underlying set of X^{op} is TX, provided with a suitable $\mathcal{T}$-structure. This structure needs to be considered in addition to the free $\mathbb{T}$-algebra structure on TX, leading to the surprising fact that the $\mathcal{T}$-equivalent of the Yoneda functor of the the familiar V-context has now two equally important facets. Once one has fully understood this "technical" part of the general theory, it is in fact rather straightforward to extend the V-categorical results on L-completion and L-separation to $\mathcal{T}$-categories, again with the help of the L-closure. We could therefore often keep the proofs in the $\mathcal{T}$-context quite short, especially when no new ideas beyond the initial "Yoneda investment" are needed.

Completeness of V-categories and the induced topology was also investigated by Flagg [Fla97, Fla92] (who called them V-continuity spaces). Its generalization to (essentially) $\mathcal{T}$-categories was introduced by Clementino and Hofmann [CH08]. We also refer the reader to Burroni [Bur71], wo presented an alternative approach to the categories of interest in this paper.

1. Preliminaries

1.1. **The quantale V.** Throughout the paper we consider a commutative and unital quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$. Hence, V is a complete lattice with a commutative binary operation $\otimes$ and neutral element k, such that $u \otimes (-)$ preserves suprema, for all $u \in \mathsf{V}$. Consequently, V has an "internal hom" $u \multimap (-)$, given by

$$z \le u \multimap v \iff z \otimes u \le v,$$

for all $z, u, v \in \mathsf{V}$. Sometimes we write $v \multimapinv u$ instead of $u \multimap v$. The quantale is *trivial* when $\mathsf{V} = 1$; equivalently, when $k = \bot$ is the bottom element of V. Non-trivial examples of quantales are the two-element chain $2 = (\{0, 1\}, \wedge, 1)$, the extended positive half-line $\mathsf{P}_+ = ([0, \infty]^{\mathrm{op}}, +, 0)$, and $\mathsf{P}_{\max} = ([0, \infty]^{\mathrm{op}}, \max, 0)$; here $[0, \infty]^{\mathrm{op}} = ([0, \infty], \ge)$, with the natural $\ge$. (We will use $\bigvee$, $\bigwedge$ to denote suprema, infima in V, but use sup, inf, max, etc. when we work in $[0, \infty]$ and refer to the natural order $\le$.)

1.2. **V-relations.** The category V-Rel has sets as objects, and a morphism $r : X \relbar\!\!\!+\!\!\!\rightarrow Y$ is simply a function $r : X \times Y \longrightarrow \mathsf{V}$; its composite with $s : Y \relbar\!\!\!+\!\!\!\rightarrow Z$ is given by

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

There is a functor

$$\mathsf{Set} \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}$$

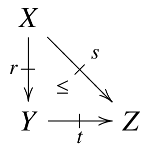
which maps objects identically and interprets a map $f : X \longrightarrow Y$ as a V-relation $f_\circ : X \nrightarrow Y$:

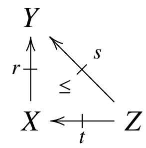
$$f_\circ(x,y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{otherwise;} \end{cases}$$

we normally write f instead of $f_\circ$. The functor is faithful precisely when $k > \bot$. The hom-sets of V-Rel carry the pointwise order of V, so that V-Rel becomes an ordered category. In fact, V-Rel is Sup-enriched (with Sup the category if complete lattices and suprema-preserving maps), hence it is a quantaloid. Consequently, for every $r : X \nrightarrow Y$, composition by r in V-Rel from either side has a right adjoint, given by *extensions* and *liftings* respectively:

$$(-)\cdot r \dashv (-) \mathrel{\bullet\!\!-} r \qquad\qquad r\cdot(-) \dashv r \mathrel{-\!\!\bullet} (-)$$

$$\frac{t\cdot r \le s}{t \le s \mathrel{\bullet\!\!-} r} \qquad\qquad \frac{r\cdot t\le s}{t \le r \mathrel{-\!\!\bullet} s}$$





$$s \mathrel{\bullet\!\!-} r(y,z) = \bigwedge_{x\in X} s(x,z) \mathrel{\circ\!\!-} r(x,y) \qquad\qquad r \mathrel{-\!\!\bullet} s(z,x) = \bigwedge_{y\in Y} r(x,y) \multimap s(z,y)$$

V-Rel has a contravariant involution

$$(\mathsf{V}\text{-}\mathsf{Rel})^{\mathrm{op}} \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}$$

which maps objects identically and assigns to $r : X \nrightarrow Y$ its opposite relation $r^\circ : Y \nrightarrow X$. When applied to a map $f = f_\circ$, one obtains $f \dashv f^\circ$ in the 2-category V-Rel.

1.3. **V-categories.** A V-*category* $X = (X, a)$ is a set X with a V-relation $a : X \nrightarrow X$ satisfying $1_X \le a$, $a \cdot a \le a$; equivalently,

$$k \le a(x,x), \qquad\qquad a(x,y)\otimes a(y,z) \le a(x,z)$$

for all $x,y,z \in X$. A V-functor $f : (X,a) \longrightarrow (Y,b)$ must satisfy $f \cdot a \le b \cdot f$; equivalently,

$$a(x,y) \le b(f(x), f(y))$$

for all $x, y \in X$. The resulting category V-Cat is the category Ord of (pre)ordered sets if V = 2, Lawvere's category Met of (pre)metric spaces if $\mathsf{V} = \mathsf{P}_+$ (see [Law73]), and the category UMet of (pre)ultrametric spaces if $\mathsf{V} = \mathsf{P}_{\max}$. For the trivial quantale one has 1-Cat = Set. Furthermore, $\mathsf{V} = (\mathsf{V}, \multimap)$ with its internal hom becomes a V-category.

V-Cat is a symmetric monoidal closed category, with tensor product

$$(X,a)\otimes(Y,b) = (X\times Y, a\otimes b), \qquad\qquad a\otimes b((x,y),(x',y')) = a(x,x')\otimes b(y,y'),$$

and internal hom

$$(X,a) \multimap (Y,b) = (\mathsf{V}\text{-}\mathsf{Cat}(X,Y), [a,b]), \qquad\qquad [a,b](f,g) = \bigwedge_{x\in X} b(f(x), g(x)).$$

The $\otimes$-neutral object is $E = (E,k)$ (with a singleton set E), which generally must be distinguished from the terminal object $1 = (1, \top)$ in V-Cat. The internal hom describes the pointwise order if V = 2, and the usual sup-metric if $\mathsf{V} = \mathsf{P}_+$ or $\mathsf{V} = \mathsf{P}_{\max}$.

1.4. **V-modules.** The category V-Mod has V-categories as objects, and a morphism $\varphi : (X, a) \mathrel{-\!\!\circ\!\!\to} (Y, b)$ is a V-relation $\varphi : X \mathrel{-\!\!\!\mapstochar\!\!\to} Y$ with $\varphi \cdot a \leq \varphi$ and $b \cdot \varphi \leq \varphi$. Since always $\varphi = \varphi \cdot 1_X \leq \varphi \cdot a$ and $\varphi = 1_Y \cdot \varphi \leq b \cdot \varphi$, one actually has $\varphi \cdot a = \varphi$ and $b \cdot \varphi = \varphi$ for a V-module $\varphi : X \mathrel{-\!\!\circ\!\!\to} Y$. In particular, the V-module $a : X \mathrel{-\!\!\circ\!\!\to} X$ assumes the role of the identity morphism on X in V-Mod, and we write $a = 1_X^*$, in order not to confuse it with 1_X in V-Cat. This notation is extended to arbitrary maps $f : X \longrightarrow Y$ by

$$f_* = b \cdot f \qquad\qquad \text{and} \qquad\qquad f^* = f^\circ \cdot b,$$

and one easily verifies:

Lemma 1.1. *The following are equivalent for a map $f : X \longrightarrow Y$ between V-categories X and Y:*

(i) *$f : X \longrightarrow Y$ is a V-functor.*
(ii) *f_* is a V-module $f_* : X \mathrel{-\!\!\circ\!\!\to} Y$.*
(iii) *f^* is a V-module $f^* : Y \mathrel{-\!\!\circ\!\!\to} X$.*

Hence there are functors which make the following diagram commute.

$$\begin{array}{ccccc} \mathsf{V\text{-}Cat} & \xrightarrow{(-)_*} & \mathsf{V\text{-}Mod} & \xleftarrow{(-)^*} & (\mathsf{V\text{-}Cat})^{\mathrm{op}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathsf{Set} & \xrightarrow[(-)_\circ]{} & \mathsf{V\text{-}Rel} & \xleftarrow[(-)^\circ]{} & \mathsf{Set}^{\mathrm{op}} \end{array}$$

Here the vertical full embeddings are given by $X \longmapsto (X, 1_X)$. Just like V-Rel also V-Mod is a quantaloid, with the same pointwise order structure. But not just suprema of V-modules formed in V-Rel are again V-modules, also extensions and liftings. For example, for $\varphi : (X, a) \mathrel{-\!\!\circ\!\!\to} (Y, b)$, $\psi : (Z, c) \mathrel{-\!\!\circ\!\!\to} (Y, b)$, the lifting $\varphi \mathrel{-\!\!\bullet} \psi$ formed in V-Rel is indeed a V-module $\varphi \mathrel{-\!\!\bullet} \psi : (Z, c) \mathrel{-\!\!\circ\!\!\to} (Y, b)$: from $\psi \cdot c \leq \psi$ and $\varphi \cdot (\varphi \mathrel{-\!\!\bullet} \psi) \leq \psi$ one obtains $\varphi \cdot (\varphi \mathrel{-\!\!\bullet} \psi) \cdot c \leq \psi$ and then $(\varphi \mathrel{-\!\!\bullet} \psi) \cdot c \leq \varphi \mathrel{-\!\!\bullet} \psi$; similarly $a \cdot (\varphi \mathrel{-\!\!\bullet} \psi) \leq \varphi \mathrel{-\!\!\bullet} \psi$. Also the contravariant involution of V-Rel extends to V-Mod (e.g., if $\varphi : X \mathrel{-\!\!\circ\!\!\to} Y$, then $\varphi : X^{\mathrm{op}} \mathrel{-\!\!\circ\!\!\to} Y^{\mathrm{op}}$, where $X^{\mathrm{op}} = (X, a^\circ)$ is the usual opposite V-category), and one has the commutative diagram

$$\begin{array}{ccc} (\mathsf{V\text{-}Mod})^{\mathrm{op}} & \xrightarrow{(-)^{\mathrm{op}}} & \mathsf{V\text{-}Mod} \\ {\scriptstyle (-)^*}\uparrow & & \uparrow{\scriptstyle (-)_*} \\ \mathsf{V\text{-}Cat} & \xrightarrow[(-)^{\mathrm{op}}]{} & \mathsf{V\text{-}Cat}. \end{array}$$

As a quantaloid, V-Mod is in particular a 2-category, and for all $f : X \longrightarrow Y$ in V-Cat one has

$$f_* \dashv f^*$$

in V-Mod. V-Cat inherits its 2-categorical structure from V-Mod via

$$\begin{aligned} f \leq f' :\iff f^* \leq (f')^* &\iff \forall x \in X, y \in Y \,.\, b(y, f(x)) \leq b(y, f'(x)) \\ \iff f'_* \leq f_* &\iff \forall x \in X, y \in Y \,.\, b(f'(x), y) \leq b(f(x), y) \\ \iff 1_X^* \leq (f')^* \cdot f_* &\iff \forall x \in X \,.\, k \leq b(f(x), f'(x)). \end{aligned}$$

Hence, the previous diagram actually shows commuting 2-functors when we add dualization w.r.t. 2-cells (indicated by co) appropriately:

$$\begin{array}{ccc} (\mathsf{V\text{-}Mod})^{\mathrm{op}} & \xrightarrow{(-)^{\mathrm{op}}} & \mathsf{V\text{-}Mod} \\ {\scriptstyle (-)^*}\uparrow & & \uparrow{\scriptstyle (-)_*} \\ \mathsf{V\text{-}Cat} & \xrightarrow[(-)^{\mathrm{op}}]{} & \mathsf{V\text{-}Cat}^{\mathrm{co}} \end{array}$$

Of course, $\mathsf{V}\text{-}\mathsf{Cat}$ being a 2-category, there is also a notion of adjointness in $\mathsf{V}\text{-}\mathsf{Cat}$:

$$\begin{aligned} f \dashv g \text{ in } \mathsf{V}\text{-}\mathsf{Cat} &\iff f \cdot g \leq 1 \text{ and } 1 \leq g \cdot f \text{ in } \mathsf{V}\text{-}\mathsf{Cat} \\ &\iff g^* \cdot f^* \leq 1^* \text{ and } 1^* \leq f^* \cdot g^* \text{ in } \mathsf{V}\text{-}\mathsf{Mod} \\ &\iff g^* \dashv f^* \text{ in } \mathsf{V}\text{-}\mathsf{Mod} \\ &\iff f_* = g^* \quad (\text{since } f_* \dashv f^* \text{ in } \mathsf{V}\text{-}\mathsf{Mod}) \\ &\iff \forall x \in X, y \in Y \,.\, a(x, g(y)) = b(f(x), y). \end{aligned}$$

1.5. **Yoneda.** V-modules give rise to V-functors, as follows.

