Exponentiability in categories of lax algebras

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Dedicated to Nico Pumplün on the occasion of his seventieth birthday

Abstract

For a complete cartesian-closed category $\mathbf{V}$ with coproducts, and for any pointed endofunctor T of the category of sets satisfying a suitable Beck-Chevalley-type condition, it is shown that the category of lax reflexive $(T, \mathbf{V})$-algebras is a quasitopos. This result encompasses many known and new examples of quasitopoi.

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Key words: lax algebra, partial product, locally cartesian-closed category, quasitopos.

0   Introduction

Failure to be cartesian closed is one of the main defects of the category of topological spaces. But often this defect can be side-stepped by moving temporarily into the quasitopos hull of **Top**, the category of pseudotopological (or Choquet) spaces, see for example [11, 14, 7]. A pseudotopology on a set X is most easily described by a relation $\mathfrak{x} \to x$ between ultrafilters $\mathfrak{x}$ on X and points x in X, the only requirement for which is the *reflexivity* condition $\overset{\bullet}{x} \to x$ for all $x \in X$, with $\overset{\bullet}{x}$ denoting the principal ultrafilter on x. In this setting, a topology on X is a pseudotopology which satisfies the *transitivity* condition

$$\mathfrak{X} \to \mathfrak{y} \ \& \ \mathfrak{y} \to z \ \Rightarrow \ m(\mathfrak{X}) \to z$$

for all $z \in X$, $\mathfrak{y} \in UX$ (the set of ultrafilters on X) and $\mathfrak{X} \in UUX$; here the relation $\to$ between UX and X has been naturally extended to a relation between UUX and UX, and $m = m_X : UUX \to UX$ is the unique map that gives U together with $e_X(x) = \overset{\bullet}{x}$ the structure of a monad $\mathsf{U} = (U, e, m)$. Barr [2] observed that the two conditions, reflexivity and transitivity, are precisely the two basic laws of a lax Eilenberg-Moore algebra when one extends the **Set**-monad U to a lax monad of Rel(**Set**), the category of sets with relations as morphisms. In [9] Barr's presentation of topological spaces was extended to include Lawvere's presentation of metric spaces as $\mathbf{V}$-categories with $\mathbf{V} = \overline{\mathbb{R}}_+$, the extended real half-line. Thus, for any symmetric monoidal category $\mathbf{V}$ with coproducts preserved by the tensor product, and for any **Set**-monad T that suitably extends from **Set**-maps to all $\mathbf{V}$-matrices (or "$\mathbf{V}$-relations", with

ordinary relations appearing for $\mathbf{V} = \mathbf{2}$, the two-element chain), the paper [9] develops the notion of reflexive and transitive $(\mathsf{T}, \mathbf{V})$-algebra, investigates the resulting category $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$, and presents many examples, in particular $\mathbf{Top} = \mathrm{Alg}(\mathsf{U}, \mathbf{2})$.

The purpose of this paper is to show that dropping the transitivity condition leads us to a quasitopos not only in the case of $\mathbf{Top}$, but rather generally. In order to define just reflexive $(\mathsf{T}, \mathbf{V})$-algebras, one indeed needs neither the tensor product of $\mathbf{V}$ (just the "unit" object) nor the "multiplication" of the monad T. Positively speaking then, we start off with a category $\mathbf{V}$ with coproducts and a distinguished object I in $\mathbf{V}$ and any pointed endofunctor T of $\mathbf{Set}$ and define the category $\mathrm{Alg}(T, \mathbf{V})$. Our main result says that when $\mathbf{V}$ is complete and locally cartesian closed and a certain Beck-Chevalley condition is satisfied, also $\mathrm{Alg}(T, \mathbf{V})$ is locally cartesian closed (Theorem 2.7).

Defining reflexive $(T, \mathbf{V})$-algebras for the "truncated" data T, $\mathbf{V}$ entails a considerable departure from [9], as it is no longer possible to talk about the bicategory $\mathrm{Mat}(\mathbf{V})$ of $\mathbf{V}$-matrices. The missing tensor product prevents us from being able to introduce the (horizontal) matrix composition; however, "whiskering" by $\mathbf{Set}$-maps (considered as 1-cells in $\mathrm{Mat}(\mathbf{V})$) is still well-defined and well-behaved, and this is all that is needed in this paper.

We explain the relevant properties of $\mathrm{Mat}(\mathbf{V})$ in Section 1 and define the needed Beck-Chevalley condition. Briefly, this condition says that the comparison map that "measures" the extent to which the T-image of a pullback diagram in $\mathbf{Set}$ still is a pullback diagram must be a lax epimorphism when considered a 1-cell in $\mathrm{Mat}(\mathbf{V})$. Having presented our main result, at the end of Section 2 we show that this condition is equivalent to asking T to preserve pullbacks or, if $\mathbf{V}$ is thin (i.e., a preordered class), to transform pullbacks into weak pullback diagrams (barring trivial choices for I and $\mathbf{V}$). In certain cases, (BC) turns out to be even a necessary condition for local cartesian closedness of $\mathrm{Alg}(T, \mathbf{V})$, see 2.10. In Section 3 we show how to construct limits and colimits in $\mathrm{Alg}(T, \mathbf{V})$ in general, and Section 4 presents the construction of partial map classifiers, leading us to the theorem stated in the Abstract. A list of examples follows in Section 5.

