PROOF OF (9.29) IN STEPS

I am including the proof of (9.29) as part exercise because the book says the proof can be done by modifying the short proof that it provides. As far as I can see it isn't that easy. Lemma 2 was a homework problem for the review questions for the final so consider this a solution to that problem. Make sure that you can fill in the reasons for each of the following steps of these proofs.

P.S. A 'Lemma' is another word for Theorem.

Lemma 1:

$$(\exists x| : true) \equiv true$$

$$\begin{aligned} & true \Rightarrow (\exists x| : true) - \text{(9.28) with } P := true \\ = & \\ & true \vee (\exists x| : true) \equiv (\exists x| : true) \\ = & \\ & true \equiv (\exists x| : true) \end{aligned}$$

Lemma 2: Assume that x does not occur in Q, then

$$(\exists x| : Q \vee P) \equiv Q \vee (\exists x| : P)$$

$$\begin{aligned} & (\exists | : true) \Rightarrow ((\exists x| : Q \vee P) \equiv Q \vee (\exists x| : P)) - \text{ this is (9.23)} \\ = & \\ & true \Rightarrow ((\exists x| : Q \vee P) \equiv Q \vee (\exists x| : P)) \\ = & \\ & (\exists x| : Q \vee P) \equiv Q \vee (\exists x| : P) \end{aligned}$$

Lemma 3: If y does not occur in R_x, then

$$(\exists x | R_x : (\forall y | : P_{xy})) \Rightarrow (\forall y | : (\exists x | R_x : P_{xy}))$$

Proof:

$$(\exists x | R_x : (\forall y | : P_{xy})) \Rightarrow (\forall y | : (\exists x | R_x : P_{xy}))$$

$$=$$

$$(\exists x | R_x : (\forall y | : P_{xy})) \vee (\forall y | : (\exists x | R_x : P_{xy})) \equiv (\forall y | : (\exists x | R_x : P_{xy}))$$

$$=$$

$$(\forall y | : (\exists x | R_x : (\forall y | : P_{xy})) \vee (\exists x | R_x : P_{xy})) \equiv (\forall y | : (\exists x | R_x : P_{xy}))$$

$$=$$

$$(\forall y | : (\exists x | R_x : P_{xy} \vee (\forall y | : P_{xy}))) \equiv (\forall y | : (\exists x | R_x : P_{xy}))$$

$= <$ (9.13) $(\forall y | : P_{xy}) \Rightarrow P_{xy}$ which is equivalent to $(\forall y | : P_{xy}) \vee P_{xy} \equiv P_{xy} >$

$$(\forall y | : (\exists x | R_x : P_{xy})) \equiv (\forall y | : (\exists x | R_x : P_{xy}))$$

Theorem (9.29): If y does not occur in R_x and x does not occur in Q_y, then

$$(\exists x | R_x : (\forall y | Q_y : P_{xy})) \Rightarrow (\forall y | Q_y : (\exists x | R_x : P_{xy}))$$

Proof:

$$(\exists x | R_x : (\forall y | Q_y : P_{xy}))$$

$$=$$

$$(\exists x | R_x : (\forall y | : \neg Q_y \vee P_{xy}))$$

$$\Rightarrow$$

$$(\forall y | : (\exists x | R_x : \neg Q_y \vee P_{xy}))$$

$$=$$

$$(\forall y | : \neg Q_y \vee (\exists x | R_x : P_{xy}))$$

$$=$$

$$(\forall y | Q_y : (\exists x | R_x : P_{xy}))$$