Proposition 1.2. *The following are equivalent for* V*-relations* $\varphi : X \relbar\!\!\!\mapsto Y$ *between* V*-categories:*

(i) $\varphi : X \relbar\!\!\circ\!\!\!\rightarrow Y$ *is a* V*-module.*
(ii) $\varphi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ *is a* V*-functor.*

With $\varphi = a = 1_X^* : X \relbar\!\!\circ\!\!\!\rightarrow X$ we obtain in particular the V-functor $a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}$ whose mate $\ulcorner a \urcorner$ is the Yoneda-V-functor

$$y : X \longrightarrow \hat{X} := (X^{\mathrm{op}} \multimap \mathsf{V}), \;\; x \longmapsto a(-, x).$$

The structure $\hat{a}$ of $\hat{X}$ is given by

$$\hat{a}(f, f') = \bigwedge_{x \in X} f(x) \multimap f'(x).$$

Lemma 1.3. *For all* $x \in X$ *and* $f \in \hat{X}$*,* $\hat{a}(y(x), f) = f(x)$*.*

One calls a V-functor $f : (X, a) \longrightarrow (Y, b)$ *fully faithful* if $a(x, x') = b(f(x), f(x'))$ for all $x, x' \in X$; equivalently, if $1_X^* = f^* \cdot f_*$ (since $f^* \cdot f_* = f^\circ \cdot b \cdot f$), or just $1_X^* \geq f^* \cdot f_*$ (since the other inequality comes for free).

Corollary 1.4. $y : X \longrightarrow \hat{X}$ *is fully faithful.*

1.6. **L-separation.** For V-functors $f, g : Z \longrightarrow X$ we write $f \cong g$ if $f \leq g$ and $g \leq f$; equivalently, if $f^* = g^*$, or $f_* = g_*$. We call X *L-separated* if $f \cong g$ implies $f = g$, for all $f, g : Z \longrightarrow X$. The full subcategory of $\mathsf{V}\text{-}\mathsf{Cat}$ consisting of all L-separated V-categories is denoted by $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$. Obviously, it suffices to consider $Z = E$ (the $\otimes$-neutral object) here: writing $x : E \longrightarrow X$ in $\mathsf{V}\text{-}\mathsf{Cat}$ instead of $x \in X$, we just note that $f_* = g_*$ implies

$$(f \cdot x)_* = f_* \cdot x_* = g_* \cdot x_* = (g \cdot x)_*.$$

This proves the equivalence of (i),(ii) of the following proposition.

Proposition 1.5. *The following statements are equivalent for a* V*-category* $X = (X, a)$*.*

(i) *X is L-separated.*
(ii) $x \cong y$ *implies* $x = y$*, for all* $x, y \in X$*.*
(iii) *For all* $x, y \in X$*, if* $a(x, y) \geq k$ *and* $a(y, x) \geq k$*, then* $x = y$*.*
(iv) *The Yoneda functor* $y : X \longrightarrow \hat{X}$ *is injective.*

Proof. For (ii) $\iff$ (iii) $\iff$ (iv) one observes

$$\begin{aligned} y(x) = y(y) &\iff x^\circ \cdot a = y^\circ \cdot a \\ &\iff x^* = y^* \\ &\iff x \leq y \text{ and } y \leq x \\ &\iff k \leq a(x, y) \text{ and } k \leq a(y, x). \end{aligned}$$

□

Corollary 1.6. *The* V*-category* V *is L-separated. For all* V*-categories* X, Y, *if* Y *is L-separated,* $X \multimap Y$ *is also L-separated. In particular,* $\hat{X}$ *is L-separated, for every* X.

Proof. $k \le u \multimap v$ and $k \le v \multimap u$ means $u \le v$ and $v \le u$ in V, hence $u = v$. For $Y = (Y, b)$ and $X = (X, a)$, $k \le [a, b](f, g)$ in $X \multimap Y$ means $k \le b(f(x), g(x))$ for all $x \in X$, which makes the second statement obvious. □

1.7. **L-completeness.** Following Lawvere [Law73] we call a V-category X *L-complete* if every adjunction $\varphi \dashv \psi : X \to Z$ in V-Mod is of the form $f_* \dashv f^*$, for a V-functor $f : Z \longrightarrow X$. Clearly, if X is L-separated, such a presentation is unique. As in 1.6, it suffices to consider $Z = E$ here; but we need the Axiom of Choice for that.

Proposition 1.7. *The following statements are equivalent for a* V*-category* X.

(i) *X is L-complete.*
(ii) *Each left adjoint* V*-module* $\varphi : E \to X$ *is of the form* $\varphi = x_*$ *for some* x *in* X.
(iii) *Each right adjoint* V*-module* $\psi : X \to E$ *is of the form* $\psi = x^*$ *for some* x *in* X.

Elements in $\hat{X}$ are V-functors $X^{\mathrm{op}} \cong X^{\mathrm{op}} \otimes E \longrightarrow \mathsf{V}$ which, by Proposition 1.2, may be considered as V-modules $\psi : X \to E$. Suppose such a V-module has a left adjoint $\varphi : E \to X$. From $\varphi \cdot \psi \le 1_X^*$ one obtains $\varphi \le 1_X^* \multimapdotinv \psi$ (see 1.2), and from $(1_X^* \multimapdotinv \psi) \cdot \psi \le 1_X^*$ and $\psi \cdot \varphi \ge 1_E^*$ one has $1_X^* \multimapdotinv \psi \le \varphi$. Hence, if ψ is right adjoint, its left adjoint must necessarily be $1_X^* \multimapdotinv \psi$; moreover $(1_X^* \multimapdotinv \psi) \cdot \psi \le 1_X^*$ always holds. Therefore:

Proposition 1.8. *A* V*-module* $\psi : X \to E$ *(with* $X = (X, a)$*) is right adjoint if, and only if,* $1_E^* \le \psi \cdot (1_X^* \multimapdotinv \psi)$*, that is, if*

$$(*) \qquad k \le \bigvee_{y \in X} \psi(y) \otimes \left(\bigwedge_{x \in X} a(x, y) \multimapinv \psi(x) \right).$$

Note that $\bigwedge_{x \in X} a(x, y) \multimapinv \psi(x) = \hat{a}(\psi, y(y))$. We call a V-functor $\psi : X^{\mathrm{op}} \longrightarrow \mathsf{V}$ *tight* ([Tho07]) if, as a V-module $X \to E$, it is right adjoint, that is, if it satisfies $(*)$. We consider

$$\tilde{X} = \{\psi \in \hat{X} \mid \psi \text{ tight}\}$$

as a full V-subcategory of $\hat{X}$. Our goal is to exhibit $\tilde{X}$ as an "L-completion" of X.

Examples 1.9. (1) $\mathsf{V} = \mathsf{2}$. A V-functor $X^{\mathrm{op}} \longrightarrow \mathsf{2}$ is the characteristic function of a down-closed set A in the (pre)ordered set X. Condition $(*)$ then reads as

$$\exists y \in A \, \forall x \in A \,.\, x \le y,$$

so that $A = \downarrow y$. In other words, $\tilde{X}$ is simply the image of the Yoneda functor $y : X \longrightarrow \hat{X}$, $y \longmapsto \downarrow y$.
(2) $\mathsf{V} = \mathsf{P}_+$. A tight V-functor $X^{\mathrm{op}} \longrightarrow \mathsf{V}$ is given by a function $\psi : X \longrightarrow [0, \infty]$ with

$$\psi(y) \le \psi(x) \;\Rightarrow\; \psi(x) - \psi(y) \le a(x, y) \quad (x, y \in X),$$
$$\inf_{y \in X} (\psi(y) + \sup_{\substack{x \in X, \\ \psi(x) \le a(x,y)}} (a(x, y) - \psi(x))) = 0,$$

here a is the metric on X. If a is symmetric (so that $a = a^\circ$), these conditions are more conveniently described as

$$|\psi(x) - \psi(y)| \le a(x, y) \le \psi(x) + \psi(y) \quad (x, y \in X),$$
$$\inf_{x \in X} \psi(x) = 0.$$

These are precisely the *supertight* maps on X considered in [LS00], where the reader finds the necessary details.
(3) $\mathsf{V} = \mathsf{P}_{\max}$. Here the two conditions of (2) change to

$$\psi(y) < \psi(x) \;\Rightarrow\; \psi(x) \leq a(x, y) \quad (x, y \in X),$$
$$\inf_{y \in Y}(\max(\psi(y), \sup_{\substack{x \in X,\\ \psi(x) < a(x,y)}} (a(x, y)))) = 0.$$

1.8. **L-injectivity.** A V-functor $f : (X, a) \longrightarrow (Y, b)$ is called *L-dense* if $f_* \cdot f^* = 1^*_Y$; that is, if $b = b \cdot f \cdot f^\circ \cdot b$, or

$$b(y, y') = \bigvee_{x \in X} b(y, f(x)) \otimes b(f(x), y')$$

for all $y, y' \in Y$. L-dense V-functors have good composition-cancellation properties.

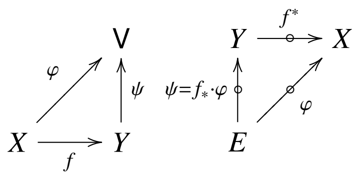
Lemma 1.10. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be V-functors. Then the following assertions hold:*

(1) *f, g L-dense $\Rightarrow$ $g \cdot f$ L-dense.*
(2) *$g \cdot f$ L-dense $\Rightarrow$ g L-dense.*
(3) *$g \cdot f$ L-dense, g fully faithful $\Rightarrow$ f L-dense.*
(4) *$g \cdot f$ fully faithful, f L-dense $\Rightarrow$ g fully faithful.*

A fully faithful L-dense V-functor is an *L-equivalence*. Hence, f is an L-equivalence if, and only if, f_* (or f^*) is an isomorphism in V-Mod. A V-category Z is *pseudo-injective* if, for every fully faithful V-functor $f : X \longrightarrow Y$ and for all V-functors $h : X \longrightarrow Z$ there is a V-functor $g : Y \longrightarrow Z$ with $g \cdot f \cong h$; if strict equality is obtainable, we call Z injective. Z is *L-injective* if this extension property is required only for L-equivalences f. Hence, injectivity implies pseudo-injectivity, and every pseudo-injective V-category is also L-injective.

Lemma 1.11. *The V-category V is injective, hence in particular L-injective.*

Proof. Let $f : X \longrightarrow Y$ be fully faithful and $\varphi : X \longrightarrow \mathsf{V}$ be any V-functor. Then the V-module $\varphi : E \multimapdotboth X$ factors as $\varphi = f^* \cdot \psi$, with $\psi = f_* \cdot \varphi$. But the V-module $f^* \cdot \psi$ corresponds to the V-functor $\psi \cdot f$, hence $\psi \cdot f = \varphi$.



□

Note that the V-functor ψ has been constructed effectively, with

$$\psi(y) = \bigvee_{x \in X} \varphi(x) \otimes b(f(x), y).$$

In case $\mathsf{V} = \mathsf{2}$, this means

$$\psi(y) = \top \iff \exists x \in X \,.\, (\varphi(x) = \top \text{ and } f(x) \leq y),$$

and for $\mathsf{V} = \mathsf{P}_+$ we have

$$\psi(y) = \inf_{x \in X}(\varphi(x) + b(f(x), y)).$$

Proposition 1.12. *For all V-categories X, Y, if Y is pseudo-injective or L-injective, $X \multimap Y$ has the respective property. In particular, $\hat{X}$ is injective.*

Proof. Let $f : A \longrightarrow B$ be a fully faithful, and consider any V-functor $\varphi : A \longrightarrow (X \multimap Y)$, with Y pseudo-injective. Since $f \otimes 1_X$ is fully faithful, the mate $\llcorner\varphi\lrcorner : A \otimes X \longrightarrow Y$ factors (up to $\cong$) as $\llcorner\varphi\lrcorner \cong \llcorner\psi\lrcorner \cdot (f \otimes 1_X)$, with $\llcorner\psi\lrcorner : B \otimes X \longrightarrow Y$ corresponding to a V-functor $\psi : B \longrightarrow (X \multimap Y)$. Since $\llcorner\psi\lrcorner \cdot (f \otimes 1_X)$ corresponds to $\psi \cdot f$, $\varphi \cong \psi \cdot f$ follows. The proof works *mutatis mutandis* for L-injectivity. □

Our goal is to show that L-injectivity and L-completeness are equivalent properties.