1 V-matrices

1.1 Let $\mathbf{V}$ be a category with coproducts and a distinguished object I. A $\mathbf{V}$-*matrix* (or $\mathbf{V}$-*relation*) r from a set X to a set Y, denoted by $r : X \nrightarrow Y$, is a functor $r : X \times Y \to \mathbf{V}$, i.e. an $X \times Y$-indexed family $(r(x, y))_{x,y}$ of objects in $\mathbf{V}$. With X, Y fixed, such $\mathbf{V}$-matrices form the objects of a category $\mathrm{Mat}(\mathbf{V})(X, Y)$, the morphisms $\varphi : r \to s$ of which are natural transformations, i.e. families $(\varphi_{x,y} : r(x, y) \to s(x, y))_{x,y}$ of morphisms in $\mathbf{V}$; briefly,

$$\mathrm{Mat}(\mathbf{V})(X, Y) = \mathbf{V}^{X \times Y}.$$

1.2 Every **Set**-map $f : X \to Y$ may be considered as a **V**-matrix $f : X \nrightarrow Y$ when one puts

$$f(x, y) = \begin{cases} I & \text{if } f(x) = y, \\ 0 & \text{else,} \end{cases}$$

with 0 denoting a fixed initial object in **V**. This defines a functor

$$\mathbf{Set}(X, Y) \longrightarrow \mathrm{Mat}(\mathbf{V})(X, Y),$$

of the discrete category $\mathbf{Set}(X, Y)$, and the question is: when do we obtain a full embedding, for all X and Y? Precisely when

(*) $\mathbf{V}(I, 0) = \emptyset$ and $|\mathbf{V}(I, I)| = 1$,

as one may easily check. In the context of a cartesian-closed category **V**, we usually pick for I a terminal object 1 in **V**, and then condition (*) is equivalently expressed as

(**) $0 \ncong 1$,

preventing **V** from being equivalent to the terminal category.

1.3 While in this paper we do not need the horizontal composition of **V**-matrices in general, we do need the composites sf and gr for maps $f : X \to Y$, $g : Y \to Z$ and **V**-relations $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$, defined by

$$\begin{aligned} (sf)(x, z) &= s(f(x), z), \\ (gr)(x, z) &= \sum_{y \,:\, g(y) = z} r(x, y), \end{aligned}$$

for $x \in X$, $z \in Z$; likewise for morphisms $\varphi : r \to r'$ and $\psi : s \to s'$. Hence, we have the "whiskering" functors

$$-f : \mathrm{Mat}(\mathbf{V})(Y, Z) \to \mathrm{Mat}(\mathbf{V})(X, Z),$$

$$g- : \mathrm{Mat}(\mathbf{V})(X, Y) \to \mathrm{Mat}(\mathbf{V})(X, Z).$$

The horizontal composition with **Set**-maps from either side is associative up to coherent isomorphisms whenever defined; hence, if $h : U \to X$ and $k : Z \to V$, then

$$(sf)h = s(fh) \quad \text{and} \quad k(gr) \cong (kg)r.$$

Although $\mathrm{Mat}(\mathbf{V})$ falls short of being a bicategory, even a sesquicategory [15], we refer to sets as 0-cells of $\mathrm{Mat}(\mathbf{V})$, **V**-matrices as its 1-cells, and natural transformations between them as its 2-cells.

1.4 The transpose $r^\circ : Y \nrightarrow X$ of a **V**-matrix $r : X \nrightarrow Y$ is defined by $r^\circ(y, x) = r(x, y)$ for all $x \in X$, $y \in Y$. Obviously $r^{\circ\circ} = r$, and with

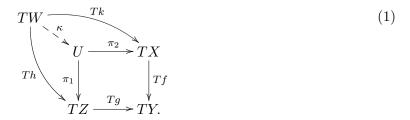$$(sf)^\circ = f^\circ s^\circ, \ (gr)^\circ = r^\circ g^\circ$$

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we can also introduce whiskering by transposes of **Set**-maps from either side, also for 2-cells.

A **Set**-map $f : X \to Y$ gives rise to 2-cells

$$\eta : 1_X \to f^\circ f, \ \ \varepsilon : f f^\circ \to 1_Y$$

satisfying the triangular identities $(\varepsilon f)(f\eta) = 1_f$, $(f^\circ \varepsilon)(\eta f^\circ) = 1_f$.

1.5 For a functor $T : \mathbf{Set} \to \mathbf{Set}$, we denote by $\kappa : TW \to U$ the comparison map from the T-image of the pullback $W := Z \times_Y X$ of (g,f) to the pullback $U := TZ \times_{TZ} TX$ of (Tg, Tf)

$$(1)$$

We say that the **Set**-functor T satisfies the *Beck-Chevalley Condition (BC)* if the 1-cell κ is a lax epimorphism; that is, if the "whiskering" functor $-\kappa : \mathrm{Mat}(\mathbf{V})(TW, S) \to \mathrm{Mat}(\mathbf{V})(U, S)$ is full and faithful, for every set S.

In the next section we will relate this condition with other known formulations of the Beck-Chevalley condition.

2 Local cartesian closedness of $\mathrm{Alg}(T, \mathbf{V})$

2.1 Let (T, e) be a pointed endofunctor of **Set** and **V** category with coproducts and a distinguished object I. A *lax (reflexive) $(T, \mathbf{V})$-algebra* (X, a, η) is given by a set X, a 1-cell $a : TX \twoheadrightarrow X$ and a 2-cell $\eta : 1_X \to a e_X$ in $\mathrm{Mat}(\mathbf{V})$. The 2-cell η is completely determined by the **V**-morphisms

$$\eta_x := \eta_{x,x} : I \longrightarrow a(e_X(x), x),$$

$x \in X$. As we shall not change the notation for this 2-cell, *we write (X, a) instead of (X, a, η)*. A *(lax) homomorphism* $(f, \varphi) : (X, a) \to (Y, b)$ of $(T, \mathbf{V})$-algebras is given by a map $f : X \to Y$ in **Set** and a 2-cell $\varphi : fa \to b(Tf)$ which must preserve the units: $(\varphi e_X)(f\eta) = \eta f$. The 2-cell φ is completely determined by a family of **V**-morphisms

$$f_{\mathfrak{x},x} : a(\mathfrak{x}, x) \longrightarrow b(Tf(\mathfrak{x}), f(x)),$$

$x \in X$, $\mathfrak{x} \in TX$, and preservation of units now reads as $f_{e_X(x),x}\eta_x = \eta_{f(x)}$ for all $x \in X$. For simplicity, *we write f instead of (f, φ)*, and when we write

$$f_{\mathfrak{x},x} : a(\mathfrak{x}, x) \longrightarrow b(\mathfrak{y}, y)$$

this automatically entails $\mathfrak{y} = Tf(\mathfrak{x})$ and $y = f(x)$; these are the **V***-components* of the homomorphism f. Composition of (f, φ) with $(g, \psi) : (Y, b) \to (Z, c)$ is defined by

$$(g, \psi)(f, \varphi) = (gf, (\psi(Tf))(g\varphi))$$

which, in the notation used more frequently, means

$$(gf)_{\mathfrak{x},x} = (a(\mathfrak{x},x) \xrightarrow{f_{\mathfrak{x},x}} b(\mathfrak{y},y) \xrightarrow{g_{\mathfrak{y},y}} c(\mathfrak{z},z)).$$

We obtain the category $\mathrm{Alg}(T, \mathbf{V})$ (denoted by $\mathrm{Alg}(T, e; \mathbf{V})$ in [9]).

