York University

Faculty of Arts, Faculty of Science

Math 1090

Midterm Test 2 Version 2

SOLUTIONS

Instructions:

1. There are 5 questions on 4 pages.
2. Answer all questions.
3. Your work must justify the answer you give.

Question	**Points**	**Marks**
1	7	
2	6	
3	4	
3	6	
4	7	
Total	30	

1. (7 points) Use the Deduction Theorem (Method of assuming the antecedent) to prove

$$\vdash (p \Rightarrow q) \quad \Rightarrow \quad (p \vee r \ \Rightarrow q \vee r) \ .$$

Answer: Assume $p \Rightarrow q$ from which one obtains by (3.57) that $p \vee q \ \equiv \ q$ is a temporary theorem. Now

$$\begin{array}{ll} & p \vee r \ \Rightarrow q \vee r \\ = & \langle \ (3.57) \ \rangle \\ & p \vee r \vee q \vee r \ \equiv \ q \vee r \\ = & \langle \ (3.24), (3.25), (3.26) \ \rangle \\ & p \vee q \vee r \ \equiv \ q \vee r \\ = & \langle \ \text{Temporary theorem}, p \vee q \ \equiv \ q \ \rangle \\ & q \vee r \ \equiv \ q \vee r \end{array}$$

2. (6 points) Prove

$$\vdash (\forall y \mid y^3 - x = 3 \vee y = 5 : y = x) \quad \Rightarrow \quad x = 5\ .$$

Answer:

$$\begin{array}{ll} & (\forall y \mid y^3 - x = 3 \vee y = 5 : y = x) \\ = & \langle\ (8.18)\ \rangle \\ & (\forall y \mid y^3 - x = 3 : y = x) \ \wedge\ (\forall y \mid y = 5 : y = x) \\ = & \langle\ (8.14),\ y \text{ d.n.o.f. } 5\ \rangle \\ & (\forall y \mid y^3 - x = 3 : y = x) \ \wedge\ (5 = x) \\ \Rightarrow & \langle\ (3.76)(\text{b})\ \rangle \\ & 5 = x \\ = & \langle\ (1.3)\ \rangle \\ & x = 5 \end{array}$$

3. (4 points) Prove (9.26),

$$\vdash (\exists x \mid R : P) \quad \Rightarrow \quad (\exists x \mid R : P \vee Q)\ .$$

You may not use any Theorems from Chapter 9 in your proof.
Answer:

$$\begin{array}{ll} & (\exists x \mid R : P) \quad \Rightarrow \quad (\exists x \mid R : P \vee Q) \\ = & \langle\ (8.15)\ \rangle \\ & (\exists x \mid R : P) \quad \Rightarrow \quad (\exists x \mid R : P) \ \vee\ (\exists x \mid R : Q) \end{array}$$

4. (6 points) Prove that

$$(\exists x \mid \,:\, Qx \vee Px) \quad \Rightarrow \quad (\exists x \mid \,:\, Qx) \;\vee\; Px$$

cannot be a theorem by finding an interpretation (state) for which it is false. Use $\{0,1\}$ as universe of discourse and indicate why your answer is correct.
Answer: We need an interpretation where $(\exists x \mid \,:\, Qx \vee Px)$ is T and $(\exists x \mid \,:\, Qx) \;\vee\; Px$ is F.
Take $\{0,1\}$ as universe of discourse.
Take Px to be "$x = 1$" and Qx to be "$false$".
Assign the value 0 to the free occurrence of x.
We can see that $(\exists x \mid \,:\, Qx \vee Px)$ is T as $false \vee P1$ is T.
As $P0$ is F and $(\exists x \mid \,:\, Qx)$ is F, $(\exists x \mid \,:\, Qx) \;\vee\; Px$ is F.

5. (7 points) Prove that

$$\vdash (+i \,|\, 0 \le i \le n : i) = (+j \,|\, 0 \le j \le n : n - j) \ .$$

Hint: $(+j \,|\, 0 \le j \le n : n - j)$ is $(+j \,|\, 0 \le j \le n : i[i := n - j])$.
Answer:

$$
\begin{array}{cl}
 & (+j \,|\, 0 \le j \le n : n - j) \\
= & \langle \ (8.14),\ i \text{ d.n.o.f. } n - j \ \rangle \\
 & (+j \,|\, 0 \le j \le n : (+i \,|\, i = n - j : i)) \\
= & \langle \ (8.20),\ i \text{ d.n.o.f. } 0 \le j \le n \ \rangle \\
 & (+j, i \,|\, 0 \le j \le n \ \wedge\ i = n - j : i) \\
= & \langle \ \vdash (\star j, i | R : P) = (\star i, j | R : P),\ \vdash\ i = n - j \ \equiv\ j = n - i \ \rangle \\
 & (+i, j \,|\, 0 \le j \le n \ \wedge\ j = n - i : i) \\
= & \langle \ (3.84)(\text{b}) \ \rangle \\
 & (+i, j \,|\, 0 \le i - n \le n \ \wedge\ j = n - i : i) \\
= & \langle \ (8.20).\ j \text{ d.n.o.f. } 0 \le i - n \le n \ \rangle \\
 & (+i \,|\, 0 \le i - n \le n : (+j \,|\, j = n - i : i)) \\
= & \langle \ (8.14),\ j \text{ d.n.o.f. } n - i \ \rangle \\
 & (+i \,|\, 0 \le n - i \le n : i) \\
= & \langle \ \vdash\ 0 \le n - i \le n \ \equiv\ 0 \le i \le n \ \rangle \\
 & (+i \,|\, 0 \le i \le n : i)
\end{array}
$$

The end