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Teaching To See Like a Mathematician

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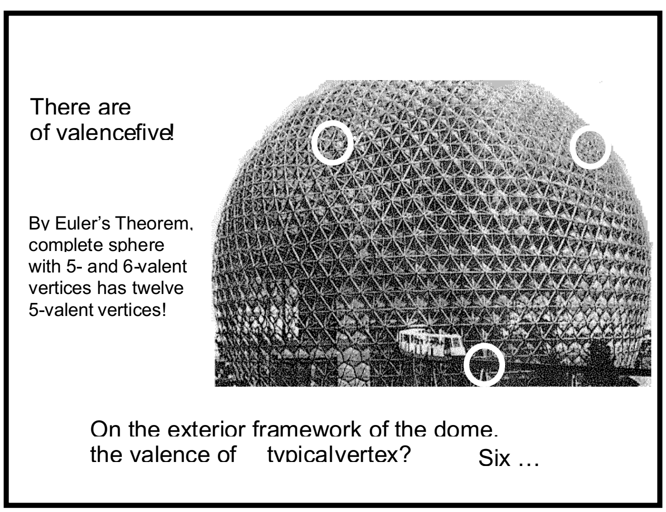
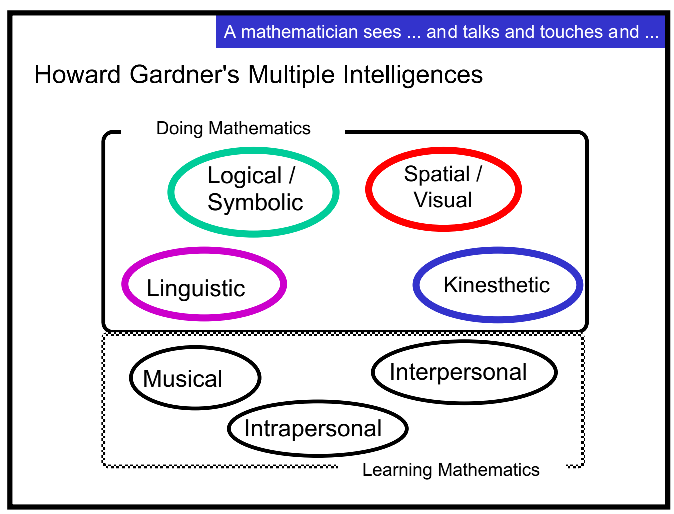
My goal with the use of visuals in mathematics

I am a research geometer, working on applied problems such as: What shapes of built frameworks will stand up or fall over? How can we program geometric constraints for CAD or CAGD? How can we reconstruct 3-D objects from a single picture or several partial views? What parts of proteins are flexible? and What patterns occur in computational origami. It will be no surprise that I make substantial use of static and dynamic diagrams as well as physical models in my research, and in my communication. However, over the last decade I have worked consistently to make the use of visuals more explicit, more precise, and more extensive in my mathematical practice. When I write, I often have a sequence of visuals (or mixed visuals and text, as in an augmented PowerPoint presentation) in mind around which the text is structured. (This is true of the current paper - see two samples below. The first is a visually presented mathematics problem.) This point form from that presentation summarizes where I begin these reflections:

There has been a difference between my public and private face in mathematics

- I do my mathematics visually, in private;
- I question, problem solve, analyze, explore, prove, answer, and communicate visually.
- Choosing when and how I use visuals has
  changed the questions I pose;
  changed the methods I use to solve them
  changed the answers I give
  changed my communication and my teaching
- I wish the same options for my students.





Visual forms in the practice of mathematics

Beyond my own reflections, there is a lot of anecdotal and observational evidence for the extensive role of visuals and diagrams in the practice of mathematics. The classical book of Hadamard records recollections of a number of well known mathematicians on how they made significant discoveries - and images are typically the form in which the solution first became conscious. The book of Brown surveys this over a longer period from a more philosophical point of view. Interviews of current mathematicians by the Mathematics Educator Leona Burton found a lot of conversation about 'intuition'. These are words which people use when there are no other words, for example when the process is visual but not yet sharp in form. The classical books of Tufte and Wainer on visual displays of information are, in large part, about the visual display of statistical data, part of the mathematical sciences in the broad sense.