2.2 Let $\mathbf{V}$ be finitely complete. The pullback (W, d) of $f : (X, a) \to (Z, c)$ and $g : (Y, b) \to (Z, c)$ in $\mathrm{Alg}(T, \mathbf{V})$ is constructed by the pullback $W = X \times_Z Y$ in $\mathbf{Set}$ and a family of pullback diagrams in $\mathbf{V}$, as follows:

$$
\begin{array}{ccc}
d(\mathfrak{w}, w) & \xrightarrow{f'_{\mathfrak{w},w}} & b(\mathfrak{y}, y) \\
{\scriptstyle g'_{\mathfrak{w},w}} \downarrow & & \downarrow {\scriptstyle g_{\mathfrak{y},y}} \\
a(\mathfrak{x}, x) & \xrightarrow{f_{\mathfrak{x},x}} & c(\mathfrak{z}, z)
\end{array}
$$

for all $w \in W$; hence,

$$d(\mathfrak{w}, w) = a(Tg'(\mathfrak{w}), g'(w)) \times_c b(Tf'(\mathfrak{w}), f'(w))$$

in $\mathbf{V}$, where $g' : W \to X$ and $f' : W \to Y$ are the pullback projections in $\mathbf{Set}$. For each $w = (x, y)$ in W, we define $\eta_w := < \eta_x, \eta_y >$.

2.3 Every set X carries the *discrete $(T, \mathbf{V})$-structure* e_X°. In fact, the 2-cell $\eta : 1_X \to e_X^\circ e_X$ making (X, e_X°) a $(T, \mathbf{V})$-algebra is just the unit of the adjunction $e_X \dashv e_X^\circ$ in $\mathrm{Mat}(\mathbf{V})$. Now $X \mapsto (X, e_X^\circ)$ defines the left adjoint of the forgetful functor

$$\mathrm{Alg}(T, \mathbf{V}) \longrightarrow \mathbf{Set}$$

since every map $f : X \to Y$ into a $(T, \mathbf{V})$-algebra (Y, b) becomes a homomorphism $f : (X, e_X^\circ) \to (Y, b)$; indeed the needed 2-cell $f e_X^\circ \to b(Tf)$ is obtained from the unit 2-cell $\eta : 1 \to b e_Y$ with the adjunction $e_X \dashv e_X^\circ$: it is the mate of $f\eta : f \to b e_Y f = b(Tf) e_X$. In pointwise notation, for

$$f_{\mathfrak{x},x} : e_X^\circ(\mathfrak{x}, x) \longrightarrow b(\mathfrak{y}, y)$$

one has $f_{\mathfrak{x},x} = 1_I$ if $e_X(x) = \mathfrak{x}$; otherwise its domain is the initial object 0 of $\mathbf{V}$, i.e. it is *trivial*.

2.4 We consider the discrete structure in particular on a one-element set 1. Then, for every $(T, \mathbf{V})$-algebra (X, a), an element $x \in X$ can be equivalently considered as a homomorphism $x : (1, e_1^\circ) \to (X, a)$ whose only non-trivial component is the unit $\eta_x : I \to a(e_X(x), x)$.

2.5 Assume $\mathbf{V}$ to be complete and locally cartesian closed. For a homomorphism $f : (X, a) \to (Y, b)$ and an additional $(T, \mathbf{V})$-algebra (Z, c) we form a substructure of the partial product of the underlying $\mathbf{Set}$-data (see [10]), namely

$$
\begin{array}{ccc}
Z \xleftarrow{\text{ev}} Q & \xrightarrow{q} & X \\
{\scriptstyle f'} \downarrow & & \downarrow {\scriptstyle f} \\
P & \xrightarrow{p} & Y,
\end{array}
\qquad (2)
$$

with

$$P = Z^f = \{(s,y) \mid y \in Y, \; s : (X_y, a_y) \to (Z, c)\},$$

$$Q = Z^f \times_Y X = \{(s,x) \mid x \in X, \; s : (X_{f(x)}, a_{f(x)}) \to (Z, c)\},$$

where $(X_y = f^{-1}y, a_y)$ is the domain of the pullback

$$i_y : (X_y, a_y) \longrightarrow (X, a)$$

of $y : (1, e_1^\circ) \to (Y, b)$ along f. Of course, p and q are projections, and ev is the evaluation map. We must find a structure $d : TP \nrightarrow P$ which, together with a 2-cell η, will make these maps morphisms in $\mathrm{Alg}(T, \mathbf{V})$.

For $(s,y) \in P$ and $\mathfrak{p} \in TP$, in order to define $d(\mathfrak{p}, (s,y))$, consider each pair $x \in X$ and $\mathfrak{q} \in TQ$ with $f(x) = y$ and $Tf'(\mathfrak{q}) = \mathfrak{p}$ and form the partial product

$$c(\mathfrak{z}, s(x)) \xleftarrow{\;\tilde{ev}_{\mathfrak{q},x}\;} c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}} \times_b a(\mathfrak{x}, x) \longrightarrow a(\mathfrak{x}, x) \qquad (3)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow f_{\mathfrak{x},x}$$

$$c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}} \xrightarrow{\;\;\tilde{p}_{\mathfrak{q},x}\;\;} b(\mathfrak{y}, y)$$

in $\mathbf{V}$, where $\mathfrak{z} = T\mathrm{ev}(\mathfrak{q})$, and then the multiple pullback $d(\mathfrak{p}, (s,y))$ of the morphisms $\tilde{p}_{\mathfrak{q},x}$ in $\mathbf{V}$, as in:

$$c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}}$$
$$\overset{\pi_{\mathfrak{q},x}}{\nearrow} \qquad \overset{\tilde{p}_{\mathfrak{q},x}}{\searrow}$$
$$d(\mathfrak{p}, (s,y)) \xrightarrow{\;\;p_{\mathfrak{p},(s,y)}\;\;} b(\mathfrak{y}, y).$$