2. L-closure

2.1. **L-dense V-functors.** We first show that L-dense V-functors are characterized as "epimorphisms up to $\cong$".

Proposition 2.1. *A V-functor $m : M \longrightarrow X$ is L-dense if, and only if, for all V-functors $f, g : X \longrightarrow Y$ with $f \cdot m = g \cdot m$ one has $f \cong g$.*

Proof. The necessity of the condition is clear since from $f_* \cdot m_* = g_* \cdot m_*$ one obtains $f_* = g_*$ when $m_* \cdot m^* = 1_X^*$. To show the converse implication, by Lemma 1.10 we may assume that m is a full embedding $M \hookrightarrow X$ and consider its cokernel pair

$$(X, a) \underset{g}{\overset{f}{\rightrightarrows}} (Y, b),$$

given by the disjoint union

$$Y = \{f(x) = g(x) \mid x \in M\} \cup \{f(x) \mid x \in X \setminus M\} \cup \{g(x) \mid x \in X \setminus M\},$$

where both f and g are full embeddings, and

$$b(f(x), g(y)) = \bigvee_{z \in Z} a(x, z) \otimes a(z, y)$$

for all $y, x \in X \setminus M$. Since $f_* = g_*$ by hypothesis, we obtain

$$a(x, y) = b(g(x), g(y)) = b(f(x), g(y)) = m_* \cdot m^*(x, y)$$

for all $x, y \in X \setminus M$. But this identity holds trivially when $x \in M$ or $y \in M$. Hence $m_* \cdot m^* = 1_X^*$. □

Since $f \cong g$ precisely when $f \cdot x \cong g \cdot x$ for all $x \in X$ (considered as $x : E \longrightarrow X$), it is now easy to identify the largest subset of X which contains M as an L-dense subset.

2.2. **L-closure.** For a V-category X and $M \subseteq X$, we define the *L-closure* of M in X by

$$\overline{M} = \{x \in X \mid \forall f, g : X \longrightarrow Y \,.\, (f|_M = g|_M \Rightarrow f \cdot x \cong g \cdot x)\}$$

and prove:

Proposition 2.2. *Let $X = (X, a)$ be a V-category, $M \subseteq X$ and $x \in X$. Then the following assertions are equivalent.*

(i) $x \in \overline{M}$.
(ii) $a(x, x) \le \bigvee_{y \in M} a(x, y) \otimes a(y, x)$.
(iii) $k \le \bigvee_{y \in M} a(x, y) \otimes a(y, x)$.
(iv) $1_E^* \le x^* \cdot m_* \cdot m^* \cdot x_*$, *where m denotes the full embedding $m : M \hookrightarrow X$.*
(v) $m^* \cdot x_* \dashv x^* \cdot m_*$.
(vi) $x_* : E \to\!\!\!\!\circ\; X$ *factors through $m_* : M \to\!\!\!\!\circ\; X$ by a map $\varphi : E \to\!\!\!\!\circ\; M$ in* V-Mod.

Proof. (i)⇒(ii) follows from the L-density of $M \hookrightarrow \overline{M}$. (ii)⇒(iii) is clear since $k \leq a(x,x)$. To see (iii)⇒(iv), just observe

$$x^* \cdot m_* \cdot m^* \cdot x_*(\star,\star) = \bigvee_{y\in M} a(x,y) \otimes a(y,x).$$

The second inequality needed for the adjunction (v) comes for free: $m^* \cdot x_* \cdot x^* \cdot m_* \leq m^* \cdot m_* = 1_M^*$. Hence,(iv)⇒(v)follows. Assuming (v), we have $m_* \cdot m^* \cdot x_* \dashv x^* \cdot m_* \cdot x^*$ as well as $m_* \cdot m^* \cdot x_* \leq x_*$ and $x^* \cdot m_* \cdot x^* \leq x^*$, which implies $m_* \cdot m^* \cdot x_* = x_*$. This shows (v)⇒(vi). Finally, assume (vi) and let $f, g : X \longrightarrow Y$ with $f|_M = g|_M$. Then

$$f_* \cdot x_* = f_* \cdot m_* \cdot \varphi = g_* \cdot m_* \cdot \varphi = g_* \cdot x_*,$$

which proves (i). □

V-functors respect the L-closure, as we show next.

Proposition 2.3. *For a* V*-functor $f : X \longrightarrow Y$ and $M, M' \subseteq X$, $N \subseteq Y$, we have:*

(1) *$M \subseteq \overline{M}$ and $M \subseteq M'$ implies $\overline{M} \subseteq \overline{M'}$.*
(2) *$\overline{\varnothing} = \varnothing$ and $\overline{\overline{M}} = \overline{M}$.*
(3) *$f(\overline{M}) \subseteq \overline{f(M)}$ and $f^{-1}(\overline{N}) \supseteq \overline{f^{-1}(N)}$.*
(4) *If k is $\vee$-irreducible (so that $k \leq u \vee v$ implies $k \leq u$ or $k \leq v$), then $\overline{M \cup M'} = \overline{M} \cup \overline{M'}$.*

Proof. (1), (2) are obvious. For (3), applying Lemma 1.10 to

$$\begin{array}{ccc} M & \longrightarrow & f(M) \\ \downarrow & & \downarrow \\ \overline{M} & \longrightarrow & f(\overline{M}) \end{array}$$

one sees that $f(M) \longrightarrow f(\overline{M})$ is L-dense, hence $f(\overline{M}) \subseteq \overline{f(M)}$. With $M = f^{-1}(N)$, this implies $\overline{f^{-1}(N)} \subseteq f^{-1}(\overline{N})$. To see (4), we just need to show that $x \in \overline{M \cup M'}$ implies $x \in \overline{M}$ or $x \in \overline{M'}$. But this follows from

$$k \leq \bigvee_{y\in M\cup M'} a(x,y) \otimes a(y,x) = \left(\bigvee_{y\in M} a(x,y) \otimes a(y,x)\right) \vee \left(\bigvee_{y\in M'} a(x,y) \otimes a(y,x)\right),$$

assuming that k is $\vee$-irreducible. □

Corollary 2.4. *If k is $\vee$-irreducible in* V*, then the L-closure operator defines a topology on X such that every* V*-functor becomes continuous. Hence, L-closure defines a functor L :* V-Cat ⟶ Top.

Examples 2.5. (1) For $X = (X, \leq)$ in 2-Cat = Ord and $M \subseteq X$, one has $x \in \overline{M}$ precisely when $x \leq z \leq x$ for some $z \in M$. Also, $M \subseteq X$ is open in LX if every $x \in M$ satisfies

$$\forall z \in X \,.\, (x \leq z \leq x \;\Rightarrow\; z \in M).$$

(2) In Met, $\overline{M} = \{x \in X = (X,a) \mid \inf_{z\in M}(a(x,z) + a(z,x)) = 0\}$, and in UMet

$$\overline{M} = \{x \in X = (X,a) \mid \inf_{z\in M}(\max(a(x,z), a(z,x))) = 0\}$$

, which for symmetric (ultra)metric spaces describes the ordinary topological closure.

2.3. **L-separatedness via L-closure.**

Proposition 2.6. *Let $X = (X, a)$ be a* V*-category and $\Delta \subseteq X \times X$ the diagonal. Then*

$$\overline{\Delta} = \{(x, y) \in X \times X \mid x \cong y\}.$$

Proof. Let first $(x, y) \in \overline{\Delta}$. With $\pi_1, \pi_2 : X \times X \longrightarrow X$ denoting the projection maps, we have $\pi_1|_\Delta = \pi_2|_\Delta$ and therefore $x = \pi_1(x, y) \cong \pi_2(x, y) = y$. Assume now $x \cong y$. Note that the canonical functor $\mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Ord}$ preserves products, hence

$$(x_1, y_1) \cong (x_2, y_2) \iff x_1 \cong x_2 \text{ and } y_1 \cong y_2,$$

for all $(x_1, y_1), (x_2, y_2) \in X \times X$. Therefore we have $(x, y) \cong (x, x)$. Let now $f, g : X \times X \longrightarrow Y$ be V-functors with $f|_\Delta = g|_\Delta$. Then $f(x, y) \cong f(x, x) = g(x, x) \cong g(x, y)$. □

Corollary 2.7. *A* V*-category X is L-separated if and only if the diagonal Δ is closed in $X \times X$.*

Theorem 2.8. $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *is an epi-reflective subcategory of* $\mathsf{V}\text{-}\mathsf{Cat}$*, where the reflection map is given by $y_X : X \longrightarrow y_X(X)$, for each* V*-category X. Hence, limits of L-separated* V*-categories are formed in* $\mathsf{V}\text{-}\mathsf{Cat}$*, while colimits are obtained by reflecting the colimit formed in* $\mathsf{V}\text{-}\mathsf{Cat}$*. The epimorphisms in* $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *are precisely the L-dense* V*-functors.*

2.4. **L-completeness via L-closure.**

Lemma 2.9. *Let $X = (X, a)$ be a* V*-category and $M \subseteq X$.*

(1) *Assume that X is L-complete and M is L-closed. Then M is L-complete.*
(2) *Assume that X is L-separated and M is L-complete. Then M is L-closed.*

Proof. (1) follows immediately from Proposition 2.2. To see (2), let $x \in X$ be such that $m^* \cdot x_* \dashv x^* \cdot m_*$. Since M is L-complete, there is some $y \in M$ such that $y_* = m^* \cdot x_*$ and $y^* = x^* \cdot m_*$. Hence $m(y)_* = m_* \cdot y_* \le x_*$ and $m(y)^* = y^* \cdot m^* \le x^*$ and therefore, $m(y)_* = x_*$. L-separation of X gives now $m(y) = x$, i.e. $x \in M$. □

Theorem 2.10. *Let $X = (X, b)$ be a* V*-category. The following assertions are equivalent.*

(i) *X is L-complete.*
(ii) *X is L-injective.*
(iii) *$y : X \longrightarrow \tilde{X}$ has a pseudo left-inverse* V*-functor $R : \tilde{X} \longrightarrow X$, i.e. $R \cdot y \cong 1_X$.*

Proof. To see (i)⇒(ii), let $i : A \longrightarrow B$ be a fully faithful dense V-functor and $f : A \longrightarrow X$ be a V-functor. Since $i_* \dashv i^*$ is actually an equivalence of V-modules, we have $f_* \cdot i^* \dashv i_* \cdot f^*$. Hence, since X is L-complete, there is a V-functor $g : B \longrightarrow X$ such that $g_* = f_* \cdot i^*$, hence $g_* \cdot i_* = f_*$.
The implication (ii)⇒(iii) is surely true since $y : X \longrightarrow \tilde{X}$ is L-dense and fully faithful.
Finally, to see (iii)⇒(i), let $R : \tilde{X} \longrightarrow X$ be a left inverse of $y : X \longrightarrow \tilde{X}$. Then $y \cdot R = 1_{\tilde{X}}$ since $y : X \longrightarrow \tilde{X}$ is dense and $\tilde{X}$ is L-separated. Hence, for each right adjoint V-module $\psi : X \,-\!\!\circ\!\!\to E$, we have $\psi = R(\psi)^*$. □

Proposition 2.11. *For a* V*-category X, as a set $\tilde{X}$ (see 1.7) coincides with the L-closure of $y(X)$ in $\hat{X}$. Hence, $y : X \longrightarrow \tilde{X}$ is fully faithful and L-dense, and $\tilde{X}$ is L-complete.*

Proof. By Proposition 2.2, a V-functor $\psi : X^{\mathrm{op}} \longrightarrow \mathsf{V}$ lies in the L-closure of $y(X)$ in $\hat{X}$ if, and only if,

$$k \le \bigvee_{y \in X} \hat{a}(\psi, y(y)) \otimes \hat{a}(y(y), \psi).$$

Since $\hat{a}(y(y), \psi) = \psi(y)$ by Lemma 1.3, this means precisely that ψ must be tight. □

Theorem 2.12. *The full subcategory* $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$ *of* $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *of L-complete* V*-categories is an epi-reflective subcategory of* $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$*. The reflection map of a L-separated* V*-category* X *is given by any L-dense embedding of* X *into a L-complete and L-separated* V*-category, for instance by* $y : X \longrightarrow \tilde{X}$*.*

3. The $\mathcal{T}$-setting

3.1. **The theory $\mathcal{T}$.** From now on we assume that the quantale V is part of a *strict topological theory* $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ as introduced in [Hof07]. Here $\mathbb{T} = (T, e, m)$ is a Set-monad where T and m satisfy (BC) (that is, T sends pullbacks to weak pullbacks and each naturality square of m is a weak pullback) and $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ is a map such that

$$1_{\mathsf{V}} = \xi \cdot e_{\mathsf{V}}, \qquad\qquad \xi \cdot T\xi = \xi \cdot m_{\mathsf{V}},$$

the diagrams

$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\otimes)} & T\mathsf{V} \\ {\scriptstyle \langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle \xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\otimes]{} & \mathsf{V}, \end{array} \qquad\qquad \begin{array}{ccc} T1 & \xrightarrow{Tk} & T\mathsf{V} \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[k]{} & \mathsf{V}, \end{array}$$

commute and

> $(\xi_X)_X : P_{\mathsf{V}} \longrightarrow P_{\mathsf{V}}T$ is a natural transformation, where P_{V} is the V-powerset functor considered as a functor from Set to Ord and the X-component $\xi_X : P_{\mathsf{V}}(X) \longrightarrow P_{\mathsf{V}}T(X)$ is given by $\varphi \longmapsto \xi\cdot T\varphi$.

