SOLUTIONS for MATH1310Q EXAM 2B

1. (20 marks) Write Maple commands to carry out each of the following operations.
(a) Find the roots of $x^4 - x^2 + 5$ as decimals.
(b) Find $\lim_{x\to 1} \frac{\sqrt{x^2-1}}{2x-1}$.
(c) Find the fourth derivative of $f(x) = x(x^2+1)\ln x$.
(d) Find the graph of the function $y = x\cos\frac{1}{x}$ for $x \in [-\frac{1}{2\pi}, \frac{1}{2\pi}]$.
Solution:

(a) $evalf(solve(x\hat{\ }4 - x\hat{\ }2 + 5 = 0, x));$
(b) $limit(sqrt(x\hat{\ }2 - 1)/(2 * x - 1), x = 1);$
(c) $f := x- > x * (x\hat{\ }2 + 1) * \ln(x);$
$(D@@4)(f)(x);$
(d) $plot(x * \cos(1/x), x = -1/(2 * Pi)..1/(2 * Pi), discont = true);$

2. (15 marks) Use the method of integration by parts to find the integral:

$$\int x \sin\sqrt{x}\, dx$$

Solution:
We apply integration by parts three times:

$$\begin{aligned}
\int x\sin\sqrt{x}\,dx &= -2\int x\sqrt{x}\left(-\frac{1}{2\sqrt{x}}\sin\sqrt{x}\right)\,dx = -2\int x\sqrt{x}(\cos\sqrt{x})'\,dx \\
&= \text{(integration by parts)} = -2\left[x\sqrt{x} - \frac{3}{2}\int\sqrt{x}\cos\sqrt{x}\,dx\right] \\
&= -2x\sqrt{x} + 6\int\sqrt{x}\left(\frac{1}{2\sqrt{x}}\cos\sqrt{x}\right)\,dx \\
&= -2x\sqrt{x} + 6\int\sqrt{x}(\sin\sqrt{x})'\,dx = \text{(integration by parts)} \\
&= -2x\sqrt{x} + 6\left[x\sin\sqrt{x} - \int\sin\sqrt{x}\,dx\right] \\
&= -2x\sqrt{x} + 6x\sin\sqrt{x} - 6\int(-2\sqrt{x})\left(-\frac{1}{2\sqrt{x}}\sin\sqrt{x}\right)\,dx \\
&= \text{(integration by parts)} \\
&= -2x\sqrt{x} + 6x\sin\sqrt{x} + 12\sqrt{x}\cos\sqrt{x} - 12\int\frac{1}{2\sqrt{x}}\cos\sqrt{x}\,dx \\
&= -2x\sqrt{x} + 6x\sin\sqrt{x} + 12\sqrt{x}\cos\sqrt{x} - 12\sin\sqrt{x} + C.
\end{aligned}$$

Another method: We use the substitution $u = \sqrt{x}$ which transforms the integral in the following way:

$$\int x\sin\sqrt{x}\,dx = 2\int u^3\sin u\,du$$

The latter integral can be computed using integration by parts. A similar example was considered in the textbook, see page 424:

$$\int u^3 \sin u \, du = -u^3 \cos u + 3u^2 \sin u + 6u \cos u - 6 \sin u + C.$$

3. (10 marks) (a) Find the recursion formula for the integral:

$$\int (\ln x)^n \, dx \qquad \text{for } n \geq 1.$$

(b) Find the integral explicitly for $n = 3$.
Solution: (a) Let us apply integration by parts to the original integral:

$$\begin{aligned}\int (\ln x)^n \, dx &= \int (x)'(\ln x)^n \, dx = (\text{integration by parts}) \\ &= x(\ln x)^n - n \int x(\ln x)^{n-1}\frac{1}{x} \, dx \\ &= x(\ln x)^n - n \int (\ln x)^{n-1} \, dx\end{aligned}$$

(b) Using the recursion formula we deduce that:

$$\begin{aligned}\int (\ln x)^3 \, dx &= x(\ln x)^3 - 3 \int (\ln x)^2 \, dx \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6 \int (\ln x) \, dx \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6 \int 1 \cdot dx \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6x + C.\end{aligned}$$

4. (15 marks) Determine whether the improper integral converges or diverges:

$$\int_1^\infty \frac{\ln x}{x^2} \, dx$$

Solution: First let us find an indefinite integral:

$$\begin{aligned}\int \frac{\ln x}{x^2} \, dx &= -\int \left(-\frac{1}{x^2}\right) \ln x \, dx = -\int \left(\frac{1}{x}\right)' \ln x \, dx \\ &= (\text{integration by parts}) = -\frac{1}{x}\ln x + \int \frac{1}{x} \cdot \frac{1}{x} \, dx = -\frac{\ln x}{x} - \frac{1}{x} + C.\end{aligned}$$

Now let us use the definition of an improper integral:

$$\int_1^\infty \frac{\ln x}{x^2} \, dx = \lim_{A \to \infty} \left(-\frac{\ln A}{A} - \frac{1}{A}\right) + \frac{\ln 1}{1} + \frac{1}{1} = 1.$$

Therefore the impoper integral is convergent.