2.6 We define the 2-cell $\eta : 1_P \to de_P$ componentwise. Let $(s,y) \in P$ and consider each $x \in X$ and $\mathfrak{q} \in TQ$ with $f(x) = y$ and $Tf'(\mathfrak{q}) = e_P(s,y) = T(s,y)e_1$ (where $(s,y) : 1 \to P$). Consider the pullback $j_y : X_y \to Q$ of $(s,y) : 1 \to P$ along f' in $\mathbf{Set}$; whence, $j_y(x) = s(x)$. By (BC) there is $\mathfrak{x} \in TX_y$ such that $Tj_y(\mathfrak{x}) = \mathfrak{q}$ and $T!(\mathfrak{x}) = e_1(*)$ (where $! : X_y \to 1$ and $*$ is the only point of 1). Since $\mathrm{ev}j_y = s$, we may form the diagram

$$c(\mathfrak{z}, s(x)) \xleftarrow{\;s_{\mathfrak{x},x}\;} a_y(\mathfrak{x}, x) \xrightarrow{\;(i_y)_{\mathfrak{x},x}\;} a(\mathfrak{x}, x)$$

$$\downarrow \qquad\qquad\qquad \downarrow f_{\mathfrak{x},x}$$

$$I \xrightarrow{\;\;\eta_y\;\;} b(e_Y(y), y)$$

in $\mathbf{V}$, where $\mathfrak{z} = T\mathrm{ev}(\mathfrak{q}) = Ts(\mathfrak{x})$, and the square is a pullback. The universal property of (3) guarantees the existence of $\tilde{\eta}_{\mathfrak{q},x} : I \to c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}}$ such that $\tilde{p}_{\mathfrak{q},x}\tilde{\eta}_{\mathfrak{q},x} = \eta_y$ and $\tilde{ev}_{\mathfrak{q},x}(\tilde{\eta}_{\mathfrak{q},x} \times_b 1) = s_{\mathfrak{x},x}$. Then, with the multiple pullback property, the morphisms $\tilde{\eta}_{\mathfrak{q},x}$ define jointly $\eta_{(s,y)} : I \to d(e_P(s,y), (s,y))$.

2.7 Theorem. *If the pointed $\mathbf{Set}$-functor T satisfies (BC) and $\mathbf{V}$ is complete and locally cartesian closed, then also $\mathrm{Alg}(T, \mathbf{V})$ is locally cartesian closed.*

Proof. Continuing in the notation of 2.5 and 2.6, we equip Q with the lax algebra structure $r : TQ \nrightarrow Q$ that makes the square of diagram (2) a pullback diagram in $\mathrm{Alg}(T, \mathbf{V})$. Then the

2-cell defined by
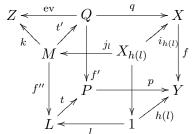
$$r(\mathfrak{q},(s,x))\xrightarrow{\pi_{\mathfrak{q},x}\times_b 1} c(\mathfrak{z},s(x))^{f_{\mathfrak{x},x}}\times_b a(\mathfrak{x},x)\xrightarrow{\tilde{\mathrm{ev}}_{\mathfrak{q},x}} c(\mathfrak{z},s(x))$$

makes $\mathrm{ev}:(Q,r)\to(Z,c)$ a homomorphism.

In order to prove the universal property of the partial product, given any other pair $(h:(L,u)\to(Y,b),k:(M,v)\to(Z,c))$, where $M:=L\times_Y X$, we consider the map $t:L\to P$, defined by $t(l):=(s_l,h(l))$, with
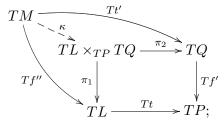
$$((X_{h(l)},a_{h(l)}))\xrightarrow{s_l}(Z,c))=((X_{h(l)},a_{h(l)})\xrightarrow{j_l}(M,v)\xrightarrow{k}(Z,c)),$$

where j_l is the pullback of $l:(1,e_1^\circ)\to(L,u)$ along $f'':(M,v)\to(L,u)$. We remark that in the commutative diagram

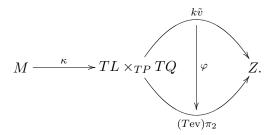every vertical face of the cube is a pullback in **Set**.

Now, for each $l\in L$ and $\mathfrak{l}\in L$ we define $t_{\mathfrak{l},l}:u(\mathfrak{l},l)\to d(Tt(\mathfrak{l}),t(l))$ componentwise. Since $\mathrm{ev}t'=k$ we observe that Tk factors through the comparison map $\kappa:TM\to TL\times_{TP}TQ$, defined by the diagram

that is $Tk=(T\mathrm{ev})(Tt')=(T\mathrm{ev})\pi_2\kappa$. Since also kv factors through κ, i.e., $kv=k\tilde{v}\kappa$, with (BC) we conclude that the 2-cell $kv\to c(Tk)$ is of the form

For each $x\in X$ and $\mathfrak{q}\in TQ$ such that $f(x)=h(l)$ and $Tf'(\mathfrak{q})=Tt(\mathfrak{l})$, let $\mathfrak{m}\in TM$ be such that $(Tf'')(\mathfrak{m})=\mathfrak{l}$ and $(Tt')(\mathfrak{m})=\mathfrak{q}$. In the diagram

$$\begin{array}{ccc}
c(\mathfrak{z},s_l(x)) \xleftarrow{k_{\mathfrak{m},(l,x)}} v(\mathfrak{m},(l,x)) & \longrightarrow & a(\mathfrak{x},x)\\
\downarrow & & \downarrow{\scriptstyle f_{\mathfrak{x},x}}\\
u(\mathfrak{l},l) \xrightarrow{\quad h_{\mathfrak{l},l}\quad} & & b(\mathfrak{y},y)
\end{array}$$