Explicitly, $P_{\mathsf{V}}(X) = \mathsf{V}^X$, and for a function $f : X \longrightarrow Y$ we have a canonical map $f^{-1} : \mathsf{V}^Y \longrightarrow \mathsf{V}^X$, $\varphi \longmapsto \varphi \cdot f$. Now $P_{\mathsf{V}}(f)$ is defined as the left adjoint to f^{-1}, explicitly, for $\varphi \in \mathsf{V}^X$ we have $P_{\mathsf{V}}(\varphi)(y) = \bigvee_{x\in f^{-1}(y)} \varphi(x)$. Furthermore, we assume $T1 = 1$.

Examples 3.1. (1) For each quantale V, $(\mathbb{1}, \mathsf{V}, 1_{\mathsf{V}})$ is a strict topological theory, where $\mathbb{1} = (\mathrm{Id}, 1, 1)$ denotes the identity monad.

(2) $\mathcal{U}_{\mathsf{2}} = (\mathbb{U}, \mathsf{2}, \xi_{\mathsf{2}})$ is a strict topological theory, where $\mathbb{U} = (U, e, m)$ denotes the ultrafilter monad and ξ_{2} is essentially the identity map.

(3) $\mathcal{U}_{\mathsf{P}_+} = (\mathbb{U}, \mathsf{P}_+, \xi_{\mathsf{P}_+})$ is a strict topological theory, where

$$\xi_{\mathsf{P}_+} : U\mathsf{P}_+ \longrightarrow \mathsf{P}_+, \quad \mathfrak{x} \longmapsto \inf\{v \in \mathsf{P}_+ \mid [0, v] \in \mathfrak{x}\}.$$

As shown in [Hof07, Lemma 3.2], the right adjoint $\multimap$ of the tensor product $\otimes$ in V is automatically compatible with the map $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ in the sense that

$$\xi \cdot T(\multimap) \leq \multimap \cdot \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2\rangle.$$

$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\multimap)} & T\mathsf{V} \\ {\scriptstyle \langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & \geq & \downarrow{\scriptstyle \xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\multimap]{} & \mathsf{V} \end{array}$$

Furthermore, our condition $T1 = 1$ implies $m_X^\circ \cdot e_X = e_{TX} \cdot e_X$ for each set X. In fact, $m_X^\circ \cdot e_X \geq e_{TX} \cdot e_X$ is true for each monad since $m_X^\circ \geq e_{TX}$. Let now $\mathfrak{X} \in TTX$ and $x \in X$ such that $m_X(\mathfrak{X}) = e_X(x)$. We consider the commutative diagram

$$\begin{array}{ccc} TT1 & \xrightarrow{TTx} & TTX \\ {\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle m_X} \\ T1 & \xrightarrow[Tx]{} & TX, \end{array}$$

where $x : 1 \longrightarrow X$. Since m satisfies (BC), there is some $\mathfrak{Y} \in TT1 = 1$ with $TTx(\mathfrak{Y}) = \mathfrak{X}$, that is, $\mathfrak{X} = e_{TX} \cdot e_X(x)$.

The functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ can be extended to a 2-functor $T_\xi : \mathsf{V}\text{-}\mathsf{Rel} \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}$ as follows. Given a V-relation $r : X \times Y \longrightarrow \mathsf{V}$, we define $T_\xi r : TX \times TY \longrightarrow \mathsf{V}$ as the left Kan-extension

$$\begin{array}{ccc} T(X \times Y) & \xrightarrow{\text{can}} & TX \times TY \\ & {\scriptstyle \xi_{X\times Y}(r)=\xi\cdot Tr}\searrow \quad \swarrow{\scriptstyle T_\xi r} & \\ & \mathsf{V} & \end{array}$$

in Ord (where $TX, TY, T(X\times Y)$ are discrete), i.e. the smallest (order-preserving) map $s : TX\times TY \longrightarrow \mathsf{V}$ such that $\xi \cdot Tr \leq s \cdot \text{can}$. Elementwise one has

$$T_\xi r(\mathfrak{x}, \mathfrak{y}) = \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in T(X \times Y), T\pi_1(\mathfrak{w}) = \mathfrak{x}, T\pi_2(\mathfrak{w}) = \mathfrak{y} \right\}$$

for each $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$. We obtain now the following properties.

Proposition 3.2 ([Hof07])**.** *The following assertions hold:*

1. *For each* V*-matrix* $r : X \nrightarrow Y$, $T_\xi(r^\circ) = T_\xi(r)^\circ$ *(and we write* $T_\xi r^\circ$*).*
2. *For each function* $f : X \longrightarrow Y$, $Tf = T_\xi f$ *(and therefore also* $Tf^\circ = T_\xi f^\circ$*).*
3. $e_Y \cdot r \leq T_\xi r \cdot e_X$ *for all* $r : X \nrightarrow Y$ *in* V-Rel.
4. $m_Y \cdot T_\xi^2 r = T_\xi r \cdot m_X$ *for all* $r : X \nrightarrow Y$ *in* V-Rel.

3.2. $\mathcal{T}$**-relations.** We define a $\mathcal{T}$*-relation* from X to Y to be a V-relation of the form $a : TX \nrightarrow Y$, and write $a : X \rightharpoonup Y$. Given also $b : Y \rightharpoonup Z$, the composite $b \circ a : X \rightharpoonup Z$ is given by the Kleisli convolution

$$b \circ a = b \cdot T_\xi a \cdot m_X^\circ.$$

Composition of $\mathcal{T}$-relations is associative, and for each $\mathcal{T}$-matrix $a : X \rightharpoonup Y$ we have $a \circ e_X^\circ = a$ and $e_Y^\circ \circ a \geq a$, hence $e_X^\circ : X \rightharpoonup X$ is a lax identity. We call a $\mathcal{T}$-relation $a : X \rightharpoonup Y$ *unitary* if $e_Y^\circ \circ a = a$, so that $e_X^\circ : X \rightharpoonup X$ is the identity on X in the category $\mathcal{T}$-URel of sets and unitary $\mathcal{T}$-relations, with the Kleisli convolution as composition. The hom-sets of $\mathcal{T}$-URel inherit the order-structure from V-Rel , and composition of (unitary) $\mathcal{T}$-relations respects this order in both variables. Many notions and arguments can be transported from the V-setting to the $\mathcal{T}$-setting by substituting relational composition by Kleisli convolution.

Given a $\mathcal{T}$-relation $c : X \rightharpoonup Z$, the composition by c from the right side has a right adjoint but composition by c from the left side in general not. Explicitly, given also $b : X \rightharpoonup Y$, we pass from

$$\begin{array}{ccc} X & \xrightarrow{b} & Y \\ {\scriptstyle c}\downarrow & & \\ Z & & \end{array} \qquad \text{to} \qquad \begin{array}{ccc} TX & \xrightarrow{b} & Y \\ {\scriptstyle m_X^\circ}\downarrow & & \\ TTX & & \\ {\scriptstyle T_\xi c}\downarrow & & \\ TZ & & \end{array}$$

in $\mathcal{T}$-URel $\qquad\qquad$ in V-Rel

and define the extension $b \multimapinv c : Z \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ as $b \bullet\!\!- (T_\xi c \cdot m_X^\circ) : TZ \relbar\joinrel\mapstochar\joinrel\rightarrow Y$.

3.3. $\mathcal{T}$**-categories.** A $\mathcal{T}$-category $X = (X, a)$ is a set X equipped with a $\mathcal{T}$-relation $a : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X$ satisfying $e_X^\circ \leq a$ and $a \circ a \leq a$; equivalently,

$$k \leq a(e_X(x), x), \qquad\qquad T_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(\mathfrak{x}, x)$$

for all $\mathfrak{X} \in TTX$, $\mathfrak{x} \in TX$ and $x \in X$. A $\mathcal{T}$-functor $f : (X, a) \longrightarrow (Y, b)$ must satisfy $f \cdot a \leq b \cdot Tf$, which in pointwise notation reads as

$$a(\mathfrak{x}, x) \leq b(Tf(\mathfrak{x}), f(x))$$

for all $\mathfrak{x} \in TX$ and $x \in X$. The resulting category of $\mathcal{T}$-categories and $\mathcal{T}$-functors is denoted by $\mathcal{T}$-Cat (see also [CH03, CT03, CHT04]). Note that the quantale V becomes in a natural way a $\mathcal{T}$-category $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ where $\hom_\xi : T\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$, $(\mathfrak{v}, v) \longmapsto (\xi(\mathfrak{v}) \multimap v)$.

Examples 3.3. (1) For each quantale V, $\mathcal{I}_{\mathsf{V}}$-categories are precisely V-categories and $\mathcal{I}_{\mathsf{V}}$-functors are V-functors.

(2) The main result of [Bar70] states that $\mathcal{U}_2$-Cat is isomorphic to the category Top of topological spaces. The $\mathcal{U}_2$-category $\mathsf{V} = \mathsf{2}$ is the Sierpinski space with $\{0\}$ open and $\{1\}$ closed. In [CH03] it is shown that $\mathcal{U}_{\mathsf{P}_+}$-$\mathsf{Cat}$ is isomorphic to the category App of approach spaces (see [Low97] for more details about App).

A $\mathcal{T}$-category $X = (X, a)$ can also be thought of as a lax Eilenberg–Moore algebra, since the two conditions above can be equivalently expressed as

$$1_X \leq a \cdot e_X, \qquad\qquad a \cdot T_\xi a \leq a \cdot m_X.$$

As a consequence, each $\mathbb{T}$-algebra (X, α) can be considered as a $\mathcal{T}$-category by simply regarding the function $\alpha : TX \longrightarrow X$ as a $\mathcal{T}$-relation $\alpha : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X$. The free Eilenberg-Moore algebra (TX, m_X) – viewed as a $\mathcal{T}$-category – is denoted by $|X|$.