Looking from the outside at young children, the cognitive scientist and educational psychologist Howard Gardner described multiple intelligences. Unfortunately, he misassociated 'mathematics' solely with the logical/symbolic intelligence. As my figure above expresses, experience and the current literature show that at least four of the intelligences play an important role in the practice of mathematics. In particular, some mathematicians and some students rely essentially of the spatial/visual and kinesthetic intelligences. In some research in intelligence testing, spatial temporal reasoning is also correlated with superior performance in mathematics.

Probing the next level inwards, the linguists and cognitive scientists Lakoff and Nunez remind us that whatever processes the brain uses to 'do mathematics', these are adapting processes evolved long before there was mathematics (or science). Their metaphors are typically visual and recent brain studies have confirmed that the visual system fires as many of those tasks are performed by adults.

Now consider even simple arithmetic skills. For example: 7x5=?. This appears to be done with language lookup in a multiplication table. However: Is 7x5>26? This appears to be done in the visual system on an analogue number line (Butterworth, Dehaeme). This logarithmic line appears to based on proportion. This appears to be the same part associated with eye hand coordination and with simple arithmetic with 1,2,3 done by three day old babies (pre language).

This visual connection is not a surprise, since external visuals can so clearly be used for these tasks. The work of people like Kosslyn confirm that the mind's eye creates internal images that resemble external diagrams. The whole image appears in the visual area V1, as well as multiple levels of decomposition, association etc. in other levels of the visual system. Moreover, something like a word in the brain can trigger the creation of such an image in V1. We really can and do work 'in the mind's eye' with the same processes we use with an external image. Although there are differences (less stable, more easily transformed) the basic modes of visual reasoning remain accessible for mental work in mathematics.

Increasingly, the opinion among people who study these issues is that mathematics is not a language. The reasoning, the thinking, appears much more centered on the visual world. How we teach, assess, and practice the visual skills has become a central issue for mathematics educators.

Do You See What I See?

The simple answer is *no*! We see with our minds, not just with our eyes (Hoffmann). What happens in our brains in response to our eyes, what we notice, what we think about, the associations we make are shaped in our brains. What we see is shaped by our experiences, the patterns we know, even differences in connections in our brains, both at birth and developed over a lifetime of learning. All mathematicians and scientists do not see the same thing even in a simple mathematical graph. Experts and apprentices do not see the same thing [Roth]. Teachers and students do not see the same thing in a diagram or an animated image on the computer or even in an algebra equation. Different students in a class do not see the same thing. This is a fundamental fact to be addressed for the use of visuals in communication, in teaching, and in learning.

Studies of change blindness confirm that many changes in images can be missed and some web sites offer telling illustrations of this (Rensik, Simons). What we say we see is stored only at a simplified level and only a few selected parts are 'seen' in detail. What people attend to makes a big difference to what they notice and what they think about. This too is part of the classroom.

Recognizing this is a critical first step for a teacher. We need to show students what we are attending to, and find ways to direct (and shift) their attention to track effective visual thinking around a diagram. The analogy would be that the student and the teacher are using different algebraic notation. We would stop to bring people together on a common set of conventions.

As an aside, the analogy from visuals to algebra is often true in reverse. Substantial portions of algebra are based on visual appearance. When students and teachers 'see' the equation differently, they behave very differently and serious confusion results. Teaching what we see is an important task in algebra, as in geometry. The alternatives leave students viewing weird 'magic'.

Difficulties with Visuals in Mathematics Teaching.

These differences and gaps between the visual experience of the teacher and the visual experience of the student are important in understanding why and how visuals in the classroom can fail. Many individuals and groups have recognized that visualization had the potential to improve student learning or student performance in mathematics education, and then been disappointed at the actual observations and difficulties in the classroom. Here are some key points extracted by the mathematics educator Tommy Dreyfus:

(a) The inability of students to 'see' a diagram in multiple ways;
(b) Difficulty recognizing transformations implied in diagrams;
(c) Incorrect or unconventional interpretation of graphs;
(d) Problems connecting visualization and analytic thought;
(e) Information is determined by rules and conventions which have not been learned;
(f) Effective use depends on intervening conceptual thought and teaching.