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in $\mathbf{V}$ one has $\mathfrak{z} = (T\mathrm{ev})(\mathfrak{q})$ and the morphism $k_{\mathfrak{m},(l,x)}$ depends only on $\mathfrak{q}$ and $\mathfrak{l}$. Moreover, the square is a pullback, hence there is a $\mathbf{V}$-morphism $\tilde{t}_{\mathfrak{l},l} : u(\mathfrak{l}, l) \to c(\mathfrak{z}, s_l(x))^{f_{\mathfrak{x},x}}$ such that $\tilde{p}_{\mathfrak{q},x}\tilde{t}_{\mathfrak{l},l} = h_{\mathfrak{l},l}$ and $k_{\mathfrak{m},(l,x)}(\tilde{t}_{\mathfrak{l},l} \times_b 1) = \tilde{\mathrm{ev}}_{\mathfrak{q},x}$. With the multiple pullback property, the morphisms $\tilde{t}_{\mathfrak{l},l}$ define the unique 2-cell that makes $t : (L, u) \to (P, d)$ a homomorphism. $\qquad\square$

If in the proof we take for (Y, b) the terminal object of $\mathrm{Alg}(T, \mathbf{V})$, that is, the pair $(1, \top)$ where the lax structure $\top$ is constantly equal to the terminal object of $\mathbf{V}$, we conclude:

2.8 Corollary. *If the pointed $\mathbf{Set}$-functor T satisfies (BC) and $\mathbf{V}$ is complete and cartesian closed, then also $\mathrm{Alg}(T, \mathbf{V})$ is cartesian closed.*

We explain now the strength of our Beck-Chevalley condition.

2.9 Proposition. *For T and $\mathbf{V}$ as in 1.5, let $\mathbf{V}(I, 0) = \emptyset$. Then:*

(a) *If T satisfies (BC), then T transforms pullbacks into weak pullbacks. The two conditions are actually equivalent when $\mathbf{V}$ is thin (i.e. a preordered class).*

(b) *If $\mathbf{V}$ is not thin, satisfaction of (BC) by T is equivalent to preservation of pullbacks by T.*

(c) *If $\mathbf{V}$ is cartesian closed, with $I = 1$ the terminal object, then T satisfies (BC) if and only if $(Tf)^\circ Tg = Tk(Th)^\circ$, for every pullback diagram*

$$
\begin{array}{ccc}
W & \xrightarrow{\ k\ } & X \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{\ g\ } & Y
\end{array}
\tag{4}
$$

in $\mathbf{Set}$.

Proof. (a) Let $\kappa : TW \to U$ be the comparison map of diagram (1). By (BC) the 2-cell $\kappa\eta : \kappa \to \kappa\kappa^\circ\kappa$ is the image by $-\kappa$ of a 2-cell $\sigma : 1_U \to \kappa\kappa^\circ$. Hence, for each $u \in U$ there is a $\mathbf{V}$-morphism $I \to \kappa\kappa^\circ(u, u) = \displaystyle\sum_{\mathfrak{w} \in TW \,:\, \kappa(\mathfrak{w})=u} \kappa(\mathfrak{w}, u)$. Therefore the set $\{\mathfrak{w} \in TW \,|\, \kappa(\mathfrak{w}) = u\}$ cannot be empty, that is, κ is surjective.

If $\mathbf{V}$ is thin and κ is surjective, there is a (necessarily unique) 2-cell $1_U \to \kappa\kappa^\circ$. Then each 2-cell $\psi : \kappa r \to \kappa s$ induces a 2-cell $\varphi : r \to s$ defined by

$$
r \xrightarrow{\ r\sigma\ } r\kappa\kappa^\circ \xrightarrow{\ \psi\kappa^\circ\ } s\kappa\kappa^\circ \xrightarrow{\ s\varepsilon\ } s
$$

whose image under $-\kappa$ is necessarily ψ.

(b) If T preserves pullbacks, then κ is an isomorphism and (BC) holds.

Conversely, let T satisfy (BC) and let $\kappa : TW \to U$ be a comparison map as in (1). We consider $\mathfrak{w}_0, \mathfrak{w}_1 \in TW$ with $\kappa(\mathfrak{w}_0) = \kappa(\mathfrak{w}_1)$ and $\mathbf{V}$-morphisms $\alpha, \beta : v \to v'$ with $\alpha \neq \beta$, and define $r : U \times U \to \mathbf{V}$ by $r(u, u') = v$ and $s : U \times U \to \mathbf{V}$ by $s(u, u') = v'$. The 2-cell $\psi : r\kappa \to s\kappa$, with $\psi_{\mathfrak{w},u} = \alpha$ if $\mathfrak{w} = \mathfrak{w}_0$ and $\psi_{\mathfrak{w},u} = \beta$ elsewhere, factors through κ only if $\mathfrak{w}_0 = \mathfrak{w}_1$.

(c) For any commutative diagram (4) there is a 2-cell $kh^\circ \to f^\circ g$, defined by

$$
kh^\circ \xrightarrow{\ \eta kh^\circ\ } f^\circ fkh^\circ = f^\circ ghh^\circ \xrightarrow{\ f^\circ g\varepsilon\ } f^\circ g,
$$

which is an identity morphism in case the diagram is a pullback.

If T satisfies (BC) and $\mathbf{V}$ is not thin, the equality $Tk(Th)^\circ = (Tf)^\circ Tg$ follows from (b). If $\mathbf{V}$ is thin, then in the diagram (1) the 2-cell $\sigma : 1 \to \kappa\kappa^\circ$ considered in (a) gives rise to a 2-cell

$$(Tf)^\circ Tg = \pi_2\pi_1^\circ \xrightarrow{\pi_2\sigma\pi_1^\circ} \pi_2\kappa\kappa^\circ\pi_1^\circ = Tk(Th)^\circ,$$

and the equality follows.