Every $\mathcal{T}$-category $X = (X, a)$ has an underlying V-category $\mathrm{S}X = (X, a \cdot e_X)$. Indeed, this defines a functor $\mathrm{S} : \mathcal{T}$-$\mathsf{Cat} \longrightarrow \mathsf{V}$-$\mathsf{Cat}$ which has a left adjoint $\mathrm{A} : \mathsf{V}$-$\mathsf{Cat} \longrightarrow \mathcal{T}$-$\mathsf{Cat}$ defined by $\mathrm{A}X = (X, e_X^\circ \cdot T_\xi r)$, for each V-category $X = (X, r)$. There is yet another interesting functor connecting $\mathcal{T}$-categories and V-categories, namely $\mathrm{M} : \mathcal{T}$-$\mathsf{Cat} \longrightarrow \mathsf{V}$-$\mathsf{Cat}$ which sends a $\mathcal{T}$-category (X, a) to the V-category $(TX, T_\xi a \cdot m_X^\circ)$. The *dual* $\mathcal{T}$*-category* X^{op} (see [CH08]) of a $\mathcal{T}$-category $X = (X, a)$ is then defined as

$$X^{\mathrm{op}} = \mathrm{A}(\mathrm{M}(X)^{\mathrm{op}}).$$

Examples 3.4. For $\mathbb{T} = \mathbb{U}$ the ultrafilter monad, the topology on $|X|$ can be described via the Zariski-closure:

$$\mathfrak{x} \in \mathrm{cl}(\mathcal{A}) \iff \mathfrak{x} \supseteq \bigcap \mathcal{A} \iff \mathfrak{x} \subseteq \bigcup \mathcal{A},$$

for $\mathfrak{x} \in UX$ and $\mathcal{A} \subseteq UX$. Furthermore, for $X \in \mathcal{U}_2$-$\mathsf{Cat} \cong \mathsf{Top}$, $M(X) = (UX, \leq)$ is the (pre)ordered set where

$$\mathfrak{x} \leq \mathfrak{y} \iff \forall A \in \mathfrak{x} \,.\, \overline{A} \in \mathfrak{y}$$

for $\mathfrak{x}, \mathfrak{y} \in UX$. Then X^{op} is the Alexandroff space induced by the dual order $\geq$. If $X \in \mathcal{U}_{\mathsf{P}_+}$-$\mathsf{Cat} \cong \mathsf{App}$ is an approach space with distance function $\mathrm{dist} : PX \times X \longrightarrow \mathsf{P}_+$, then $M(X) = (UX, d)$ is the (generalized) metric space with

$$d(\mathfrak{x}, \mathfrak{y}) = \inf\{\varepsilon \in [0, \infty] \mid \forall A \in \mathfrak{x} \,.\, \overline{A}^{(\varepsilon)} \in \mathfrak{y}\},$$

where $\mathfrak{x}, \mathfrak{y} \in UX$ and $\overline{A}^{(\varepsilon)} = \{x \in X \mid \mathrm{dist}(A, x) \leq \varepsilon\}$.

The tensor product of V can be transported to $\mathcal{T}$-Cat by putting $(X,a)\otimes(Y,b)=(X\times Y,c)$ with

$$c(\mathfrak{w},(x,y))=a(\mathfrak{x},x)\otimes b(\mathfrak{y},y),$$

where $\mathfrak{w}\in T(X\times Y)$, $x\in X$, $y\in Y$, $\mathfrak{x}=T\pi_1(\mathfrak{w})$ and $\mathfrak{y}=T\pi_2(\mathfrak{w})$. The $\mathcal{T}$-category $E=(E,k)$ is a $\otimes$-neutral object, where E is a singleton set and k the constant relation with value $k\in\mathsf{V}$. Unlike in the V-case, in general this does not result in a closed structure on $\mathcal{T}$-Cat. However, as shown in [Hof07], if a $\mathcal{T}$-category $X=(X,a)$ satisfies $a\cdot T_{\xi}a=a\cdot m_X$, then $X\otimes\_:\mathcal{T}\text{-}\mathsf{Cat}\longrightarrow\mathcal{T}\text{-}\mathsf{Cat}$ has a right adjoint $\_^X:\mathcal{T}\text{-}\mathsf{Cat}\longrightarrow\mathcal{T}\text{-}\mathsf{Cat}$. Explicitly, for a $\mathcal{T}$-category $Y=(Y,b)$, the exponential $X\multimap Y$ is given by the set

$$\{f:X\longrightarrow Y\mid f\text{ is a }\mathcal{T}\text{-functor}\},$$

equipped with the structure-relation $[\![a,b]\!]$ defined by

$$[\![a,b]\!](\mathfrak{p},h)=\bigvee\left\{v\in\mathsf{V}\;\middle|\;\forall\mathfrak{q}\in T\pi_2^{-1}(\mathfrak{p}),x\in X\,.\,a(T\pi_1(\mathfrak{q}),x)\otimes v\le b(T\mathrm{ev}(\mathfrak{q}),h(x))\right\},$$

where $\mathfrak{p}\in T(Y^X)$, $h\in Y^X$, $\pi_1:X\times(X\multimap Y)\longrightarrow X$ and $\pi_2:X\times(X\multimap Y)\longrightarrow Y^X$. Using the adjunction $u\otimes\_\dashv u\multimap\_$ in V, we see that

$$[\![a,b]\!](\mathfrak{p},h)=\bigwedge_{\substack{\mathfrak{q}\in T(X\times(X\multimap Y)),x\in X\\ \mathfrak{q}\longmapsto\mathfrak{p}}}a(T\pi_1(\mathfrak{q}),x)\multimap b(T\mathrm{ev}(\mathfrak{q}),h(x)).$$

Lemma 3.5. *Let $X=(X,a)$, $Y=(Y,b)$ be $\mathcal{T}$-categories with $a\cdot T_{\xi}a=a\cdot m_X$ and $h,h'\in(X\multimap Y)$. Then*

$$[\![a,b]\!](e_{Y^X}(h),h')=\bigwedge_{x\in X}b(e_Y(h(x)),h'(x)).$$

3.4. **$\mathcal{T}$-modules.** Let $X=(X,a)$ and $Y=(Y,b)$ be $\mathcal{T}$-categories and $\varphi:X\,{-\!\!\!\rightharpoonup}\,Y$ be a $\mathcal{T}$-relation. We call φ a *$\mathcal{T}$-module*, and write $\varphi:X\,{-\!\!\circ\!\!\rightharpoonup}\,Y$, if $\varphi\circ a\le\varphi$ and $b\circ\varphi\le\varphi$. Note that we always have $\varphi\circ a\ge\varphi$ and $b\circ\varphi\ge\varphi$, so that the $\mathcal{T}$-module condition above actually implies equality. It is easy to see that the extension as well as the lifting (if it exists) in $\mathcal{T}$-URel of $\mathcal{T}$-modules is again a $\mathcal{T}$-module. Furthermore, we have $a:X\,{-\!\!\circ\!\!\rightharpoonup}\,X$ for each $\mathcal{T}$-category $X=(X,a)$; in fact, a is the identity $\mathcal{T}$-module on X for the Kleisli convolution. The category of $\mathcal{T}$-categories and $\mathcal{T}$-modules, with Kleisli convolution as composition is denoted by $\mathcal{T}$-Mod. In fact, $\mathcal{T}$-Mod is an ordered category, with the structure on hom-sets inherited from $\mathcal{T}$-URel.

Let now $X=(X,a)$ and $Y=(Y,b)$ be $\mathcal{T}$-categories and $f:X\longrightarrow Y$ be a Set-map. We define $\mathcal{T}$-relations $f_*:X\,{-\!\!\!\rightharpoonup}\,Y$ and $f^*:Y\,{-\!\!\!\rightharpoonup}\,X$ by putting $f_*=b\cdot Tf$ and $f^*=f^\circ\cdot b$ respectively. Hence, for $\mathfrak{x}\in TX$, $\mathfrak{y}\in TY$, $x\in X$ and $y\in Y$, $f_*(\mathfrak{x},y)=b(Tf(\mathfrak{x}),y)$ and $f^*(\mathfrak{y},x)=b(\mathfrak{y},f(x))$. Given now $\mathcal{T}$-modules φ and ψ, we have

$$\varphi\circ f_*=\varphi\cdot Tf\qquad\text{and}\qquad f^*\circ\psi=f^\circ\cdot\psi.$$

The latter equality follows from

$$f^*\circ\psi=f^\circ\cdot b\cdot T_{\xi}\psi\cdot m_Z^\circ=f^\circ\cdot\psi,$$

whereas the first equality follows from

$$\varphi\circ f_*=\varphi\circ(b\cdot Tf)=\varphi\cdot T_{\xi}b\cdot T^2f\cdot m_X^\circ=\varphi\cdot T_{\xi}b\cdot m_Y^\circ\cdot Tf=\varphi\cdot Tf.$$

In particular we have $b\circ f_*=f_*$ and $f^*\circ b=f^*$, as well as $f_*\circ f^*=b\cdot Tf\cdot Tf^\circ\cdot T_{\xi}b\cdot m_Y^\circ\le b$. The latter inequality becomes even an equality provided that f is surjective. As before, one easily verifies:

Proposition 3.6. *The following assertions are equivalent for a* Set*-map f between $\mathcal{T}$-categories:*

(i) *$f:X\longrightarrow Y$ is a $\mathcal{T}$-functor.*
(ii) *f_* is a $\mathcal{T}$-module $f_*:X\,{-\!\!\circ\!\!\rightharpoonup}\,Y$.*

(iii) f^* *is a* $\mathcal{T}$*-module* $f^* : Y \multimap X$.

As in the V-case, there are functors

$$\mathcal{T}\text{-Cat} \xrightarrow{(-)_*} \mathcal{T}\text{-Mod} \xleftarrow{(-)^*} \mathcal{T}\text{-Cat}^{\mathrm{op}}.$$

We can transport the order on the hom-sets from $\mathcal{T}$-Mod to $\mathcal{T}$-Cat via the functor $(\_)^* : \mathcal{T}\text{-Cat}^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-Mod}$. That is, we can define $f \leq g$ if $f^* \leq g^*$, or equivalently, if $g_* \leq f_*$. With this definition we turn $\mathcal{T}$-Cat into an *ordered category*. As usual, we call $\mathcal{T}$-functors $f, g : X \longrightarrow Y$ *equivalent*, and write $f \cong g$, if $f \leq g$ and $g \leq f$. Hence, $f \cong g$ if and only if $f^* = g^*$, which in turn is equivalent to $f_* = g_*$.

Lemma 3.7. *Let* $f, g : X \longrightarrow Y$ *be* $\mathcal{T}$*-functors between* $\mathcal{T}$*-categories* $X = (X, a)$ *and* $Y = (Y, b)$*. Then*

$$f \leq g \iff \forall x \in X \,.\, k \leq b(e_Y(f(x)), g(x)).$$

Proof. If $g_* \leq f_*$, then

$$k \leq g_*(e_X(x), g(x)) \leq f_*(e_X(x), g(x)) = b(e_Y(f(x)), g(x)).$$

On the other hand, if $k \leq b(e_Y(g(x)), f(x))$ for each $x \in X$, then

$$f^*(\mathfrak{y}, x) = b(\mathfrak{y}, f(x)) \leq T_{\xi} b(e_{TY}(\mathfrak{y}), e_Y(f(x))) \otimes b(e_Y(f(x)), g(x)) \leq b(\mathfrak{y}, g(x)) = g^*(\mathfrak{y}, x). \qquad \square$$

In particular, for $\mathcal{T}$-functors $f, g : X \longrightarrow \mathsf{V}$, we have $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. Assume now that $X = (X, a)$, $Y = (Y, b)$ and $Z = (Z, c)$ are $\mathcal{T}$-categories where $a \cdot T_{\xi} a = a \cdot m_X$. By combining the previous lemma with Lemma 3.5, we obtain $f \leq g \iff \ulcorner f \urcorner \leq \ulcorner g \urcorner$ for all $\mathcal{T}$-functors $f, g : X \otimes Y \longrightarrow Z$, where $\ulcorner f \urcorner, \ulcorner g \urcorner : Y \longrightarrow Z^X$.

3.5. **Yoneda.** Also $\mathcal{T}$-modules give rise to $\mathcal{T}$-functors, but in addition to X^{op} we must also take the $\mathcal{T}$-category $|X|$ (see 3.3) into consideration.

Theorem 3.8 ([CH08])**.** *For* $\mathcal{T}$*-categories* (X, a) *and* (Y, b)*, and a* $\mathcal{T}$*-relation* $\psi : X \rightharpoonup Y$*, the following assertions are equivalent:*

(i) $\psi : (X, a) \multimap (Y, b)$ *is a* $\mathcal{T}$*-module.*
(ii) *Both* $\psi : |X| \otimes Y \longrightarrow \mathsf{V}$ *and* $\psi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ *are* $\mathcal{T}$*-functors.*

Since we have $a : X \multimap X$ for each $\mathcal{T}$-category $X = (X, a)$, the theorem above provides us with two $\mathcal{T}$-functors

$$a : |X| \otimes X \longrightarrow \mathsf{V} \qquad \text{and} \qquad a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}.$$

We refer to the mate $y = \ulcorner a \urcorner : X \longrightarrow (|X| \multimap \mathsf{V})$ of the first $\mathcal{T}$-functor as the Yoneda functor of X.

