SOLUTIONS FOR THE LAST HOMEWORK

I am not going to provide a solution to problems (4) and (5) since you should have those in your notes. Also, the solution to problem (9) is on the two page proof of (9.29) (but you will need to fill in the reasons for that proof).

(1) (a) $p \wedge (p \Rightarrow q) \Rightarrow (p \Rightarrow q)$

This one is a tautology. In fact, it is an instance of (3.76) (b) and so it is already a theorem.

(b) $p \wedge (q \Rightarrow p) \Rightarrow (p \Rightarrow q)$

This one is not a tautology. Let $p = true$ and $q = false$. Then, $p \wedge (q \Rightarrow p) \equiv true$ and $p \Rightarrow q \equiv false$. Therefore $(p \wedge (q \Rightarrow p) \Rightarrow (p \Rightarrow q)) \equiv (true \Rightarrow false) \equiv false$.

(2) Since $P \Rightarrow Q$ is a theorem then $P \vee Q \equiv Q$ is a theorem by (3.57), and since $Q \Rightarrow R$ is a theorem then $Q \vee R \equiv R$ is a theorem.

$$
\begin{aligned}
& R \\
= &< \text{ since } Q \vee R \equiv R > \\
& Q \vee R \\
= &< \text{ since } P \vee Q \equiv Q > \\
& P \vee Q \vee R \\
= &< \text{ since } Q \vee R \equiv R > \\
& P \vee R
\end{aligned}
$$

therefore $R \equiv P \vee R$ is a theorem and by (3.57) this is equivalent to $P \Rightarrow R$.

(3) Assume that $P \wedge Q$ is a theorem, then $P \wedge Q \equiv true$ is a theorem by (3.3).

$$
\begin{aligned}
& P \\
= &< (3.39) > \\
& P \wedge true \\
= &< P \wedge Q \equiv true > \\
& P \wedge P \wedge Q \\
= &< (3.38) > \\
& P \wedge Q
\end{aligned}
$$

Therefore P is a theorem since it is equivalent to $P \wedge Q$.

(4) We did this one in class

(5) We did this one in class

(6) All counter-examples to this question will be given in the domain of integers.

(a) $(\forall x| : Px \Rightarrow Qx) \Rightarrow (Px \Rightarrow (\forall y| : Qy))$ - $Px := (x = 0)$, $Qx := (x = 0)$, and the free variable $x := 0$. This statement is $false$ in this case.

(b) $(\forall y| : Py \Rightarrow Qy) \Rightarrow (Qx \Rightarrow (\forall y| : Qy))$ - $Px := (x = 0)$, $Qx := (x = 0)$, $x := 0$. This statement is $false$ in this case.

(c) $(Px \Rightarrow Qx) \Rightarrow ((\exists y| : Py) \Rightarrow (\exists y| : Qy))$ - $Px := x = 0$, $Qx := false$, $x := 1$. The statement is $false$ in this case.

(d) $(Px \equiv Qx) \Rightarrow ((\exists y| : Py) \equiv (\exists y| : Qy))$ - $Px := x = 0$, $Qx := false$, $x := 1$. The statement is $false$ in this case.

(e) $(Px \Rightarrow Qx) \Rightarrow ((\exists y| : Py) \Rightarrow (\exists y| : Qy))$ - this is the same as (c)

(f) $(Px \equiv Qx) \Rightarrow ((\exists y| : Py) \equiv (\exists y| : Qy))$ -this is the same as (d)

(g) $(Px \Rightarrow Qx) \Rightarrow (Px \Rightarrow (\exists y| : Qy))$ - this one is a tautology and a proof is given below.

(h) $(Px \Rightarrow Qx) \Rightarrow ((\forall y| : Py) \Rightarrow (\forall y| : Qy))$ - $Px := true$, $Qx := x = 1$, $x := 1$. The statement is $false$ in this case.

To prove $(Px \Rightarrow Qx) \Rightarrow (Px \Rightarrow (\exists y| : Qy))$, we assume that $(Px \Rightarrow Qx)$ is a theorem.

$$
\begin{array}{c}
Px \\
\Rightarrow < assumption > \\
Qx \\
= < (9.28) > \\
(\exists y| : Qy)
\end{array}
$$

Therefore $(Px \Rightarrow Qx) \Rightarrow (Px \Rightarrow (\exists y| : Qy))$.

(7) (a)

$$(+x, y|((x = 3) \vee (x = 4) \vee (x = 5)) \wedge (y = x - 1) : y^2) = 2^2 + 3^2 + 4^2$$

$$
\begin{array}{l}
\quad (+x, y|((x = 3) \vee (x = 4) \vee (x = 5)) \wedge (y = x - 1) : y^2) \\
= < (8.20) > \\
\quad (+x|((x = 3) \vee (x = 4) \vee (x = 5)) : (+y|(y = x - 1) : y^2)) \\
= < (8.14) > \\
\quad (+x|((x = 3) \vee (x = 4) \vee (x = 5)) : (x - 1)^2) \\
= < (8.16) > \\
\quad (+x|((x = 3) \vee (x = 4)) : (x - 1)^2) + (+x|(x = 5) : (x - 1)^2) \\
= < (8.16) > \\
\quad (+x|(x = 3) : (x - 1)^2) + (+x|(x = 4) : (x - 1)^2) + (+x|(x = 5) : (x - 1)^2) \\
= < (8.14)\ 3\ times > \\
\quad 2^2 + 3^2 + 4^2
\end{array}
$$

$$(\cdot x, y|((x = 5) \vee (y = 2)) \wedge (y = x^2) : y^2) = 5^4 \cdot 2^2$$

$$\begin{aligned}
&(\cdot x, y|((x = 5) \vee (y = 2)) \wedge (y = x^2) : y^2) \\
= &< (3.84)\ (a) > \\
&(\cdot x, y|((x = 5) \vee (x^2 = 2)) \wedge (y = x^2) : y^2) \\
= &< (8.20) > \\
&(\cdot x|((x = 5) \vee (x^2 = 2)) : (\cdot y|(y = x^2) : y^2)) \\
= &< (8.14) > \\
&(\cdot x|((x = 5) \vee (x^2 = 2)) : (x^2)^2) \\
= &< (8.16) > \\
&(\cdot x|(x = 5) : (x^2)^2) \cdot (\cdot x|(x^2 = 2) : (x^2)^2) \\
= &< (8.14) > \\
&(5^2)^2 \cdot 2^2
\end{aligned}$$

(8) Let P_x be a statement that depends on the variable x. Prove that

$$(\forall x| : \neg P_x \vee (x \neq y)) \equiv \neg P_y$$

$$\begin{aligned}
&(\forall x| : \neg P_x \vee (x \neq y)) \\
= &< (9.3)\ (a) > \\
&(\forall x|\neg(x \neq y) : \neg P_x) \\
= &< arithmetic > \\
&(\forall x|x = y : \neg P_x) \\
= &< (8.14) > \\
&\neg P_y
\end{aligned}$$

(9) See the file on the proof of theorem (9.29).