

## THEOREMS OF THE PROPOSITIONAL CALCULUS

### EQUIVALENCE AND TRUE

- (3.1) **Axiom, Associativity of  $\equiv$ :**  $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$   
(3.2) **Axiom, Symmetry of  $\equiv$ :**  $p \equiv q \equiv q \equiv p$   
(3.3) **Axiom, Identity of  $\equiv$ :**  $true \equiv q \equiv q$   
(3.4) *true*  
(3.5) **Reflexivity of  $\equiv$ :**  $p \equiv p$

### NEGATION, INEQUIVALENCE, AND FALSE

- (3.6) **Axiom, Definition of false:**  $false \equiv \neg true$   
(3.9) **Axiom, Distributivity of  $\neg$  over  $\equiv$ :**  $\neg(p \equiv q) \equiv \neg p \equiv q$   
(3.10) **Axiom, Definition of  $\neq$ :**  $(p \neq q) \equiv \neg(p \equiv q)$   
(3.11)  $\neg p \equiv q \equiv p \equiv \neg q$   
(3.12) **Double negation:**  $\neg\neg p \equiv p$   
(3.13) **Negation of false:**  $\neg false \equiv true$   
(3.14)  $(p \neq q) \equiv \neg p \equiv q$   
(3.15)  $\neg p \equiv p \equiv false$   
(3.16) **Symmetry of  $\neq$ :**  $(p \neq q) \equiv (q \neq p)$   
(3.17) **Associativity of  $\neq$ :**  $((p \neq q) \neq r) \equiv (p \neq (q \neq r))$   
(3.18) **Mutual associativity:**  $((p \neq q) \equiv r) \equiv (p \neq (q \equiv r))$   
(3.19) **Mutual interchangeability:**  $p \neq q \equiv r \equiv p \equiv q \neq r$

### DISJUNCTION

- (3.24) **Axiom, Symmetry of  $\vee$ :**  $p \vee q \equiv q \vee p$   
(3.25) **Axiom, Associativity of  $\vee$ :**  $(p \vee q) \vee r \equiv p \vee (q \vee r)$   
(3.26) **Axiom, Idempotency of  $\vee$ :**  $p \vee p \equiv p$   
(3.27) **Axiom, Distributivity of  $\vee$  over  $\equiv$ :**  $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$   
(3.28) **Axiom, Excluded Middle:**  $p \vee \neg p$   
(3.29) **Zero of  $\vee$ :**  $p \vee true \equiv true$   
(3.30) **Identity of  $\vee$ :**  $p \vee false \equiv p$   
(3.31) **Distributivity of  $\vee$  over  $\vee$ :**  $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$   
(3.32)  $p \vee q \equiv p \vee \neg q \equiv p$

### CONJUNCTION

- (3.35) **Axiom, Golden rule:**  $p \wedge q \equiv p \equiv q \equiv p \vee q$   
(3.36) **Symmetry of  $\wedge$ :**  $p \wedge q \equiv q \wedge p$

- (3.37) **Associativity of  $\wedge$ :**  $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$   
(3.38) **Idempotency of  $\wedge$ :**  $p \wedge p \equiv p$   
(3.39) **Identity of  $\wedge$ :**  $p \wedge true \equiv p$   
(3.40) **Zero of  $\wedge$ :**  $p \wedge false \equiv false$   
(3.41) **Distributivity of  $\wedge$  over  $\wedge$ :**  $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$   
(3.42) **Contradiction:**  $p \wedge \neg p \equiv false$   
(3.43) **Absorption:** (a)  $p \wedge (p \vee q) \equiv p$   
(b)  $p \vee (p \wedge q) \equiv p$   
(3.44) **Absorption:** (a)  $p \wedge (\neg p \vee q) \equiv p \wedge q$   
(b)  $p \vee (\neg p \wedge q) \equiv p \vee q$