Conversely, the equality $(Tf)^\circ Tg = Tk(Th)^\circ$ guarantees the surjectivity of κ, hence (BC) follows in case $\mathbf{V}$ is thin, by (a). If $\mathbf{V}$ is not thin, we first observe that a coproduct $\sum_X I$ is isomorphic to I only if X is a singleton, due to the cartesian closedness of $\mathbf{V}$. Now, $(Tf)^\circ Tg = Tk(Th)^\circ$ means that, for every $\mathfrak{z} \in TZ$ and $\mathfrak{x} \in TX$ with $Tg(\mathfrak{z}) = Tf(\mathfrak{x})$,

$$I = Tf(\mathfrak{x}, Tg(\mathfrak{z})) = Tf^\circ Tg(\mathfrak{z}, \mathfrak{x}) = TkTh^\circ(\mathfrak{z}, \mathfrak{x}) = \sum\{I \mid \mathfrak{w} \in TW \,:\, Tk(\mathfrak{w}) = \mathfrak{x} \ \& \ Th(\mathfrak{w}) = \mathfrak{z}\}.$$

From this equality we conclude that there exists exactly one such $\mathfrak{w}$, i.e. $TW = TZ \times_{TY} TX$. $\square$

2.10 Finally we remark that, in some circumstances, the 2-categorical part of (BC) is essential for local cartesian-closedness of $\mathrm{Alg}(T, \mathbf{V})$. Indeed, if $\mathbf{V}$ is extensive [4], T transforms pullback diagrams into weak pullback diagrams and $\mathrm{Alg}(T, \mathbf{V})$ is locally cartesian closed, then T satisfies (BC), as we show next. To check (BC) we consider a 2-cell $\psi : r\kappa \to s\kappa$, with $\kappa : TW \to U$ the comparison map of diagram (1) and $r, s : U \to S$. We need to check that $\psi = \varphi\kappa$ for a unique 2-cell $\varphi : r \to s$. This 2-cell exists, and it is unique if and only if

$$\forall \mathfrak{w}_0, \mathfrak{w}_1 \in TW \ \ \forall s \in S \ \ \kappa(\mathfrak{w}_0) = \kappa(\mathfrak{w}_1) \ \Rightarrow \ \psi_{\mathfrak{w}_0,s} = \psi_{\mathfrak{w}_1,s}.$$

For $v := r(\kappa(\mathfrak{w}_0), s)$ and $v' := s(\kappa(\mathfrak{w}_0), s)$, and $\alpha := \psi_{\mathfrak{w}_0,s}$ and $\beta = \psi_{\mathfrak{w}_1,s}$, we want to show that $\alpha = \beta$.

For that, in the pullback diagram (4) we consider structures a, b, c, d, on X, Y, Z and W respectively, constantly equal to $I + v$, with $\eta : I \to I + v$ the coproduct injection. For d' constantly equal to $I + v'$, in the diagram

$$
\begin{array}{ccc}
(W, d') & \xleftarrow{\ (\mathrm{id},\varepsilon)\ } (W, d) & \xrightarrow{\ (k,1)\ } (X, a) \\
& {\scriptstyle (h,1)}\Big\downarrow & \Big\downarrow{\scriptstyle (f,1)} \\
& (Z, c) \xrightarrow{\ (g,1)\ } & (Y, b)
\end{array}
$$

we define ε by:

$$
\varepsilon_{\mathfrak{w},w} = \begin{cases} 1 + \alpha & \text{if } \mathfrak{w} = \mathfrak{w}_0, \\ 1 + \beta & \text{elsewhere.} \end{cases}
$$

The square is a pullback. Hence the morphism $(\mathrm{id}, \varepsilon)$ factors through the partial product via $t \times_Y \mathrm{id}$, with $t : Z \to P$. Since the 2-cell of $t \times_Y \mathrm{id}$ is obtained by a pullback construction and $\kappa(\mathfrak{w}_0) = \kappa(\mathfrak{w}_1)$, its 2-cell "identifies" $\mathfrak{w}_0$ and $\mathfrak{w}_1$, hence $\varepsilon_{\mathfrak{w}_0,w} = \varepsilon_{\mathfrak{w}_1,w}$, that is, $1 + \alpha = 1 + \beta$. Therefore $\alpha = \beta$, by extensitivity of $\mathbf{V}$.

3 (Co)completeness of the category $\mathrm{Alg}(T, \mathbf{V})$

3.1 We assume $\mathbf{V}$ to be complete and cocomplete. The construction of limits in $\mathrm{Alg}(T, \mathbf{V})$ reduces to a combined construction of limits in $\mathbf{Set}$ and $\mathbf{V}$, as we show next.

The limit of a functor

$$
\begin{aligned}
F : \mathbf{D} &\to \mathrm{Alg}(T, \mathbf{V}) \\
D &\mapsto (FD, a_D) \\
D \xrightarrow{f} E &\mapsto (FD, a_D) \xrightarrow{Ff} (FE, a_E)
\end{aligned}
$$

is constructed in two steps.

First we consider the composition of F with the forgetful functor into $\mathbf{Set}$

$$
\mathbf{D} \xrightarrow{\quad F \quad} \mathrm{Alg}(T, \mathbf{V}) \xrightarrow{\qquad} \mathbf{Set}, \tag{5}
$$

and construct its limit in $\mathbf{Set}$

$$
(L \xrightarrow{\quad p^D \quad} FD)_{D \in \mathbf{D}}.
$$

Then, we define the $(T, \mathbf{V})$-algebra structure $a : TL \twoheadrightarrow L$, that is the map $a : TX \times X \to \mathbf{V}$, pointwise. For every $\mathfrak{l} \in TL$ and $l \in L$, we consider now the functor

$$
\begin{aligned}
F_{\mathfrak{l}, l} : \mathbf{D} &\to \mathbf{V} \\
D &\mapsto a_D(Tp^D(\mathfrak{l}), p^D(l)) \\
D \xrightarrow{f} E &\mapsto a_D(Tp^D(\mathfrak{l}), p^D(l)) \xrightarrow{Ff_{Tp^D(\mathfrak{l}), p^D(l)}} a_E(Tp^E(\mathfrak{l}), p^E(l))
\end{aligned}
$$

and its limit in $\mathbf{V}$

$$
(a(\mathfrak{l}, l) \xrightarrow{\quad p^D_{\mathfrak{l}, l} \quad} a_D(Tp^D(\mathfrak{l}), p^D(l)))_{D \in \mathbf{D}}.
$$

This equips $p^D : (L, a) \to (FD, a_D)$ with a 2-cell $p^D a \to a_D Tp^D$.