Theorem 3.9 ([CH08])**.** *Let* $X = (X, a)$ *be a* $\mathcal{T}$*-category. Then the following assertions hold:*

(1) *For all* $\mathfrak{x} \in TX$ *and* $\varphi \in (|X| \multimap \mathsf{V})$*,* $[\![m_X, \hom_{\xi}]\!](Ty(\mathfrak{x}), \varphi) \leq \varphi(\mathfrak{x})$*.*
(2) *Let* $\varphi \in (|X| \multimap \mathsf{V})$*. Then*

$$\forall \mathfrak{x} \in TX \,.\, \varphi(\mathfrak{x}) \leq [\![m_X, \hom_{\xi}]\!](Ty(\mathfrak{x}), \varphi) \iff \varphi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \text{ is a } \mathcal{T}\text{-functor}.$$

Consequently, we put $\hat{X} = (\hat{X}, \hat{a})$ where

$$\hat{X} = \{\psi \in (|X| \multimap \mathsf{V}) \mid \psi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \text{ is a } \mathcal{T}\text{-functor}\}$$

considered as a subcategory of $|X| \multimap \mathsf{V}$, so that $\hat{a}$ is the restriction of $[\![m_X, \hom_{\xi}]\!]$ to $\hat{X}$ (see Subsection 3.3). In particular, $y : X \longrightarrow \hat{X}$ is full and faithful.

Example 3.10. For $X \in \mathcal{U}_2\text{-}\mathsf{Cat} \cong \mathsf{Top}$, every $\psi \in \hat{X}$ is the characteristic function of a Zariski-closed and down-closed subset $\mathcal{A} \subseteq UX$ (see Examples 3.4). Using the ψ-$\mathcal{A}$-exchange, we will now give an alternative description of $\hat{X}$, as the set $F_0(X)$ of (possibly improper) filters on the lattice τ of open sets of X, in terms of the bijective maps

$$\hat{X} \xrightarrow{\Phi} F_0(X) \qquad \text{and} \qquad F_0(X) \xrightarrow{\Pi} \hat{X},$$

where $\Phi(\mathcal{A}) = \bigcap \mathcal{A} \cap \tau$ and $\Pi(\mathfrak{f}) = \{\mathfrak{x} \in UX \mid \mathfrak{f} \subseteq \mathfrak{x}\}$. Clearly, $\mathcal{A} = \Pi(\mathfrak{f})$ is Zariski-closed. If $\mathfrak{x} \leq \mathfrak{y}$ for some $\mathfrak{x} \in UX$ and $\mathfrak{y} \in \mathcal{A}$, then, for each $A \in \mathfrak{x}$ and $B \in \mathfrak{f}$, one has

$$\overline{A} \cap B \neq \emptyset$$

which, since B is open, gives $A \cap B \neq \emptyset$. Hence $\mathfrak{f} \subseteq \mathfrak{x}$, that is, $\mathfrak{x} \in \mathcal{A}$. Furthermore, one easily proves $\mathfrak{f} = \Phi\Pi(\mathfrak{f})$ and $\mathcal{A} \subseteq \Pi\Phi(\mathcal{A})$. On the other hand, for $\mathfrak{x} \supseteq \bigcap \mathcal{A} \cap \tau$ and $A \in \mathfrak{x}$ we have $X \setminus \overline{A} \notin \bigcap \mathcal{A}$, and therefore $X \setminus \overline{A} \notin \mathfrak{x}$ for some $\mathfrak{x} \in \mathcal{A}$, hence $\overline{A} \in \mathfrak{x}$. Consequently, $\overline{A} \subseteq \bigcup \mathcal{A}$ and, since $\mathcal{A}$ is Zariski-closed, $\mathfrak{x} \leq \mathfrak{y}$ for some $\mathfrak{y} \in \mathcal{A}$. But $\mathcal{A}$ is also down-closed, hence $\mathfrak{x} \in \mathcal{A}$. Similarly and, in fact, more easily one can show that there are bijective maps

$$\check{X} \xrightarrow{\Phi'} F_1(X) \qquad \text{and} \qquad F_1(X) \xrightarrow{\Pi'} \check{X},$$

where $\check{X} = \{\mathcal{A} \subseteq UX \mid \mathcal{A} \text{ is Zariski-closed and up-closed}\}$, $F_1(X)$ is the set of all (possibly improper) filters on the lattice σ of closed sets of X, $\Phi'(\mathcal{A}) = \bigcap \mathcal{A} \cap \sigma$ and $\Pi'(\mathfrak{f}) = \{\mathfrak{x} \in UX \mid \mathfrak{f} \subseteq \mathfrak{x}\}$. Furthermore, for any Zariski-closed $\mathcal{A} \subseteq UX$, its down-closure $\downarrow\!\mathcal{A}$ is Zariski-closed as well. To see this, let $\mathfrak{x} \in \mathrm{cl}(\downarrow\!\mathcal{A})$. Hence $\mathfrak{x} \in \bigcup \downarrow\!\mathcal{A}$ and therefore, for any $A \in \mathfrak{x}$, we have $\overline{A} \in \bigcup \mathcal{A}$. The set

$$\mathfrak{j} = \{B \subseteq X \mid \forall \mathfrak{a} \in \mathcal{A}\,.\, B \notin \mathfrak{a}\}.$$

is an ideal, and $\mathfrak{j} \cap \{\overline{A} \mid A \in \mathfrak{x}\} = \varnothing$. Hence, there is some $\mathfrak{y} \in UX$ such that $\mathfrak{x} \leq \mathfrak{y}$ and $\mathfrak{j} \cap \mathfrak{y} = \varnothing$. But the latter fact gives us $\mathfrak{y} \subseteq \bigcup \mathcal{A}$, that is, $\mathfrak{y} \in \mathrm{cl}\,\mathcal{A} = \mathcal{A}$. We conclude $\mathfrak{x} \in \downarrow\!\mathcal{A}$. Similarly one can show that $\uparrow\!\mathcal{A}$ is Zariski-closed for each Zariski-closed subset $\mathcal{A} \subseteq UX$ (but now use $\mathfrak{x} \in \mathrm{cl}(\uparrow\!\mathcal{A}) \iff \bigcap \uparrow\!\mathcal{A} \subseteq \mathfrak{x}$). The topology of $\hat{X}$ is the *compact-open topology*, which has as basic open sets

$$B(\mathcal{B}, \{0\}) = \{\mathcal{A} \in \hat{X} \mid \mathcal{A} \cap \mathcal{B} = \varnothing\} \qquad (\mathcal{B} \subseteq UX \text{ Zariski-closed}).$$

Since $B(\mathcal{B}, \{0\}) = B(\uparrow\!\mathcal{B}, \{0\})$, it is enough to consider Zariski-closed and up-closed subsets $\mathcal{B} \subseteq UX$. Hence, using the bijections $\hat{X} \cong F_0(X)$ and $\check{X} \cong F_1(X)$, $F_0(X)$ has

$$\{\mathfrak{f} \in F_0(X) \mid \exists A \in \mathfrak{f}, B \in \mathfrak{g}\,.\, A \cap B = \varnothing\} \qquad (\mathfrak{g} \in F_1(X))$$

as basic open sets. Clearly, it is enough to consider $\mathfrak{g} = \dot{B}$ the principal filter induced by a closed set B, so that all sets

$$\{\mathfrak{f} \in F_0(X) \mid \exists A \in \mathfrak{f}\,.\, A \cap B = \varnothing\} = \{\mathfrak{f} \in F_0(X) \mid X \setminus B \in \mathfrak{f}\} \qquad (B \subseteq X \text{ closed})$$

form a basis for the topology on $F_0(X)$. But this is precisely the topology on $F_0(X)$ considered in [Esc97].

3.6. **L-separation.** We call a $\mathcal{T}$-category $X = (X, a)$ *L-separated* whenever, for every $\mathcal{T}$-category Y, the ordered set $\mathcal{T}\text{-}\mathsf{Cat}(Y, X)$ is separated, that is: its preorder is anti-symmetric. The full subcategory of $\mathcal{T}\text{-}\mathsf{Cat}$ consisting of all L-separated $\mathcal{T}$-categories is denoted by $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$.

Proposition 3.11. *Let $X = (X, a)$ be a $\mathcal{T}$-category. Then the following assertions are equivalent:*

(i) *X is L-separated.*
(ii) *$x \cong y$ implies $x = y$, for all $x, y \in X$.*
(iii) *For all $x, y \in X$, if $a(e_X(x), y) \geq k$ and $a(e_X(y), x) \geq k$, then $x = y$.*
(iv) *$y : X \longrightarrow \hat{X}$ is injective.*

Proof. As for Proposition 1.5. □

Corollary 3.12. (1) *The $\mathcal{T}$-category $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ is separated.*

(2) *For all $\mathcal{T}$-categories $Y = (Y, b)$ and $X = (X, a)$ where Y is L-separated and $a \cdot T_\xi a = a \cdot m_X$, Y^X is L-separated. In particular, $|X| \multimap \mathsf{V}$ is L-separated, for each $\mathcal{T}$-category X.*

(3) *Any subcategory of an L-separated $\mathcal{T}$-category is L-separated. In particular, $\hat{X}$ is L-separated, for every $\mathcal{T}$-category X.*

Examples 3.13. A topological space is L-separated if, and only if, it is T_0. An approach space $X = (X, d)$ with distance function $d : PX \times X \longrightarrow \mathsf{P}_+$ is L-separated if and only if

$$d(\{x\}, y) = 0 = d(\{y\}, x) \;\Rightarrow\; x = y$$

for all $x, y \in X$.

3.7. **L-completeness.** As in 1.7, we call a $\mathcal{T}$-category $X = (X, a)$ *L-complete* if every adjunction $\varphi \dashv \psi$ with $\varphi : Z \,{-\!\!\circ\!\!\rightharpoonup}\, X$ and $\psi : X \,{-\!\!\circ\!\!\rightharpoonup}\, Z$ is of the form $f_* \dashv f^*$ for a $\mathcal{T}$-functor $f : Z \longrightarrow X$. Of course, Up to equivalence, f is uniquely determined by $\varphi \dashv \psi$, and is indeed unique if X is L-separated. As before, it is enough to consider $Z = E$ (see also [CH08]).

Proposition 3.14. *Let $X = (X, a)$ be a $\mathcal{T}$-category. The following assertions are equivalent:*

(i) *X is L-complete.*

(ii) *Each left adjoint $\mathcal{T}$-module $\varphi : E \,{-\!\!\circ\!\!\rightharpoonup}\, X$ is of the form $\varphi = x_*$ for some x in X.*

(iii) *Each right adjoint $\mathcal{T}$-module $\psi : X \,{-\!\!\circ\!\!\rightharpoonup}\, E$ is of the form $\psi = x^*$ for some x in X.*

A topological space is L-complete precisely if it is weakly-sober, that is, if every irreducible closed set is the closure of a point. A similar result holds for approach spaces: L-completeness is equivalent to the condition that every irreducible closed variable set A be representable (see [CH08] for details). Furthermore, in both cases we obtain that L-complete and L-separated objects are precisely the fixed objects of the dual adjunction between topological (approach) spaces and (approach) frames, with dualizing object $\mathsf{V} = 2$ ($\mathsf{V} = \mathsf{P}_+$ respectively; see [VO05] for details).