If we want to use diagrams and other visuals for communication, shared problem solving, etc. it becomes critical whether the two people are seeing the same thing. When the student does not 'see what the teacher is talking about' in an image, there are things we can do, interventions we can make. There are now tools and approaches that move us the next step on this path. The next sections will investigate these tools and some issues related to them. Even in mathematics, the general visual aphorism applies:

> What we see is both a window onto the world and a mirror of our experiences and learning.

We can change what we see.

Since we see with our minds and we can change our minds, we can change what we see. A classic book by Edwards on learning to draw says that if you change how you see, all the rest of drawing will be easy. The same may be true in mathematics. New computer tools, play with objects and images, and guided practice, can change how students 'see' both external visuals and images in their mind's eye. The shift from simple images to insight is not a matter of luck, it is a matter of learning. Changing what we see becomes an important educational task. How do we communicate what is worth attention? What do we offer, as cues and conventions in images, to shift attention or to provoke a transformation of the image? There are levels of visual performance, of seeing, thinking and communicating, just like there are levels of performance in algebra. These are skills to be understood, shared, practiced and valued in the mathematics classroom and in assessment.

Dynamic Geometry Sketches.

Dynamic Geometry programs, such as Cabri Geometrie, Geometers SketchPad and Cinderella are now used for the teaching of school geometry. Much experimentation and educational research is underway on their impact (Sinclair). In some school system these programs (and related data visualization tools such as Fathom) are universally licensed and mandated. This is true where I live in Ontario, Canada.

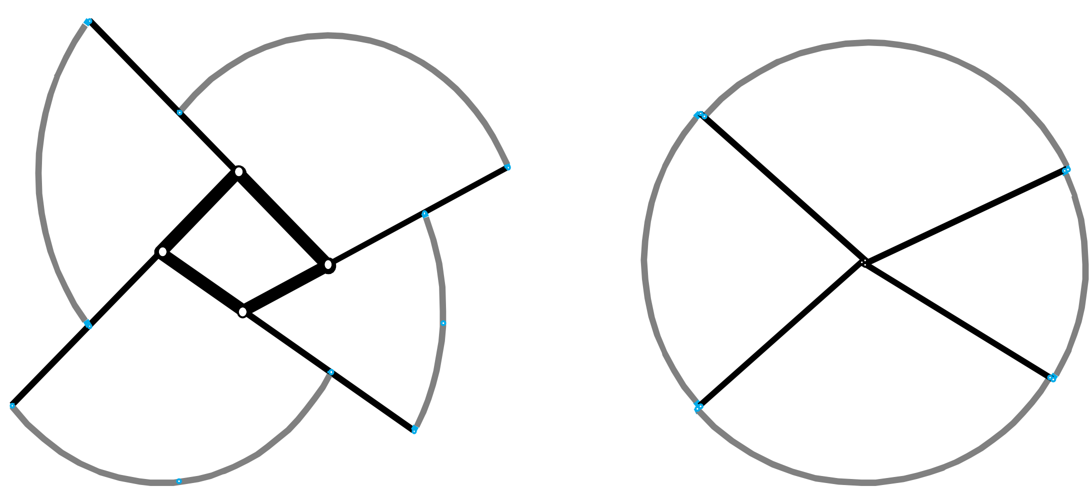
In a dynamic geometry program, a few initial choices (e.g. points) are made and further objects are constructed with geometric constraints: lines through two existing points, points where two existing lines meet, circles with given centers and fixed (or variable) radii, the line through a point perpendicular to another line, the translated copy of a prior piece to a new point, etc. The typical program includes both the electronic equivalent of the ruler and compass constructions, but the electronic version of basic transformations (translation, rotation, dilation, reflection).

