FAMILY NAME: FIRST NAME:

STUDENT NO: SIGNATURE:

This test has (11) problems on (5) pages. Make sure you have them all!

!! NO NOTES, BOOKS, PHONES, CALCULATORS OR ANY ELECTRONICS ALLOWED !!

PROBLEM	1	2	3	4	5	6	7	8	9	10	11	TOTAL
MAX # PTS	2	6	2	5	5	2	4	6	6	6	6	50
PTS OBTAINED												

PROBLEM 1 List all elements of $\mathcal{P}(\{1,2,3\})$ (the power set of $\{1,2,3\}$):

$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$

PROBLEM 2 Let $A =_{def} \{n \in \mathbb{Z} \mid n^2 = 2n\}$, $B =_{def} \{n \in \mathbb{Z} \mid (n \neq 0 \rightarrow n = 2)\}$, and $C =_{def} \mathbb{Z} - \{n \in \mathbb{Z} \mid n \neq 0 \text{ and } n \neq 2\}$.

(a) Is $A = \{0,2\}$? ☑ YES ☐ NO (Check)

Prove your claim:

"$\subseteq$": $n \in A \rightarrow n^2 = 2n$. Case 1: $n = 0 \rightarrow n \in \{0,2\}$
Case 2: $n \neq 0 \rightarrow n = 2$ (from $n^2 = 2n$) $\rightarrow n \in \{0,2\}$.

"$\supseteq$": $n \in \{0,2\} \rightarrow n = 0$ or $n = 2$. Since $0^2 = 2 \cdot 0$ and $2^2 = 2 \cdot 2$: $n \in A$ in both cases.

(b) Is $B = \{0,2\}$? ☑ YES ☐ NO (Check)

Prove your claim:

$n \in B \leftrightarrow (n \neq 0 \rightarrow n = 2) \xleftrightarrow{\text{(NONAME RULE)}} n = 0 \text{ or } n = 2 \leftrightarrow n \in \{0,2\}$

(c) Is $C = \{0,2\}$? ☑ YES ☐ NO (Check)

Prove your claim:

$n \in C \leftrightarrow n \notin \{n \in \mathbb{Z} \mid n \neq 0 \text{ and } n \neq 2\} \leftrightarrow \text{not}\,(n \neq 0 \text{ and } n \neq 2)$

$\leftrightarrow n = 0 \text{ or } n = 2 \leftrightarrow n \in \{0,2\}$

PROBLEM 3 List all elements of $\{1,2,3\} \times \{1,2,3\}$

$(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$

PROBLEM 4 Let A, B, C be arbitrary sets.

(a) Is $A \times (B \cap C) = (A \times B) \cap (A \times C)$? ☑ YES ☐ No (check)

(b) Is $A \times (B \cup C) = (A \times B) \cup (A \times C)$? ☑ YES ☐ No (check)

(c) Prove one of the assertions you made in (a) and (b). Check which one you are proving (a) ☐ or (b) ☐ :

(a) $(x,y) \in A \times (B \cap C) \iff x \in A \wedge y \in B \cap C$

$$\iff x \in A \wedge (y \in B \wedge y \in C)$$
$$\iff (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C)$$
$$\iff (x,y) \in A \times B \wedge (x,y) \in A \times C$$
$$\iff (x,y) \in (A \times B) \cap (A \times C)$$

(b)

PROBLEM 5 Let A, B be any sets. Prove:

$$(A-B) \cup (B-A) = (A \cup B) - (A \cap B).$$

$x \in (A-B) \cup (B-A) \iff x \in A-B \vee x \in B-A$

$$\iff (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$$
$$\iff ((x \in A \wedge x \notin B) \vee x \in B) \wedge ((x \in A \wedge x \notin B) \vee x \notin A)$$
$$\iff (x \in A \vee x \in B) \wedge (x \notin B \vee x \in B) \wedge ((x \in A \vee x \notin A) \wedge (x \notin B \vee x \notin A)$$
$$\iff x \in A \cup B \wedge T \wedge T \wedge \text{not}(x \in B \wedge x \in A)$$
$$\iff x \in A \cup B \wedge x \notin A \cap B$$
$$\iff x \in (A \cup B) - (A \cap B)$$