- (3.45) **Distributivity of  $\vee$  over  $\wedge$ :**  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$   
(3.46) **Distributivity of  $\wedge$  over  $\vee$ :**  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$   
(3.47) **De Morgan:** (a)  $\neg(p \wedge q) \equiv \neg p \vee \neg q$   
(b)  $\neg(p \vee q) \equiv \neg p \wedge \neg q$   
(3.48)  $p \wedge q \equiv p \wedge \neg q \equiv \neg p$   
(3.49)  $p \wedge (q \equiv r) \equiv p \wedge q \equiv p \wedge r$   
(3.50)  $p \wedge (q \equiv p) \equiv p \wedge q$   
(3.51) **Replacement:**  $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$   
(3.52) **Definition of  $\equiv$ :**  $p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$   
(3.53) **Exclusive or:**  $p \neq q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$   
(3.55)  $(p \wedge q) \wedge r \equiv p \equiv q \equiv r \equiv p \vee q \equiv q \vee r \equiv r \vee p \equiv p \vee q \vee r$

### IMPLICATION

- (3.57) **Axiom, Definition of Implication:**  $p \Rightarrow q \equiv p \vee q \equiv q$   
(3.58) **Axiom, Consequence:**  $p \leftarrow q \equiv q \Rightarrow p$   
(3.59) **Definition of implication:**  $p \Rightarrow q \equiv \neg p \vee q$   
(3.60) **Definition of implication:**  $p \Rightarrow q \equiv p \wedge q \equiv p$   
(3.61) **Contrapositive:**  $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$   
(3.62)  $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$   
(3.63) **Distributivity of  $\Rightarrow$  over  $\equiv$ :**  $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$   
(3.64)  $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$   
(3.65) **Shunting:**  $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$   
(3.66)  $p \wedge (p \Rightarrow q) \equiv p \wedge q$   
(3.67)  $p \wedge (q \Rightarrow p) \equiv p$   
(3.68)  $p \vee (p \Rightarrow q) \equiv true$   
(3.69)  $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$

- (3.70)  $p \vee q \Rightarrow p \wedge q \equiv p \equiv q$   
(3.71) **Reflexivity of  $\Rightarrow$ :**  $p \Rightarrow p \equiv true$   
(3.72) **Right zero of  $\Rightarrow$ :**  $p \Rightarrow true \equiv true$   
(3.73) **Left identity of  $\Rightarrow$ :**  $true \Rightarrow p \equiv p$   
(3.74)  $p \Rightarrow false \equiv \neg p$   
(3.75)  $false \Rightarrow p \equiv true$   
(3.76) **Weakening/strengthening:** (a)  $p \Rightarrow p \vee q$   
(b)  $p \wedge q \Rightarrow p$   
(c)  $p \wedge q \Rightarrow p \vee q$   
(d)  $p \vee (q \wedge r) \Rightarrow p \vee q$   
(e)  $p \wedge q \Rightarrow p \wedge (q \vee r)$

- (3.77) **Modus ponens:**  $p \wedge (p \Rightarrow q) \Rightarrow q$   
(3.78)  $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$   
(3.79)  $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$   
(3.80) **Mutual implication:**  $(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \equiv q)$   
(3.81) **Antisymmetry:**  $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow (p \equiv q)$   
(3.82) **Transitivity:** (a)  $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$   
(b)  $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$   
(c)  $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

### LEIBNIZ AS AN AXIOM

- (3.83) **Axiom, Leibniz:**  $e = f \Rightarrow E_e^x = E_f^x$   
(3.84) **Substitution:** (a)  $(e = f) \wedge E_e^x \equiv (e = f) \wedge E_f^x$   
(b)  $(e = f) \Rightarrow E_e^x \equiv (e = f) \Rightarrow E_f^x$   
(c)  $q \wedge (e = f) \Rightarrow E_e^x \equiv q \wedge (e = f) \Rightarrow E_f^x$   
(3.85) **Replace by true:** (a)  $p \Rightarrow E_p^x \equiv p \Rightarrow E_{true}^x$   
(b)  $q \wedge p \Rightarrow E_p^x \equiv q \wedge p \Rightarrow E_{true}^x$   
(3.86) **Replace by false:** (a)  $E_p^x \Rightarrow p \equiv E_{false}^x \Rightarrow p$   
(b)  $E_p^x \Rightarrow p \vee q \equiv E_{false}^x \Rightarrow p \vee q$   
(3.87) **Replace by true:**  $p \wedge E_p^x \equiv p \wedge E_{true}^x$   
(3.88) **Replace by false:**  $p \vee E_p^x \equiv p \vee E_{false}^x$   
(3.89) **Shannon:**  $E_p^x \equiv (p \wedge E_{true}^x) \vee (\neg p \wedge E_{false}^x)$   
(4.1)  $p \Rightarrow (q \Rightarrow p)$   
(4.2) **Monotonicity of  $\vee$ :**  $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$   
(4.3) **Monotonicity of  $\wedge$ :**  $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$