What makes this dynamic is that one can then vary the sketch by moving the initial choices. This is either automatic (e.g. animation of a point along a predetermined line) or under the control of the user (dragging with the mouse). In these motions, and constraints are preserved, including transformational constraints. In particular, when an object is transformed, the linked before and after images become 'siblings' – changing one changes the other, with the original and the reflection on an equal footing. At their best, these tools present us with images and connections we had not anticipated or considered before. They change what we see.

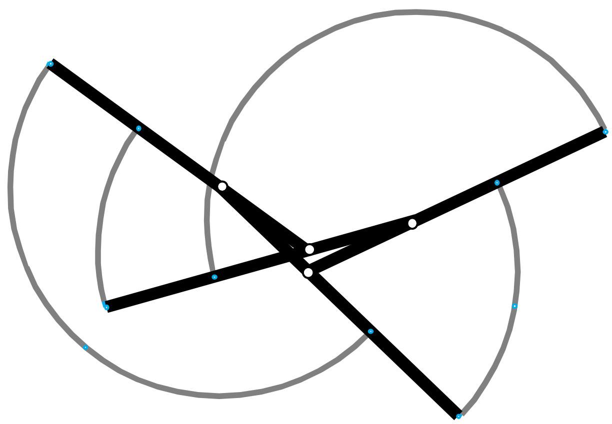
Before illustrating (as much as a few static images can) some reasoning embedded in these forms, it is useful to recall some of the history of these programs. Two of them (Cabri, GSP) were developed explicitly for teaching high school mathematics (primarily geometry) with new tools for visualization and exploration. Cinderella was developed to assist researchers with tools for exploring unsolved problems and conjectures. It was a tool for making mental images and possibilities into external, accurate, dynamic diagrams. These streams rapidly converged, with all programs being used for explorations and modeling in the full range of research and educational settings. The programs should not be viewed as 'crutches' to be used then abandoned by students, but as tools for explicit apprenticeship in modes of thinking inherent in our practice of mathematics.

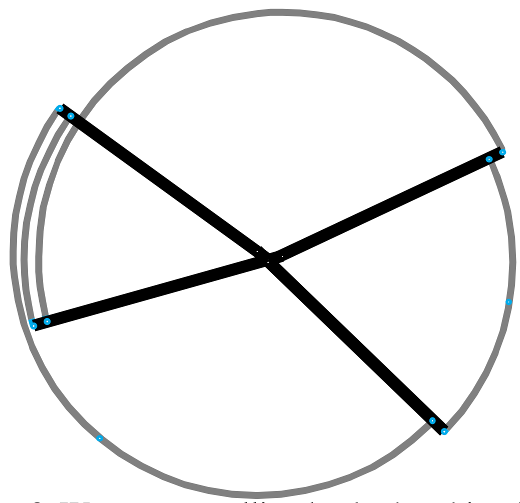
Examples of Dynamic Geometry Reasoning.

In this paper, I will only present a couple of examples, through additional examples are central to my presentation.



Example 1. Exterior angles of a plane polygon. Consider the four exterior angles of a quadrilateral, taken in a counterclockwise sweep around the polygon. If we zoom back, keeping the marked angles scaled so we can see them (alternatively, we dilate the quadrilateral in with a dilation tool of a dynamic geometry program) the image transforms into a mild perturbation of a circle with the quadrilateral shrunken to the center. Clearly in our vision, the sum of the exterior angles is a full circle (360 degrees). Moreover, with a bit of visual thought, one can 'see' that this would hold for any convex quadrilateral, just moving the four points on the circle around a bit. In fact, it would work for any convex polygon of any size, just by changing the number of spokes on the final wheel.