For a pair $\psi : X \,{-\!\!\circ\!\!\rightharpoonup}\, Y$ and $\varphi : Y \,{-\!\!\circ\!\!\rightharpoonup}\, X$ of adjoint $\mathcal{T}$-modules $\varphi \dashv \psi$, the same calculation as in 1.7 shows that $\varphi = 1_X^* \multimapinv \psi$. Since for each $\mathcal{T}$-module $\psi : X \,{-\!\!\circ\!\!\rightharpoonup}\, Y$ one obtains $(1_X^* \multimapinv \psi) \circ \psi \leq 1_X^*$, ψ is right adjoint if and only if $\psi \circ (1_X^* \multimapinv \psi) \geq (1_Y)_*$. Considering in particular $Y = E$, a $\mathcal{T}$-module $\psi : X \,{-\!\!\circ\!\!\rightharpoonup}\, E$ is right adjoint if, and only if,

$$k \leq \bigvee_{\mathfrak{x} \in TX} \psi(\mathfrak{x}) \otimes \xi \cdot T\varphi(\mathfrak{x})$$

where $\varphi = 1_X^* \multimapinv \psi$. Note that $\bigvee\{\xi \cdot T\psi(\mathfrak{X}) \mid \mathfrak{X} \in TTX, m_X(\mathfrak{X}) = \mathfrak{x}\} = \psi(\mathfrak{x})$ since $\psi : |X| \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor Hence, with the help of Lemma 3.5, we see that

$$\begin{aligned}
\varphi(x) &= \bigwedge_{\mathfrak{x} \in TX} \left(\left(\bigvee_{\substack{\mathfrak{X} \in TTX, \\ m_X(\mathfrak{X}) = \mathfrak{x}}} \xi \cdot T\psi(\mathfrak{X}) \right) \multimap a(\mathfrak{x}, x) \right) \\
&= \bigwedge_{\mathfrak{x} \in TX} (\psi(\mathfrak{x}) \multimap a(\mathfrak{x}, x)) \\
&= \hat{a}(e_{\hat{X}}(\psi), y(x)).
\end{aligned}$$

Lemma 3.15. *Let $\psi : X \,{-\!\!\circ\!\!\rightharpoonup}\, E$ be a $\mathcal{T}$-module and put $\varphi = 1_X^* \multimapinv \psi$. Then, for every $\mathfrak{x} \in TX$,*

$$\xi \cdot T\varphi(\mathfrak{x}) = T_\xi \hat{a}(e_{T\hat{X}} \cdot e_{\hat{X}}(\psi), T\, y(\mathfrak{x})).$$

Proof. Since $\xi \cdot T\varphi(\mathfrak{x}) = T_\xi\varphi(\mathfrak{x})$, one obtains the result by applying T_ξ to the equality above. □

Hence, we now have:

Proposition 3.16. *Let $X = (X, a)$ be $\mathcal{T}$-category. A $\mathcal{T}$-module $\psi : X \multimap E$ is right adjoint if and only if*

$$(\dagger) \qquad k \leq \bigvee_{\mathfrak{x} \in TX} \psi(\mathfrak{x}) \otimes T_\xi\hat{a}(e_{T\hat{X}} \cdot e_{\hat{X}}(\psi), T\,y(\mathfrak{x})).$$

Given a $\mathcal{T}$-category $X = (X, a)$, we call a $\mathcal{T}$-functor $\psi : |X| \longrightarrow \mathsf{V}$ *tight* if $\psi : X^{\mathrm{op}} \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor and if, considered as a $\mathcal{T}$-module $\psi : X \multimap E$, it is right adjoint, that is, if it satisfies (†).

Example 3.17. For a topological space X and $\psi \in \hat{X}$, as in Example 3.10 we can identify ψ with a Zariski-closed and down-closed subset $\mathcal{A} \subseteq UX$, and then $1_X^* \multimap \psi$ with

$$A = \{x \in X \mid \forall \mathfrak{a} \in \mathcal{A}\,.\, \mathfrak{a} \longrightarrow x\}.$$

Then ψ is tight if, and only if, there exists some $\mathfrak{a} \in \mathcal{A}$ with $A \in \mathfrak{a}$. Furthermore, under the bijection $\hat{X} \cong F_0(X)$ (see Example 3.10), a tight map ψ corresponds to a filter $\mathfrak{f} \in F_0(X)$ with $(\operatorname{Lim}\mathfrak{f})\,\#\,\mathfrak{f}$, where $\operatorname{Lim}\mathfrak{f}$ denotes the set of all limit points of $\mathfrak{f}$, and where $A\,\#\,\mathfrak{g}$ if for all $B \in \mathfrak{f}\,.\,A \cap B \neq \varnothing$. Furthermore, for each $\mathfrak{f} \in F_0(X)$ one has

$$(\operatorname{Lim}\mathfrak{f})\,\#\,\mathfrak{f} \iff \mathfrak{f} \text{ is completely prime},$$

that is: if $\bigcup_{i \in I} U_i \in \mathfrak{f}$, then $U_i \in \mathfrak{f}$ for some $i \in I$. In fact, if $(\operatorname{Lim}\mathfrak{f})\,\#\,\mathfrak{f}$ and $\bigcup_{i \in I} U_i \in \mathfrak{f}$ for some family of open subsets of X, then $(\operatorname{Lim}\mathfrak{f}) \cap \bigcup_{i \in I} U_i \neq \varnothing$. Therefore, for some $i \in I$, U_i contains a limit point of $\mathfrak{f}$. Hence $U_i \in \mathfrak{f}$. Conversely, assume that $\mathfrak{f}$ is completely prime. Suppose that $U \in \mathfrak{f}$ does not contain a limit point of $\mathfrak{f}$. Then, for each $x \in U$, there is an open neiborhood U_x of x with $U_x \notin \mathfrak{f}$. But $\bigcup_{x \in X} U_x \in \mathfrak{f}$ and, since $\mathfrak{f}$ is completely prime, $U_x \in \mathfrak{f}$ for some $x \in U$, a contradiction.

3.8. **L-injectivity.** The notions of L-dense $\mathcal{T}$-functor, L-equivalence as well as L-injective $\mathcal{T}$-category can now be introduced as in 1.8. More precise, we call a $\mathcal{T}$-functor $f : (X, a) \longrightarrow (Y, b)$ *L-dense* if $f_* \circ f^* = 1_X^*$, which amounts to $b \cdot Tf \cdot Tf^\circ \cdot T_\xi b \cdot m_Y^\circ = b$. L-dense $\mathcal{T}$-functors have the same composition-cancellation properties as V-functors (see 1.8). A fully faithful L-dense $\mathcal{T}$-functor is an *L-equivalence*, which can be equivalently expressed by saying that f_* is an isomorphism in $\mathcal{T}$-Mod. A $\mathcal{T}$-category Z is called *pseudo-injective* if, for all $\mathcal{T}$-functors $f : X \longrightarrow Z$ and fully faithful $\mathcal{T}$-functors $i : X \longrightarrow Y$, there exists a $\mathcal{T}$-functor $g : Y \longrightarrow Z$ such that $g \cdot i \cong f$. Z is called *L-injective* if this extension property is only required along L-equivalences $i : X \longrightarrow Y$. Of course, for an L-separated $\mathcal{T}$-category Z, $g \cdot i \cong f$ implies $g \cdot i = f$, and then pseudo-injectivity coincides with the usual notion of injectivity. The following two results can be proven as in 1.8.

Lemma 3.18. *The $\mathcal{T}$-category V is injective.*

Proposition 3.19. *For all $\mathcal{T}$-categories $Y = (Y, b)$ and $X = (X, a)$ where Y is L-injective (pseudo-injective) and $a \cdot T_\xi a = a \cdot m_X$, Y^X is L-injective (pseudo-injective).*

In particular, we obtain the injectivity of the $\mathcal{T}$-category $|X| \multimap \mathsf{V}$. Lateron we will see that $\hat{X}$ and $\tilde{X}$ are also L-injective.

4. L-closure

4.1. **L-dense $\mathcal{T}$-functors.** As in 2.1, L-dense $\mathcal{T}$-functors can be characterized as "epimorphisms up to $\cong$". However, we will use here a different proof.

Lemma 4.1. *Let $X = (X, a)$ be a $\mathcal{T}$-category, $M \subseteq X$ and $i : M \hookrightarrow X$ the embedding of M into X. Then i is dense if, and only if,*

$$(\ddagger) \qquad k \leq \bigvee_{\mathfrak{a}\in TM} a(\mathfrak{a}, x) \otimes T_{\xi}a(Te_X \cdot e_X(x), \mathfrak{a})$$

for all $x \in X$.

Proof. Recall that i is L-dense whenever $i_* \circ i^* \geq a$, that is,

$$a(\mathfrak{x}, x) \leq \bigvee_{\mathfrak{a}\in TM} \bigvee_{\substack{\mathfrak{X}\in TTX \\ m_X(\mathfrak{X})=\mathfrak{x}}} a(\mathfrak{a}, x) \otimes T_{\xi}a(\mathfrak{X}, \mathfrak{a})$$

for all $\mathfrak{x} \in TX$ and $x \in X$. If i is dense, then ($\ddagger$) follows from the inequality above by putting $\mathfrak{x} = e_X(x)$ and using $m_X^{\circ} \cdot e_X = e_{TX} \cdot e_X$ (see Subsection 3.1). On the other hand, from ($\ddagger$) we obtain

$$\begin{aligned} a(\mathfrak{x}, x) &\leq \bigvee_{\mathfrak{a}\in TM} a(\mathfrak{a}, x) \otimes T_{\xi}a(Te_X \cdot e_X(x), \mathfrak{a}) \otimes a(\mathfrak{x}, x) \\ &\leq \bigvee_{\mathfrak{a}\in TM} a(\mathfrak{a}, x) \otimes T_{\xi}T_{\xi}a(e_{TTX} \cdot e_{TX}(\mathfrak{x}), e_{TX} \cdot e_X(x)) \otimes T_{\xi}a(Te_X \cdot e_X(x), \mathfrak{a}) \\ &\leq \bigvee_{\mathfrak{a}\in TM} a(\mathfrak{a}, x) \otimes T_{\xi}a(e_{TX}(\mathfrak{x}), \mathfrak{a}) \\ &\leq \bigvee_{\mathfrak{a}\in TM} \bigvee_{\substack{\mathfrak{X}\in TTX \\ m_X(\mathfrak{X})=\mathfrak{x}}} a(\mathfrak{a}, x) \otimes T_{\xi}a(\mathfrak{X}, \mathfrak{a}). \end{aligned}$$

□

Proposition 4.2. *For a $\mathcal{T}$-functor $i : M \longrightarrow X$, the following assertions are equivalent:*

(i) *$i : M \longrightarrow X$ is L-dense.*
(ii) *For all $\mathcal{T}$-functors $f, g : X \longrightarrow Y$, with $f \cdot i = g \cdot i$ one has $f \cong g$.*
(iii) *For all $\mathcal{T}$-functors $f, g : X \longrightarrow \mathsf{V}$, with $f \cdot i = g \cdot i$ one has $f = g$.*

Proof. Assuming (i), so that $i : M \longrightarrow X$ is L-dense, from $f \cdot i = g \cdot i$ we obtain $f_* = g_*$ and therefore (ii) since $i_* \circ i^* = 1_X^*$. The implication (ii)⇒(iii) holds trivially since V is L-separated. Now assume (iii). According to the remarks made above, we can assume that $i : M \longrightarrow X$ is the embedding of a subset $M \subseteq X$. For $x \in X$, First note that the map

$$\varphi : X \longrightarrow \mathsf{V},\ y \longmapsto a(e_X(x), y)$$

is a $\mathcal{T}$-functor since $a : |X| \otimes X \longrightarrow \mathsf{V}$ is one. Using the same argument as in [Hof07, Lemma 6.8], we see that also

$$\psi : X \longrightarrow \mathsf{V},\ y \longmapsto \bigvee_{\mathfrak{x}\in TM} T_{\xi}a(Te_X \cdot e_X(x), \mathfrak{x}) \otimes a(\mathfrak{x}, y)$$

is a $\mathcal{T}$-functor. Clearly, for each $y \in X$ we have $\psi(y) \leq \varphi(y)$. If $y \in M$, we can choose $\mathfrak{x} = e_X(y) \in TM$ and therefore, using $Te_X \cdot e_X = e_{TX} \cdot e_X$ and op-laxness of e, obtain $\varphi(y) \leq \psi(y)$. Hence $\varphi|_M = \psi|_M$, and from our assumption (iii) we deduce $k \leq \varphi(x) = \psi(x)$. □

4.2. **L-closure.** For a $\mathcal{T}$-category $X = (X, a)$ and $M \subseteq X$, we define the *L-closure* of M in X by

$$\overline{M} = \{x \in X \mid \forall f, g : X \longrightarrow Y \,.\, (f|_M = g|_M \ \Rightarrow\ f(x) \cong g(x))\}.$$

Hence $\overline{M}$ is the largest subset N of X making the inclusion map $i : M \hookrightarrow N$ dense.

Proposition 4.3. *Let $X = (X, a)$ be a $\mathcal{T}$-category, $M \subseteq X$ and $x \in X$. Then the following assertions are equivalent.*

(i) *$x \in \overline{M}$.*

(ii) $k \le \bigvee_{\mathfrak{x}\in TM} a(\mathfrak{x},x)\otimes T_{\xi}a(Te_X\cdot e_X(x),\mathfrak{x})$.
(iii) $i^*\circ x_* \dashv x^*\circ i_*$, *where* $i: M\hookrightarrow X$ *is the inclusion map.*
(iv) $1_E^* \le x^*\circ i_*\circ i^*\circ x_*$,
(v) $i^*\circ x_* \dashv x^*\circ i_*$.
(vi) $x_*: E \multimap X$ *factors through* $i_*: M\multimap X$ *by a map* $\varphi: E\multimap M$ *in* $\mathcal{T}$-Mod.

Proof. Using Lemma 4.1 one can proceed as in Proposition 2.2. □

We can now proceed as in Subsection 2.2.