5. (25 marks) Find the integral using a trigonometric substitution:

$$\int \frac{\sqrt{x^2+2}}{x}\,dx$$

Solution: The integral is of the type x^2+a^2 with $a=\sqrt{2}$. Therefore the trigonometric substitution is $x=\sqrt{2}\tan\theta$ with differential $dx=\sqrt{2}\sec^2\theta d\theta$.

$$\int \frac{\sqrt{x^2+2}}{x}\,dx = \int \frac{\sqrt{2}\sec\theta}{\sqrt{2}\tan\theta}\,\sqrt{2}\sec^2\theta\;d\theta = \sqrt{2}\int \frac{d\theta}{\sin\theta\cos^2\theta}.$$

Here we use the substitution $u=\cos\theta$ similarly to the way considered in Trigonometric Identities subsection.

$$\sqrt{2}\int \frac{d\theta}{\sin\theta\cos^2\theta} = -\sqrt{2}\int \frac{du}{\sin^2\theta\cos^2\theta} = -\sqrt{2}\int \frac{du}{u^2(1-u^2)}.$$

The latter integral is the integral of rational function. Let us use the method of partial fractions to find the coefficients A, B, C, D such that:

$$\begin{aligned}\frac{1}{u^2(1-u^2)} &= \frac{A}{u}+\frac{B}{u^2}+\frac{C}{1-u}+\frac{D}{1+u}\\ &= \frac{Au(1-u^2)+B(1-u^2)+Cu^2(1+u)+Du^2(1-u)}{u^2(1-u^2)}.\end{aligned}$$

Substitution of $u=0$ gives us $1=B\cdot 1$ and hence $B=1$. Similarly, we substitute $u=1$, $u=-1$ and $u=2$ to get:

$$\begin{aligned}&1 = 2\cdot C\\ &1 = 2\cdot D\\ &1 = A\cdot 2\cdot(1-4)+B\cdot(1-4)+C\cdot 4\cdot(1+2)+D\cdot 4\cdot(1-2)\end{aligned}$$

Therefore $A=0$, $B=1$, $C=1/2$, $D=1/2$. Hence our integral transforms to:

$$\begin{aligned}-\sqrt{2}\int \frac{du}{u^2(1-u^2)} &= -\sqrt{2}\int\left(\frac{0}{u}+\frac{1}{u^2}+\frac{1/2}{1-u}+\frac{1/2}{1+u}\right)du\\ &= \frac{\sqrt{2}}{u}+\frac{\sqrt{2}}{2}\ln|u-1|-\frac{\sqrt{2}}{2}\ln|u+1|+C.\end{aligned}$$

And, finally, to express the final answer in terms of original variable x we notice that in a right triangle with sides x, $\sqrt{2}$ and $\sqrt{x^2+2}$ the tangent is given by $\tan\theta = x/\sqrt{2}$ and $\cos\theta = \sqrt{2}/\sqrt{x^2+2}$. That is $u=\cos\theta=\sqrt{2}/\sqrt{x^2+2}$. We substitute it into the expression for the integral to get:

$$\int \frac{\sqrt{x^2+2}}{x}\,dx = \sqrt{x^2+2}+\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}}{\sqrt{x^2+2}}-1\right|-\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}}{\sqrt{x^2+2}}+1\right|+C.$$

6. (15 marks) Find the integral using the method of partial fractions:

$$\int \frac{x^2 + x - 1}{x^4 - 1}\, dx$$

<u>Solution:</u> By the method of partial fractions:

$$\begin{aligned}
\frac{x^2 + x - 1}{x^4 - 1} &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx + D}{x^2 + 1} \\
&= \frac{A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)}{x^4 - 1} \\
&= \frac{(A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + (A-B-D)}{x^4 - 1}
\end{aligned}$$

Comparing the coefficients of the corresponding powers of x we get:

$$\begin{aligned}
x^3 &: \quad 0 = A + B + C \\
x^2 &: \quad 1 = A - B + D \\
x^1 &: \quad 1 = A + B - C \\
1 &: \quad -1 = A - B - D
\end{aligned}$$

This system of equations has a solution $A = 1/4$, $B = 1/4$, $C = -1/2$ and $D = 1$. Therefore:

$$\begin{aligned}
\int \frac{x^2 + x - 1}{x^4 - 1}\, dx &= \int \left(\frac{1/4}{x-1} + \frac{1/4}{x+1} + \frac{(-1/2)x + 1}{x^2 + 1} \right) dx \\
&= \frac{1}{4} \int \frac{dx}{x-1} + \frac{1}{4} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{x}{x^2+1}\, dx + \int \frac{dx}{x^2+1}\, dx \\
&= \frac{1}{4} \ln|x-1| + \frac{1}{4} \ln|x+1| - \frac{1}{4} \ln|x^2+1| + \arctan x + C.
\end{aligned}$$