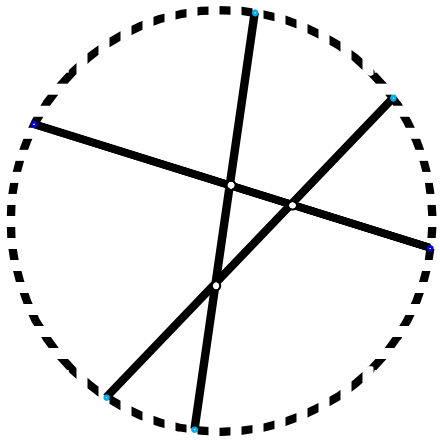


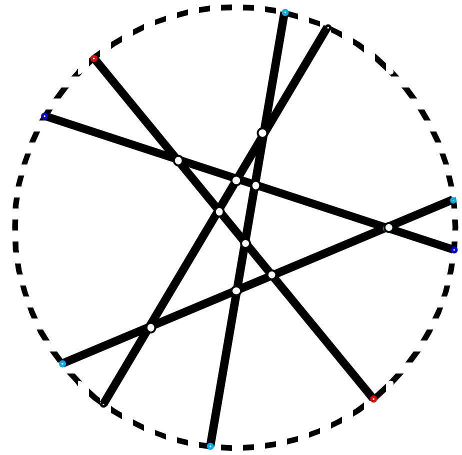
What if the quadrilateral becomes non-convex? We see, on pulling back, that this still involves a net full circle, but there is some 'back tracking' in which certain segments of the circle are covered multiple times. Provided that the angles are measured with signs (+ for counterclockwise, - for clockwise) then the results still holds - the sum of the exterior angles is 360 degrees. One can (and one does) go one to explore what happens if the polygon is self-intersecting, etc. Clearly, this dynamic image introduces a 'big picture' that shifts us from a detailed accounting with angles of triangles which decompose the polygon, into a simple visual reasoning which is both much simpler and much more powerful. It is possible to use such reasoning without the dynamic programs, but it is difficult to ensure that students will create the correct mental image, or make the correct modifications for non-convex, self-intersecting etc. situations. The dynamic diagram embodies the reasoning in a shared image we can gesture with, and speculate about as a mathematical community.

I note that recent research on visual reasoning shows the importance of ‘timing’ in working with animations. In many situations, people actually think better (and learn better) from a sequence of stills, at a pace they control, than from an animation at a pace they do not control.

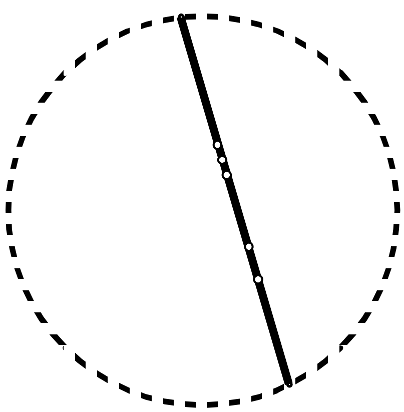
Example Polya's plane cutting problem.

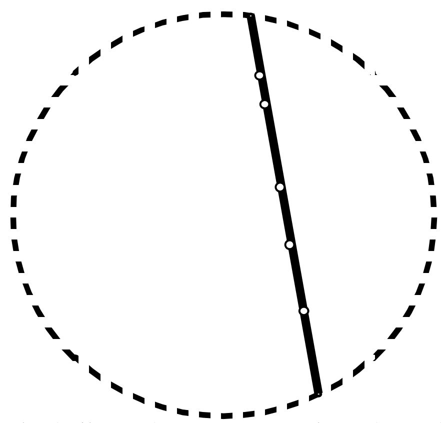
This is the plane version of a 3-D problem famously explored by Georg Polya in his video tape 'Let us now teach guessing'. What is the maximum number of pieces into which the plane can be decomposed by n lines? For the first few numbers, it is clear: n=0: 1 piece; n=1: 2 pieces; n=2: 4 pieces. The first mild surprise is n=3: 7 pieces. [For simplicity in these images, we are cutting up the interior of a circle - a Pizza as I tell my students. The reasoning is the same.] For





more lines, the pattern continues: n=4: 11 pieces; n=5: 16 pieces. The pattern of numbers may become obvious to people who play with number patterns: n lines: 1+ n(n+1)/2 pieces. Even with this formula, there is the key problem of reasoning: Why?