By construction

$$
(L, a) \xrightarrow{\quad p^D \quad} (FD, a_D) \tag{6}
$$

is a cone for F. To check that it is a limit, let

$$
(Y, b) \xrightarrow{\quad g^D \quad} (FD, a_D)
$$

be a cone for F. By construction of (L, p^D), there exists a map $t : Y \to L$ such that $p^D t = g^D$ for each $D \in \mathbf{D}$. For each $\mathfrak{y} \in TY$ and $y \in Y$,

$$
b(\mathfrak{y}, y) \xrightarrow{\quad g^D_{\mathfrak{y}, y} \quad} a_D(Tp^D(Tt(\mathfrak{y})), p^D(t(y)))
$$

is a cone for the functor $F_{Tt(\mathfrak{y}), t(y)}$. Hence, by construction of $a(Tt(\mathfrak{y}), t(y))$, there exists a unique $\mathbf{V}$-morphism $t_{\mathfrak{y}, y}$ making the diagram

$$
\begin{array}{ccc}
a(Tt(\mathfrak{y}), t(y)) & \xrightarrow{\quad p^D_{\mathfrak{y}, y} \quad} & a_D(Tp^D(Tt(\mathfrak{y})), p^D(t(y))) \\
\Big\uparrow{\scriptstyle t_{\mathfrak{y}, y}} & \nearrow{\scriptstyle g^D_{\mathfrak{y}, y}} & \\
b(\mathfrak{y}, y) & &
\end{array}
$$

commutative. These **V**-morphisms define pointwise the unique 2-cell $gb \to p^D a$.

For each $l \in L$, $\eta_l : I \to a(e_L(l), l)$ is the morphism induced by the cone

$$(\eta^D_{p^D(l), p^D(l)} : I \to a_D(e_{FD}(p^D(l)), p^D(l)))_{D \in \mathbf{D}}.$$

3.2 Cocompleteness. To construct the colimit of a functor $F : \mathbf{D} \to \mathrm{Alg}(T, \mathbf{V})$ we first proceed analogously to the limit construction. That is, we form the colimit in **Set**

$$(FD \xrightarrow{\;\;i^D\;\;} Q)_{D \in \mathbf{D}}$$
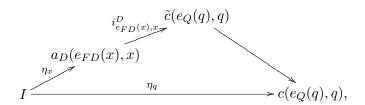
of the functor (5).

To construct the structure $c : TQ \twoheadrightarrow Q$, for each $\mathfrak{q} \in TQ$ and $q \in Q$, we consider the functor $F^{\mathfrak{q}, q} : \mathbf{D} \to \mathbf{V}$, with

$$F^{\mathfrak{q}, q}(D) = \sum \{a_D(\mathfrak{x}, x) \,|\, Ti^D(\mathfrak{x}) = \mathfrak{q}, \, i^D(x) = q\},$$

and, for $f : D \to E$, the morphism $F^{\mathfrak{q}, q}(f) : F^{\mathfrak{q}, q}(D) \to F^{\mathfrak{q}, q}(E)$ is induced by

$$a_D(\mathfrak{x}, x) \xrightarrow{Ff_{\mathfrak{x}, x}} a_E(Tf(\mathfrak{x}), f(x)) \longrightarrow \sum \{a_E(\mathfrak{y}, y) \,|\, Ti^E(\mathfrak{y}) = \mathfrak{q}, \, i^E(y) = q\} = F^{\mathfrak{q}, q}(E).$$

and denote by $\tilde{c}(\mathfrak{q}, q)$ the colimit of $F^{\mathfrak{q}, q}$. If $\mathfrak{q} \neq e_Q(q)$ for $q \in Q$, then $\tilde{c}(\mathfrak{q}, q)$ is in fact the structure $c(\mathfrak{q}, q)$ on the colimit. For $\mathfrak{q} = e_Q(q)$, the multiple pushout

defines $c(e_Q(q), q)$, with $D \in \mathbf{D}$ and $x \in FD$ such that $i^D(x) = q$.

4 Representability of partial morphisms

4.1 Let $\mathcal{S}$ be a pullback-stable class of morphisms of a category $\mathbf{C}$. An $\mathcal{S}$-*partial map from* X *to* Y is a pair ($X \xleftarrow{\;s\;} U \longrightarrow Y$) where $s \in \mathcal{S}$. We say that $\mathcal{S}$ *has a classifier* if there is a morphism $\mathrm{true} : 1 \to \tilde{1}$ in $\mathcal{S}$ such that every morphism in $\mathcal{S}$ is, in a unique way, a pullback of true; $\mathbf{C}$ *has* $\mathcal{S}$-*partial map classifiers* if, for every $Y \in \mathbf{C}$, there is a morphism $\mathrm{true}_Y : Y \to \tilde{Y}$ in $\mathcal{S}$ such that every $\mathcal{S}$-partial map ($X \xleftarrow{\;s\;} U \longrightarrow Y$) from X to Y can be uniquely completed so that the diagram

is a pullback.