## PROOF TECHNIQUES

- (4.4) **Deduction:** To prove  $P \Rightarrow Q$ , assume  $P$  and prove  $Q$ .
- (4.5) **Case analysis:** If  $E_{true}^P$ ,  $E_{false}^P$  are theorems, then so is  $E_P^P$ .
- (4.6) **Case analysis:**  $(p \vee q \vee r) \wedge (p \Rightarrow s) \wedge (q \Rightarrow s) \wedge (r \Rightarrow s) \Rightarrow s$
- (4.7) **Mutual implication:** To prove  $P \equiv Q$ , prove  $P \Rightarrow Q$  and  $Q \Rightarrow P$ .
- (4.9) **Proof by contradiction:** To prove  $P$ , prove  $\neg P \Rightarrow false$ .
- (4.12) **Proof by contrapositive:** To prove  $P \Rightarrow Q$ , prove  $\neg Q \Rightarrow \neg P$

## GENERAL LAWS OF QUANTIFICATION

For symmetric and associative binary operator  $\star$  with identity  $u$ .

- (8.13) **Axiom, Empty range:**  $(\star x \mid false : P) = u$
- (8.14) **Axiom, One-point rule:** Provided  $\neg occurs('x', 'E')$ ,  
 $(\star x \mid x = E : P) = P[x := E]$
- (8.15) **Axiom, Distributivity:** Provided each quantification is defined,  
 $(\star x \mid R : P) \star (\star x \mid R : Q) = (\star x \mid R : P \star Q)$
- (8.16) **Axiom, Range split:** Provided  $R \wedge S \equiv false$  and each quantification is defined,  
 $(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$
- (8.17) **Axiom, Range split:** Provided each quantification is defined,  
 $(\star x \mid R \vee S : P) \star (\star x \mid R \wedge S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$
- (8.18) **Axiom, Range split for idempotent  $\star$ :** Prov. each quant. is defined,  
 $(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$
- (8.19) **Axiom, Interchange of dummies:** Provided each quantification is defined,  $\neg occurs('y', 'R')$ , and  $\neg occurs('x', 'Q')$ ,  
 $(\star x \mid R : (\star y \mid Q : P)) = (\star y \mid Q : (\star x \mid R : P))$
- (8.20) **Axiom, Nesting:** Provided  $\neg occurs('y', 'R')$ ,  
 $(\star x, y \mid R \wedge Q : P) = (\star x \mid R : (\star y \mid Q : P))$
- (8.21) **Axiom, Dummy renaming:** Provided  $\neg occurs('y', 'R, P')$ ,  
 $(\star x \mid R : P) = (\star y \mid R[x := y] : P[x := y])$
- (8.22) **Change of dummy:** Provided  $\neg occurs('y', 'R, P')$ , and  $f$  has an inverse,  $(\star x \mid R : P) = (\star y \mid R[x := f, y] : P[x := f, y])$
- (8.23) **Split off term:**  $(\star i \mid 0 \leq i < n + 1 : P) = (\star i \mid 0 \leq i < n : P) \star P_n^i$

## THEOREMS OF THE PREDICATE CALCULUS

### UNIVERSAL QUANTIFICATION

- (9.2) **Axiom, Trading:**  $(\forall x \mid R : P) \equiv (\forall x \mid \neg R \Rightarrow P)$
- (9.3) **Trading:** (a)  $(\forall x \mid R : P) \equiv (\forall x \mid \neg R \vee P)$   
 (b)  $(\forall x \mid R : P) \equiv (\forall x \mid R \wedge P \equiv R)$   
 (c)  $(\forall x \mid R : P) \equiv (\forall x \mid \neg R \vee P \equiv P)$