Proposition 4.4. *For a* $\mathcal{T}$*-functor* $f: X\longrightarrow Y$ *and* $M, M'\subseteq X$, $N\subseteq Y$, *one has:*

(1) $M\subseteq \overline{M}$ *and* $M\subseteq M'$ *implies* $\overline{M}\subseteq\overline{M'}$.
(2) $\overline{\overline{M}}=\overline{M}$ *and, if* $T\varnothing=\varnothing$, *then* $\overline{\varnothing}=\varnothing$.
(3) $f(\overline{M})\subseteq\overline{f(M)}$ *and* $f^{-1}(\overline{N})\supseteq\overline{f^{-1}(N)}$.
(4) *If* k *is* $\vee$*-irreducible and* T *preserves binary sums, then* $\overline{M\cup M'}=\overline{M}\cup\overline{M'}$.

Corollary 4.5. *If* k *is* $\vee$*-irreducible in* V *and* T *preserves finite sums, then the L-closure operator defines a topology on* X *such that every* $\mathcal{T}$*-functor becomes continuous. Hence, L-closure defines a functor* $L: \mathcal{T}\text{-Cat}\longrightarrow \mathsf{Top}$.

Example 4.6. For a topological space X, $x\in X$ lies in the L-closure of $A\subseteq X$ precisely if there exists some ultrafilter $\mathfrak{x}\in UA$ with $\overline{x}\in\mathfrak{x}$ and which converges to x; in other words, for every neighborhood U of x we have $U\cap\overline{x}\cap A\neq\varnothing$. Hence the L-closure of a topological space X coincides with the so called *b-closure* [Bar68].

4.3. **L-separation via the L-closure.**

Proposition 4.7. *Let* $X=(X,a)$ *be a* $\mathcal{T}$*-category and* $\Delta\subseteq X\times X$ *the diagonal. Then*

$$\overline{\Delta}=\{(x,y)\in X\times X \mid x\cong y\}.$$

Proof. As for Proposition 2.6. □

Corollary 4.8. *A* $\mathcal{T}$*-category* X *is L-separated if and only if the diagonal* Δ *is closed in* $X\times X$.

Theorem 4.9. $\mathcal{T}\text{-Cat}_{\text{sep}}$ *is an epi-reflective subcategory of* $\mathcal{T}$-Cat, *where the reflection map is given by* $y_X: X\longrightarrow y_X(X)$, *for each* $\mathcal{T}$*-category* X. *Hence, limits of L-separated* $\mathcal{T}$*-categories are formed in* $\mathcal{T}$-Cat, *while colimits are obtained by reflecting the colimit formed in* $\mathcal{T}$-Cat. *The epimorphisms in* $\mathcal{T}\text{-Cat}_{\text{sep}}$ *are precisely the L-dense* $\mathcal{T}$*-functors.*

4.4. **L-completeness via the L-closure.**

Lemma 4.10. *Let* $X=(X,a)$ *be a* $\mathcal{T}$*-category and* $M\subseteq X$.

(1) *Assume that* X *is L-complete and* M *be L-closed. Then* M *is L-complete.*
(2) *Assume that* X *is L-separated and* M *is L-complete. Then* M *is L-closed.*

Proof. As for Lemma 2.9. □

Theorem 4.11. *Let* $X=(X,b)$ *be a* $\mathcal{T}$*-category. The following assertions are equivalent.*

(i) *X is L-complete.*
(ii) *X is L-injective.*
(iii) $y: X\longrightarrow\tilde{X}$ *has a pseudo left-inverse* $\mathcal{T}$*-functor* $R:\tilde{X}\longrightarrow X$, *i.e.* $R\cdot y\cong 1_X$.

Proof. As for Theorem 2.10. □

Therefore we have that $|X| \multimap \mathsf{V}$ is L-complete. Our next result shows that $\hat{X}$ is also L-complete.

Proposition 4.12. *$\hat{X}$ is L-closed in $|X| \multimap \mathsf{V}$, for each $\mathcal{T}$-category X.*

Proof. Let $X = (X, a)$ be a $\mathcal{T}$-category and assume that $\varphi \in (|X| \multimap \mathsf{V})$ belongs to the closure of $\hat{X}$, that is,

$$k \le \bigvee_{\mathfrak{u} \in T\hat{X}} [\![m_X, \hom_\xi]\!](\mathfrak{u}, \varphi) \otimes T_{\!\xi}[\![m_X, \hom_\xi]\!](Te_{|X|\multimap\mathsf{V}} \cdot e_{|X|\multimap\mathsf{V}}(\varphi), \mathfrak{u}).$$

We wish to show that $r(\mathfrak{x}, \mathfrak{y}) \otimes \varphi(\mathfrak{y}) \le \varphi(\mathfrak{x})$ for all $\mathfrak{x}, \mathfrak{y} \in TX$, where $r = T_{\!\xi} a \cdot m_X^\circ$.
First note that, for all $\alpha, \beta \in (|X| \multimap \mathsf{V})$,

$$e_{|X|\multimap\mathsf{V}}^\circ \cdot [\![m_X, \hom_\xi]\!](\alpha, \beta) = \bigwedge_{\mathfrak{x} \in TX} (\alpha(\mathfrak{x}) \multimap \beta(\mathfrak{x})).$$

Hence, with $h_\mathfrak{x} : (|X| \multimap \mathsf{V}) \to\!\!\!\!\!\mapsto (|X| \multimap \mathsf{V})$, $h_\mathfrak{x}(\alpha, \beta) = (\alpha(\mathfrak{x}) \multimap \beta(\mathfrak{x}))$, we have $Te_{|X|\multimap\mathsf{V}}^\circ \cdot T_{\!\xi}[\![m_X, \hom_\xi]\!] \le T_{\!\xi} h_\mathfrak{x}$. Since the diagram

$$\begin{array}{ccc}
(|X| \multimap \mathsf{V}) \times (|X| \multimap \mathsf{V}) & \xrightarrow{\mathrm{ev}_\mathfrak{x} \times \mathrm{ev}_\mathfrak{x}} & \mathsf{V} \times \mathsf{V} \\
 & \searrow^{h_\mathfrak{x}} & \downarrow{\scriptstyle \multimap} \\
 & & \mathsf{V}
\end{array}$$

commutes, and since in the diagram

$$\begin{array}{ccccc}
T((|X| \multimap \mathsf{V}) \times (|X| \multimap \mathsf{V})) & \xrightarrow{T(\mathrm{ev}_\mathfrak{x} \times \mathrm{ev}_\mathfrak{x})} & T(\mathsf{V} \times \mathsf{V}) & \xrightarrow{T(\multimap)} & T\mathsf{V} \\
{\scriptstyle \mathrm{can}}\downarrow & & {\scriptstyle \mathrm{can}}\downarrow & & \\
T(|X| \multimap \mathsf{V}) \times T(|X| \multimap \mathsf{V}) & \xrightarrow[T\,\mathrm{ev}_\mathfrak{x} \times T\,\mathrm{ev}_\mathfrak{x}]{} & T\mathsf{V} \times T\mathsf{V} & \ge & \downarrow{\scriptstyle \xi} \\
 & & {\scriptstyle \xi\times\xi}\downarrow & & \\
 & & \mathsf{V} \times \mathsf{V} & \xrightarrow[\multimap]{} & \mathsf{V}
\end{array}$$

the left hand rectangle commutes whereas in the right hand rectangle the "lower path" is greater or equal to the "upper path", we obtain

$$T_{\!\xi} h_\mathfrak{x}(\mathfrak{u}, \mathfrak{v}) \le (\mathfrak{u}(\mathfrak{x}) \multimap \mathfrak{v}(\mathfrak{x}))$$

for every $\mathfrak{x} \in TX$ and $\mathfrak{u}, \mathfrak{v} \in T(|X| \multimap \mathsf{V})$, where $\mathfrak{u}(\mathfrak{x}) = \xi \cdot T\,\mathrm{ev}_\mathfrak{x}(\mathfrak{u})$. Accordingly, $e_{|X|\multimap\mathsf{V}}(\varphi)(\mathfrak{x}) = \varphi(\mathfrak{x})$, and we obtain

$$\forall \mathfrak{x} \in TX\ .\ T_{\!\xi}[\![m_X, \hom_\xi]\!](Te_{|X|\multimap\mathsf{V}} \cdot e_{|X|\multimap\mathsf{V}}(\varphi), \mathfrak{u}) \le (\varphi(\mathfrak{x}) \multimap \mathfrak{u}(\mathfrak{x})).$$

Furthermore, for all $\mathfrak{x}, \mathfrak{y} \in TX$ one has

$$\begin{array}{ccccc}
\hat{X} & \xrightarrow{\Delta} & \hat{X} \times \hat{X} & \xrightarrow{\mathrm{ev}_\mathfrak{x} \times \mathrm{ev}_\mathfrak{y}} & \mathsf{V} \times \mathsf{V} \\
{\scriptstyle !}\downarrow & & \le & & \downarrow{\scriptstyle \multimap} \\
1 & & \xrightarrow[r(\mathfrak{x},\mathfrak{y})]{} & & \mathsf{V}
\end{array}$$

and obtains

$$r(\mathfrak{x}, \mathfrak{y}) \le \xi \cdot T(\multimap) \cdot T(\mathrm{ev}_\mathfrak{x} \times \mathrm{ev}_\mathfrak{y}) \cdot T\Delta(\mathfrak{u}) \le (\mathfrak{u}(\mathfrak{y}) \multimap \mathfrak{u}(\mathfrak{x}))$$

for all $\mathfrak{u} \in T\hat{X}$. We conclude that

$$r(\mathfrak{x},\mathfrak{y}) \otimes \varphi(\mathfrak{y}) \leq \bigvee_{\mathfrak{u}\in T\hat{X}} r(\mathfrak{x},\mathfrak{y}) \otimes \varphi(\mathfrak{y}) \otimes [\![m_X, \hom_\xi]\!](\mathfrak{u},\varphi) \otimes T_\xi[\![m_X, \hom_\xi]\!](Te_{|X|\multimap\mathsf{V}} \cdot e_{|X|\multimap\mathsf{V}}(\varphi),\mathfrak{u})$$
$$\leq \bigvee_{\mathfrak{u}\in T\hat{X}} r(\mathfrak{x},\mathfrak{y}) \otimes \varphi(\mathfrak{y}) \otimes (\varphi(\mathfrak{y}) \multimap \mathfrak{u}(\mathfrak{y})) \otimes [\![m_X, \hom_\xi]\!](\mathfrak{u},\varphi)$$
$$\leq \bigvee_{\mathfrak{u}\in T\hat{X}} r(\mathfrak{x},\mathfrak{y}) \otimes \mathfrak{u}(\mathfrak{y}) \otimes [\![m_X, \hom_\xi]\!](\mathfrak{u},\varphi)$$
$$\leq \bigvee_{\mathfrak{u}\in T\hat{X}} \mathfrak{u}(\mathfrak{x}) \otimes [\![m_X, \hom_\xi]\!](\mathfrak{u},\varphi) \leq \bigvee_{\mathfrak{u}\in T\hat{X}} \mathfrak{u}(\mathfrak{x}) \otimes (\mathfrak{u}(\mathfrak{x}) \multimap \varphi(\mathfrak{x})) \leq \varphi(\mathfrak{x}).$$

□

Proposition 4.13. *Let $X = (X, a)$ be a $\mathcal{T}$-category and $\psi \in \hat{X}$. Then ψ is a right adjoint $\mathcal{T}$-module if and only if $\psi \in \overline{y(X)}$.*

Proof. By Proposition 3.16 and Theorem 3.9, ψ is right adjoint if, and only if,

$$k \leq \bigvee_{\mathfrak{x}\in TX} \hat{a}(T\,y(\mathfrak{x}),\psi) \otimes T_\xi \hat{a}(Te_{\hat{X}} \cdot e_{\hat{X}}(\psi), T\,y(\mathfrak{x})),$$

which means precisely that $\psi \in \overline{y(X)}$. □

The proposition above identifies $\tilde{X}$ as the L-closure of $y(X)$ in $\hat{X}$, and therefore as an L-complete $\mathcal{T}$-category. Furthermore, $y : X \longrightarrow \tilde{X}$ is fully faithful and L-dense. Hence we can state:

Theorem 4.14. *The full subcategory $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{cpl}}$ of $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$ of L-complete $\mathcal{T}$-categories is an epi-reflective subcategory of $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$. The reflection map of an L-separated $\mathcal{T}$-category X is given by any full L-dense embedding of X into an L-complete and L-separated $\mathcal{T}$-category, for instance by $y : X \longrightarrow \tilde{X}$.*

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