The images above, formed by adding the sixth line, then suppressing the original five lines (but not their intersections) contains the essential pattern. The last line added cuts each of the previous lines - indicated by the five points of intersection. Provided that these five points are distinct, they divide this line into six segments. Each of these six segments is a marker for the action that splits a previous piece into two pieces. Therefore, the added line has added six new pieces to our total (provided three lines do not meet at a point). When this images was shown in class, and the last line was wiggled (dynamically) there was absolute silence. The students could 'see' what I had been talking about, 'see' that most details did not matter. One can 'see why it works' by focusing on only the important features.

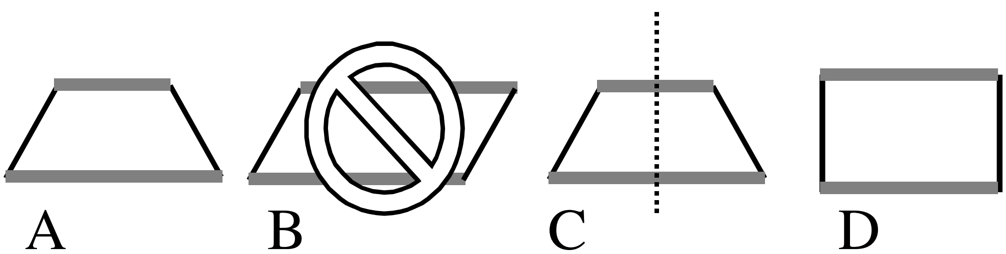
Example 3. Inclusive definitions. In elementary schools, texts, teachers, and students face a standard issue around inclusive vs. exclusive definitions. In part, the push towards exclusive definitions is a reversion to language based concepts rather than visual concepts. Let me illustrate with two typical examples:

(i) Is a square a rectangle? If a rectangle is defined as a quadrilateral with four right angles, then it is clear that a square is a rectangle. If a rectangle is defined as a parallelogram with opposite sides equal, but adjacent sides not equal, and one right angle, a square is not a rectangle. Why should one choose the inclusive definition? One reason, rooted in the practices of mathematics, shows up if we make a dynamic geometry sketch of any or these definitions. In the construction of four right angles, or even of 'opposite sides equal' we will create a sketch such that dragging the initial choices makes a square appear somewhere along the sweep of examples. Constraints such as 'adjacent sides not equal' are not constructible with these tools. Moreover, in any reasoned argument towards properties of a rectangle, the results will apply immediately to the square. It is a waste of effort to exclude, and then give a distinct proof when the reasoning applies to both classes.

(ii) Is a rectangle an isosceles trapezoid? [I use trapezoid in the North American sense – a quadrilateral with at least one pair of parallel sides, see below.] If we give a standard definition of isosceles trapezoid, by measured properties, a likely starting point might be:
(a) one pair of sides is parallel;
(b) the other pair of sides is of equal length.

These words are ambiguous, as the figure below (A,B) indicates. How do we exclude the typical parallelogram? The language based exclusive definition (very tempting) is to say only one pair of sides is parallel (B). This clearly excludes all the parallelograms. Unfortunately, it also excludes the rectangle (D)! This is visually wrong, and would prevent anyone from generating a dynamic geometry sketch which includes all isosceles trapezoids, since these will inevitably include the rectangle (and the square) as examples within their range of animation.