From Corollary 4.6 of [10] it follows that:

4.2 Proposition. *If $\mathbb{S}$ is a pullback-stable class of morphisms in a finitely complete locally cartesian-closed category $\mathbf{C}$, then the following assertions are equivalent:*

  (i) $\mathbb{S}$ *has a classifier;*

  (ii) $\mathbf{C}$ *has $\mathbb{S}$-partial map classifiers.*

4.3 Our goal is to investigate whether the category $\mathrm{Alg}(T, \mathbf{V})$ has $\mathbb{S}$-partial map classifiers, for the class $\mathbb{S}$ of extremal monomorphisms. For that we first observe:

4.4 Lemma. *An $\mathrm{Alg}(T, \mathbf{V})$-morphism $s : (U, c) \to (X, a)$ is an extremal monomorphism if and only if the map $s : U \to X$ is injective and, for each $\mathfrak{u} \in TU$ and $u \in U$, $s_{\mathfrak{u}, u} : c(\mathfrak{u}, u) \to a(\mathfrak{x}, x)$ is an isomorphism in $\mathbf{V}$.*

4.5 Proposition. *In $\mathrm{Alg}(T, \mathbf{V})$ the class of extremal monomorphisms has a classifier.*

Proof. For $\tilde{1} = (1 + 1, \tilde{\top})$, where $\tilde{\top}$ is pointwise terminal, we consider the inclusion $\mathrm{true} : 1 \to \tilde{1}$ onto the first summand. For every extremal monomorphism $s : (U, c) \to (X, a)$, we define $\chi_U : (X, a) \to \tilde{1}$ with $\chi_U : X \to 1 + 1$ the characteristic map of $s(U)$, and the 2-cell constantly $! : a(\mathfrak{x}, x) \to 1$. Then the diagram below

$$
\begin{array}{ccc}
(U, s) & \xrightarrow{\ !\ } & 1 \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mathrm{true}} \\
(X, a) & \xrightarrow{\ \chi_U\ } & \tilde{1}.
\end{array}
$$

is a pullback diagram; it is in fact the unique possible diagram that presents s as a pullback of true. $\qquad\square$

Using Theorem 2.7 and Proposition 4.5, we conclude that:

4.6 Theorem. *If the pointed $\mathbf{Set}$-functor T satisfies (BC) and $\mathbf{V}$ is a complete and cocomplete locally cartesian closed category, then $\mathrm{Alg}(T, \mathbf{V})$ is a quasitopos.*

4.7 Remark. Representability of (extremal mono)-partial maps can also be proved directly, and in this way one obtains a slight improvement of Theorem 4.6: $\mathrm{Alg}(T, \mathbf{V})$ is a quasi-topos whenever T satisfies (BC) and $\mathbf{V}$ is a complete and cocomplete cartesian closed category, not necessarily locally so.

5 Examples.

5.1 We start off with the trivial functor T which maps every set to a terminal object 1 of $\mathbf{Set}$. T preserves pullbacks. Choosing for I the top element of any (complete) lattice $\mathbf{V}$ we obtain with $\mathrm{Alg}(T, \mathbf{V})$ nothing but the topos $\mathbf{Set}$. This shows that local cartesian closedness of $\mathbf{V}$ is

not a necessary condition for local cartesian closedness of $\mathrm{Alg}(T, \mathbf{V})$. We also note that T does not carry the structure of a monad.

If, for the same T, we choose $\mathbf{V} = \mathbf{Set}$, then $\mathrm{Alg}(T, \mathbf{Set})$ is the formal coproduct completion of the category $\mathbf{Set}_*$ of pointed sets, i.e. $\mathrm{Alg}(T, \mathbf{Set}) \cong \mathrm{Fam}(\mathbf{Set}_*)$.

5.2 Let $T = \mathrm{Id}$, $e = \mathrm{id}$. Considering for $\mathbf{V}$ as in [9] the two-element chain $\mathbf{2}$, the extended half-line $\overline{\mathbb{R}}_+ = [0, \infty]$ (with the natural order reversed), and the category $\mathbf{Set}$, one obtains with $\mathrm{Alg}(T, \mathbf{V})$ the category of

- sets with a reflexive relation

- sets with a fuzzy reflexive relation

- reflexive directed graphs,

respectively.

More generally, if we let $TX = X^n$ for a non-negative integer n, with the same choices for $\mathbf{V}$ one obtains

- sets with a reflexive $(n + 1)$-ary relation

- sets with a fuzzy reflexive $(n + 1)$-ary relation

- reflexive directed "multigraphs" given by sets of vertices and of edges, with an edge having an ordered n-tuple of vertices as its source and a single edge as its target; reflexivity means that there is a distinguished edge $(x, \cdots, x) \to x$ for each vertex x.

Note that the case $n = 0$ encompasses Example 5.1.

5.3 For a fixed monoid M, let T belong to the monad T arising from the adjunction

$$\mathbf{Set}^M \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Set},$$

i.e. $TX = M \times X$ with $e_X(x) = (0, x)$, with 0 neutral in M (writing the composition in M additively). T preserves pullbacks. The quasitopos $\mathrm{Alg}(T, \mathbf{Set})$ may be described as follows. Its objects are "M-normed reflexive graphs", given by a set X of vertices and sets $a(x, y)$ of edges from x to y which come with a "norm" $v_{x,y} : a(x, y) \to M$ for all $x, y \in X$; there is a distinguished edge $1_x : x \to x$ with $v_{x,x}(1_x) = 0$. Morphisms must preserve the norm. Of course, for trivial M we are back to directed graphs as in 5.2.

It is interesting to note that if one forms $\mathrm{Alg}(\mathsf{T}, \mathbf{Set})$ for the (untruncted) monad T (see [9]), then $\mathrm{Alg}(\mathsf{T}, \mathbf{Set})$ is precisely the comma category $\mathbf{Cat}/M$, where M is considered a one-object category; its objects are categories which come with a norm function v for morphisms satisfying $v(gf) = v(g) + v(f)$ for composable morphisms f, g.

5.4 Let $T = U$ be the ultrafilter functor, as mentioned in the Introduction. U transforms pullbacks into weak pullback diagrams. Hence, for $\mathbf{V} = \mathbf{2}$ we obtain with $\mathrm{Alg}(T, \mathbf{2})$ the quasitopos of pseudotopological spaces, and for $\mathbf{V} = \overline{\mathbb{R}}_+$ the quasitopos of (what should be called) quasi-approach spaces (see [9, 8]). If we choose for $\mathbf{V}$ the extensive category $\mathbf{Set}$, then the resulting

category $\mathrm{Alg}(U, \mathbf{Set})$ is a rather naturally defined supercategory of the category of ultracategories (as defined in [9]) but fails to be locally cartesian closed, according to 2.9(b) and 2.10.

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