$(x,y) \in A \times (B \cup C) \iff x \in A \wedge y \in B \cup C$

$$\iff x \in A \wedge (y \in B \vee y \in C)$$
$$\iff (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$$
$$\iff (x,y) \in A \times B \vee (x,y) \in A \times C$$
$$\iff (x,y) \in (A \times B) \cup (A \times C)$$

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PROBLEM 6 For every integer $n \in \mathbb{Z}$, let $A_n =_{def} \{n, 0, -n\}$.

(a) Is $\bigcup_{n \in \mathbb{Z}} A_n = \bigcup_{n \in \mathbb{N}} A_n$? ☑ YES ☐ NO (Check)

(b) Is $\bigcap_{n \in \mathbb{Z}} A_n = \bigcap_{n \in \mathbb{N}} A_n$? ☑ YES ☐ NO (Check)

(No proofs required. Here $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$.)

PROBLEM 7 Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$ for all $n \in \mathbb{N}$.

(a) List all elements of $f(\{0, 1, 2, \ldots, 9\})$:

$0, 1, 4, 9, 16, 25, 36, 49, 64, 81$

(b) List all elements of $f^{-1}(\{0, 1, 2, \ldots, 9\})$:

$0, 1, 2, 3$

PROBLEM 8 Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $f((n,m)) = n + m$ for all $(n, m) \in \mathbb{N} \times \mathbb{N}$.

(a) Is f injective? ☐ YES ☑ NO (Check)

Prove your claim:

$f((0,1)) = 1 = f((1,0))$, but $(0,1) \neq (1,0)$

(b) Is f surjective? ☑ YES ☐ NO (Check)

Prove your claim:

Given $k \in \mathbb{N}$, let $(n,m) =_{def} (k, 0)$. Then $f((n,m)) = k + 0 = k$.

PROBLEM 9 Let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ be the set of positive real numbers and define $f: \mathbb{R}_+ \to \mathbb{R}_+$ by $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}_+$. Prove that f is bijective and find a formula for $f^{-1}(y)$ for all $y \in \mathbb{R}_+$.

Since $f \circ f(x) = f(f(x)) = f(\frac{1}{x}) = \frac{1}{\frac{1}{x}} = x$ for all $x \in \mathbb{R}_+$, one has $f \circ f =$ Identity map of $\mathbb{R}_+$. Hence, f is injective and surjective (see Problem 11), and $f^{-1} = f$, that is: $f^{-1}(y) = \frac{1}{y}$ for all $y \in \mathbb{R}_+$.

PROBLEM 10 For $a, b \in \mathbb{R}$ with $a \leq b$, let $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.

(a) Find a bijective function $f: [0,100] \to [0,1]$. (Just give the definition of $f(x)$):

$$f(x) = \frac{x}{100} \quad (\text{for all } x \in [0,100])$$

(b) Are the following sets countable?

	YES	NO (Check)
– The set of all rational numbers:	☑	☐
– The set of all integers divisible by 100:	☑	☐
– The set $[0, \frac{1}{100}]$	☐	☑
– The set of molecules that make up your body.	☑	☐

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PROBLEM 11 For functions $f: A \to B$ and $g: B \to C$, prove:

(a) If $g \circ f$ is injective, f is also injective:

For $x_1, x_2 \in A$, let $f(x_1) = f(x_2)$

$$\to g(f(x_1)) = g(f(x_2))$$
$$\to g \circ f(x_1) = g \circ f(x_2)$$
$$\to x_1 = x_2. \quad \text{(since } g \circ f \text{ is injective)}$$

Hence, f is injective.

(b) If $g \circ f$ is surjective, g is also surjective:

Let $z \in C \to \exists x \in A$ with $g \circ f(x) = z$ (since $g \circ f$ is surjective)

$$\to g(f(x)) = z$$
$$\to \exists y \in B : g(y) = z \quad \text{(just take } y = f(x)\text{)}.$$

Hence, g is surjective.