A visually appropriate (and mathematically superior) approach is to abandon the



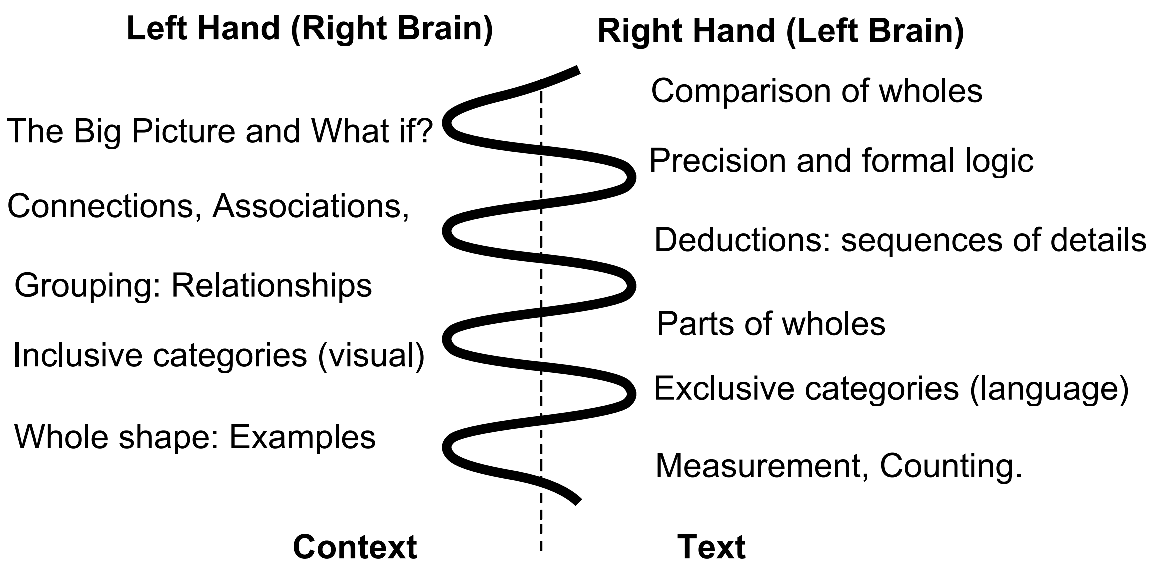
definition via measurements, and give a definition based on symmetry (a visual concept). Specifically, an isosceles trapezoid is a quadrilateral with a mirror of symmetry joining the mid points of a pair of opposite sides (C). Check it out. This works, this generates immediately all of the standard (and less standard) properties of the object, and it is inclusive of the rectangle and the square. Note that in the same spirit, a rectangle would be better defined as having two mirrors of symmetry, each joining the midpoints of a pair of opposite sides! In fact, all the quadrilaterals have superior, inclusive definitions in terms of mirrors and half turns. These inclusive definitions also work well on the sphere.

Explorations with dynamic geometry programs play several roles in visual thinking. These activities make visible various kinds and levels of thinking (see the diagram below). The explorations and constraints also give feedback and afford opportunities to move to alternative levels of visual thinking. They are a shared external medium that modifies our internal visual practices. They are one of the tools to change the way we see (and think) in mathematics.

An old saw says: "geometric thinking is the art of accurate reasoning from inaccurate diagrams". Students have typically been taught not to rely on the diagram, or the accuracy of the diagram. Something interesting happens to students with a dynamic geometry sketch. The sketch is accurate (to an unprecedented level). If the program measures an angle as 89.5 degrees, that is direct evidence it is not a right angle. Students need to relearn how to reason with accurate diagrams as 'close enough' no longer applies (Sinclair)! The use of such diagrams does change the search for counter-examples and the associated exploration of conjectures. It also, over time, changes the accuracy with which we make and reason with mental models that have evolved from the dynamic geometry play.

Visuals in the 'hierarchy of learning'.

The van Hiele model of learning geometry presents visuals as the basic, essential experience that all students must move through. Along with the kinesthetic, it is the layer which students must have access to in order not to become lost. The standard version is that students (and mathematicians) move beyond this to higher, non-visual levels of reasoning. What actually happens in my experience (see also de Villiers) is that the visual just ceases to be noticed since it is not recorded expressed in words. It is, in fact fully capable of sustaining the practice of mathematics at the highest levels. The visual is an important route for 'mathematical intuition' - pointing to something we currently do not teach.



This figure is my image of how one weaves between alternate modes of text and context, detail and bigger picture, visual and verbal, in the use of dynamic geometry programs. In this chart, 'Left Hand' and 'Right Hand' are understood, in part, as metaphors for packages of skills and locations (see Ornstein). One understands that processing is not that clearly localized, not uniformly localized across individuals. Moreover the movement of text and context, whole and part, applies to more general processing, including language processing. Nevertheless, I think the image is evocative of some important issues for our attention.