- (9.4) **Trading:** (a)  $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \Rightarrow P)$   
 (b)  $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : \neg R \vee P)$   
 (c)  $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \wedge P \equiv R)$   
 (d)  $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \vee P \equiv P)$
- (9.5) **Axiom, Distributivity of  $\vee$  over  $\forall$ :** Prov.  $\neg occurs('x', 'P')$ ,  
 $P \vee (\forall x \mid R : Q) \equiv (\forall x \mid R : P \vee Q)$

- (9.6) Provided  $\neg occurs('x', 'P')$ ,  $(\forall x \mid R : P) \equiv P \vee (\forall x \mid \neg R)$
- (9.7) **Distributivity of  $\wedge$  over  $\forall$ :** Provided  $\neg occurs('x', 'P')$ ,  
 $\neg(\forall x \mid \neg R) \Rightarrow ((\forall x \mid R : P \wedge Q) \equiv P \wedge (\forall x \mid R : Q))$
- (9.8)  $(\forall x \mid R : true) \equiv true$
- (9.9)  $(\forall x \mid R : P \equiv Q) \Rightarrow ((\forall x \mid R : P) \equiv (\forall x \mid R : Q))$
- (9.10) **Range weakening/strengthening:**  $(\forall x \mid Q \vee R : P) \Rightarrow (\forall x \mid Q : P)$
- (9.11) **Body weakening/strengthening:**  $(\forall x \mid R : P \wedge Q) \Rightarrow (\forall x \mid R : P)$
- (9.12) **Monotonicity of  $\forall$ :**  
 $(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\forall x \mid R : Q) \Rightarrow (\forall x \mid R : P))$
- (9.13) **Instantiation:**  $(\forall x \mid \cdot : P) \Rightarrow P[x := e]$
- (9.16)  $P$  is a theorem iff  $(\forall x \mid \cdot : P)$  is a theorem.

### EXISTENTIAL QUANTIFICATION

- (9.17) **Axiom, Generalized De Morgan:**  
 $(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$
- (9.18) **Generalized De Morgan:** (a)  $\neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P)$   
 (b)  $\neg(\exists x \mid R : P) \equiv (\forall x \mid R : \neg P)$   
 (c)  $(\exists x \mid R : \neg P) \equiv \neg(\forall x \mid R : P)$
- (9.19) **Trading:**  $(\exists x \mid R : P) \equiv (\exists x \mid \neg R \wedge P)$
- (9.20) **Trading:**  $(\exists x \mid Q \wedge R : P) \equiv (\exists x \mid Q : R \wedge P)$
- (9.21) **Distributivity of  $\wedge$  over  $\exists$ :** Provided  $\neg occurs('x', 'P')$ ,  
 $P \wedge (\exists x \mid R : Q) \equiv (\exists x \mid R : P \wedge Q)$
- (9.22) Provided  $\neg occurs('x', 'P')$ ,  $(\exists x \mid R : P) \equiv P \wedge (\exists x \mid R)$
- (9.23) **Distributivity of  $\vee$  over  $\exists$ :** Provided  $\neg occurs('x', 'P')$ ,  
 $(\exists x \mid R) \Rightarrow ((\exists x \mid R : P \vee Q) \equiv P \vee (\exists x \mid R : Q))$
- (9.24)  $(\exists x \mid R : false) \equiv false$
- (9.25) **Range weakening/strengthening:**  $(\exists x \mid R : P) \Rightarrow (\exists x \mid Q \vee R : P)$
- (9.26) **Body weakening/strengthening:**  $(\exists x \mid R : P) \Rightarrow (\exists x \mid R : P \vee Q)$
- (9.27) **Monotonicity of  $\exists$ :**  
 $(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\exists x \mid R : Q) \Rightarrow (\exists x \mid R : P))$
- (9.28)  **$\exists$ -Introduction:**  $P[x := E] \Rightarrow (\exists x \mid \cdot : P)$
- (9.29) **Interchange of quantifications:**  
 Provided  $\neg occurs('y', 'R')$  and  $\neg occurs('x', 'Q')$ ,  
 $(\exists x \mid R : (\forall y \mid Q : P)) \Rightarrow (\forall y \mid Q : (\exists x \mid R : P))$
- (9.30) Provided  $\neg occurs('x', 'Q')$ ,  
 $(\exists x \mid R : P) \Rightarrow Q$  is a theorem iff  $(R \wedge P)[x := \hat{x}] \Rightarrow Q$  is a theorem