This image evolved from conversations with Michael de Villers on the van-Hiele model, and with Margaret Sinclair, as she explored student use of dynamic geometry sketches in high school classrooms. It is a specific presentation of the stages of dynamic geometry work of a pattern I have for more general mathematical work. This image tracks, in a rough way, the zig-zag path which students can use to pull together the standard (left brain, language appearance) classroom activities with the other less public complementary visual activities which mathematicians rely on to do mathematics. The only chance we have of making the effective use of visuals into something taught and learned is to develop this type of analysis into layers, to train ourselves to recognize what layer the student is working with, and to intervene to help them move on to higher layers. If these levels are invisible, and not attended to, many students will be lost in the gaps/ In a broad sense, this was the original motivation of the van Hieles when they broke down to levels of learning geometry proofs. It is, however, much more widely applicable.

Some comments on the hierarchy of dimensions.

Much of our teaching is based on the illusion that 2-D is easier than 3-D. In terms of the human visual system, and related kinesthetic and other experience, this is false. It might be easier to do analytic geometry, and axiomatic proofs, in the plane. However, children enter school living in a 3-D world. 2-D is artificial, and representation of 3-D in 2-D is a learned skill, with many conventions and pitfalls.

There are some studies which suggest students abilities with 3-D tasks (such as reflections and sectioning 3-D objects) is actually decreasing within the North American style education system from grades 3 – 11. There are also some studies which suggest that the real power of visual thinking in creative work in science is elevated by skills at moving among dimensions between 2 and 3, between 3 and 4, … .

Teaching to see in mathematics.

The individual steps in the previous diagram (and in any other decomposition of how experts work with diagrams in mathematics) are not present for students without guided experience (apprenticeship) and explicit teaching. Many failures in the use of such visuals are due in part to the gap between what the student focuses on and what the expert focuses on. The gap between what the student can imagine (image) as a next step and what the expert imagines can be huge. Far too often, we present an image and say 'behold'.

It is claimed that this was Euclid's expression for a classical, visual proof of the Pythagorean theorem. Having played with that proof, I can see many layers of decomposition and analysis, typically visual, which are need to go from the image to an understanding of why this result is true. As is typical of diagrams I have analyzed in mathematics, there are implicit steps to be tracked. These steps are probably best done with animations and sequences of diagrams, at least for a novice. Of course the expert has learned to do the animations and sequences in the mind's eye, with shifting attention, with mental movements and comparisons of pieces, with the shifts from parts to whole and back to parts illustrated in my earlier image. Too often, we do not teach the skills, or even explicitly model the skills in a way that the apprentices can observe and imitate.

However, we can teach these skills.

1. A first step is an evolving awareness of how visuals are or could be used, and an explicit encouragement of their uses.
2. A second step is paying attention to when students don't see what we see, seeking those occasions out and exploring them.
3. A third step is developing and sharing diverse examples, and diverse ways to see individual examples, along with tools which let students experience what we are seeing.

I have been engaged in this in my own teaching for a number of years. Recently, I have been offering a first year seminar on Information in Visual Form which explored some of these general themes with a cross-section of science students (see the web site below). I have been presenting examples and thematic reflections at meetings of mathematicians and mathematics educators for several years. I look forward to a wider community effort to develop and share these examples and the responses of students and other practitioners to these examples.

One such community response will shortly appear in the SIGGRAPH White Paper on Visual Learning in Science and Engineering, giving the collective reflections of an interdisciplinary group spanning cognitive science, visual and media artists, computer scientists, and science educations.

Dynamic images, which direct the eye with colour, with change and with the challenge of noticing what does not change are now available with tools such as PowerPoint or Geometer's Sketchpad. We should encourage students to use these resources and all forms of visuals and manipulatives not as a crutch, but as an apprenticeship into what mathematicians do. We should change what students see, and change their experience of mathematics with that. Mathematics is the search for patterns and the visual is an essential element of this search, as it is in much of science (see Roth and Zee).

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